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Automorphic Forms, Shimura Varieties, and L-functions

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Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6–16, 1988

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FOREWORD

In 1977, the AMS Summer Research Institute was devoted to "Automorphic Forms, Representations, and L-Functions". One of its central topics was the relation between automorphic forms (in their modern guise as automorphic forms on adèle groups) and various objects arising from algebraic geometry, most notably the Hasse-Weil zeta functions of varieties, Galois representations, and Grothendieck's motives. These conjectural relations had been explored by Shimura and others, but Langlands had formulated a systematic program to study them for Shimura varieties. At the time of the conference, Deligne and Langlands stated several fundamental conjectures concerning Shimura varieties, Galois representations, and L-functions.

The decade following the conference saw substantial progress on many of these problems, and the conference was organized in Ann Arbor in 1988 to review this progress and to explore new avenues of research and new questions. In the theory of automorphic forms, advances have been made in the study of the Arthur-Selberg trace formula, the analytic properties of automorphic L-functions including in some cases their analytic continuation, Langlands's functoriality principle including its proof in some important instances, the structure and properties of the discrete spectrum for classical groups, and the p-adic interpolation of certain L-functions. Moreover the baffling problems raised by L-indistinguishability are better understood. As regards Shimura varieties, the basic conjecture of Shimura and Deligne about their canonical models has been proved in the strenthened form conjectured by Langlands at the Corvallis conference, and a combinatorial conjecture of Langlands, allowing one to express their local zeta functions in automorphic terms, proved in some cases. Certain questions that arise in the comparison of the ℓ-adic representations associated with Shimura varieties and automorphic forms (Zucker's conjecture, formulas for the traces of Hecke operators in L^2- cohomology spaces) have been solved. Important arithmetic consequences of the theory of automorphic forms and the functoriality principle have been obtained or seem more accessible, for example, the construction of the Galois representations associated with Maass forms and proofs of the Tate conjecture for certain arithmetic varieties. There has also been progress in the study of the local zeta function of a Shimura variety at a prime of bad reduction. Finally, starting with Drinfeld, the analogues of these problems have been studied for function fields.

The articles in these Proceedings, which are expansions of the lectures given at the conference, are intended to reflect these advances. They are divided, in a somewhat arbitrary manner, between two volumes. The first volume contains expository articles on the trace formula (Labesse) and on the progress since Corvallis in understanding the analytic properties of L-functions (Shahidi). The articles of Milne and Clozel develop two different aspects of Langlands's paper at Corvallis: while Milne's article explains results on Langlands's conjecture on conjugates of Shimura varieties and how they should extend to holomorphic automorphic forms and mixed Shimura varieties, that of Clozel takes up the more speculative question of defining a category of automorphic representations that has the structure of a Tannakian category. The article of Laumon is concerned with finding a geometric interpretation for certain Eisenstein series in the function field case. The papers of Arthur and Kottwitz concern, *inter alia,* the conjectural Hecke-Galois relations for Shimura varieties in the most general case; Kottwitz's paper also includes a conjectural description of the number of points on a Shimura variety over a finite field.

The second volume contains papers on Galois representations associated with automorphic forms (Blasius, Carayol, Taylor); bad reduction of Shimura varieties (Rapoport); higher L-functions (Jacquet-Shalika); coherent cohomology and automorphic forms (Harris); a Lefschetz trace formula, conjectured by Deligne, of importance for zeta functions of Shimura varieties (Zink); the conjectures of Tate and Beilinson in the context of Shimura varieties and a review of the progress that has been made on them and related questions (Ramakrishnan); the p-adic L-functions associated with Shimura convolutions (Hida); and finally, the proof of the Zucker conjecture (Zucker).

The conference was supported by generous grants from the National Science Foundation through the Presidential Young Investigator and Special Projects programs. We are indebted to the Mathematics Department of the University of Michigan, and especially Don Lewis, for

its assistance, and to the School of Business Administration for providing us with an air-conditioned auditorium during one of the hottest spells of the century in Ann Arbor. The manuscripts not submitted in TeX were TeX-ed by Steve Tinney and Chris Weider, and Steve Tinney had the difficult task of preparing a uniform manuscript for the publisher from files in submitted in every known dialect of TeX. Finally, the nonmathematical organization of the conference would not have been possible without the exceptional efforts of Lee Zukowski.

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CONTENTS OF VOLUME II

Unipotent Automorphic Representations: Global Motivation

JAMES ARTHUR

Contents

§1. INTRODUCTION

In the paper [**3**], we gave a conjectural description of the discrete spectrum attached to the automorphic forms on a general reductive group. The main qualitative feature of this description was a Jordan decomposition into semisimple and unipotent constituents. This is in keeping with the dual nature of conjugacy classes and characters, and in fact, with a general parallelism between geometric objects and spectral objects that is observed in many mathematical contexts. Such a decomposition for automorphic representations would of course be parallel to the Jordan decomposition for rational conjugacy classes. It would also be analogous to the Jordan decomposition that is an essential part of the representation theory of finite algebraic groups.

The decomposition should actually apply uniformly to the automorphic representations in certain families. The families or "packets" are indexed by certain parameters which are the source of the decomposition. The quantitative side of the conjectures in [**3**] is a formula for the multiplicity with which a representation in any packet occurs in the

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discrete spectrum. It is a generalization of the formula for *tempered* representations which is implicit in the examples in [**15**]. In terms of the Jordan decomposition, tempered automorphic representations are semisimple. The multiplicity formula for nontempered automorphic representations contains some new signs. These are constructed out of the root numbers of certain L-functions, attached to the semisimple part of the given automorphic representation.

In this paper, we shall try to give some motivation for the conjectures. Some version of the conjectures, at least for many classical groups, ought to follow from the stable trace formula. This is certainly so in the few cases where the stable trace formula has been established [**15**], [**22**]. In general, one would need to combine the theory of endoscopy with the ordinary (or twisted) trace formula to obtain a stable trace formula. There are still a number of problems to be solved, but one can guess what the final answer will be. The purpose of this paper is to show that it is compatible with the conjectures of [**3**].

For purposes of introduction, let G be a connected, simply connected group over a number field F. We shall be interested in the spectral side of the trace formula. The essential ingredient we shall study is a certain distribution

$$I_{\mathrm{disc},t}(f)\ , \qquad\qquad f \in C_c^\infty(G(\mathbb{A})),$$

which is discrete in the parameters which describe the representations of $G(\mathbb{A})$. It is given by an explicit formula (3.1), one term of which involves the trace of f on the discrete spectrum. When the stable trace formula has been established, the payoff will be an identity

$$I_{\mathrm{disc},t}(f) = \sum_H \iota(G,H) S\hat{I}^H_{\mathrm{disc}}(f^H)\ , \tag{1.1}$$

in which H ranges over elliptic endoscopic groups, $\iota(G,H)$ is a certain constant, and $f \to S\hat{I}^H_{\mathrm{disc}}(f^H)$ is a pullback to G of a stable distribution on $H(\mathbb{A})$. (Recall that the endoscopic groups are a natural family of quasi-split groups attached to G. Recall too that a stable distribution is a special case of an invariant distribution, which arises as a natural consequence of the difference between rational conjugacy and geometric conjugacy. We refer the reader to [3, §3] for a brief discussion of these notions and of the Langlands-Shelstad transfer mapping f^H.) As a distribution on $G(\mathbb{A})$, $S\hat{I}^H_{\mathrm{disc}}(f^H)$ is not generally stable.

However, the trace of f on the discrete spectrum is also usually not stable. Endoscopic groups were actually invented by Langlands with the aim of measuring this lack of stability.

The endoscopic groups on the right hand side of (1.1) should all contribute to the multiplicity formula for representations in the discrete spectrum. However, the trace of f on the discrete spectrum is only one of several terms in the explicit formula for $I_{\mathrm{disc},t}(f)$. The other terms are the surviving remnants of Eisenstein series, and are parametrized by (conjugacy classes of) proper Levi subgroups of G. Each such term is a linear combination of distributions, which are obtained by taking the trace of a product of two operators, one being the action of f on the induced discrete spectrum, and the other being an intertwining operator that comes from Eisenstein series. These additional terms have one important function. They account for that part of the discrete spectrum of a given H which under functoriality maps into the continuous spectrum of G. However, the additional terms also contribute irrelevant information, which complicates the study of (1.1). The attempt to separate the extraneous information from the contribution of the discrete spectrum leads to combinatorial difficulties. The main point of this paper is to solve these combinatorial problems.

The results are given in §5–§8. In §5 we expand $I_{\mathrm{disc},t}(f)$ into a linear combination of irreducible characters. This hinges on the conjectures of [**3**]. However, we have only the modest goal of showing that the conjectures are compatible with (1.1), so we are free to assume them. Each coefficient in the expansion contains a certain quotient of L-functions, which comes from the global intertwining operators. If the irreducible character is tempered, this quotient should equal the parity of the pole of the L-function at $s = 1$. If the irreducible character is nontempered, however, it will have a unipotent part. When the corresponding unipotent element is not *even*, the quotient must also be expressed in terms of the order of the L-function at the center of the critical strip. The exact relation is given by Proposition 5.1, which we prove in §6. Together with Lemma 7.1, it provides the justification for the sign characters which appear in the general multiplicity formula.

In §7 we establish a parallel expansion of the right hand side of (1.1) into irreducible characters. This requires various properties from endoscopy, some known and others which are expected to hold, which we discuss in §2 and §3. The endoscopic groups H consist of the quasi-split form of G, together with groups of smaller dimension. By reasons of induction, then, the stable distributions $S\hat{I}^H_{\mathrm{disc}}$ are uniquely determined by (1.1). However, we must derive the expansion in §7 without reference to the left hand side of (1.1). The coefficients in the expansion have to be given as certain undetermined constants, which can be regarded as "stable multiplicities", and which only later are tied precisely to the sign characters discussed above. For a parameter which contributes to the tempered discrete spectrum, the corresponding coefficient will be familiar from [15, §6,7] and [12, §12]. It is then just equal to 1, divided by the order of a certain finite group.

Our aim is to show that with the assumption of the conjectures of [**3**], the left and right hand sides of (1.1) are equal. We would thus like to establish a term by term identification of the two parallel expansions. However, this is not immediately obvious. What remains to be proved at the end of §7 is a sort of analogue for Weyl groups of the endoscopy identity (1.1). The expansion of $I_{\mathrm{disc},t}(f)$ contains certain constants $i(x)$, which are defined if x is any connected component of a complex reductive group. The expansion for the right hand side of (1.1) is identical, except that $i(x)$ is replaced by another constant $e(x)$. In the first case, $i(x)$ is given by a finite sum over elements in the Weyl set of x. It is the analogue for Weyl groups of the left hand side of (1.1). The second constant $e(x)$ is the analogue of the right hand side of (1.1), and is given as a finite sum over the isolated conjugacy classes in x. In §8 we prove that $i(x)$ equals $e(x)$ for every component x. This establishes the term by term identification of the expansions of each side of (1.1).

At the end of §8 the reader might be wondering whether the paper has provided the global motivation claimed in the title. It is true that the identity (1.1) is weaker than the conjectural multiplicity formula (and the local conjectures on which it is based). However, the identity can still provide significant information about the discrete spectrum, for either G or its endoscopic groups. This is especially so if for one of the groups, the conjectures are known to hold. The group $GL(n)$ is such an example, thanks to recent work of Moeglin and Waldspurger [**21**]. The twisted version of (1.1), applied to $GL(n)$, will relate the discrete spectrum of many classical groups to that of $GL(n)$. In particular, it should yield some version of the multiplicity formula for the quasi-split orthogonal and symplectic groups. We shall finish the paper in §9 with an informal discussion of these questions.

Throughout the paper we shall adopt the following notational conventions. Suppose that Σ is a set on which a group Γ acts. We shall denote the set of orbits of Γ on Σ by either $\mathrm{Orb}(\Gamma, \Sigma)$ or Σ/Γ. In general, if A and B are subsets of a group Γ, we shall write

$$\mathrm{Cent}(A, B) = \{b \in B : \ b^{-1}ab = a, \quad \text{for all } a \in A\}$$

for the pointwise centralizer of A in B, and

$$\mathrm{Norm}(A, B) = \{b \in B : \ b^{-1}Ab = A\}$$

for the normalizer of A in B. Next, suppose that C is a finite union of connected components in a (nonconnected) algebraic group. Then C^+ denotes the algebraic group generated by C, and C^0 is the connected component of 1 in C^+. If s is any element in C, we set

$$C_s = \mathrm{Cent}(s, C^0) .$$

Then C_s is also an algebraic group, with identity component

$$C_s^0 = (C_s)^0 = \mathrm{Cent}(s, C^0)^0 .$$

(This differs from the notation of [2] and some other papers, in which the symbol C_s was reserved for the identity component of the centralizer.) We shall also write

$$Z(C) = \mathrm{Cent}(C, C^0) .$$

This group is the intersection of C^0 with the center of C^+, and is contained in $Z(C^0)$. Finally, if X is any topological space, $\pi_0(X)$ denotes the set of connected components of X.

§2. Endoscopic data

Suppose that G is a connected component of a reductive algebraic group over a number field F. Then G^+ stands for the group generated by G, and G^0 is the connected component of 1 in G^+. We shall assume that $G(F)$ is not empty. As in [**3**, §**6**], we shall also assume that G is an inner twist of a component in a quasi-split group. More precisely, we assume that there is a map

$$\eta : G \to G^* ,$$

where G^* is a component such that $(G^*)^0$ is quasi-split, and such that $G^*(F)$ contains an element which preserves some F-splitting of $(G^*)^0$ under conjugation. It is required that η extend to an isomorphism of G^+ with $(G^*)^+$ such that for any $\sigma \in \mathrm{Gal}(\bar{F}/F)$, the map

$$\eta\sigma(\eta^{-1}): G^* \to G^*$$

is an inner automorphism by an element in $(G^*)^0$.

The standard situation is when $G^+ = G^0$. By allowing G to be a more general component, we are providing for applications of the twisted trace formula [**5**]. Associated to the connected component G^0 we have the L-group

$${}^LG^0 = \hat{G}^0 \rtimes W_F .$$

It is a semidirect product of a complex connected group $\hat{G}^0$ with the Weil group W_F of F. (As in [**3**], we follow the notation of Kottwitz [**12**], so that $\hat{G}^0$ stands for the identity coset of the L-group. The symbol ${}^LG^0$ can then be reserved for the full L-group of G^0.) We have not assumed that G^+ is a semidirect product of G^0 with a finite cyclic group, but this does not seem to be a serious concern. In particular, it is reasonable to define the L-group ${}^LG^+$ of G^+ simply as a semidirect product of ${}^LG^0$ by the cyclic group $\pi_0(G^+)$ of connected components in G^+. The action of $\pi_0(G^+)$ on $\hat{G}^0$ is dual to its action by outer autmorphisms on G^0. The action of $\pi_0(G^+)$ on W_F could be defined by some map of $\pi_0(G^+)$ into $H^1(W_F, Z(\hat{G}^0))$. However, for simplicity we shall assume that $\pi_0(G^+)$ and W_F (as subgroups of ${}^LG^+$) commute. Associated to the component G we have an "L-coset"

$${}^LG = \hat{G} \rtimes W_F ,$$

in which $\hat{G}$ is a coset of $\hat{G}^0$ in a group $\hat{G}^+$ such that

$${}^LG^+ = \hat{G}^+ \rtimes W_F .$$

Notice that

$$Z(\hat{G}) = \mathrm{Cent}(\hat{G}, \hat{G}^0)$$

is in general a proper subgroup of the center

$$Z(\hat{G}^0) = \mathrm{Cent}(\hat{G}^0, \hat{G}^0)$$

of $\hat{G}^0$. We must always be careful to distinguish between these two groups. The Galois group $\Gamma = \Gamma_F$ of $\bar{F}$ over F acts on both $Z(\hat{G})$ and $Z(\hat{G}^0)$. The subgroups of Γ-invariant elements are given by

$$Z(\hat{G})^\Gamma = \mathrm{Cent}({}^LG, \hat{G}^0)$$

and

$$Z(\hat{G}^0)^\Gamma = \mathrm{Cent}({}^LG^0, \hat{G}^0) .$$

These too are not generally equal. Observe that

$$A_{\hat{G}} = \left(Z(\hat{G})^\Gamma\right)^0$$

is the maximal Γ-invariant torus in the center of $\hat{G}^+$. It is of course not the dual group of the maximal split torus A_G in the center of G^+. It is associated, rather, to the dual of the real vector space

$$\underline{\mathfrak{a}}_G = \mathrm{Hom}\big(X^*(G)_F, \mathbb{R}\big) .$$

($X^*(G)_F$ denotes the module of F-rational characters on G^+.) More precisely,

$$X^*(G)_F \cong X_*(A_{\hat{G}}) ,$$

so that the complex dual space $\underline{\mathfrak{a}}^*_{G,\mathbb{C}} = X^*(G)_F \otimes \mathbb{C}$ is the Lie algebra of $A_{\hat{G}}$. We shall write

$$\kappa_G = (A_{\hat{G}^0})^{\hat{G}} = Z(\hat{G})^\Gamma \cap \left(Z(\hat{G}^0)^\Gamma\right)^0$$

for the group of fixed points of $\hat{G}$ in $A_{\hat{G}^0}$. It is a closed subgroup of $A_{\hat{G}^0}$ whose identity component equals $A_{\hat{G}}$.

The theory of endoscopy for nonconnected groups is the subject of work in progress by Kottwitz and Shelstad. As in **[3, §6]**, we shall guess at the ultimate form of some of this theory by extrapolating from the connected case. Thus, an endoscopic datum $(H, \mathcal{H}, s, \xi)$ should consist of a connected quasi-split group H over F, an extension

$$I \longrightarrow \hat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1 ,$$

a semisimple coset s in $\hat{G}/Z(\hat{G}^0)$, and an L-embedding ξ of $\mathcal{H}$ into ${}^LG^0$. The definition is similar to the one given in **[3, §3,§6]** except that s is now a coset of $Z(\hat{G}^0)$ instead of a single element in $\hat{G}$. It is required that

$$\xi(\hat{H}) = \mathrm{Cent}(s, \hat{G}^0)^0 ,$$

the connected centralizer in $\hat{G}^0$ of any element in the coset s, and that

$$s\xi(h)s^{-1} = a(w_h)\xi(h) , \qquad h \in \mathcal{H}, \tag{2.1}$$

where w_h is the image of h in W_F and $a(\cdot)$ represents a locally trivial element in $H^1(W_F, Z(\hat{G}^0))$. In other words, $a(\cdot)$ belongs to $\ker^1(W_F, Z(\hat{G}^0))$, the kernel of the map

$$H^1(W_F, Z(\hat{G}^0)) \longrightarrow \bigoplus_v H^1(W_{F_v}, Z(\hat{G}^0)) ,$$

in which v runs over the valuations of F. It is further required that the two extensions $\mathcal{H}$ and ${}^LH$ define the same map of W_F into $\mathrm{Out}(\hat{H})$, the group of outer automorphisms of $\hat{H}$.

Recall that an endoscopic datum is said to be *elliptic* if the set $\xi(\mathcal{H})s$ is not contained in any proper parabolic subset of ${}^LG$. Equivalently, the datum is elliptic if and only if the group

$$\xi(Z(\hat{H})^\Gamma)/\xi(Z(\hat{H})^\Gamma) \cap Z(\hat{G})^\Gamma$$

is finite, or again, if and only if $\xi(A_{\hat{H}})$ equals $A_{\hat{G}}$. Finally, two elliptic endoscopic data $(H, \mathcal{H}, s, \xi)$ and $(H', \mathcal{H}', s', \xi')$ are equivalent if there exist dual isomorphisms $\alpha : H \to H'$ and $\beta : \mathcal{H}' \to \mathcal{H}$, together with an element $g \in \hat{G}^0$ such that

$$g\xi(\beta(h'))g^{-1} = \xi'(h') , \qquad h' \in \mathcal{H}',$$

and

$$gsg^{-1} = s' .$$

Suppose that $(H, \mathcal{H}, s, \xi)$ is an elliptic endoscopic datum. We shall write $\mathrm{Aut}(H)$ for the group of elements g in $\hat{G}^0$ such that $gsg^{-1} = s$, and $g\xi(\mathcal{H})g^{-1} = \xi(\mathcal{H})$. Then $\mathrm{Aut}(H)$ is a reductive subgroup of $\hat{G}^0$. Notice that $\xi(\hat{H})Z(\hat{G}^0)^\Gamma$ is a closed subgroup of $\mathrm{Aut}(H)$. We shall need to know later that it is of finite index. Equivalently, we must establish

LEMMA 2.1. *The identity component of* $\mathrm{Aut}(H)$ *equals*

$$\xi(\hat{H})\big(Z(\hat{G}^0)^\Gamma\big)^0 \;=\; \xi(\hat{H})A_{(\hat{G}^0)}\,.$$

Let s_1 be a fixed element in the coset s, and write

$$\widetilde{C}_s \;=\; \{g \in \hat{G}^0 : s_1 g s_1^{-1} g^{-1} \in Z(\hat{G}^0)\} \;=\; \{g \in \hat{G}^0 : gsg^{-1} = s\}$$

and

$$C_s \;=\; \{g \in \hat{G}^0 : s_1 g s_1^{-1} g^{-1} = 1\}\,.$$

Then

$$g \longrightarrow s_1 g s_1^{-1} g^{-1}$$

is an injective map from $\widetilde{C}_s/C_s$ onto a closed subgroup $\hat{Z}(s)$ of $Z(\hat{G}^0)$.

LEMMA 2.2. *The subgroup*

$$\hat{Z}'(s) \;=\; \{s_1 z s_1^{-1} z^{-1} : z \in Z(\hat{G}^0)\}$$

is of finite index in $\hat{Z}(s)$.

PROOF: Suppose that g belongs to $\widetilde{C}_s$. We can write $g = g_1 z$, where g_1 belongs to the derived subgroup $\hat{G}^0_{\mathrm{der}}$ of $\hat{G}^0$ and z belongs to $Z(\hat{G}^0)$. Then

$$s_1 g s_1^{-1} g^{-1} \;=\; s_1 g_1 s_1^{-1} g_1^{-1} \cdot s_1 z s_1^{-1} z^{-1}\,.$$

In particular, both g_1, and z belong to $\widetilde{C}_s$. But the element $s_1 g_1 s_1^{-1} g_1^{-1}$ lies in G^0_{der}. The lemma follows from the fact that $\hat{G}^0_{\mathrm{der}}$ has finite center. □

PROOF OF LEMMA 2.1: According to the first condition in its definition, $\mathrm{Aut}(H)$ is contained in $\widetilde{C}_s$. Let $\mathrm{Aut}'(H)$ be the subgroup of elements $g \in \mathrm{Aut}(H)$ such that $s_1 g s_1^{-1} g^{-1}$ belongs to $\hat{Z}'(s)$. The last lemma tells us that $\mathrm{Aut}'(H)$ is of finite index in $\mathrm{Aut}(H)$.

Let g be an element in $\mathrm{Aut}'(H)$. Then we can write

$$g \;=\; g_1 z_1\,, \qquad\qquad g_1 \in C_s\,,\ z_1 \in Z(\hat{G}^0)\,.$$

Suppose also that h is an element in $\xi(\mathcal{H})$. The second condition in the definition of $\mathrm{Aut}(H)$ implies that ghg^{-1} equals $h_1^{-1}h$, for some element $h_1 \in \xi(\hat{H})$. We can write this as

$$hz_1h^{-1}z_1^{-1} \;=\; (hg_1h^{-1})^{-1}h_1 g_1\,.$$

Both h_1 and g_1 commute with s_1. It follows easily from (2.1) that hg_1h^{-1} also commutes with s_1. Therefore $hz_1h^{-1}z_1^{-1}$ commutes with s_1, and belongs to the subgroup $Z(\hat{G})$ of $Z(\hat{G}^0)$. Now

$$hz_1h^{-1}z_1^{-1} = \sigma(z_1)z_1^{-1},$$

where σ is the projection of h onto $\Gamma = \mathrm{Gal}(\bar{F}/F)$. The action of Γ on $Z(\hat{G}^0)$ factors through a finite quotient $\mathrm{Gal}(E/F)$, and this action preserves the subgroup $Z(\hat{G})$. We obtain a homomorphism

$$g \longrightarrow \sigma(z_1)z_1^{-1}$$

from $\mathrm{Aut}'(H)$ to the finite group $H^1(\mathrm{Gal}(E/F),\, Z(\hat{G}))$. Suppose that g lies in the kernel of this map. Then

$$\sigma(z_1)z_1^{-1} = \sigma(z)z^{-1}, \qquad \sigma \in \Gamma,$$

for some element $z \in Z(\hat{G})$. In other words, there is a decomposition $z_1 = zz_1'$, for elements z in $Z(\hat{G})$ and z_1' in $Z(\hat{G}^0)^\Gamma$. We can therefore write $g = g_1'z_1'$, where the element $g_1' = g_1z$ lies in the centralizer C_s. In other words, g belongs to the subgroup $C_sZ(\hat{G}^0)^\Gamma$.

It remains only to observe that $\xi(\hat{H})$ is the identity component of C_s. We obtain an embedded chain

$$\xi(\hat{H})Z(\hat{G}^0)^\Gamma \subset C_sZ(\hat{G}^0)^\Gamma \subset \mathrm{Aut}'(H) \subset \mathrm{Aut}(H)$$

of normal subgroups of finite index. Therefore $\xi(\hat{H})Z(\hat{G}^0)^\Gamma$ is of finite index in $\mathrm{Aut}(H)$, and the two groups have the same identity component. $\square$

Let $(H, \mathcal{H}, s, \xi)$ be a fixed endoscopic datum. One is interested in the L-homomorphisms of W_F into ${}^LG$ whose image is contained in $\xi(\mathcal{H})$. (Recall that an L-homomorphism between two extensions of W_F is a homomorphism which commutes with the projection onto W_F.) One might like to be able to identify such objects with L-homomorphisms of W_F into the L-group ${}^LH$ of H. However, this is not always possible. The L-group is a semidirect product $\hat{H} \rtimes W_F$ relative to an L-action of W_F on $\hat{H}$ [**20, 1.4**]. (The action of W_F of course factors through the quotient Γ of W_F.) But the two extensions $\mathcal{H}$ and ${}^LH$ of W_F by $\hat{H}$ need not be isomorphic. In other words, there might not be an L-embedding of ${}^LH$ into ${}^LG$ which co-incides with the image of ξ. Fortunately the problem is not serious. In the case that $G = G^0$, the question can be resolved by taking a z-extension of G, as has been explained in [**20, (4.4)**]. In the general case, Shelstad has pointed out that it is necessary to work directly with extensions of the endoscopic groups H. Suppose, then, that

$$1 \longrightarrow Z_1 \longrightarrow H_1 \longrightarrow H \longrightarrow 1 \tag{2.2}$$

is a central extension of quasi-split groups over F. We shall review the question of whether there exists an L-embedding

$$\xi_1 : \mathcal{H} \longrightarrow {}^LH_1$$

which extends the canonical embedding $\hat{H} \hookrightarrow \hat{H}_1$ of dual groups.

Consider first the kernel K_F of the projection $W_F \rightarrow \Gamma_F$, a connected group. It would be no trouble to construct an embedding for the preimage $\mathcal{H}'$ of K_F in $\mathcal{H}$. For it follows easily from (2.1) that $\xi(\mathcal{H}')$ equals the subgroup $\xi(\hat{H}) \times K_F$ of ${}^LG^0$. In other words, there is a splitting $\theta : K_F \rightarrow \mathcal{H}'$ such that

$$h\theta(k)h^{-1} = \theta(w_h k w_h^{-1}) \,, \qquad h \in \mathcal{H}\,,\ k \in K_F\,, \tag{2.3}$$

where w_h is the image of h in W_F. Now by assumption, the map of W_F into $\mathrm{Out}(\hat{H})$ defined by $\mathcal{H}$ is the same as the L-action

$$h \longrightarrow w(h) \,, \qquad h \in \hat{H}\,,\ w \in W_F\,,$$

used to define ${}^LH$. It follows that θ can be extended to a section from W_F to $\mathcal{H}$ such that

$$\theta(w)h\theta(w)^{-1} = w(h) \,, \qquad h \in \hat{H}\,,\ w \in W_F\,.$$

Keep in mind that it is only the restriction of θ to K_F which is a homomorphism. However, θ is uniquely determined up to multiplication by elements in the center $Z(\hat{H})$ of $\hat{H}$. Therefore

$$\theta(w_1)\theta(w_2) = b(w_1, w_2)\theta(w_1, w_2) \,, \qquad w_1, w_2 \in W_F\,,$$

where $b(w_1, w_2)$ is a 2-cocycle from W_F to $Z(\hat{H})$. By (2.3), $b(w_1, w_2)$ depends only on the images of w_1 and w_2 in Γ_F. We shall write β

for the image of b in $H^2\big(W_F, Z(\hat{H}_1)\big)$, relative to the embedding of $Z(\hat{H})$ into $Z(\hat{H}_1)$. Then β is the inflation of a class in $H^2\big(\Gamma_F, Z(\hat{H}_1)\big)$ which is independent of θ. Suppose that β is trivial. That is,

$$\beta(w_1, w_2) = z(w_1)w_1\big(z(w_2)\big)z(w_1w_2)^{-1}, \qquad w_1, w_2 \in W_F,$$

for a function $z : W_F \to Z(\hat{H}_1)$ which is uniquely determined up to a 1-cocycle. Every element in $\mathcal{H}$ can be represented uniquely in the form

$$h\theta(w), \qquad h \in \hat{H},\ w \in W_F,$$

and the map

$$\xi_1\big(h\theta(w)\big) = hz(w) \rtimes w \tag{2.4}$$

is then an L-embedding of $\mathcal{H}$ into ${}^LH_1$. Conversely, if an embedding ξ_1 exists, the function $z(w)$ in (2.4) will split the class β.

Assume that the embedding ξ_1 exists. Suppose also that the central subgroup Z_1 of H_1 is connected. Then we can form the L-group ${}^LZ_1 = \hat{Z}_1 \times W_F$, and there is a canonical projection ${}^LH_1 \to {}^LZ_1$. We also have an exact sequence

$$1 \longrightarrow Z(\hat{H}) \longrightarrow Z(\hat{H}_1) \longrightarrow \hat{Z}_1 \longrightarrow 1$$

of complex abelian groups. Let $z_1(w)$ be the projection of $z(w)$ onto $\hat{Z}_1$. Then z_1 is a 1-cocycle from W_F to $\hat{Z}_1$. In fact, if we agree not to distinguish between a cocycle and its corresponding cohomology class, z_1 is just the preimage of the class $b \in H^2\big(W_F, Z(\hat{H})\big)$ determined by the long exact sequence

$$\begin{aligned} \cdots \to H^1\big(W_F, Z(\hat{H}_1)\big) &\to H^1(W_F, \hat{Z}_1) \\ &\to H^2\big(W_F, Z(\hat{H})\big) \to H^2\big(W_F, Z(\hat{H}_1)\big) \to \cdots \end{aligned}$$

It is uniquely determined modulo the image of $H^1\big(W_F, Z(\hat{H}_1)\big)$ in $H^1(W_F, \hat{Z}_1)$. The map

$$\alpha_1(w) = z_1(w) \rtimes w, \qquad w \in W_F,$$

is an L-homomorphism of W_F to ${}^LZ_1$.

Suppose that $L_F \to W_F$ is some extension of W_F. Suppose also that $\psi : L_F \to {}^L G$ is an L-homomorphism whose image is contained in $\xi(\mathcal{H})$. That is, $\psi = \xi \circ \psi_H$, for some L-homomorphism $\psi_H : L_F \to \mathcal{H}$. Then $\psi_1 = \xi_1 \circ \psi_H$ is an L-homomorphism of L_F into ${}^L H_1$. Set

$$\psi_H(t) = \gamma(t)\theta(w_t) , \qquad t \in L_F ,$$

where w_t is the image of t in W_F and $\gamma(t)$ belongs to $\hat{H}$. Then

$$\xi_1\big(\psi_H(t)\big) = \gamma(t)z(w_t) \times w_t , \qquad t \in L_F .$$

It follows that the composition of ψ_1 with the projection ${}^L H_1 \to {}^L Z_1$ equals α_1 (or rather, the pullback of α_1 to L_F.) Conversely, any L-homomorphism $\psi_1 : L_F \to {}^L H_1$ whose projection to ${}^L Z_1$ equals α_1 is easily seen to be of the form $\xi_1 \circ \psi_H$. We can summarize these remarks in a commutative diagram

$$\begin{tikzcd}
& & L_F \arrow[dll, dashed, "\alpha_1"'] \arrow[dl, "\psi_1"] \arrow[d, "\psi_H"] \arrow[dr, "\psi"] & \\
{}^L Z_1 & {}^L H_1 \arrow[l] & \mathcal{H} \arrow[l, "\xi_1"'] \arrow[r, "\xi"] & {}^L G
\end{tikzcd}$$

In conclusion, we want to associate pairs (H_1, ξ_1) to endoscopic data, where H_1 is a central extension (2.2) and ξ_1 is an L-embedding (2.4). We shall call such a pair a *splitting* for the endoscopic datum. We shall say that (H_1, ξ_1) is a *distinguished* splitting if, in addition, the map $H_1(\mathbb{A}) \to H(\mathbb{A})$ between adèle groups is surjective, and the central subgroup Z_1 is an induced torus. That is, Z_1 is a product of tori of the form $\mathrm{Res}_{E/F}(G_m)$. In particular, Z_1 is connected, as we assumed in the discussion above. Any endoscopic datum has a distinguished splitting. For example, the cocycle $b(w_1, w_2) \in Z(\hat{H})$ that we described above often splits. In this case, we can simply take $(H_1, \xi_1) = (H, Id)$. In general, we can always take H_1 to be a z-extension of H **[11, §1]**, the existence of which is established in **[17, pp. 721–722]**. The first condition follows from **[11, Lemma 1.1(3)]**, while the second is part of the definition of a z-extension. It is also part of the definition that the derived group of H_1 is simply connected. This in turn implies that $Z(\hat{H}_1)$ is a complex torus. It follows from **[17, Lemma 4]** that the class $\beta \in H^2(W_F, Z(\hat{H}_1))$ is trivial. The embedding ξ_1 therefore exists, and (H_1, ξ_1) becomes a distinguished splitting. In general, if (H_1, ξ_1) is any distinguished splitting, one needs to know that the canonical map

$$\ker^1(F, Z(\hat{H})) \longrightarrow \ker^1(F, Z(\hat{H}_1))$$

is an isomorphism. (As before, $\ker^1(F, Z(\hat{H}))$ denotes the kernel of the map

$$H^1(F, Z(\hat{H})) \longrightarrow \bigoplus_v H^1(F_v, Z(\hat{H}))\,.)$$

This follows from the proof of **[12, Lemma 4.3.2(a)]**. We will also use the injectivity of the map

$$H^1(F, Z(\hat{H})) \longrightarrow H^1(F, Z(\hat{H}_1))\ ,$$

which is a consequence of the long exact sequence of cohomology, and the fact the group $H^0(F, \hat{Z}_1) = \pi_0(\hat{Z}_1^\Gamma)$ is trivial.

§3. The discrete part of the trace formula

We are going to study a piece of the trace formula. It consists of those distributions on the spectral side of the trace formula which are discrete with respect to the natural measure on the relevant automorphic representations. This part of the formula contains the actual trace on the discrete spectrum. It is thus the payload, the part which will eventually be used to compare automorphic representations on different groups. Of course, there are serious problems relating to the other terms in the trace formula which will have to be overcome first. Our intention in this paper is simply to see what can be learned once these other problems have been solved.

Let $\mathbb{A}$ be the adèle ring of F. We should first identify our space of test functions on $G(\mathbb{A})$, the set of $\mathbb{A}$-valued points in G. Consider the diagonalizable group $Z(G) = \mathrm{Cent}(G, G^0)$. We shall fix a closed subgroup X of the group $Z(G, \mathbb{A})$ of adèle points such that $X \cap Z(G, F)$ is closed, and such that $XZ(G,F)\backslash Z(G, \mathbb{A})$ is compact. Let χ be a character on X which is trivial on $X \cap Z(G, F)$. Then $C_c^\infty(G(\mathbb{A}), \chi)$ will denote the space of smooth functions f on $G(\mathbb{A})$, of compact support modulo X, such that

$$f(zx) = \chi(z)^{-1} f(x)\ , \qquad z \in \chi,\ z \in G(\mathbb{A})\,.$$

Let t be an arbitrary but fixed nonnegative real number. The corresponding discrete part of the trace formula is the distribution

$$I_{\mathrm{disc},t}(f)\ , \qquad f\in C_c^\infty\big(G(\mathbb{A}),\chi\big)\,,$$

on $C_c^\infty\big(G(\mathbb{A}),\chi\big)$ which is given by the expression

$$(3.1)\quad \sum_{\{M\}}\ \sum_{w\in W^G(\underline{\mathfrak{a}}_M)_{\mathrm{reg}}} |\pi_0(G^+)|^{-1}|W^G(\underline{\mathfrak{a}}_M)|^{-1} |\det(w-1)_{\underline{\mathfrak{a}}_M^G}|^{-1}\mathrm{tr}\big(M(w,0)\rho_{P,t}(0,f)\big)\,.$$

(See **[2**, §4], **[4**, §**II.9]**.) We shall describe very briefly the terms in this expression. The outer sum is over the finite set of $G^0(F)$-orbits of Levi components M of F-rational parabolic subgroups P of G^0. The inner sum is over the regular elements

$$W^G(\underline{\mathfrak{a}}_M)_{\mathrm{reg}} \;=\; \{w\in W^G(\underline{\mathfrak{a}}_M):\ \det(w-1)_{\underline{\mathfrak{a}}_M^G}\neq 0\}$$

in the Weyl set

$$W^G(\underline{\mathfrak{a}}_M) \;=\; \mathrm{Norm}(A_M,G)/M$$

of (G,M). As in earlier papers, we regard the Weyl elements as operators on the real vector space

$$\underline{\mathfrak{a}}_M \;=\; \mathrm{Hom}\big(X(M)_F,\mathbb{R}\big)$$

which leave invariant the kernel $\underline{\mathfrak{a}}_M^G$ of the projection of $\underline{\mathfrak{a}}_M$ onto $\underline{\mathfrak{a}}_G$. For each M there is canonical isomorphism from

$$A_{M,\infty} \;=\; A_{M_{\mathbb{Q}}}(\mathbb{R})^0\ , \qquad M_{\mathbb{Q}}=\mathrm{Res}_{F/\mathbb{Q}}(M)\,,$$

onto $\underline{\mathfrak{a}}_M$. If $A^G_{M,\infty}$ denotes the preimage of $\underline{\mathfrak{a}}_M^G$ in $A_{M,\infty}$, we can extend χ uniquely to a character χ_M on $X_M = A^G_{M,\infty}X$ which is trivial on $A^G_{M,\infty}$. Let $L^2_{\mathrm{disc},t}\big(M(F)\backslash M(\mathbb{A}),\chi_M^{-1}\big)$ be the subspace of $L^2\big(M(F)\backslash M(\mathbb{A}),\chi_M^{-1}\big)$ which decomposes under $M(\mathbb{A})$ as a direct sum of irreducible represention whose Archimedean infinitesimal character has norm t. Then

$$\rho_{P,t}(0):\ f\ \longrightarrow\ \rho_{P,t}(0,f) \;=\; \int_{X\backslash G(\mathbb{A})} f(x)\rho_{P,t}(0,x)dx$$

stands for the corresponding representation induced from $P(\mathbb{A})$ to the group $G(\mathbb{A})^+$ generated by $G(\mathbb{A})$. It acts on a Hilbert space $\mathcal{H}_{P,t}$ of χ_M^{-1}-equivariant functions on $G(\mathbb{A})^+$. Finally,

$$M(w,0): \mathcal{H}_{P,t} \longrightarrow \mathcal{H}_{P,t}, \qquad w \in W^G(\mathfrak{a}_M)_{\mathrm{reg}},$$

is the global intertwining operator which comes from the theory of Eisenstein series. For a given conductor, $I_{\mathrm{disc},t}(f)$ is a finite linear combination of irreducible characters on $G(\mathbb{A})^+$.

There are some minor discrepancies between (3.1) and the original definition **[2, (4.3)]**. In (3.1) we have summed over the orbits $\{M\}$ instead of all Levi components which contain a given minimal one. This is why $|W^G(\mathfrak{a}_M)|^{-1}$ appears instead of the normalizing constant $|W_0^M||W_0^G|^{-1}$ from **[2]**. The operator $\rho_{P,t}(0,f)$ here comes from a representation of $G(\mathbb{A})^+$ induced from a subgroup of the connected component $G^0(\mathbb{A})$. It is a direct sum of $|\pi_0(G^+)|$ copies of the corresponding operator from **[2]**, which comes essentially from the induced representation of $G^0(\mathbb{A})$. Hence the constant $|\pi_0(G^+)|^{-1}$ in (3.1). The difference between taking a χ-equivariant function on $G(\mathbb{A})$, as we have done here, and a function defined on the subset $G(\mathbb{A})^1$ of $G(\mathbb{A})$, as in **[2]**, is purely formal. In **[2]**, there was also the additional assumption that f was K-finite, but this was only for dealing with other terms in the trace formula.

The program for comparing trace formulas on different groups, as it is presently conceived, falls into the general framework of stabilizing the trace formula. The basic references for this problem are **[18]**, **[12]**, and **[13]**. The problem was solved completely for $G = \mathrm{SL}(2)$ in **[15]**. A general solution would include: a transfer map from functions for G to functions for endoscopic data, a stable distribution analogous to $I_{\mathrm{disc},t}$ for any quasi-split group, and an identity relating $I_{\mathrm{disc},t}$ to the corresponding stable distributions for endoscopic data. We shall discuss the transfer first, and then describe the expected properties of the other objects in the form of a hypothesis.

Suppose that $(H, \mathcal{H}, s, \xi)$ is an elliptic endoscopic datum for G. Assume also that we have fixed a distinguished splitting (H_1, ξ_1) for the endoscopic datum. As we recall from §1, ξ_1 determines an L-homomorphism $\alpha_1 : W_F \to {}^L Z_1$. Let

$$\zeta_1 : Z_1(F)\backslash Z_1(\mathbb{A}) \longrightarrow \mathbb{C}^*$$

be the character associated to α_1 by the Langlands correspondence for tori. Now, in the special case that $G = G^0$, the results **[20]** of Langlands and Shelstad imply the existence of a canonical map $f \to f^{H_1}$ from functions $f \in C_c^\infty(G(\mathbb{A})$ to functions $f^{H_1}(\gamma_{H_1})$ on suitable stable conjugacy classes in $H(\mathbb{A})$, with the property that

$$f^{H_1}(z_1\gamma_{H_1}) = \zeta_1(z_1)^{-1} f^{H_1}(\gamma_{H_1}) , \qquad z_1 \in Z_1(\mathbb{A}) .$$

(See also [12], [13] and [3].) The map must be constructed as a tensor product of the local maps $f_v \to f_v^{H_1}$, $f_v \in C_c^\infty(G(F_v))$, which are defined explicitly in [20]. Langlands and Shelstad expect that f^{H_1} is the set of stable orbital integrals on $H_1(\mathbb{A})$ of a function g in $C_c^\infty(H_1(\mathbb{A}), \zeta_1)$. We shall assume that this is so. In fact, we shall assume that the transfer map

$$f \longrightarrow f^{H_1} , \qquad f \in C_c^\infty(G(\mathbb{A})) ,$$

has been defined, and has this property, for general G.

We should actually modify the transfer mapping so that its domain is the space $C_c^\infty(G(\mathbb{A}, \chi)$ considered earlier. Lemma 4.4A of [20] suggests how the functions

$$f_z(x) = f(zx) , \qquad z \in Z(G, \mathbb{A}),\ x \in G(\mathbb{A}),\ f \in C_c^\infty(G(\mathbb{A})) ,$$

should behave under the transfer map. In general, there will be a norm mapping $z \to z'$ from $Z(G,F)\backslash Z(G,\mathbb{A})$ into $Z(H,F)\backslash Z(H,\mathbb{A})$. We also have the exact sequence

$$1 \longrightarrow Z_1 \longrightarrow Z(H_1) \longrightarrow Z(H) \longrightarrow 1 .$$

We can then expect a formula

$$(f_z)^{H_1}(\gamma_{H_1}) = \zeta_1(z_1) f^{H_1}(z_1\gamma_{H_1}) , \tag{3.2}$$

where ζ_1 is an extension to $Z(H_1, F)\backslash Z(H_1, \mathbb{A})$ of the character on $Z_1(F)\backslash Z_1(\mathbb{A})$, and z_1 is any point in $Z(H_1, \mathbb{A})$ whose image in $Z(H, \mathbb{A})$ equals z'. Recall that χ is a character on the closed subgroup X of $Z(G, \mathbb{A})$. We shall assume that

$$\chi(z) = \chi'(z') , \qquad z \in X ,$$

where χ' is a character on the image X' of X in $Z(H, \mathbb{A})$. To define the transfer mapping for functions in $C_c^\infty(G(\mathbb{A}), \chi)$, we simply multiply each side of (3.2) by $\chi(z)$, and integrate over z in $X \cap Z(G,F)\backslash X$. Let X_1 be the preimage of X' in $Z(H_1, \mathbb{A})$, and set

$$\chi_1(z_1) = \zeta_1(z_1)\chi'(z') ,$$

for any point $z_1 \in X_1$ with image z' in X'. Then χ_1 is a character on X_1, and the triple (H_1, X_1, χ_1) satisfies the conditions we imposed on (G, X, χ). In this context our assumption is that for any function $f \in C_c^\infty(G(\mathbb{A}), \chi)$ there is a function $g \in C_c^\infty(H_1(\mathbb{A}), \chi_1)$ whose stable orbital integrals are given by f^{H_1}. The function g is of course not uniquely determined by f. However, if SI is any stable distribution on $C_c^\infty(H_1(\mathbb{A}), \chi_1)$, $SI(g)$ will be uniquely determined by f. We shall therefore write

$$S\hat{I}(f^{H_1}) = SI(g) .$$

The ultimate goal is to give an expansion of $I_{\mathrm{disc},t}$ as a linear combination of stable distributions on the equivalence classes of elliptic endoscopic data $\{H\}$ for G. The coefficients will be certain constants $\iota(G,H)$, which in the case $G = G^0$ were introduced by Langlands [**18**]. (Following the usual convention of metonymy, we shall often write H in place of a full endoscopic datum $(H, \mathcal{H}, s, \xi)$.) Kottwitz has established a simple formula for these constants [**12, Theorem 8.3.1**], again when $G = G^0$. Let

$$\tau_1(G^0) = \tau(G^0)\tau(G^0_{\mathrm{sc}})^{-1}$$

be the relative Tamagawa number of G^0 [**12**, §5]. ($\tau(G^0)$ denotes the ordinary Tamagawa number of G^0, and G^0_{sc} is the simply connected cover of the derived group of G^0. Thus according to Weil's conjecture, which has been established by Kottwitz [**14**] for groups without E_8 factors, $\tau_1(G^0)$ simply equals $\tau(G^0)$.) Kottwitz' formula is then

$$\iota(G,H) = \tau_1(G^0)\tau_1(H)^{-1}|\pi_0(\mathrm{Aut}(H))|^{-1} .$$

In the general case, the constants have not yet been defined. We shall have to get by with a makeshift definition that reduces to Kottwitz' formula when $G = G^0$.

If we are given an equivalence class $\{H\}$ of elliptic endoscopic data, we shall usually assume implicitly that H is a representative of the class such that ξ is the identity. That is, $\mathcal{H}$ is an embedded subgroup of ${}^LG^0$. Then $Z(\hat{H})^\Gamma$ is a subgroup of $\hat{G}^0$ whose identity component equals $A_{\hat{G}}$. The subgroup $\kappa_G = (A_{\hat{G}^0})^{\hat{G}}$ of $\hat{G}^0$ also has $A_{\hat{G}}$ as its identity component, so that $\kappa_G \cap Z(\hat{H})^\Gamma$ is a subgroup of finite index in κ_G. For general G we shall simply define

$$(3.3)\ \iota(G,H) = \tau_1(G^0)\tau_1(H)^{-1}|\pi_0(\mathrm{Aut}(H))|^{-1}|\kappa_G/\kappa_G\cap Z(\hat{H})^\Gamma|^{-1}.$$

The fourth factor in the product on the right, which of course equals 1 when $G = G^0$, is suggested by the calculations in §7.

We can now state the hypothesis. Part of it applies to any (G, X, χ) as above, and part applies to triples (G_1, X_1, χ_1) with the restriction that G_1 is a connected quasi-split group over F.

HYPOTHESIS 3.1. *For any (G_1, X_1, χ_1) there is a stable distribution $SI^{G_1}_{\mathrm{disc},t}$ on $C_c^\infty(G_1(\mathbb{A}), \chi_1)$ with the property that for any (G, X, χ), the distribution*

$$E_{\mathrm{disc},t}(f) = \sum_H \iota(G,H) S\hat{I}^{H_1}_{\mathrm{disc},t}(f^{H_1}) \tag{3.4}$$

equals $I_{\mathrm{disc},t}(f)$. Here f stands for any function in $C_c^\infty(G(\mathbb{A}), \chi)$ and H is summed over the equivalence classes of elliptic endoscopic data for G. □

Remarks. 1. It is understood that we have fixed a distinguished splitting (H_1, ξ_1) for each H. The distribution $S\hat{I}^{H_1}_{\mathrm{disc},t}(f^{H_1})$ should then depend only on H and not on the splitting.

2. The stable distributions $SI^{G_1}_{\mathrm{disc},t}$ are uniquely determined by the condition that $E_{\mathrm{disc},t}(f)$ equals $I_{\mathrm{disc},t}(f)$. For suppose that they have been defined inductively for any group whose semisimple part has dimension less than that of G_1. Setting $G = G_1$, one simply defines

$$SI^{G_1}_{\mathrm{disc},t}(f) = I_{\mathrm{disc},t}(f) - \sum_{H \neq G_1} \iota(G,H) S\hat{I}^{H_1}_{\mathrm{disc},t}(f^{H_1}).$$

In this case, the hypothesis becomes the assertion that the right hand side is a stable distribution in f. This of course is highly nontrivial. It is likely to be resolved only by proving a similar assertion for all the other terms in the trace formula. There is a discussion of this question in the paper [**19**].

We shall need a slightly different formula for $\iota(G,H)$ in §7. For H as above, set

$$\overline{Z}(\hat{H})^\Gamma = Z(\hat{H})^\Gamma Z(\hat{G}^0)/Z(\hat{G}^0) \cong Z(\hat{H})^\Gamma / Z(\hat{H})^\Gamma \cap Z(\hat{G})^\Gamma.$$

Since H represents an elliptic endoscopic datum, $\overline{Z}(\hat{H})^\Gamma$ is a finite (abelian) group.

LEMMA 3.2. *The constant* $\iota(G,H)$ *equals*

$$(3.5)\quad |\ker^1(F,Z(\hat{G}^0))|^{-1}|\pi_0(\kappa_G)|^{-1} \\ |\ker^1(F,Z(\hat{H}))||\overline{Z}(\hat{H})^\Gamma|^{-1}|\mathrm{Aut}(H)/\hat{H}Z(\hat{G}^0)^\Gamma|^{-1}.$$

PROOF: The main point is the formula

$$\tau_1(G^0) \;=\; |\pi_0(Z(\hat{G}^0)^\Gamma)||\ker^1(F,Z(\hat{G}^0))|^{-1}$$

of Sansuc and Kottwitz for the relative Tamagawa number **[12, (5.1.1)]**. From this, it will be a routine matter to derive the expression (3.5) from (3.3). For Lemma 2.1 tells us that

$$|\pi_0(\mathrm{Aut}(H))|^{-1} \;=\; |\mathrm{Aut}(H)/\hat{H}Z(\hat{G}^0)^\Gamma|^{-1}|\hat{H}Z(\hat{G}^0)^\Gamma/\hat{H}A_{\hat{G}^0}|^{-1}\,.$$

Keeping in mind that $A_{\hat{G}^0}$ is the identity component of $Z(\hat{G}^0)^\Gamma$, we deduce that

$$\begin{aligned} &|\hat{H}Z(\hat{G}^0)^\Gamma/\hat{H}A_{\hat{G}^0}|^{-1} \\ &= |Z(\hat{G}^0)^\Gamma/Z(\hat{G}^0)^\Gamma\cap(\hat{H}A_{\hat{G}^0})|^{-1} \\ &= |\pi_0(Z(\hat{G}^0)^\Gamma)|^{-1}|(Z(\hat{G}^0)^\Gamma\cap(\hat{H}A_{\hat{G}^0}))/A_{\hat{G}^0}|\,. \end{aligned}$$

Moreover,

$$\begin{aligned} &|Z(\hat{G}^0)^\Gamma\cap(\hat{H}A_{\hat{G}^0}))/A_{\hat{G}^0}| \\ &= |\hat{H}\cap Z(\hat{G}^0)^\Gamma/\hat{H}\cap A_{\hat{G}^0}| \\ &= |Z(\hat{H})^\Gamma\cap Z(\hat{G})^\Gamma/Z(\hat{H})^\Gamma\cap\kappa_G| \\ &= |\pi_0(Z(\hat{H})^\Gamma\cap Z(\hat{G})^\Gamma)||\pi_0(Z(\hat{H})^\Gamma\cap\kappa_G)|^{-1} \\ &= |\pi_0(Z(\hat{H})^\Gamma)||\overline{Z}(\hat{H})^\Gamma|^{-1}|\pi_0(\kappa_G)|^{-1}|\kappa_G/Z(\hat{H})^\Gamma\cap\kappa_G|\,. \end{aligned}$$

The lemma follows from the formula above for $\tau_1(G^0)$ and its analogue for $\tau_1(H)$. □

§4. THE CONJECTURAL MULTIPLICITY FORMULA

Our goal is to provide some motivation for the conjectures on non-tempered automorphic representations stated in **[1]** and **[3]**. The main global ingredient of the conjectures is a multiplicity formula for automorphic representations in the discrete spectrum. It is a generalization of similar formula for tempered automorphic representations which was implicit in the examples of **[15]** and was stated explicitly in **[12]**. We shall recall the various objects from **[3,** §8] needed to state the formula.

The automorphic representations which occur in the spectral decomposition should be attached to maps

$$\psi : L_F \times SL(2,\mathbb{C}) \rightarrow {}^L G^0 . \tag{4.1}$$

such that the projection onto $\hat{G}^0$ of the image L_F is bounded. Here L_F is hypothetical Langlands group, which we shall assume is an extension of the Weyl group W_F by a compact connected group. The maps themselves are subject to certain conditions. For example, ψ should be globally relevant, in the sense that its image must not lie in a parabolic subgroup of ${}^L G^0$ unless the corresponding parabolic subgroup of G^0 is defined over F. Another condition is designed to insure that ψ parametrizes representations of $G^0(\mathbb{A})$ which lift to $G(\mathbb{A})^+$. Let

$$S_\psi = S_\psi(G)$$

be the set of elements $s \in \hat{G}$ such that each point

$$s\psi(t')s^{-1}\psi(t')^{-1} , \qquad t' \in L_F \times SL(2,\mathbb{C}),$$

belongs to $Z(\hat{G}^0)$, and such that the class of the 1-cocycle

$$t \longrightarrow s\psi(t)s^{-1}\psi(t)^{-1} , \qquad t \in L_F,$$

lies in the subgroup $\ker^1(L_F, Z(\hat{G}^0))$ of $H^1(L_F, Z(\hat{G}^0))$. The condition on ψ is that S_ψ be nonempty. Recall also that two parameters ψ_1 and ψ_2 are equivalent if there is an element $g \in \hat{G}^0$ such that

$$\psi_2(t,u) = g^{-1}\psi_1(t,u)ga(t) , \qquad (t,u) \in L_F \times SL(2,\mathbb{C}), \tag{4.2}$$

where $a(t)$ is a 1-cocycle of L_F in $Z(\hat{G}^0)$ whose class in $H^1(L_F, Z(\hat{G}^0))$ lies in $\ker^1(L_F, Z(\hat{G}^0))$.

Let $\Psi(G)$ denote the set of equivalence classes of maps (4.1) which satisfy the required conditions **[3,** §8]. Let $\Psi_0(G)$ denote the subset of (equivalence classes of) maps $\psi \in \Psi(G)$ such that the set

$$\bar{S}_\psi = \bar{S}_\psi(G) = S_\psi(G)/Z(\hat{G}^0)$$

is finite. In [4] we called these maps *elliptic.* They should parametrize automorphic representations which occur in the discrete spectrum. It will be convenient to define two other subsets of $\Psi(G)$. Let us say that ψ is *weakly elliptic* if the group $\bar{S}_\psi(G^0)$ (obtained by replacing G with the identity component G^0) has finite center. We shall say that ψ is *discrete* if it satisfies the weaker condition that the group

$$\bar{S}_\psi^+ = S_\psi^+/Z(\hat{G}^0)$$

generated by $\bar{S}_\psi$ has finite center. (Keep in mind that $\bar{S}_\psi^+$, S_ψ^+, $\bar{S}_\psi(G^0)$, $S_\psi(G^0)$, etc., are complex reductive Lie groups which are generally not connected.) Let $\Psi_0'(G)$ and $\Psi_{\mathrm{disc}}(G)$ denote the set of (equivalence classes of) maps $\psi \in \Psi(G)$ which are weakly elliptic and discrete, respectively. Then we have embeddings

$$\Psi_0(G) \subset \Psi_0'(G) \subset \Psi_{\mathrm{disc}}(G) \subset \Psi(G) .$$

Let χ be a fixed character on a subgroup X of $Z(G, \mathbb{A})$ which satisfies the conditions of §3. We may as well assume that X is contained in $Z^0(G, \mathbb{A})$, the adèle group of the identity component of $Z(G)$. There is a canonical map from ${}^LG^0$ onto the L-group ${}^LZ^0(G)$ of $Z^0(G)$. The composition of any parameter $\psi \in \Psi(G)$ with the map gives a parameter in $\Psi(Z^0(G))$, and therefore a dual character

$$\zeta_\psi : Z^0(G, F)\backslash Z^0(G, \mathbb{A}) \longrightarrow \mathbb{C}^* .$$

We shall write $\Psi(G, \chi)$, $\Psi_0(G, \chi)$, etc., for the set of parameters ψ in $\Psi(G)$, $\Psi_0(G)$, etc., such that the character ζ_ψ coincides with χ on X.

Suppose that $\psi \in \Psi(G, \chi)$. As in [**3**, §**8**], we can form the finite set

$$\mathcal{S}_\psi = \mathcal{S}_\psi(G) = S_\psi/S_\psi^0 Z(\hat{G}^0) .$$

It is a coset of

$$\mathcal{S}_\psi(G^0) = S_\psi(G^0)/S_\psi^0 Z(\hat{G}^0)$$

in the finite group

$$\mathcal{S}_\psi^+ = \mathcal{S}_\psi(G^+) = S_\psi(G^+)/S_\psi^0 Z(\hat{G}^0) .$$

Now the local conjectures in [**3**, §**6**] assert that there is a set Π_ψ of representations attached to ψ. The elements in Π_ψ should in fact belong to $\Pi_{\mathrm{unit}}(G(\mathbb{A}), \chi)$, the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})^+$ whose restrictions to $G^0(\mathbb{A})$ remain irreducible, and whose central character on X coincides with χ. There should also be a canonical pairing

$$< x, \pi > , \qquad\qquad x \in \mathcal{S}_\psi^+, \ \pi \in \Pi_\psi ,$$

such that the functions $x \to < x, \pi >$ are characters of nonzero finite dimensional representations of $\mathcal{S}_\psi^+$. Finally, the conjectures assert the existence of stable distributions

$$f_1 \longrightarrow f_1^{G_1}(\psi_1) , \qquad\qquad \psi_1 \in \Psi(G_1, \chi_1), \tag{4.3}$$

on $C_c^\infty(G_1(\mathbb{A}), \chi_1)$, for each (G_1, X_1, χ_1) with G_1 connected and quasi-split.

Let us recall how the distributions (4.3) are supposed to behave with respect to endoscopic data. Suppose that s is a semisimple element in $\bar{S}_\psi$. Take $\hat{H}$ to be the connected centralizer in $\hat{G}^0$ of any point in s, and set

$$\mathcal{H} = \hat{H}\psi(L_F \times SL(2, \mathbb{C})) .$$

There is obviously an injection $\hat{H} \to \mathcal{H}$ and a surjection $\mathcal{H} \to W_F$. We are assuming that the kernel of the map $L_F \to W_F$ is connected, and it follows that ψ maps both the kernel and $SL(2, \mathbb{C})$ into $\hat{H}$. Therefore

$$1 \longrightarrow \hat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1$$

is a short exact sequence. We can identify $\hat{H}$, equipped with the canonical L-action of W_F induced by $\mathcal{H}$, with the dual of a well defined quasi-split group H over F. If ξ is the inclusion of $\mathcal{H}$ into ${}^LG^0$, then $(H, \mathcal{H}, s, \xi)$ is an endoscopic datum for G. It has the property that ψ equals $\xi \circ \psi_H$ for some L-homomorphism ψ_H of $L_F \times SL(2, \mathbb{C})$ into $\mathcal{H}$. Now, let (H_1, ξ_1) be any distinguished splitting for the endoscopic datum. We can construct the character χ_1 on a closed subgroup X_1 of $Z(H_1, \mathbb{A})$ as in §2, and from our remarks in §2, we see that the parameter

$$\psi_1 = \xi_1 \circ \psi_H$$

belongs to $\Psi(H_1, \chi_1)$. According to our assumptions on the transfer map $f \to f^{H_1}$, the distribution

$$f \longrightarrow f^{H_1}(\psi_1) , \qquad\qquad f \in C_c^\infty(G(\mathbb{A}), \chi),$$

makes sense. It should satisfy the formula

$$f^{H_1}(\psi_1) = \sum_{\pi \in \{\Pi_\psi\}} < \bar{s}_\psi \bar{s}, \pi > f_G(\pi) , \tag{4.4}$$

where $\bar{s}$ is the image of s in $\mathcal{S}_\psi$, s_ψ is the element

$$\psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

in $S_\psi(G^0)$, and

$$f_G(\pi) = \mathrm{tr}\Big(\int_{X\backslash G(\mathbb{A})} f(x)\pi(x)dx\Big) .$$

As in [**3**], $\{\Pi_\psi\}$ denotes the set of orbits in Π_ψ under $\pi_0(G^+)^*$, the dual of the finite component group, which acts in the obvious way on $\Pi_{\mathrm{unit}}\big(G(\mathbb{A}), \chi\big)$. Recall that the element s_ψ was introduced in [**3, §4**] to describe the signs which occurred on the right hand side of (4.4).

The objects we have just described, namely the packets Π_ψ, the pairings $< x, \pi >$, the stable distributions (4.3), and the formula (4.4), are all consequences of the local conjectures [**3, Conjectures 6.1 and 6.2**]. The adèlic versions described here are simply restricted tensor products of the local versions in [**3**]. We shall assume their existence in what follows.

We should also recall the sign character

$$\varepsilon_\psi : \mathcal{S}_\psi^+ \longrightarrow \{\pm 1\}$$

which occurs in the conjectural multiplicity formula. Set

$$L'_F = L_F \times SL(2, \mathbb{C}) ,$$

and consider the representation

$$\tau_\psi(s, t') = \mathrm{Ad}\big(s\psi(t')\big) , \qquad s \in \bar{S}_\psi^+,\ t' \in L'_F ,$$

of $\bar{S}_\psi^+ \times L'_F$ on the Lie algebra $\hat{\underline{\mathbf{g}}}$ of $\hat{G}$. Let

$$\tau_\psi = \bigoplus_k \tau_k = \bigoplus_k (\lambda_k \otimes \mu_k \otimes \nu_k)$$

be the decomposition of τ_ψ in which λ_k, μ_k and ν_k are irreducible (finite dimensional) representations of $\bar{S}_\psi^+$, L_F and $SL(2,\mathbb{C})$ respectively. The global L-function $L(s,\mu_k)$ will be defined as a product of local L-functions. We shall assume it has analytic continuation and satisfies the functional equation

$$L(s,\mu_k) \;=\; \varepsilon(s,\mu_k)L(1-s,\widetilde{\mu}_k)\ ,$$

where $\varepsilon(s,\mu_k)$ is a finite product of local root numbers. It follows from the functional equation that if μ_k is equivalent to its contragredient $\widetilde{\mu}_k$, then $\varepsilon(\frac{1}{2},\mu_k)=\pm 1$. Let us write $\hat{\underline{\mathbf{g}}}_\psi$ for the direct sum of those irreducible constituents τ_k such that (i) $\mu_k \cong \widetilde{\mu}_k$, (ii) $\varepsilon(\frac{1}{2},\mu_k)=-1$, and (iii) $\dim \nu_k$ is even. The sign character is then given by

$$\varepsilon_\psi(x) \;=\; \varepsilon_\psi^G(x) \;=\; \prod_k \det\bigl(\lambda_k(s)\bigr)\ , \qquad x\in\mathcal{S}_\psi^+\ , \tag{4.5}$$

where the product is taken over those k such that τ_k is contained in $\hat{\underline{\mathbf{g}}}_\psi$, and s is any element in $\bar{S}_\psi^+$ which projects onto x. In other words,

$$\varepsilon_\psi^G(x) \;=\; \det\bigl(s,\ \mathrm{End}_{L_F'}(\hat{\underline{\mathbf{g}}}_\psi)\bigr)\ . \tag{4.5$'$}$$

We could actually have replaced the first condition in the definition of $\hat{\underline{\mathbf{g}}}_\psi$ by the stronger assertion (i$'$) $\tau_k \cong \widetilde{\tau}_k$. Indeed ν_k is always equal to its contragredient, and

$$\det\bigl(\widetilde{\lambda}_k(s)\bigr) \;=\; \det\lambda_k(s)^{-1}\ .$$

Therefore, the contribution to (4.5) of the distinct pairs $(\tau_k,\widetilde{\tau}_k)$ equals 1. It should also be noted that the condition (iii) above is not really necessary. For suppose that τ_k satisfies (i$'$) and (ii), but that $\dim(\nu_k)$ is odd. Then ν_k corresponds to the principal unipotent in an odd orthogonal group. Since μ_k is self-contragredient, its image must be contained in either the orthogonal or the symplectic group. We shall assume the generalization of the theorem of Fröhlich and Queyrot **[6]** which, in view of the sign $\varepsilon(\frac{1}{2},\mu_k)=-1$, implies that μ_k is actually symplectic. Finally, since the representation τ_k is self contragredient and preserves the Killing form, it must be orthogonal. For this to be so, the third representation in the tensor product must actually be symplectic. Therefore $\det\,\lambda_k(s)\ =1$, and τ_k contributes nothing to (4.5). This explains the apparent discrepancy between the present definition (4.5) and the earlier one **[3, (8.4)]**.

If ϕ is any vector in the Hilbert space $L^2(G^0(F)\backslash G^0(\mathbb{A}), \chi^{-1})$, set

$$(R(y)\phi)(x) = \phi(\xi^{-1}xy), \qquad x \in G^0(F)\backslash G^0(\mathbb{A}),$$

for any points $y \in G(\mathbb{A})^+$ and $\xi \in G^+(F)$ such that $\xi^{-1}y$ belongs to $G^0(\mathbb{A})$. This gives an extension of the regular representation to $G(\mathbb{A})^+$. For any representation $\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}), \chi)$, let $m_0(\pi)$ be the multiplicity with which π occurs as a discrete summand of R. Now, suppose that π belongs to a packet Π_ψ, $\psi \in \Psi(G, \chi)$. Then we have the nonnegative integer

$$m_\psi(\pi) = |\mathcal{S}_\psi^+|^{-1} \sum_{x \in \mathcal{S}_\psi^+} \varepsilon_\psi(x) < x, \pi > , \tag{4.6}$$

given explicitly in terms of the pairing. The multiplicity formula amounts to the global component of our conjecture, and will be stated formally as a hypothesis.

HYPOTHESIS 4.1. *For any representation* $\pi \in \Pi_{\mathrm{unit}}(G(\mathbb{A}), \chi)$, *we have the multiplicity formula*

$$m_0(\pi) = \sum_{\psi \in \Psi_0(G,\chi)} m_\psi(\pi) . \tag{4.7}$$

□

Before discussing the conjectures, we shall collect a few simple observations for our later use. Let ψ be a fixed map in $\Psi(G)$. (We shall sometimes not distinguish between a map and its equivalence class.) Let C_ψ denote the centralizer in $\hat{G}^0$ of the image of ψ. Then $C_\psi Z(\hat{G}^0)$ is a subgroup of $S_\psi(\hat{G}^0)$. The quotient

$$\bar{C}_\psi = C_\psi Z(\hat{G}^0)/Z(\hat{G}^0)$$

is a subgroup of $\bar{S}_\psi(G^0)$. Now, the image of the cocycle

$$t \to s\psi(t)s^{-1}\psi(t^{-1}), \qquad t \in L_F,\ s \in S_\psi(G^0),$$

in $H^1(L_F, Z(\hat{G}^0))$ gives a map from $S_\psi(G^0)$ into $\ker^1(L_F, Z(\hat{G}^0))$ whose kernel is easily seen to equal $C_\psi Z(\hat{G}^0)$. We therefore obtain a continuous injection

$$S_\psi(G^0)/C_\psi Z(\hat{G}^0) \cong \bar{S}_\psi(G^0)/\bar{C}_\psi \hookrightarrow \ker^1(L_F, Z(\hat{G}^0)) .$$

According to Lemma 11.2.2 of [**12**], or rather its extension to the hypothetical group L_F, $\ker^1(L_F, Z(\hat{G}^0))$ is isomorphic to $\ker^1(F, Z(\hat{G}^0))$. In particular, $\ker^1(L_F, Z(\hat{G}^0))$ is a finite discrete group. Therefore, the connected component $\bar{S}^0_\psi$ of $\bar{S}_\psi$ maps to the identity element in $\ker^1(L_F, Z(\hat{G}^0))$. We obtain an identity

$$\bar{S}^0_\psi = \bar{C}^0_\psi \tag{4.8}$$

of connected components. In particular, if we set

$$\mathcal{C}_\psi = C_\psi Z(\hat{G}^0)/C^0_\psi Z(\hat{G}^0) = \bar{C}_\psi/\bar{C}^0_\psi ,$$

we can write the injection above as

$$\mathcal{S}_\psi(G^0)/\mathcal{C}_\psi \hookrightarrow \ker^1(L_F, Z(\hat{G}^0)) . \tag{4.9}$$

Suppose that s is a semisimple element in $\bar{S}_\psi$. According to our conventions, $\bar{S}_{\psi,s}$ denotes the centralizer of s in $\bar{S}^0_\psi$, and $\bar{S}^0_{\psi,s}$ is the connected component of 1 in $\bar{S}_{\psi,s}$. We can also take the centralizer $\bar{C}_{\psi,s}$ of s in $\bar{C}^0_\psi$, and its identity component $\bar{C}^0_{\psi,s}$. In §7 we shall use the identities $\bar{S}_{\psi,s} = \bar{C}_{\psi,s}$ and $\bar{S}^0_{\psi,s} = \bar{C}^0_{\psi,s}$. These of course follow immediately from (4.8). We shall also have occasion to consider some slightly different centralizers. Keeping in mind that s is a coset in $\hat{G}/Z(\hat{G}^0)$, we write $S_{\psi,s}$ for the centralizer in S^0_ψ of any element in the coset s. Then

$$S_{\psi,s}Z(\hat{G}^0)/Z(\hat{G}^0)$$

is a subgroup of $\bar{S}_{\psi,s}$, which by Lemma 2.2 is of finite index. In particular, we have an equality

$$\bar{S}^0_{\psi,s} = S^0_{\psi,s}Z(\hat{G}^0)/Z(\hat{G}^0) \tag{4.10}$$

of identity components. Similarly, if $C_{\psi,s}$ denotes the centralizer in C^0_ψ of any element in the coset s, we have

$$\bar{S}^0_{\psi,s} = \bar{C}^0_{\psi,s} = C^0_{\psi,s}Z(\hat{G}^0)/Z(\hat{G}^0) . \tag{4.11}$$

§5. The expansion of $I_{\mathrm{disc},t}(f)$

We have now stated two global hypotheses. As we have already noted, Hypothesis 3.1 should be a consequence of a stable trace formula. Once this is established, one could try to combine the formulas

(3.1) and (3.4) to deduce something approaching the multiplicity formula in Hypothesis 4.1. Our aims in this paper are more modest. We shall simply show that the two hypotheses are compatible. We are actually going to establish that Hypothesis 4.1, together with the local assumptions of §3, §4 and [**3**, §7], implies Hypothesis 3.1. More precisely, we shall show that the formula for $I_{\mathrm{disc},t}(f)$ obtained by combining Hypothesis 4.1 with (3.1) equals the formula for $E_{\mathrm{disc},t}(f)$ provided by the definition (3.4). In the process we shall gain some insight into the role of the sign characters ε_ψ^G.

In this section we shall derive a formula for $I_{\mathrm{disc},t}$ from Hypothesis 4.1. By combining (4.7) with (3.1) we will obtain an expansion for

$$I_{\mathrm{disc},t}(f)\,, \qquad\qquad f \in C_c^\infty\big(G(\mathbb{A}),\chi\big)\,,$$

as a linear combination of irreducible characters. In doing this we will need to apply a local conjecture from [**3**, §7] for the values of normalized intertwining operators.

According to (3.1), $I_{\mathrm{disc},t}(f)$ equals the sum over $\{M\}$ and over $w \in W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}$, of the product of

$$|\pi_0(G^+)|^{-1}|W^G(\underline{\mathbf{a}}_M)|^{-1}|\det(w-1)_{\underline{\mathbf{a}}_M^G}|^{-1}$$

with

$$\mathrm{tr}\big(M(w,0)\rho_{P,t}(0,f)\big)\,. \tag{5.1}$$

Our first task is to expand (5.1) into a linear combination of irreducible characters.

For any M, and $w \in W^G(\underline{\mathbf{a}}_M)$, we can form the component $M_w = M\cdot w$. It satisfies the same conditions as G. Now, recall that $\rho_{P,t}(0)$ is the representation of $G(\mathbb{A})^+$ obtained by parabolic induction from the action of $M(\mathbb{A})$ on

$$L^2_{\mathrm{disc},t}\big(M(F)\backslash M(\mathbb{A}),\chi_M^{-1}\big)\,. \tag{5.2}$$

This representation of $M(\mathbb{A})$ has a canonical extension to the group $M_w(\mathbb{A})^+$ generated by the coset $M_w(\mathbb{A}) = M(\mathbb{A})w$. In particular, the space (5.2) can be decomposed into a direct sum of subspaces corresponding to irreducible representations σ_w of $M_w(\mathbb{A})^+$. There is a similar decomposition

$$\mathcal{H}_{P,t} = \bigoplus_{\sigma_w}\mathcal{H}_P(\sigma_w)$$

of the induced space into subspaces which are invariant under the operator

$$(5.3) \qquad M(w,0)\rho_{P,t}(0,f) \; .$$

If the restriction of σ_w to $M(\mathbb{A})$ is reducible, one sees easily that the trace of the operator (5.3) on $\mathcal{H}_P(\sigma_w)$ vanishes. Therefore, in computing the full trace (5.1), we need only consider representations σ_w which belong to the space we have denoted by $\Pi_{\text{unit}}\big(M_w(\mathbb{A}),\chi_M^{-1}\big)$.

According to Hypothesis 4.1 (applied to M_w rather than G), the multiplicity with which a representation $\sigma_w \in \Pi_{\text{unit}}\big(M_w(\mathbb{A}),\chi_M^{-1}\big)$ occurs in (5.2) equals

$$\sum_{\psi_w \in \Psi_0(M_w,\chi_M,t)} m_{\psi_w}(\sigma_w) \; ,$$

where $m_{\psi_w}(\sigma_w)$ is the nonnegative integer defined by (4.6). We have written $\Psi_0(M_w,\chi_M,t)$ to denote the set of parameters in $\Psi_0(M_w,\chi_M)$ whose Archimedean infinitesimal character has absolute value t. Any pair ψ_w and σ_w, with $m_{\psi_w}(\sigma_w) \neq 0$, determines a subspace of (5.2), and also a subspace of the induced space $\mathcal{H}_{P,t}$. The restriction of (5.3) to this latter subspace can be expressed in terms of the operators studied in [**3**, §7]. It equals an expression

$$(5.4) \qquad m_{\psi_w}(\sigma_w)r(\psi_w)\big(R_P(\sigma_w,\psi_w)\mathcal{I}_P(\sigma,f)\big) \; ,$$

whose constituents we shall describe in a moment. The trace (5.1) becomes the sum over $\psi_w \in \Psi_0(M_w,\chi_M,t)$ and $\sigma_w \in \Pi_{\psi_w}$ of the trace of the expression (5.4).

Given M and P, it is convenient to fix a dual parabolic subgroup ${}^LP = \hat{P} \rtimes W_F$ in ${}^LG^0$ with Levi component ${}^LM = \hat{M} \rtimes W_F$. The choice of P and ${}^LP$ determines an embedding of the L-group ${}^LM$ into ${}^LG^0$. It also allows us to identify $W^G(\underline{\mathrm{a}}_M)$ with the dual Weyl set

$$\hat{W}^G(\underline{\mathrm{a}}_M) \;=\; \mathrm{Norm}(A_{\hat{M}},\hat{G})/\hat{M} \; .$$

Returning to (5.4), we note that $\mathcal{I}_P(\sigma)$ stands for the induced representation of $G(\mathbb{A})^+$ obtained from the restriction σ of σ_w to $M(\mathbb{A})$. The operator

$$R_P(\sigma_w,\psi_w) \;=\; \bigotimes_v R_P(\sigma_{w,v},\psi_{w,v})$$

is a tensor product of local normalized intertwining operators defined in [**3, (7.4)**]. When this operator is evaluated at a smooth vector in $\mathcal{H}_{P,t}$, almost all the terms in the product reduce to 1. Finally, the scalar $r(\psi_w)$ in (5.4) is obtained from an infinite product of local normalizing functions of the form [**3, (7.2)**]. It equals

$$\lim_{\lambda\to 0}\left(L(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})\varepsilon(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})^{-1}L(1,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})^{-1}\right),$$

where $\phi_{\psi_w,\lambda}$ is the twist of the global parameter

$$\phi_{\psi_w}\,:\, t\ \longrightarrow\ \psi_w\left(t,\begin{pmatrix}|t|^{\frac{1}{2}} & 0\\ 0 & |t|^{-\frac{1}{2}}\end{pmatrix}\right), \qquad t\in L_F,$$

by the vector λ in

$$\underline{\mathbf{a}}^*_{M,\mathbb{C}} \;=\; X(M)_F\otimes\mathbb{C}\;\cong\; X_*(A_{\hat{M}})\otimes\mathbb{C},$$

and $\widetilde{\rho}_{P,w}$ is the contragredient of the adjoint representation of ${}^LM$ on

$$w^{-1}\underline{\hat{\mathfrak{n}}}_P w/w^{-1}\underline{\hat{\mathfrak{n}}}_P w\,\cap\,\underline{\hat{\mathfrak{n}}}_P\,.$$

Here $\underline{\hat{\mathfrak{n}}}_P$ stands for the Lie algebra of the unipotent radical of ${}^LP$. Applying the anticipated functional equation

$$L(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda}) \;=\; \varepsilon(0,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})L(1,\rho_{P,w}\circ\phi_{\psi_w,\lambda}),$$

we write

$$r(\psi_w) \;=\; \lim_{\lambda\to 0}\left(L(1,\rho_{P,w}\circ\phi_{\psi_w,\lambda})L(1,\widetilde{\rho}_{P,w}\circ\phi_{\psi_w,\lambda})^{-1}\right). \tag{5.5}$$

(See [**16, Appendix 2**].)

Having described the terms in (5.4), we go back to the expression we have obtained for (5.1). Recall [**3, §7**] that

$$R_P(\zeta\sigma_w,\psi_w) \;=\; \zeta(M_w)R_P(\sigma_w,\psi_w),$$

for any character ζ in

$$\pi_0(M_w^+)^* \;=\; \mathrm{Hom}(M_w^+/M_w^0,\mathbb{C}^*)\,.$$

This allows us to write (5.1) as the sum over $\psi_w \in \Psi_0(M_w, \chi_M, t)$ and over the orbits $\{\sigma_w\} \in \{\Pi_{\psi_w}\}$ of $\pi_0(M_w^+)^*$ in Π_{ψ_w}, of the expression

$$m'_{\psi_w}(\sigma_w) r(\psi_w) \mathrm{tr}\big(R_P(\sigma_w, \psi_w)\mathcal{I}_P(\sigma, f)\big) .$$

where

$$m'_{\psi_w}(\sigma_w) = \sum_{\zeta \in \pi_0(M_w^+)^*} m_{\psi_w}(\zeta\sigma_w)\zeta(M_w) .$$

Applying Fourier inversion on $\pi_0(M_w^+)$ to the formula (4.6) (with G replaced by M_w), while taking into account the property (i) of the local Conjecture 6.1 in [**3**], we obtain

$$m'_{\psi_w}(\sigma_w) = |\mathcal{S}_{\psi_w}|^{-1} \sum_{u \in \mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) < u, \sigma_w > .$$

Therefore (5.1) equals

$$\sum_{\psi_w} \sum_{\{\sigma_w\}} |\mathcal{S}_{\psi_w}|^{-1} \sum_{u \in \mathcal{S}_{\psi_w}}$$
$$\varepsilon_{\psi_w}(u) r(\psi_w) < u, \sigma_w > \mathrm{tr}\big(R_P(\sigma_w, \psi_w)\mathcal{I}_P(\sigma, f)\big) .$$

Suppose that ψ_w belongs to $\Psi_0(M_w, \chi_M, t)$. Let ψ denote the composition of ψ_w with our embedding ${}^LM \subset {}^LG^0$. We claim that ψ is well defined (as an equivalence class of parameters) in $\Psi(G, \chi, t)$. Recalling (§4) the definition of equivalent parameters, we note that it is enough to show that the map

$$\mathrm{ker}^1(F, Z(\hat{G}^0)) \longrightarrow \mathrm{ker}^1(F, Z(\hat{M})) \tag{5.6}$$

is an isomorphism. By the obvious transitivity property, we can in fact assume that M is minimal, and hence a torus. Then $Z(\hat{M})/Z(\hat{G}^0)$ is a maximal torus in an adjoint group, on which the Galois action is dual to a direct sum of permutation representations. The bijectivity of (5.6) then follows from the exact sequence

$$\pi_0\big((Z(\hat{M})/Z(\hat{G}^0))^\Gamma\big) \to H^1(F, Z(\hat{G}^0))$$
$$\to H^1(F, Z(\hat{M})) \to H^1(F, Z(\hat{M})/Z(\hat{G}^0)) ,$$

and its analogues for the completions of F. (See the proof of Lemma 4.3.2(a) of [**12**].) This proves the claim.

Thus, ψ_w maps to an element ψ in $\Psi(G,\chi,t)$, to which we can associate the objects $\mathcal{S}_\psi = \mathcal{S}_\psi(G)$, $\Pi_\psi = \Pi_\psi(G)$ and $\varepsilon_\psi = \varepsilon_\psi^G$ for G. The next step is to apply a conjectural formula [**3**, §7] for the trace of the normalized intertwining operators in terms of the pairing on $\mathcal{S}_\psi \times \Pi_\psi$. As it is stated in [**3**], the formula applies to the local intertwining operators and pairings, but the product over all valuations gives a formula for the global objects. In fact, certain constants in the local formula (namely, $c(\sigma_\chi, n_w)$, $\lambda_w(\psi_F)$ and $c(\pi_\chi, n_G)$, in the notation of [**3**, §7]) have the property that their products over all valuations equal 1. The global formula is therefore simpler. If ψ_M denotes the parameter ψ_w, but regarded as an element in $\Psi(M)$ rather than $\Psi(M_w)$, then the orbits $\{\sigma_w\}$ above will be in bijective correspondence with the representations $\sigma \in \Pi_{\psi_M}$ which extend to $M_w(\mathbb{A})^+$. It follows from Conjecture 7.1 of [**3**] (and also the two remarks made after the conjecture), that

$$\sum_\sigma < u, \sigma_w > \operatorname{tr}\big(R_P(\sigma_w, \psi_w)\mathcal{I}_P(\sigma, f)\big)$$

equals

$$\sum_{\pi\in\Pi_\psi} < x_u, \pi > f_G(\pi) \ ,$$

where x_u stands for the image in $\mathcal{S}_\psi$ of the point $u \in \mathcal{S}_{\psi_w}$.

We have now obtained an expansion

$$\sum_{\psi_w\in\Psi_0(M_w,\chi_M,t)} |\mathcal{S}_{\psi_w}|^{-1} \sum_{u\in\mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) r(\psi_w) \sum_{\pi\in\Pi_\psi} < x_u, \pi > f_G(\pi) \ ,$$

for the trace (5.1). We shall substitute this into our formula for $I_{\mathrm{disc},t}(f)$. Observe that

$$\sum_{\pi\in\Pi_\psi} < x_u, \pi > f_G(\pi) = |\pi_0(G^+)| \sum_{\pi\in\{\Pi_\psi\}} < x_u, \pi > f_G(\pi) \ .$$

Therefore, $I_{\mathrm{disc},t}(f)$ equals the triple sum over $\{M\}$, $w \in W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}$ and $\psi_w \in \Psi_0(M_w, \chi_M, t)$ of the product of

$$|W^G(\underline{\mathbf{a}}_M)|^{-1} |\det(w-1)_{\underline{\mathbf{a}}_M^G}|^{-1}$$

with

$$(5.7) \qquad |\mathcal{S}_{\psi_w}|^{-1} \sum_{u\in\mathcal{S}_{\psi_w}} \varepsilon_{\psi_w}(u) r(\psi_w) \sum_{\pi\in\{\Pi_\psi\}} < x_u, \pi > f_G(\pi) \ .$$

We propose to interchange the sum over the parameters with the sums over M and w. The outer sum will then have to be over all parameters $\psi \in \Psi(G, \chi, t)$. For any ψ there will be an M, unique up to conjugacy, such that ψ is the composition of a parameter $\psi_M \in \Psi_0(M)$ with the embedding ${}^LM \subset {}^LG^0$. The condition that ψ_M also belong to $\Psi_0(M_w)$, for a given $w \in W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}$, is that the set $S_{\psi_M}(M_w)$ be nonempty. There is another way to state this. Recall that we have identified $W^G(\underline{\mathbf{a}}_M)$ with the dual Weyl set $\hat{W}^G(\underline{\mathbf{a}}_M)$. Then $S_{\psi_M}(M_w)$ is nonempty if and only if w belongs to the subset $W_\psi = W_\psi(G)$ of elements in $\hat{W}^G(\underline{\mathbf{a}}_M)$ which, modulo the isomorphic groups (5.6), centralize the image of ψ. It will be convenient for us to regard this subset W_ψ as the full Weyl set associated to $\bar{S}_\psi = S_\psi/Z(\hat{G}^0)$. It acts on the maximal torus

$$\bar{T}_\psi = A_{\hat{M}} Z(\hat{G}^0)/Z(\hat{G}^0)$$

of the connected component

$$\bar{S}^0_\psi = S^0_\psi Z(\hat{G}^0)/Z(\hat{G}^0) .$$

For any $w \in W_\psi$, we shall write $\det(w-1)$ for the determinant of $(w-1)$, acting on the Lie algebra of $\bar{T}_\psi$. One sees easily that

$$|\det(w-1)| = |\det(w-1)_{\underline{\mathbf{a}}_M^{G^0}}| = |\det(w-1)_{\underline{\mathbf{a}}_M^G}||\det(w-1)_{\underline{\mathbf{a}}_G^{G^0}}|^{-1} .$$

Now it is well known that $|\det(w-1)_{\underline{\mathbf{a}}_G^{G^0}}|$ equals the order of the kernel of w, acting on the dual torus

$$\left(Z(\hat{G}^0)^\Gamma\right)^0/\left(Z(\hat{G})^\Gamma\right)^0 .$$

(See **[23, II.1.7]**.) The action of w on this torus is of course independent of w, and the kernel is just the finite group of components in

$$\kappa_G = Z(\hat{G})^\Gamma \cap \left(Z(\hat{G}^0)^\Gamma\right)^0 .$$

Therefore

$$|\det(w-1)_{\underline{\mathbf{a}}_M^G}|^{-1} = |\det(w-1)|^{-1}|\pi_0(\kappa_G)|^{-1} .$$

In particular, w belongs to $W^G(\underline{\mathbf{a}}_M)_{\mathrm{reg}}$ if and only if it lies in the set

$$W_{\psi,\mathrm{reg}} = \{w \in W_\psi : \det(w-1) \neq 0\}$$

of regular elements in W_ψ. When this is so, the associated parameter in $\Psi_0(M_w)$ in fact belongs to $\Psi_0(M_w, \chi_M, t)$. We shall denote it by ψ_w, as above.

Actually, ψ_w is not uniquely determined by ψ and w. We must decide how many parameters in $\Psi_0(M_w, \chi_M, t)$ lie in the equivalence class of ψ. Keeping in mind the isomorphism (5.6), we see that two parameters ψ_w map to the same ψ if and only if they are conjugate by an element in $\hat{W}^G(\underline{\mathfrak{a}}_M)$. Moreover, two such conjugates are equivalent in $\Psi_0(M_w, \chi_M, t)$ if and only if they differ by an element in $W_\psi(G^0)$. The number of ψ_w associated to ψ is therefore

$$|\hat{W}^G(\underline{\mathfrak{a}}_M)||W_\psi(G^0)|^{-1} = |W^G(\underline{\mathfrak{a}}_M)||W_\psi|^{-1} .$$

Thus, our interchange of summation expresses $I_{\mathrm{disc},t}(f)$ as the sum over $\psi \in \Psi(G, \chi, t)$ and $w \in W_{\psi,\mathrm{reg}}$ of the product of

$$|\pi_0(\kappa_G)|^{-1}|W_\psi|^{-1}|\det(w-1)|^{-1}$$

with (5.7).

Suppose that $\psi \in \Psi(G)$. As in the case of a local parameter, we can define the finite set

$$\begin{aligned} \mathcal{N}_\psi = \mathcal{N}_\psi(G) &= \mathrm{Norm}(\bar{T}_\psi, \bar{S}_\psi)/\bar{T}_\psi \\ &= \mathrm{Norm}(A_{\hat{M}}, S_\psi)/A_{\hat{M}} Z(\hat{G}^0) . \end{aligned}$$

Let $\mathcal{S}^1_\psi$ be the subgroup of elements in $\mathcal{N}_\psi(G^0)$ which act trivially on $\bar{T}_\psi$. This group acts freely by translation on $\mathcal{N}_\psi$, and the set of orbits can be identified canonically with W_ψ. One sees easily from the isomorphism (5.6) that

$$\mathcal{S}_{\psi_w} = \mathcal{S}_{\psi_M} \cdot w = \mathcal{S}^1_\psi \cdot w , \qquad w \in W_\psi ,$$

for M as above. We also have the Weyl group

$$\begin{aligned} W^0_\psi &= \mathrm{Norm}(\bar{T}_\psi, \bar{S}^0_\psi)/\bar{T}_\psi \\ &= \mathrm{Norm}(A_{\hat{M}}, S^0_\psi)/A_{\hat{M}} Z(\hat{G}^0) \end{aligned}$$

of the connected component $\bar{S}^0_\psi$. This too acts freely on $\mathcal{N}_\psi$, and the set of orbits can be identified canonically with $\mathcal{S}_\psi$. We obtain a commutative diagram

$$
\begin{tikzcd}
& & 1 \arrow[d] & 1 \arrow[d] & \\
& & W^0_\psi \arrow[r, equal] \arrow[d] & W^0_\psi \arrow[d] & \\
1 \arrow[r] & \mathcal{S}^1_\psi \arrow[r] \arrow[d, equal] & \mathcal{N}_\psi \arrow[r] \arrow[d, shift right] & W_\psi \arrow[r] \arrow[d, shift right] & 1 \\
1 \arrow[r] & \mathcal{S}^1_\psi \arrow[r] & \mathcal{S}_\psi \arrow[r] \arrow[d] \arrow[u, dashed, shift right] & R_\psi \arrow[r] \arrow[d] \arrow[u, dashed, shift right] & 1 \\
& & 1 & 1 &
\end{tikzcd}
$$

as in the local case [**3, (7.1)**]. The dotted arrows stand for splittings of short exact sequences determined by a fixed Borel subgroup of $\bar{S}^0_\psi$ containing $\bar{T}_\psi$. Similarly, one obtains a commutative diagram of groups if one replaces $\mathcal{N}_\psi$, $\mathcal{S}_\psi$, W_ψ and R_ψ by the respective finite groups $\mathcal{N}^+_\psi$, $\mathcal{S}^+_\psi$, W^+_ψ and R^+_ψ they generate. We shall write $u \to x_u$ and $u \to w_u$ for the projections of $\mathcal{N}^+_\psi$ onto $\mathcal{S}^+_\psi$ and W^+_ψ. Notice that if x is any element in $\mathcal{S}_\psi$, and $\mathcal{N}(x)$ is the corresponding orbit of W^0_ψ in $\mathcal{N}_\psi$, the second projection maps $\mathcal{N}(x)$ *bijectively* onto a subset $W(x)$ of W_ψ. We shall set

$$W(x)_{\mathrm{reg}} = W(x) \cap W_{\psi,\mathrm{reg}}$$

and

$$\mathcal{N}(x)_{\mathrm{reg}} = \{u \in \mathcal{N}(x) : \ w_u \in W(x)_{\mathrm{reg}}\} .$$

We apply these observations to our formula for $I_{\mathrm{disc},t}(f)$. According to the horizontal exact sequence for $\mathcal{N}_\psi$ in the diagram, the double sum over $w \in W_{\psi,\mathrm{reg}}$ and $u \in \mathcal{S}_{\psi_w} = \mathcal{S}^1_\psi w$ can be combined into a simple sum over the regular elements in $\mathcal{N}_\psi$. We shall write

$$\varepsilon^M_\psi(u) = \varepsilon_{\psi_w}(u) \tag{5.8}$$

for any point $u \in \mathcal{N}_\psi$ whose projection onto W_ψ equals w. We also set

$$r_\psi(w) = r(\psi_w) . \tag{5.9}$$

Then ε_ψ^M and r_ψ extend to well defined characters on $\mathcal{N}_\psi^+$ and W_ψ^+ respectively. The simple sum in its turn can be decomposed by the corresponding vertical exact sequence into a double sum over $x \in \mathcal{S}_\psi$ and $u \in \mathcal{N}(x)_{\mathrm{reg}}$. Observe that

$$\begin{aligned} |W_\psi||\mathcal{S}_{\psi_w}| \;=\; |W_\psi||\mathcal{S}_\psi^1| &= |\mathcal{N}_\psi| \\ &= |\mathcal{S}_\psi||W_\psi^0| \\ &= |\mathcal{S}_\psi||W(x)|\;. \end{aligned}$$

It follows that $I_{\mathrm{disc},t}(f)$ equals the sum over ψ in $\Psi(G,\chi,t)$ of the product of

$$|\pi_0(\kappa_G)|^{-1}|\mathcal{S}_\psi|^{-1}$$

with

$$\sum_{x\in\mathcal{S}_\psi} |W(x)|^{-1} \sum_{u\in\mathcal{N}(x)_{\mathrm{reg}}}$$
$$\varepsilon_\psi^M(u) r_\psi(w_u)|\det(w_u-1)|^{-1} \sum_{\pi\in\{\Pi_\psi\}} <x,\pi> f_G(\pi)\;.$$

Any element $w \in W_\psi^+$ operates on $\bar{T}_\psi$. It preserves the set Σ_ψ of roots of $(\bar{S}_\psi^0, \bar{T}_\psi)$. We shall simply write $\varepsilon(w)$ for the usual sign attached to this permuation, namely the number (-1) raised to the power

$$|(-\Sigma_\psi^+) \cap (w\Sigma_\psi^+)|\;,$$

where Σ_ψ^+ is the set of positive roots in Σ_ψ relative to some order.

PROPOSITION 5.1. *We have*

$$r_\psi(w_u) \;=\; \varepsilon(w_u)\varepsilon_\psi^G(x_u)\varepsilon_\psi^M(u)^{-1}$$

for any element $u \in \mathcal{N}_\psi^+$.

This proposition is the motivation for the introduction of the characters ε_ψ into the multiplicity formula of Hypothesis 4.1. We shall prove it in the next section. In the meantime, we can combine it with our formula for $I_{\mathrm{disc},t}(f)$.

PROPOSITION 5.2. *The distribution $I_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with*

$$(5.10)\qquad \sum_{\psi\in\Psi(G,\chi,t)}\ \sum_{\pi\in\{\Pi_\psi\}} |\mathcal{S}_\psi|^{-1} \sum_{x\in\mathcal{S}_\psi} \varepsilon_\psi^G(x) i(x) < x, \pi > f_G(\pi)\ ,$$

where

$$(5.11)\quad i(x) \ =\ |W(x)|^{-1} \sum_{w\in W(x)_{\mathrm{reg}}} \varepsilon(w)|\det(w-1)|^{-1}\ . \qquad \square$$

§6. THE SIGN CHARACTERS ε_ψ AND $\mathbf{r}_\psi$

In this section we shall pause to study the characters ε_ψ and r_ψ. Our goal is to prove Proposition 5.1. Recall that $\varepsilon_\psi = \varepsilon_\psi^G$ is the one-dimensional character (4.5) on $\mathcal{S}_\psi^+ = \mathcal{S}_\psi(G^+)$ which comes into the conjectural multiplicity formula. The function r_ψ is the one dimensional character ((5.5), (5.9)) on $W_\psi^+ = W_\psi(G^+)$ defined by the global normalizing factors. We have seen that $\mathcal{S}_\psi^+$ and W_ψ^+ are both quotients of the finite group $\mathcal{N}_\psi^+$. We can therefore identify ε_ψ and r_ψ with characters on $\mathcal{N}_\psi^+$. Proposition 5.1 can be regarded as a formula for the quotient of these two characters.

We shall begin by expressing r_ψ in terms of the orders of certain L-functions at $s = 1$. Let $\hat{\Sigma}_M$ denote the set of roots of $(\hat{G}^0, A_{\hat{M}})$. For each $\hat{\alpha} \in \hat{\Sigma}_M$ there is a representation $\rho_{\hat{\alpha}}$ of ${}^LM$ on the root space $\hat{\underline{\mathbf{g}}}_{\hat{\alpha}}$. Having already fixed the dual parabolic subgroups P and ${}^LP = \hat{P} \rtimes W_F$, we shall write $\hat{\Sigma}_P \subset \hat{\Sigma}_M$ for the set of roots of $(\hat{P}, A_{\hat{M}})$. Fix an element $w \in W_\psi^+$, and set

$$\hat{\Sigma}_{P,w} \ =\ \{\hat{\alpha} \in \hat{\Sigma}_P :\ w\hat{\alpha} \in (-\Sigma_P)\}\ .$$

Then there is a decomposition

$$\rho_{P,w} \ =\ \bigoplus_{\hat{\alpha}\in\hat{\Sigma}_{P,w}} \rho_{-\hat{\alpha}}$$

for the representation of ${}^LM$ which occurs in (5.5). Notice that the Killing form provides an isomorphism between $\rho_{-\hat{\alpha}}$ and the contragredient $\widetilde{\rho}_{\hat{\alpha}}$. The formula (5.5) becomes

$$(6.1)\ r_\psi(w) \ =\ \lim_{\lambda\to 0} \prod_{\alpha\in\hat{\Sigma}_{P,w}} L\big(1-\lambda(\hat{\alpha}), \rho_{\hat{\alpha}}\circ\phi_\psi\big) L\big(1+\lambda(\hat{\alpha}), \rho_{\hat{\alpha}}\circ\phi_\psi\big)^{-1}\ ,$$

since

$$L(1, \rho_{\hat{\alpha}} \circ \phi_{\psi,\lambda}) = L\big(1 + \lambda(\hat{\alpha}), \rho_{\hat{\alpha}} \circ \phi_\psi\big) .$$

We are going to show that $r_\psi(w)$ equals the character

$$\prod_{\hat{\alpha} \in \hat{\Sigma}_{P,w}} (-1)^{\mathrm{ord}_{s=1}\left(L(s,\rho_{\hat{\alpha}} \circ \phi_\psi)\right)} . \tag{6.2}$$

We claim that for every root $\hat{\alpha} \in \Sigma_{P,w}$, there is also a root $\hat{\alpha}_1 \in \hat{\Sigma}_{P,w}$ such that

$$\widetilde{\rho}_{\hat{\alpha}} \circ \psi \cong \rho_{\hat{\alpha}_1} \circ \psi .$$

To this end, observe that

$$\rho_{\hat{\alpha}} \circ \psi \cong \rho_{w\hat{\alpha}} \circ \mathrm{ad}(w) \circ \psi \cong \rho_{w\hat{\alpha}} \circ \psi .$$

The first of these isomorphisms is given by the intertwining map

$$\mathrm{Ad}(w) : \underline{\mathfrak{g}}_{\hat{\alpha}} \longrightarrow \underline{\mathfrak{g}}_{w\hat{\alpha}} ,$$

and the second follows from the fact that the image of $w \in W_\psi^+$ under the adjoint representation commutes with the image of $L_F \times SL(2, \mathbb{C})$. Now, consider the orbit

$$\mathcal{O}_w(\hat{\alpha}) = \{w^j \hat{\alpha} : j \in \mathbb{Z}\}$$

of $\hat{\alpha}$ under the cyclic group generated by w. The representations

$$\{\rho_{\hat{\beta}} \circ \psi : \hat{\beta} \in \mathcal{O}_w(\hat{\alpha})\}$$

are all equivalent, and are also equivalent to the contragredients

$$\{\widetilde{\rho}_{\hat{\beta}} \circ \psi : -\beta \in \mathcal{O}_w(\hat{\alpha})\} .$$

But after a moment's thought, we see that the intersections of $\mathcal{O}_w(\hat{\alpha})$ with $\hat{\Sigma}_{P,w}$ and $(-\hat{\Sigma}_{P,w})$ contain an equal number of roots. The claim follows. In particular, the terms in the product in (6.1) can be grouped in such a way that $\rho_{\hat{\alpha}}$ appears in the numerator as well as the denominator. This leads directly to the formula (6.2) for $r_\psi(w)$.

Recall that

$$(\rho_{\hat{\alpha}} \circ \phi_\psi)(t) = \rho_{\hat{\alpha}}\left(\psi\left(t, \begin{pmatrix} |t|^{\frac{1}{2}} & 0 \\ 0 & |t|^{-\frac{1}{2}} \end{pmatrix}\right)\right) , \qquad t \in L_F .$$

Now there is a decomposition

$$(\rho_{\hat{\alpha}} \circ \psi) = \bigoplus_{j \in J(\hat{\alpha})} (\mu_j \otimes \nu_j) ,$$

where each μ_j is an irreducible *unitary* representation of L_F and ν_j is an irreducible representation of $SL(2,\mathbb{C})$. Therefore, (6.2) can be written as a product

$$\prod_{\hat{\alpha} \in \hat{\Sigma}_{P,w}} \prod_{j \in J(\hat{\alpha})} (-1)^{\mathrm{ord}_{s=1}\left(L(s,\mu_j \otimes \nu_j)\right)} \tag{6.3}$$

where $L(s,\mu_j \times \nu_j)$ stands for the L-function of the representation

$$t \longrightarrow \mu_j(t)\nu_j\begin{pmatrix} |t|^{\frac{1}{2}} & 0 \\ 0 & |t|^{-\frac{1}{2}} \end{pmatrix} , \qquad t \in L_F ,$$

of L_F. From the discussion above we see that the contragredient acts as an involution $\mu_j \otimes \nu_j \to \widetilde{\mu}_j \otimes \widetilde{\nu}_j$ on the constituents of $\rho_{P,w}$. It is, moreover, an easy consequence of the unitarity of μ_j that

$$\overline{L(s,\mu_j \otimes \nu_j)} = L(\bar{s},\widetilde{\mu}_j \otimes \widetilde{\nu}_j) ,$$

so that

$$\mathrm{ord}_{s=1}\big(L(s,\mu_j \otimes \nu_j)\big) = \mathrm{ord}_{s=1}\big(L(s,\widetilde{\mu}_j \otimes \widetilde{\nu}_j)\big) .$$

In particular, the contribution to (6.3) of a distinct pair of contragredient constituents cancels. The product (6.3) need only be taken over those constituents with

$$\mu_j \otimes \nu_j \cong \widetilde{\mu}_j \otimes \widetilde{\nu}_j .$$

Since any finite dimensional representation of $SL(2,\mathbb{C})$ is self contragredient, the condition is just $\mu_j \cong \widetilde{\mu}_j$.

The question then is to determine the sign

$$(-1)^{\mathrm{ord}_{s=1}\; L(s,\mu\otimes\nu)} , \tag{6.4}$$

for any irreducible representation $\mu \otimes \nu$ of $L_F \times SL(2,\mathbb{C})$ such that μ is unitary, and $\widetilde{\mu} \cong \mu$. Set $m = \deg(\mu)$ and $n = \deg(\nu)$. Then ν maps the matrix

$$\begin{pmatrix} |t|^{\frac{1}{2}} & 0 \\ 0 & |t|^{-\frac{1}{2}} \end{pmatrix}, \qquad t \in L_F,$$

to the diagonal matrix

$$\mathrm{diag}(|t|^{\frac{1}{2}(n-1)}, |t|^{\frac{1}{2}(n-3)}, \dots, |t|^{-\frac{1}{2}(n-1)})$$

in $GL(n,\mathbb{C})$. Therefore

$$L(s, \mu \otimes \nu) = \prod_{i=1}^{n} L(s + \frac{1}{2}(n-2i+1), \mu).$$

We must therefore describe the order of zero or pole of $L(s,\mu)$ at any real half-integer.

Hypothesis 4.1 includes the global Langlands correspondence for $GL(m)$, which asserts that

$$L(s,\mu) = L(s,\pi)$$

for some unitary, cuspidal automorphic representation π of $GL(m,\mathbb{A})$. (See [**3, §2**].) Then $L(s,\mu)$ can have a real pole only if μ is the trivial one dimensional representation, in which case there is a simple pole at $s = 0$ and $s = 1$ [**7, Corollary 13.8**]. Results of Jacquet and Shalika [**8, Theorem (1.3)**], [**9, Theorem 5.3**] imply further that the only possible zero of $L(s,\mu)$ at a real half integer is at $s = \frac{1}{2}$, the center of the critical strip. The poles of $L(s,\mu)$ will contribute to (6.4) if n is odd. However, if μ is trivial and n is of odd dimension greater than 1, the poles at 0 and 1 will both contribute, and their effect will cancel. The zeros of $L(s,\mu)$ will contribute to (6.4) if n is even. From the functional equation

$$L(s,\mu) = \varepsilon(s,\mu)L(1-s,\mu),$$

we see that $L(s,\mu)$ has a zero at $s = \frac{1}{2}$ of even or odd order, according to whether $\varepsilon(\frac{1}{2},\mu)$ equals $+1$ or -1.

We have established

LEMMA 6.1. *If $n = \deg(\nu)$ is even, the sign (6.4) equals $\varepsilon(\frac{1}{2}, \mu)$. If n is odd, (6.4) equals 1 unless $\mu \otimes \nu$ is the trivial representation of $L_F \times SL(2, \mathbb{C})$, in which case (6.4) equals (-1).* $\square$

If we substitute the formula of Lemma 6.1 into the product (6.3), we obtain a new expression for $r_\psi(w)$. To describe this in a convenient way, we shall define a character $\varepsilon_\psi^{G/M}$ which is closely related to the original characters ε_ψ^G and ε_ψ^M. Let $\hat{\underline{\mathfrak{m}}}$ denote the Lie algebra of $\hat{M}$, and let $\mathrm{Ad}_{\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}}}$ denote the adjoint representation of ${}^L M$ on $\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}}$. The group

$$\bar{N}_\psi^+ = \mathrm{Norm}(\bar{T}_\psi, \bar{S}_\psi^+) = \mathrm{Norm}(A_{\hat{M}}, S_\psi^+)/Z(\hat{G}^0)$$

also acts by the adjoint action on $\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}}$, and it commutes with the composite representation $\mathrm{Ad}_{\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}}} \circ \psi$ of $L'_F = L_F \times SL(2, \mathbb{C})$. Now, we have a decomposition

$$\mathrm{Ad}_{\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}}} \circ \psi = \bigoplus_{\hat{\alpha} \in \hat{\Sigma}_M} \bigoplus_{j \in J(\hat{\alpha})} (\mu_j \otimes \nu_j)$$

into irreducible representations of L'_F. Let us write $(\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}})_\psi$ for the direct sum of those irreducible constituents $\mu_j \otimes \nu_j$ such that (i) $\widetilde{\mu}_j \cong \mu_j$, (ii) $\varepsilon(\frac{1}{2}, \mu_j) = -1$, and (iii) $\deg(\nu_j)$ is even. Then $(\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}})_\psi$ is an invariant subspace of both L'_F and $\bar{N}_\psi^+$. Define

$$\varepsilon_\psi^{G/M}(u) = \det\Big(\widetilde{u}, \mathrm{End}_{L'_F}\big((\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}})_\psi\big)\Big), \qquad u \in \mathcal{N}_\psi^+, \tag{6.5}$$

where $\widetilde{u}$ is any element in $\bar{N}_\psi^+$ whose projection onto $\mathcal{N}_\psi^+ = \bar{N}_\psi^+/\bar{T}_\psi$ equals u. Observe that

$$(\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}})_\psi = \bigoplus_{\hat{\alpha} \in \hat{\Sigma}_M} \hat{\underline{\mathfrak{g}}}_{\hat{\alpha},\psi},$$

where

$$\hat{\underline{\mathfrak{g}}}_{\hat{\alpha},\psi} = \hat{\underline{\mathfrak{g}}}_{\hat{\alpha}} \cap (\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}})_\psi .$$

The subgroup $\mathcal{S}_\psi^1$ of $\mathcal{N}_\psi^+$ leaves invariant each of the subspaces $\hat{\underline{\mathfrak{g}}}_{\hat{\alpha},\psi}$ of $(\hat{\underline{\mathfrak{g}}}/\hat{\underline{\mathfrak{m}}})_\psi$. Since the actions of $\mathcal{S}_\psi^1$ on $\hat{\underline{\mathfrak{g}}}_{\hat{\alpha},\psi}$ and $\hat{\underline{\mathfrak{g}}}_{-\hat{\alpha},\psi}$ are contragredient, $\varepsilon_\psi^{G/M}$ is trivial on $\mathcal{S}_\psi^1$, and descends to a character on the quotient

$$\mathcal{N}_\psi^+/\mathcal{S}_\psi^1 \cong W_\psi^+ .$$

Of course the main reason for defining $\varepsilon_{\psi}^{G/M}$ is the formula

$$\varepsilon_{\psi}^{G}(u) = \varepsilon_{\psi}^{G/M}(u)\varepsilon_{\psi}^{M}(u), \qquad u \in \mathcal{N}_{\psi}^{+}, \tag{6.6}$$

which follows easily from (4.5′), (6.5) and the corresponding formula for ε_{ψ}^{M}.

To express r_{ψ} in terms of $\varepsilon_{\psi}^{G/M}$, let $\hat{\Sigma}_{M,\psi}$ be the set of roots $\hat{\alpha} \in \hat{\Sigma}_M$ such that the dimension of $\mathrm{End}_{L'_F}(\underline{\hat{\mathfrak{g}}}_{\hat{\alpha},\psi})$ is odd. It follows from properties of the determinant that

$$\varepsilon_{\psi}^{G/M}(w) = (-1)^{|\hat{\Sigma}_{M,\psi}\cap\hat{\Sigma}_{P,w}|}, \qquad w \in W_{\psi}^{+}.$$

This is just the contribution from the even dimensional representations ν_j to the expression for $r_{\psi}(w)$ given by (6.3) and Lemma 6.1. The contribution from the odd dimensional representations ν_j is simply the usual sign character $\varepsilon(w)$ attached to the group $\bar{S}_{\psi}^{+}$. Thus

$$r_{\psi}(w) = \varepsilon(w)\varepsilon_{\psi}^{G/M}(w), \qquad w \in W_{\psi}^{+}.$$

The required formula

$$r_{\psi}(u) = \varepsilon(w_u)\varepsilon_{\psi}^{G}(u)\varepsilon_{\psi}^{M}(u)^{-1}, \qquad u \in \mathcal{N}_{\psi}^{+},$$

of Proposition 5.1 then follows directly from (6.6). □

The formula (6.6) can be regarded as motivation for the definition of ε_{ψ}^{G}. The introduction of this character might have seemed odd at first. However, we now have a direct connection between ε_{ψ}^{G} and the more familiar function r_{ψ} obtained from the normalizing factors of global intertwining operators.

§7. The expansion of $E_{\mathrm{disc},t}(f)$

We turn now to the distribution $E_{\mathrm{disc},t}$. It was defined in Hypothesis 3.1 as the sum

$$\sum_{H} \iota(G,H)\widehat{SI}_{\mathrm{disc},t}^{H_1}(f^{H_1}), \qquad f \in C_c^{\infty}(G(\mathbb{A}),\chi), \tag{7.1}$$

over equivalence classes of elliptic endoscopic data. We shall convert this into an expression which is parallel to the expansion (5.10) for $I_{\mathrm{disc},t}(f)$.

Hypothesis 3.1 can be regarded as a general existence assertion. There should be a stable distribution on any quasi-split group with the property that (7.1) equals $I_{\text{disc},t}(f)$ for any component G at all. Our ultimate goal is to show that this assertion is compatible with the formula (5.10) for $I_{\text{disc},t}(f)$. Since the stable distributions are uniquely determined by the property, the problem is simply to show that they exist. For a given quasi-split group G_1, and a suitable character χ_1 on a subgroup X_1 of $Z(G, \mathbb{A})$, we shall try to construct the associated stable distribution $SI^{G_1}_{\text{disc},t}$ in terms of the parameters $\psi_1 \in \Psi(G_1, \chi_1, t)$. Our local assumptions in §4 attach a stable distribution

$$f_1 \longrightarrow f_1^{G_1}(\psi_1), \qquad\qquad f_1 \in C_c^\infty(G_1(\mathbb{A}), \chi_1),$$

on $G_1(\mathbb{A})$ to each parameter $\psi_1 \in \Psi(G_1, \chi_1)$. Let us therefore set

$$SI^{G_1}_{\text{disc},t}(f_1) = \sum_{\psi_1 \in \Psi(G_1, \chi_1, t)} SI^{G_1}_{\psi_1}(f_1), \tag{7.2}$$

where

$$SI^{G_1}_{\psi_1}(f_1) = \sigma(G_1, \psi_1) f_1^{G_1}(\psi_1),$$

for constants $\sigma(G_1, \psi_1)$ to be determined. We shall assume that the constants vanish unless ψ_1 belongs to $\Psi'_0(G_1, \chi_1, t)$, a countable subset of $\Psi(G_1, \chi_1, t)$. We shall attempt to define them so that the formula obtained by equating (7.1) with the right hand side of (5.10) is universally valid.

We fix a representative $(H, \mathcal{H}, s, \xi)$, for each equivalence class of endoscopic data for G, such that $\mathcal{H}$ is a subgroup of ${}^L G^0$ and ξ is the inclusion mapping. We also fix a distinguished splitting (H_1, ξ_1) of $(H, \mathcal{H}, s, \xi)$. The character χ_1 is then defined on a subgroup X_1 of $Z(H_1, \mathbb{A})$ as in §3. We begin with the formula

$$E_{\text{disc},t}(f) = \sum_H \iota(G, H) \sum_{\psi_1 \in \Psi'_0(H_1, \chi_1, t)} \widehat{SI}^{H_1}_{\psi_1}(f^{H_1}) \tag{7.3}$$

obtained by applying the definition (7.2) to the groups H_1 in (7.1). Our immediate goal is to convert the double sum over H and ψ_1 to a single sum over the orbits of $\widehat{G}^0$ on a certain set. In the process, we will need to apply the formula (3.5) for the coefficients $\iota(G, H)$.

Recall that $\Psi(G)$ denotes the set of maps

$$\psi : L_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG^0$$

satisfying certain conditions, and taken modulo the equivalence relation (4.2). Let us write $\widetilde{\Psi}(G)$ for the same set of parameters, but without the equivalence relation, and let $\widetilde{\Psi}(G)/\widehat{G}^0$ denote the set of $\widehat{G}^0$-orbits in $\widetilde{\Psi}(G)$. (We can also write $\widetilde{\Psi}_{\mathrm{disc}}(G)$, $\widetilde{\Psi}(G,\chi,t)$, etc., for the obvious subsets of $\widetilde{\Psi}(G)$.) We shall describe the order of the covering projection $\widetilde{\Psi}(G)/\widehat{G}^0 \longrightarrow \Psi(G)$. According to the definition (4.2), the group $\ker^1(F, Z(\widehat{G}^0))$ acts transitively on the fibres of the projection. The isotropy subgroup is just the image of $\mathcal{S}_\psi(G^0)/\mathcal{C}_\psi$ under the injection (4.9). But the finite group $\mathcal{S}_\psi(G^0)$ is bijective with the set $\mathcal{S}_\psi$. Therefore, the order of each fibre in the projection equals

$$|\ker^1(F, Z(\widehat{G}^0))|\ |\mathcal{S}_\psi|^{-1}\ |\mathcal{C}_\psi|. \tag{7.4}$$

We shall apply this remark to the quasi-split groups H_1 which occur in (7.3). We can replace the sum over $\psi_1 \in \Psi'_0(H_1, \chi_1, t)$ by the sum over $\widetilde{\Psi}'_0(H_1, \lambda_1, t)/\hat{H}_1$, provided that we divide by

$$|\ker^1(F, Z(\hat{H}_1))|\ |\mathcal{S}_{\psi_1}|^{-1}\ |\mathcal{C}_{\psi_1}|,$$

the analogue for H_1 of the integer (7.4). Since (H_1, ξ_1) is assumed to be a distinguished splitting, $\ker^1(F, Z(\hat{H}_1))$ equals $\ker^1(F, Z(\hat{H}))$. Combining this with the formula (3.5) for $\iota(G,H)$, we are able to write $E_{\mathrm{disc,t}}(f)$ as the sum over H and over $\psi_1 \in \widetilde{\Psi}'_0(H_1, \chi_1, t)/\hat{H}_1$ of

$$\begin{aligned} &|\ker^1(F, Z(\widehat{G}^0))|^{-1} |\pi_0(\kappa_G)^{-1}| \bar{Z}(\hat{H})^\Gamma|^{-1} |\mathcal{S}_{\psi_1}| \\ &\qquad \times |\mathcal{C}_{\psi_1}|^{-1} |\mathrm{Aut}(H)/\hat{H}Z(\widehat{G}^0)^\Gamma|^{-1} S\hat{I}^{H_1}_{\psi_1}(f^{H_1}). \end{aligned} \tag{7.5}$$

Keep in mind that H really stands for the equivalence class of an endoscopic datum $(H, \mathcal{H}, s, \xi)$. Now, suppose that we are given a parameter $\psi_1 \in \widetilde{\Psi}(H_1, \chi_1)$. Then ψ_1 factors to an L-homomorphism from W_F into $\mathcal{H}$, which may then be composed with the embedding $\xi : \mathcal{H} \longrightarrow {}^LG^0$. In this way we obtain a parameter $\psi \in \widetilde{\Psi}(G,\chi)$. It follows from the property (2.1) of endoscopic data that the coset $s \in \widehat{G}/Z(\widehat{G}^0)$ lies in the set

$$\bar{S}_\psi = S_\psi/Z(G^0) = S_\psi(G)/Z(G^0).$$

Conversely, suppose that we are given a parameter $\psi \in \widetilde{\Psi}(G,\chi)$ and a coset $s \in \bar{S}_\psi$ consisting of semisimple elements. Then we can define an endoscopic datum $(H, \mathcal{H}, s, \xi)$ as in §4. Recall that H is the quasi-split group whose dual group is

$$\hat{H} = \mathrm{Cent}(s, \widehat{G}^0)^0,$$

equipped with the L-action induced by

$$\mathcal{H} = \hat{H}\psi(L_F \times SL(2,\mathbb{C})),$$

and ξ is the inclusion of $\mathcal{H}$ into ${}^LG^0$. The parameter ψ then factors through $\mathcal{H}$. For any distinguished splitting (H_1, ξ_1) of the endoscopic datum, we obtain the character $\chi_1 : X_1 \longrightarrow \mathbb{C}^*$ as in §3, and ψ then yields a parameter $\psi_1 \in \widetilde{\Psi}(H_1, \chi_1)$.

We have just established a correspondence between the pairs (H, ψ_1) and (ψ, s). We want the datum H to be elliptic and the parameter ψ_1 to be weakly elliptic. We ought to describe these conditions in terms of (ψ, s). Since ψ_1 factors through $\mathcal{H}$, and $\hat{H}_1$ equals $\xi_1(\hat{H})Z(\hat{H}_1)$, we have

$$C_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong \mathrm{Cent}(\mathrm{Image}(\psi), \hat{H})Z(\hat{H})/Z(\hat{H}).$$

In other words,

$$C_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong (C_\psi \cap \hat{H})Z(\hat{H})/Z(\hat{H}). \tag{7.6}$$

In particular, there is an isomorphism

$$C^0_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong (C_\psi \cap \hat{H})^0 Z(\hat{H})/Z(\hat{H})$$

of the two identity components. Notice that $(C_\psi \cap \hat{H})^0$ equals $C^0_{\psi,s}$, the connected centralizer in C^0_ψ of any element in the coset s. Consequently

$$C^0_{\psi_1}Z(\hat{H}_1)/Z(\hat{H}_1) \cong C^0_{\psi,s}Z(\hat{H})/Z(\hat{H}) \cong C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\hat{H})^\Gamma.$$

Thus, ψ_1 is weakly elliptic if and only if the center of $C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\hat{H})^\Gamma$ is finite. Now $C^0_{\psi,s} \cap Z(\hat{H})^\Gamma$ is a central subgroup of $C^0_{\psi,s}$ which contains $A_{\hat{H}} = (Z(\hat{H})^\Gamma)^0$. Therefore, the conditions that ψ_1

be weakly elliptic and H be elliptic, taken together, are equivalent to the condition that $C^0_{\psi,s}$ has finite center modulo $A_{\widehat{G}} = (Z(\widehat{G})^\Gamma)^0$. We can describe this more simply in terms of the set

$$\bar{S}_{\psi,\mathrm{fin}} = \{s \in \bar{S}_\psi : |Z(\bar{S}^0_{\psi,s})| < \infty\}.$$

For by (4.11) we have

$$\begin{aligned}\bar{S}^0_{\psi,s} = C^0_{\psi,s}Z(\widehat{G}^0)/Z(\widehat{G}^0) &\cong C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\widehat{G}^0)\\ &= C^0_{\psi,s}/C^0_{\psi,s} \cap Z(\widehat{G})^\Gamma.\end{aligned}$$

Thus, the correspondence is between elliptic pairs (H, ψ_1) and pairs (ψ, s) such that s belongs to $\bar{S}_{\psi,\mathrm{fin}}$.

The foregoing discussion will enable us to interchange the order of summation in the original double sum over H and ψ_1. Keep in mind that $(H, \mathcal{H}, s, \xi)$ stands for a representative of an equivalence class of endoscopic data for which $\mathcal{H}$ is a subgroup of ${}^LG^0$ and ξ is the inclusion mapping. The equivalence classes themselves can be identified with the $\widehat{G}^0$-orbits of such data. The stabilizer in $\widehat{G}^0$ of $(H, \mathcal{H}, s, Id)$ is the group $\mathrm{Aut}(H)$ which appears in the expression (7.5). The group $\mathrm{Aut}(H)$ in turn acts on the set of parameters $\psi \in \widetilde{\Psi}(G, \chi, t)$ such that s belongs to $\bar{S}_{\psi,\mathrm{fin}}$. The stabilizer in $\mathrm{Aut}(H)$ of a given ψ is simply the group

$$\widetilde{C}^+_{\psi,s} = \{c \in C_\psi : csc^{-1} = s\}$$

of elements in C_ψ which fix the coset s. On the other hand, we can identify the orbits $\{\psi_1\} \in \widetilde{\Psi}(H_1, \chi_1, t)/\hat{H}_1$ with the $\hat{H}$-orbits of $\{\psi\}$. This is easily seen from the injectivity of the map

$$H^1(\Gamma, Z(\hat{H})) \longrightarrow H^1(\Gamma, Z(\hat{H}_1)),$$

noted in §2, and the fact that $\hat{H}_1 = Z(\hat{H}_1)\xi_1(\hat{H})$. We can actually take $\hat{H}Z(\widehat{G}^0)^\Gamma$-orbits of $\{\psi\}$, since $Z(\widehat{G}^0)^\Gamma$ centralizes the image of ψ. But the group $\hat{H}Z(\widehat{G}^0)^\Gamma$ has finite index in $\mathrm{Aut}(H)$, by Lemma 2.1, and the stabilizer of ψ in $\hat{H}Z(\widehat{G}^0)^\Gamma$ is the subgroup

$$C^+_{\psi,s} \cap (\hat{H}Z(G^0)^\Gamma)$$

of finite index in $\widetilde{C}^+_{\psi,s}$. Therefore, we can replace the original double sum over H and ψ_1 by the sum over the $\widehat{G}^0$-orbits in the set

$$\{(\psi, s) : \psi \in \widetilde{\Psi}(G, \chi, t),\ s \in \bar{S}_{\psi,\mathrm{fin}}\},$$

if we multiply the summand (7.5) by

$$|\mathrm{Aut}(H)/\hat{H}Z(\widehat{G}^0)^\Gamma|\ |\ \widetilde{C}^+_{\psi,s}/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\hat{G}^0)^\Gamma)|^{-1}. \tag{7.7}$$

The stabilizer in $\widehat{G}^0$ of a given parameter $\psi \in \widetilde{\Psi}(G, \chi, t)$ is the group C_ψ. We can therefore replace the sum over $\widehat{G}^0$-orbits in $\{(\psi, s)\}$ by a double sum over $\psi \in \widetilde{\Psi}(G, \chi, t)/\widehat{G}^0$ and over the set $\mathrm{Orb}(C_\psi, \bar{S}_{\psi,\mathrm{fin}})$ of orbits of C_ψ in $\bar{S}_{\psi,\mathrm{fin}}$. Obviously, $\bar{S}_{\psi,\mathrm{fin}}$ has the same set of orbits under C_ψ as under the group

$$\bar{C}_\psi = C_\psi Z(\widehat{G}^0)/Z(\widehat{G}^0).$$

The stabilizer of s in $\bar{C}_\psi$ equals

$$\bar{C}^+_{\psi,s} = \widetilde{C}^+_{\psi,s} Z(\widehat{G}^0)/Z(\widehat{G}^0) = \mathrm{Cent}(s, \bar{C}_\psi).$$

However, we would prefer to take the orbits in $\bar{S}_{\psi,\mathrm{fin}}$ under the connected component

$$\bar{C}^0_\psi = C^0_\psi Z(\widehat{G}^0)/Z(\widehat{G}^0).$$

The $\bar{C}_\psi$-orbit of s is bijective with $\bar{C}_\psi/\bar{C}^+_{\psi,s}$, while the $\bar{C}^0_\psi$-orbit is in bijective correspondence with the quotient of $\bar{C}^0_\psi$ by the group

$$\bar{C}_{\psi,s} = \mathrm{Cent}(s, \bar{C}^0_\psi).$$

Therefore, we can indeed take the second sum over $\bar{C}^0_\psi$-orbits, provided that we multiply the summand by

$$|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\ |\bar{C}_\psi/\bar{C}^0_\psi|^{-1},$$

or what is the same thing,

$$|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\ |\mathcal{C}_\psi|^{-1}. \tag{7.8}$$

Finally, we can take the first sum over $\psi \in \Psi(G,\chi,t)$ instead of $\widetilde{\Psi}(G,\chi,t)/\widehat{G}^0$, if we multiply the summand by the integer (7.4). We have shown that $E_{\mathrm{disc},t}(f)$ equals the sum over $\psi \in \Psi(G,\chi,t)$ and $s \in \mathrm{Orb}(\bar{C}^0_\psi, \bar{S}_{\psi,\mathrm{fin}})$ of the expression obtained by multiplying (7.4), (7.5), (7.7) and (7.8) together. We can write this last expression as the product of

$$(7.9) \qquad |\widetilde{C}^+_{\psi,s}/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\widehat{G}^0)^\Gamma|^{-1}|\mathcal{C}_{\psi_1}|^{-1}|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\;|\bar{Z}(\hat{H})^\Gamma|^{-1}$$

and

$$(7.10) \qquad |\pi_0(\kappa_G)|^{-1}\;|\mathcal{S}_\psi|^{-1}\;|\mathcal{S}_{\psi_1}|\;S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}).$$

The term (7.9) can be simplified. We begin by writing

$$\begin{aligned}
\mathcal{C}_{\psi_1} &\cong (C_\psi \cap \hat{H})Z(\hat{H})/(C_\psi \cap \hat{H})^0 Z(\hat{H}) \\
&\cong (C_\psi \cap \hat{H})/(C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma \\
&\cong (C_\psi \cap \hat{H})Z(\widehat{G}^0)/((C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma \cdot Z(\widehat{G}^0)).
\end{aligned}$$

The first isomorphism follows from (7.6), while the second is trivial and the third is a consequence of the fact that

$$(C_\psi \cap \hat{H}) \cap Z(\widehat{G}^0) = Z(\widehat{G})^\Gamma \cap Z(\hat{H})^\Gamma \subset Z(\hat{H})^\Gamma.$$

We also observe that

$$\begin{aligned}
\widetilde{C}^+_\psi/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\widehat{G}^0)^\Gamma) &= \widetilde{C}^+_{\psi,s}/(C_\psi \cap \hat{H})Z(\widehat{G}^0)^\Gamma \\
&\cong \widetilde{C}^+_{\psi,s}Z(\widehat{G}^0)/(C_\psi \cap \hat{H})Z(\widehat{G}^0).
\end{aligned}$$

This allows us to write

$$\begin{aligned}
&|\widetilde{C}^+_\psi/\widetilde{C}^+_{\psi,s} \cap (\hat{H}Z(\widehat{G}^0)^\Gamma)|^{-1}|\mathcal{C}_{\psi_1}|^{-1} \\
&\qquad\qquad = |\widetilde{C}^+_{\psi,s}Z(\widehat{G}^0)/(C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma Z(\widehat{G}^0)|^{-1},
\end{aligned}$$

for the first two factors in the product (7.9). Let us divide both groups in the quotient on the right by $Z(\widehat{G}^0)$. The numerator becomes

$$C^+_{\psi,s}Z(G^0)/Z(G^0) = \bar{C}^+_{\psi,s},$$

and the denominator may be written

$$\begin{aligned}(C_\psi \cap \hat{H})^0 Z(\hat{H})^\Gamma Z(\widehat{G}^0)/Z(\widehat{G}^0) \\ &= ((C_\psi \cap \hat{H})^0 Z(\widehat{G}^0)/Z(\widehat{G}^0))(Z(\hat{H})^\Gamma Z(\widehat{G}^0)/Z(\widehat{G}^0)) \\ &= (C^0_{\psi,s} Z(\widehat{G}^0)/Z(\widehat{G}^0))\bar{Z}(\hat{H})^\Gamma \\ &= \bar{C}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma,\end{aligned}$$

by (4.11). Therefore, (7.9) equals

$$\begin{aligned}|\bar{C}^+_{\psi,s}/\bar{C}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma|^{-1}|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}|\;|\bar{Z}(\hat{H})^\Gamma|^{-1} \\ &= |\bar{C}^+_{\psi,s}/\bar{C}^0_{\psi,s}|^{-1}|\bar{C}^0_{\psi,s}\cap\bar{Z}(\hat{H})^\Gamma|^{-1}|\bar{C}^+_{\psi,s}/\bar{C}_{\psi,s}| \\ &= |\bar{C}_{\psi,s}/\bar{C}^0_{\psi,s}|^{-1}|\bar{C}^0_{\psi,s}\cap\bar{Z}(\hat{H})^\Gamma|^{-1}.\end{aligned}$$

We noted in §4 that $\bar{C}_{\psi,s} = \bar{S}_{\psi,s}$. In particular

$$|\bar{C}_{\psi,s}/\bar{C}^0_{\psi,s}|^{-1} = |\bar{S}_{\psi,s}/\bar{S}^0_{\psi,s}|^{-1} = |\pi_0(\bar{S}_{\psi,s})|^{-1}.$$

The term (7.9) can by consequence by written as

$$|\pi_0(\bar{S}_{\psi,s})|^{-1}|\bar{S}^0_{\psi,s}\cap\bar{Z}(\hat{H})^\Gamma|^{-1}.$$

We have now established that $E_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi\in\Psi(G,\chi,t)}|\mathcal{S}_\psi|^{-1}\sum_{s\in\mathrm{Orb}(\bar{S}^0_\psi,\bar{S}_{\psi,\mathrm{fin}})}|\pi_0(\bar{S}_{\psi,s})|^{-1}$$
$$|\mathcal{S}_{\psi_1}|\;|\bar{S}^0_{\psi,s}\cap\bar{Z}(\hat{H})^\Gamma|^{-1}S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}).$$

By assumption,

$$S\widehat{I}^{H_1}_{\psi_1}(f^{H_1}) = \sigma(H_1,\psi_1)f^{H_1}(\psi_1).$$

Part of our local assumption in §4 is that $f^{H_1}(\psi_1)$ depends only on the image $x = \bar{s}$ of s in the set

$$\mathcal{S}_\psi = S_\psi/S^0_\psi Z(G^0) = \bar{S}_\psi/\bar{S}^0_\psi = \pi_0(\bar{S}_\psi).$$

More precisely, formula (4.4) asserts that

$$f^{H_1}(\psi_1) = \sum_{\pi\in\{\Pi_\psi\}} < \bar{s}_\psi x, \pi > f_G(\pi),$$

where $< \cdot, \cdot >$ is the global pairing on $\mathcal{S}_\psi \times \Pi_\psi$. We can therefore write $E_{\text{disc},t}(f)$ as the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi} \sum_{\pi\in\{\Pi_\psi\}} |\mathcal{S}_\psi|^{-1} \sum_{x\in\pi_0(\bar{S}_\psi)} \Big(\sum_{s\in\mathrm{Orb}(\bar{S}^0_\psi, x_{\mathrm{fin}})} |\pi_0(\bar{S}_{\psi,s})|^{-1}\tau(\psi,s)\Big) < \bar{s}_\psi x, \pi > f_G(\pi), \tag{7.11}$$

where

$$x_{\mathrm{fin}} = x \cap \bar{S}_{\psi,\mathrm{fin}}$$

and

$$\tau(\psi, s) = |\mathcal{S}_{\psi_1}|\ |\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|^{-1}\sigma(H_1, \psi_1).$$

The similarity with the formula in Proposition 5.2 appears promising. We must try to define the constants $\sigma(H_1, \psi_1)$ so that the two formulas for $I_{\text{disc},t}(f)$ always match.

Suppose that G_1 is an arbitrary connected quasi-split group over F, and that $\psi_1 \in \Psi(G_1)$. Then we shall set

$$\sigma(G_1, \psi_1) = |\mathcal{S}_{\psi_1}|^{-1}\varepsilon_{\psi_1}(\bar{s}_{\psi_1})\sigma(\bar{S}^0_{\psi_1}), \tag{7.12}$$

where $\varepsilon_{\psi_1} = \varepsilon^{G_1}_{\psi_1}$ is the sign character (4.5) and $\sigma(\bar{S}^0_{\psi_1})$ is a constant, to be determined, which depends only on the isomorphism class of the complex, connected reductive group

$$\bar{S}^0_{\psi_1} = (S_{\psi_1}/Z(\widehat{G}_1))^0.$$

We also ask that this latter constant have the property that

$$\sigma(S_1) = \sigma(S_1/Z_1)|Z_1|^{-1}, \tag{7.13}$$

for any complex connected group S_1, and any subgroup Z_1 of the center of S_1. In particular, $\sigma(S_1)$ is going to have to vanish if S_1 has infinite center. This implies that $\sigma(G_1, \psi_1) = 0$ unless $\psi_1 \in \Psi'_0(G_1)$, as we would expect.

We of course want to set $G_1 = H_1$. Then

$$\begin{aligned}\bar{S}^0_{\psi_1} = \bar{C}^0_{\psi_1} &= C^0_{\psi_1} Z(\hat{H}_1)/Z(\hat{H}_1)\\ &= (C_\psi \cap \hat{H})^0 Z(\hat{H})/Z(\hat{H})\\ &= C^0_{\psi,s} Z(\hat{H})/Z(\hat{H}),\end{aligned}$$

by the formula (7.6). Since

$$C^0_{\psi,s} \cap Z(\hat{H}) = C^0_{\psi,s} \cap Z(\hat{H})^\Gamma Z(\widehat{G}^0),$$

we obtain

$$\begin{aligned}C^0_{\psi,s} Z(\hat{H})/Z(\hat{H}) &\cong C^0_{\psi,s} Z(\hat{H})^\Gamma Z(\widehat{G}^0)/Z(\hat{H})^\Gamma Z(\widehat{G}^0)\\ &\cong (C^0_{\psi,s} Z(\widehat{G})^0/Z(\widehat{G}^0))\bar{Z}(\hat{H})^\Gamma/\bar{Z}(\hat{H})^\Gamma\\ &\cong \bar{C}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma/\bar{Z}(\hat{H})^\Gamma,\end{aligned}$$

from (4.11). But $\bar{C}^0_{\psi,s} = \bar{S}^0_{\psi,s}$, so that

$$\bar{S}^0_{\psi_1} \cong \bar{S}^0_{\psi,s}\bar{Z}(\hat{H})^\Gamma/\bar{Z}(\hat{H})^\Gamma \cong \bar{S}^0_{\psi,s}/\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma.$$

It follows from the property (7.13) that

$$\sigma(\bar{S}^0_{\psi_1}) = \sigma(\bar{S}^0_{\psi,s})|\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|.$$

Therefore,

$$\sigma(H_1, \psi_1) = |\mathcal{S}_{\psi_1}|^{-1}\varepsilon^{H_1}_{\psi_1}(\bar{s}_{\psi_1})|\bar{S}^0_{\psi,s} \cap \bar{Z}(\hat{H})^\Gamma|\sigma(\bar{S}^0_{\psi,s}),$$

so that

$$\tau(\psi, s) = \varepsilon^{H_1}_{\psi_1}(\bar{s}_{\psi_1})\sigma(\bar{S}^0_{\psi,s}).$$

LEMMA 7.1. *For H_1 and $x \in \mathcal{S}_\psi$ as in (7.11), we have*

$$\varepsilon^{H_1}_{\psi_1}(\bar{s}_{\psi_1}) = \varepsilon^G_\psi(\bar{s}_\psi x).$$

PROOF: As in §4, let

$$\tau_\psi = \bigoplus_k (\lambda_k \otimes \mu_k \otimes \nu_k)$$

be the decomposition of the representation

$$\tau_\psi : \bar{S}_\psi^+ \times L_F \times SL(2,\mathbb{C}) \longrightarrow GL(\hat{\mathfrak{g}})$$

into irreducible constituents. If I denotes the set of indices k in the direct sum, let I' denote the subset of k such that (i) $\mu_k \cong \widetilde{\mu}_k$, (ii) $\varepsilon(\frac{1}{2}, \mu_k) = -1$, and (iii) $\dim(\nu_k)$ is even. Then

$$x \in \mathcal{S}_\psi, \qquad \varepsilon_\psi^G(x) = \prod_{k \in I'} \det(\lambda_k(s)),$$

where s is any element in $\bar{S}_\psi$ which projects onto x. Notice that the element s_ψ lies in both S_ψ^+ and $SL(2,\mathbb{C})$. If k belongs to I', we obtain

$$\lambda_k(s_\psi) = \nu_k(s_\psi) = -1,$$

since $\dim \nu_k$ is even. It follows that

$$\varepsilon_\psi^G(\bar{s}_\psi) = \prod_{k \in I'} \det(\lambda_k(s_\psi)) = \prod_{k \in I'} (-1)^{\dim(\lambda_k)}.$$

Now H_1 is a central extension of the endoscopic group H attached to s. The Lie algebra of $\hat{H}$ equals the centralizer of $\mathrm{Ad}(s)$ in $\hat{\mathfrak{g}}$, and the Lie algebra of $\hat{H}_1$ can be identified with the direct sum of this algebra and a central ideal. For each k, let λ_k^s be the space of s-fixed vectors for λ_k. This of course is just the intersection of the underlying space of λ_k with the Lie algebra of $\hat{H}$. Recalling the relation between ψ and ψ_1, and applying the formula for $\varepsilon_\psi^G(\bar{s}_\psi)$ to H_1, we obtain

$$\varepsilon_{\psi_1}^{H_1}(\bar{s}_{\psi_1}) = \prod_{k \in I'} (-1)^{\dim(\lambda_k^s)}.$$

Finally, we observe that the number

$$\varepsilon_\psi^G(x) = \prod_{k \in I'} \det(\lambda_k(s))$$

equals the product of all the eigenvalues, counting multiplicities, of the operators

$$\{\lambda_k(s) : \ k \in I'\}.$$

Now the contragradient $\lambda_k \longrightarrow \widetilde{\lambda}_k$ defines an involution on the representations λ_k with $k \in I'$. In particular, if ξ is an eigenvalue, not equal to ± 1, then ξ^{-1} is also an eigenvalue with the same multiplicity. Therefore,

$$\varepsilon_\psi^G(x) = (-1)^{m(-1)},$$

where $m(-1)$ is the total multiplicity of the eigenvalue (-1). By the same token,

$$\sum_{k \in I'} (\dim(\lambda_k) - \dim(\lambda_k^s)) - m(-1)$$

is an even integer. Consequently,

$$\varepsilon_\psi^G(\bar{s}_\psi)\varepsilon_{\psi_1}^{H_1}(\bar{s}_{\psi_1}) = (-1)^{m(-1)} = \varepsilon_\psi^G(x).$$

We obtain

$$\varepsilon_\psi^G(\bar{s}_\psi x) = \varepsilon_\psi^G(x)\varepsilon_\psi^G(\bar{s}_\psi) = \varepsilon_\psi^G(x)\varepsilon_\psi^G(\bar{s}_\psi)^{-1} = \varepsilon_{\psi_1}^{H_1}(\bar{s}_{\psi_1}),$$

as required. □

The lemma allows us to write

$$\tau(\psi, s) = \varepsilon_\psi^G(\bar{s}_\psi x)\sigma(\bar{S}_{\psi,s}^0).$$

Substituting this into (7.11), and setting

$$e(x) = \sum_{s \in \mathrm{Orb}(\bar{S}_\psi^0, x_{\mathrm{fin}})} |\pi_0(\bar{S}_{\psi,s})|^{-1}\sigma(\bar{S}_{\psi,s}^0), \tag{7.14}$$

we see that $E_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi \in \Psi(G,\chi,t)} \sum_{\pi \in \{\Pi_\psi\}} |\mathcal{S}_\psi|^{-1} \sum_{x \in \pi_0(\bar{S}_\psi)} \varepsilon_\psi^G(\bar{s}_\psi x)e(x) < \bar{s}_\psi x, \pi > f_G(\pi).$$

The point $s_\psi \in S_\psi(G^0)$ belongs to the center of $S_\psi(G^+)$. Consequently, for any point s in the component x, the group $\bar{S}_{\psi,s}^0$ equals $\bar{S}_{\psi,s_\psi s}^0$. It follows that $e(x)$ equals $e(\bar{s}_\psi x)$. Substituting this into the formula above, and changing variables in the sum over $x \in \pi_0(\bar{S}_\psi)$, we see that $E_{\mathrm{disc},t}(f)$ equals the product of $|\pi_0(\kappa_G)|^{-1}$ with

$$\sum_{\psi \in \Psi(G,\chi,t)} \sum_{\pi \in \{\Pi_\psi\}} |\mathcal{S}_\psi|^{-1} \sum_{x \in \pi_0(S_\psi)} \varepsilon_\psi^G(x)e(x) < x, \pi > f_G(\pi). \tag{7.15}$$

We have now reached the stage in §7 at which we concluded §5. Taking the two sections together, we see that Hypotheses 3.1 and 4.1 yield two parallel expansions for $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$ into irreducible characters. Our goal is to show that these two expansions are in fact the same. The expansions are given by (5.10) and (7.15). They differ only in the coefficients $i(x)$ and $e(x)$, which are defined for any component $x \in \pi_0(\bar{S}_\psi)$ by (5.11) and (7.14). We must then show that the coefficients are equal. Recall that $e(x)$ depends on a constant $\sigma(S_1)$, which is to be defined for any complex, connected reductive group S_1 and which satisfies (7.13). We must show that $\sigma(S_1)$ can be defined for each S_1 in such a way that $i(x)$ and $e(x)$ are equal for any x. This is a property of Weyl groups which we shall establish in the next section.

§8. A COMBINATORIAL FORMULA FOR WEYL GROUPS

Suppose that S is a union of connected components in an arbitrary complex, reductive algebraic group. Then S^+ is the reductive group generated by S, and S^0 is the connected component of 1 in S^+. Recall also that we are writing S_s for the centralizer in S^0 of any element $s \in S$. This group is of course not always connected. As a slight generalization of (5.11), we set

$$i(S) = |W^0|^{-1} \sum_{w \in W_{\mathrm{reg}}} \varepsilon(w) |\det(w-1)|^{-1}, \tag{8.1}$$

where

$$W^0 = W(S^0) = \mathrm{Norm}(T, S^0)/T$$

is the Weyl group of S^0 relative to a fixed maximal torus T, and $W_{\mathrm{reg}} = W(S)_{\mathrm{reg}}$ is the set of elements w in the Weyl set

$$W = W(S) = \mathrm{Norm}(T, S)/T$$

such that $\det(w-1) \neq 0$. The determinant can be taken on the real vector space $\underline{\mathbf{a}}_T = \mathrm{Hom}(X(T), \mathbb{R})$. As in §5, $\varepsilon(w) = \pm 1$ is the parity of the number of positive roots of (S^0, T) which are mapped by w to negative roots.

As in §7, we shall write $\mathrm{Orb}(S^0, \Sigma)$ instead of Σ/S^0 for the set of orbits under conjugation by S^0 on an invariant subset Σ of S. This will prevent any confusion of orbits with cosets. The main example is when Σ equals the subset

$$S_{\mathrm{fin}} = \{s \in S : |Z(S_s^0)| < \infty\},$$

in which case the set $\mathrm{Orb}(S^0, S_{\mathrm{fin}})$ is finite.
Our object is to prove

THEOREM 8.1. *There are unique constants $\sigma(S_1)$, defined for each connected and semisimple complex group S_1, such that for any S the number*

$$e(S) = \sum_{s \in \mathrm{Orb}(S^0, S_{\mathrm{fin}})} |\pi_0(S_s)|^{-1} \sigma(S_s^0) \tag{8.2}$$

equals $i(S)$. The constants have the further property that

$$\sigma(S_1) = \sigma(S_1/Z_1)|Z_1|^{-1} \tag{8.3}$$

for any central subgroup Z_1 of S_1.

Remarks. 1. It is obviously enough to prove the theorem when S is just one connected component. We shall assume this in what follows.
2. Let us agree to write $\sigma(S_1) = 0$ if S_1 is *any* complex, connected algebraic group which is not semisimple. In particular, this constant vanishes if S_1 is a reductive group with infinite center. The equation (8.2) can then be written

$$e(S) = \sum_{s \in \mathrm{Orb}(S^0, S)} |\pi_0(S_s)|^{-1} \sigma(S_s^0) .$$

3. Theorem 8.1 is what remains to be proved of the comparison of $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$ that we began in §5 and §7. It is interesting to observe that Theorem 8.1 is actually a miniature replica of the original problem. It is a formal analogue for Weyl groups of the problem of comparing $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$, and indeed of many of the comparison problems, both local and global, that arise from endoscopy. I do not know whether it is part of a larger theory of endoscopy for Weyl groups, or whether results of this nature are already implicit in the representation theory of Weyl groups and finite Chevalley groups.

We shall begin the proof of Theorem 8.1 by taking note of the uniqueness of the constants $\sigma(S_1)$. For a given semisimple S_1, assume inductively that $\sigma(S_1')$ has been defined for any S_1' of dimension smaller than S_1. Then $\sigma(S_1)$ is determined by the formula,

$$\sigma(S_1)|Z(S_1)| = i(S_1) - \sum_{s \in \mathrm{Orb}(S_1^0, S_1 - Z(S_1))} |\pi_0(S_{1,s})|^{-1} \sigma(S_{1,s}^0) ,$$

which follows from the required equality of $e(S_1)$ with $i(S_1)$. In other words, the special case of (8.2) that $S = S^0 = S_1$ provides a definition of the constant $\sigma(S_1)$.

Having defined the constants $\sigma(S_1)$ we shall next establish the property (8.3). The argument is similar to part of the discussion of §7. Suppose that S is an arbitrary component, and that Z is a finite subgroup of $Z(S^0)$ which is invariant under conjugation by S. Then $\bar{S} = S/Z$ is a connected component of the reductive group $\bar{S}^+ = S^+/Z$, of which the identity component $\bar{S}^0$ equals S^0/Z.

LEMMA 8.2. *(i)* $i(S) = i(\bar{S})$.
(ii) $e(S) = e(\bar{S})$.
(iii) If $S^0 = S$, *then* $\sigma(S) = \sigma(\bar{S})|Z|^{-1}$.

PROOF: The property (i) follows easily from the definition (8.1). We shall establish the other two properties together. To this end we shall assume inductively that (ii) holds for any connected group of dimension smaller than S.

If the group

$$Z(S) = \mathrm{Cent}\big(S, Z(S^0)\big)$$

is infinite, the quantities $e(\bar{S})$, $e(S)$, $\sigma(\bar{S}^0)$ and $\sigma(S^0)$ all vanish, and there is nothing to prove. We can therefore assume that $Z(S)$ is finite. This implies that the group $Z(S^+) \cap S$ is also finite. Let $\bar{s}$ be a coset in $\bar{S}_{\mathrm{fin}} = S_{\mathrm{fin}}/Z$ which does not lie in $Z(\bar{S}^+) \cap S$. Then

$$\bar{S}_{\bar{s}} = \mathrm{Cent}(\bar{s}, \bar{S}^0)$$

is a proper subgroup of $\bar{S}^0$. Since

$$\bar{S}^0_{\bar{s}} = S^0_s Z/Z = S^0_s / S^0_s \cap Z$$

for any element s in the coset $\bar{s}$, our induction assumption implies that

$$\sigma(\bar{S}^0_{\bar{s}}) = \sigma(S^0_s)|S^0_s \cap Z| \ .$$

Let $\widetilde{S}_{\bar{s}}$ be the normalizer in S^0 of the coset $\bar{s}$. The set of orbits in $\mathrm{Orb}(S^0, S)$ which meet $\bar{s}$ can be identified with $\mathrm{Orb}(\widetilde{S}_{\bar{s}}, \bar{s})$, a set of cardinality

$$|Z||\widetilde{S}_{\bar{s}}/S_s|^{-1} \ .$$

Observe that

$$\begin{aligned}
&\sum_{s\in \mathrm{Orb}(\widetilde{S}_{\bar{s}},\bar{s})} |\pi_0(S_s)|^{-1}\sigma(S_s^0) \\
&= |Z||\widetilde{S}_{\bar{s}}/S_s|^{-1}|S_s/S_s^0|^{-1}\sigma(\bar{S}_{\bar{s}}^0)|S_s^0 \cap Z|^{-1} \\
&= |(\widetilde{S}_{\bar{s}}/Z)/(S_s^0 Z/Z)|^{-1}\sigma(\bar{S}_{\bar{s}}^0) \\
&= |\pi_0(\bar{S}_{\bar{s}})|^{-1}\sigma(\bar{S}_{\bar{s}}^0)\ .
\end{aligned}$$

Summing over all such $\bar{s}$, we obtain

$$e(S) - \sigma(S^0)|Z(S^+)\cap S| \;=\; e(\bar{S}) - \sigma(\bar{S}^0)|Z(\bar{S}^+)\cap \bar{S}|\ .$$

If $Z(\bar{S}^+)\cap S$ is empty, it follows immediately that $e(S)$ equals $e(\bar{S})$. Suppose that $Z(\bar{S}^+)\cap S$ is not empty. Then S acts on the group S^0 by inner automorphisms, and we have

$$e(S) \;=\; e(S^0) \;=\; i(S^0) \;=\; i(S)$$

from the definitions. (The equality of $e(S^0)$ and $i(S^0)$ was part of the definition of $\sigma(S^0)$.) Similarly $e(\bar{S}) = i(\bar{S})$. The property (i) then implies that $e(S)$ equals $e(\bar{S})$ in this case as well. This is the required property (ii). Suppose that $S = S^0$. Then $Z(S^+)\cap S$ equals $Z(S)$, a group which of course is not empty. The property (iii) follows from the fact that $|Z(S)| = |Z(\bar{S})||Z|$. □

The property (iii) of the lemma is the required condition (8.3) We still have the main part of the proof of the theorem, which is to show that $e(S)$ equals $i(S)$. This of course is a problem only if S is not equal to S^0.

As a warm-up, let us verify the equality of $e(S)$ and $i(S)$ in the special case that $S^0 = T$ is a torus. Then W consists of one element w, the adjoint operation of S on T. We can assume that this element is regular. Recall **[23, II.1.7]** that

$$|\det(w-1)| \;=\; |T^w|\ ,$$

where T^w denotes the kernel of w in T. Since $\varepsilon(w) = 1$, we obtain

$$i(S) \;=\; |T^w|^{-1}\ .$$

On the other hand,

$$T^w = \mathrm{Cent}(s,T^0) = S_s ,$$

for any element $s \in S$. The regularity of w means that s belongs to S_{fin}, and that $S_s^0 = \{1\}$. Therefore $\sigma(S_s^0)$ equals 1. But the T-orbit of s equals the product of s with $\{t^{-1}w(t) : t \in T\}$, a subtorus of T. This subtorus has the same dimension as T, and must therefore equal T. In other words, the orbit of s equals S, so there is only one summand on the right hand side of (8.2). We obtain

$$e(S) = |\pi_0(S_s)|^{-1} = |T^w|^{-1} = i(S) ,$$

as required.

Now suppose that S is arbitrary. We shall use Lemma 8.2 to effect a simplification. First, observe that $i(S)$ and $e(S)$ depend only on S^0 and the set of automorphisms of S^0 induced from conjugation by S. We may therefore assume that S^+ is the semidirect product of S^0 with $\pi_0(S^+)$. Now, let S_{sc}^0 be the simply-connected covering of the derived group of S^0, and let $S_{\mathrm{cent}}^0 = Z(S^0)^0$ be the connected component of the center of S^0. Then

$$\widetilde{S}^0 = S_{\mathrm{sc}}^0 \times S_{\mathrm{cent}}^0$$

is a finite covering group of S^0. In particular, S^0 equals $\widetilde{S}^0/Z$, where Z is the finite central subgroup of $\widetilde{S}^0$. It is then readily verified that $S = \widetilde{S}/Z$, where $\widetilde{S} = S_{\mathrm{sc}} \times S_{\mathrm{cent}}$ is a component which normalizes Z and such that the identity components $(\widetilde{S})^0$, $(S_{\mathrm{sc}})^0$ and $(S_{\mathrm{cent}})^0$ equal the respective groups $\widetilde{S}^0$, S_{sc}^0 and S_{cent}^0 above. Applying Lemma 8.2 and the calculaion above for tori, we obtain

$$\begin{aligned} e(S) - i(S) &= e(S_{\mathrm{sc}} \times S_{\mathrm{cent}}) - i(S_{\mathrm{sc}} \times S_{\mathrm{cent}}) \\ &= e(S_{\mathrm{sc}})e(S_{\mathrm{cent}}) - i(S_{\mathrm{sc}})i(S_{\mathrm{cent}}) \\ &= \big(e(S_{\mathrm{sc}}) - i(S_{\mathrm{sc}})\big)i(S_{\mathrm{cent}}) . \end{aligned}$$

(We have also used the fact, easily verified from the definitions, that e and i are multiplicative on products.)

It is therefore enough to show that $e(S)$ equals $i(S)$ in the special case that S^0 is semisimple and simply connected. We shall assume this from now on. If s is any semisimple element in S, the group

$$S_s = \mathrm{Cent}(s, S^0)$$

is then connected, by **[24, Theorem 8.1]**. In this case, it is part of our definition that $e(S_s)$ equals $i(S_s)$. If t is a semisimple element in S^0, the connectedness of S_t implies that the set

$$S^t = \mathrm{Cent}(t, S)$$

is either connected or empty. We can assume inductively that if $\dim(S^t) < \dim(S)$, then $e(S^t)$ equals $i(S^t)$.

LEMMA 8.3. *The required equality of $e(S)$ and $i(S)$ is equivalent to the formula*

$$\sum_{s \in \mathrm{Orb}(S^0, S)} i(S_s) = \sum_{t \in \mathrm{Orb}(S^0, S^0)} i(S^t) . \tag{8.4}$$

PROOF: If $s \in S$ and $t \in S^0$ are elements that commute, we write

$$S_{s,t} = \mathrm{Cent}(\{s, t\}, S^0) .$$

It is obvious that

$$\pi_0\big((S_s)_t\big) = \pi_0(S_{s,t}) = \pi_0\big((S_t)_s\big) .$$

The left hand side of (8.4) then equals

$$\begin{aligned}
&\sum_{s \in \mathrm{Orb}(S^0, S)} i(S_s) \\
&= \sum_{s \in \mathrm{Orb}(S^0, S)} e(S_s) \\
&= \sum_{s \in \mathrm{Orb}(S^0, S)} \sum_{t \in \mathrm{Orb}(S_s, S_s)} |\pi_0(S_{s,t})|^{-1} \sigma(S^0_{s,t}) \\
&= \sum_{\{(s,t) \in S \times S^0 : st = ts\}/S^0} |\pi_0(S_{s,t})|^{-1} \sigma(S^0_{s,t}) \\
&= \sum_{t \in \mathrm{Orb}(S^0, S^0)} \sum_{s \in \mathrm{Orb}(S_t, S^t)} |\pi_0(S_{s,t})|^{-1} \sigma(S^0_{s,t}) \\
&= \sum_{t \in \mathrm{Orb}(S^0, S^0)} e(S^t) .
\end{aligned}$$

This last expression would just be the right hand side of (8.4) if $e(S^t)$ were replaced by $i(S^t)$. But if t does not belong to $Z(S)$, $\dim(S^t)$ is smaller than $\dim(S)$, and $e(S^t)$ equals $i(S^t)$ by our induction assumption. It t belongs to $Z(S)$, S^t is just S itself. Therefore, the equality of $e(S)$ and $i(S)$ is indeed equivalent to the identity (8.4). $\square$

It remains for us to establish the formula (8.4), in which S is a component such that S^0 is semisimple and simply connected. We shall deal with each side separately. According to **[24, Theorem 7.5]**, any semisimple element in S normalizes some pair (T_1, B_1) of groups, where T_1 is a maximal torus in S^0 and B_1 is a Borel subgroup of S^0 which contains T_1. Let B be a fixed Borel subgroup of S^0 which contains our fixed maximal torus T. Then any semisimple orbit of S^0 in S contains an element which normalizes T and B. The normalizer of T and B in S can be written Tw_B, where w_B is a fixed semisimple element in S which preserves some splitting of B. Let S' and T' denote the centralizers of w_B in S^0 and T. Since S^0 is simply connected, S' is a connected reductive group which contains T' as a maximal torus. In particular, we can form the usual sign character ε' on the Weyl group W' of (S', T').

LEMMA 8.4. *The left hand side of (8.4) equals the number*

$$\Delta(W', \varepsilon') = |W'|^{-1} \sum_{w \in W'_{\mathrm{reg}}} \varepsilon'(w) . \tag{8.5}$$

PROOF: Let N' denote the normalizer of T' in S'. Then

$$W' = N'/T' \cong TN'/T .$$

We claim that

$$\mathrm{Norm}(T', S^0) = TN' . \tag{8.6}$$

To see this, we shall consider the (open) chambers in the real vector spaces

$$\underline{\mathrm{a}}_{T'} = \mathrm{Hom}(X(T'), \mathbf{R}) \subset \underline{\mathrm{a}}_T = \mathrm{Hom}(X(T), \mathbf{R})$$

determined by the roots of S' and S^0. The fact that w_B preserves a splitting in (B, T) implies that the simple roots of $(B \cap S', T')$ are just the orbits under powers of $\mathrm{ad}(w_B)$ of the simple roots of (B, T). The corresponding positive chambers are therefore related by $\underline{\mathrm{a}}^+_{T'} = \underline{\mathrm{a}}_{T'} \cap \underline{\mathrm{a}}^+_T$. By an argument of symmetry, any chamber in $\underline{\mathrm{a}}_{T'}$ becomes the intersection of $\underline{\mathfrak{a}}_{T'}$ with a uniquely determined chamber in $\underline{\mathfrak{a}}_T$. Suppose that n is an element in $\mathrm{Norm}(T', S^0)$. Then n also normalizes T, since T is the centralizer of T' in S^0. The chamber $\mathrm{Ad}(n)(\underline{\mathfrak{a}}_T^+)$ contains an open subset of $\underline{\mathfrak{a}}_{T'}$, and therefore a chamber

$$\mathrm{Ad}(n')(\underline{\mathfrak{a}}_{T'}^+) \ , \qquad\qquad n' \in N' ,$$

in $\underline{\mathfrak{a}}_{T'}$. The map $\mathrm{Ad}(n')^{-1}\mathrm{Ad}(n)$ will then send the chambers $\underline{\mathfrak{a}}_{T'}^+$ and $\underline{\mathfrak{a}}_T^+$ to themselves. This justifies the claim (8.6).

We have agreed that any semisimple orbit of S^0 in S intersects Tw_B. Suppose that two elements s_1 and s in Tw_B are S^0-conjugate. Since T' is a maximal torus of both S_s and S_{s_1}, s_1 and s are conjugate by an element in the group (8.6). From this it follows that there is a canonical bijection from $\mathrm{Orb}(TN', Tw_B)$ to the semisimple elements in $\mathrm{Orb}(S^0, S)$. It is of course only semisimple orbits which are relevant to (8.4). We can therefore write the left hand side of (8.4) as

$$\begin{aligned} &\sum_{s\in \mathrm{Orb}(S^0,S)} i(S_s) \\ &= \sum_{s\in\mathrm{Orb}(TN',Tw_B)} i(S_s) \\ &= \sum_{s\in\mathrm{Orb}(TN',Tw_B)} |W(S_s)|^{-1} \sum_{w\in W(S_s)_{\mathrm{reg}}} \varepsilon_s(w)|\det(w-1)|^{-1} \ , \end{aligned}$$

where ε_s stands for the sign character on the Weyl group $W(S_s)$ of S_s.

If s belongs to Tw_B, S_s need not be a subgroup of S'. However, the elements in $W(S_s)$ normalize T', and are induced from the group (8.6). Therefore

$$W(S_s) \ = \ \mathrm{Cent}(s, N')/T' \ \cong \ T\mathrm{Cent}(s, N')/T \ .$$

In particular, $W(S_s)$ is a subgroup of W'. Thus, the simple reflections in $W(S_s)$ are also reflections in W', and since ε_s and ε' both take the value (-1) on any such reflection, we see that ε_s equals the restriction of ε' to $W(S_s)$. We can substitute this into the expression above. Our characterization of $W(S_s)$ also suggests that we should change the sum over $\mathrm{Orb}(TN', Tw_B)$ to a sum over the smaller set

$$\bar{T}_B \ = \ \mathrm{Orb}(T, Tw_B) \ = \ \{t^{-1}w_B t w_B^{-1} : \ t \in T\} \setminus Tw_B \ .$$

The expression for the left hand side of (8.4) becomes

$$|W'|^{-1} \sum_{s \in \bar{T}_B} \sum_{w \in W(S_s)_{\mathrm{reg}}} \varepsilon'(w)|\det(w-1)|^{-1} .$$

The group W' operates on $\bar{T}_B$. It is easy to check that

$$W(S_s)_{\mathrm{reg}} = \{w \in W'_{\mathrm{reg}} : w(s) = s\} .$$

The last expression can therefore be written

$$|W'|^{-1} \sum_{\{(s,w)\in \bar{T}_B \times W'_{\mathrm{reg}} : w(s)=s\}} \varepsilon'(w)|\det(w-1)|^{-1} .$$

Now T' is a finite covering of the torus

$$\{t^{-1} w_B t w_B^{-1} : t \in T\}\backslash T ,$$

and $|\det(w-1)|$ equals the number of fixed points of w in either torus. In particular, this number equals the order of the fixed point set $\bar{T}_B^w$ of w in $\bar{T}_B$. We can therefore write our expression as

$$\begin{aligned} & |W'|^{-1} \sum_{w \in W'_{\mathrm{reg}}} \sum_{s \in \bar{T}_B^w} \varepsilon'(w)|\bar{T}_B^w|^{-1} \\ &= |W'|^{-1} \sum_{w \in W'_{\mathrm{reg}}} \varepsilon'(w) \\ &= \Delta(W', \varepsilon') . \qquad \square \end{aligned}$$

LEMMA 8.5. *The right hand side of (8.4) equals the number*

$$\Delta(W,\varepsilon) = |W^0|^{-1} \sum_{w \in W_{\mathrm{reg}}} \varepsilon(w) . \tag{8.7}$$

PROOF: Since any semisimple conjugacy class in S^0 meets T, we have a bijection from $\mathrm{Orb}(W^0, T)$ to the set of semisimple elements in $\mathrm{Orb}(S^0, S^0)$. The right hand side of (8.4) can then be written

$$\begin{aligned} & \sum_{t \in \mathrm{Orb}(S^0,S^0)} i(S^t) \\ &= \sum_{t \in \mathrm{Orb}(W^0,T)} |W(S_t)|^{-1} \sum_{w \in W(S^t)_{\mathrm{reg}}} \varepsilon^t(w)|\det(w-1)|^{-1} , \end{aligned}$$

where ε^t stands for the sign character on the Weyl set $W(S^t)$.

We need only consider elements $t \in T$ such that S^t is not empty. For any such t, T is a maximal torus in the connected group $(S^t)^0 = S_t$, and $W(S^t)$ is a subset of $W = W(S)$. We claim that ε^t is the restriction of ε to $W(S^t)$. The group $W(S_t)$ is generated by reflections which lie in $W(S^0)$. Since this group acts simply transitively on $W(S^t)$, it suffices to check that ε and ε^t coincide on one element in $W(S^t)$. Let s be a semisimple element in S^t. Then there is a conjugate

$$s_1 = gsg^{-1}, \qquad g \in S^0,$$

of s which lies in Tw_B. We can in fact choose g so that $t_1 = gtg^{-1}$ lies in the maximal torus T' of S_{s_1}. It then follows that t_1 equals $w_1(t)$ for some $w_1 \in W^0$, and that t is fixed by the element $w_1^{-1}w_Bw_1$ in $W(S)$. In other words, $w_1^{-1}w_Bw_1$ belongs to $W(S^t)$. Since this element normalizes the Borel subgroups $w_1^{-1}Bw_1$ and $w_1^{-1}Bw_1 \cap S_t$ of S^0 and S_t, we have

$$\varepsilon(w_1^{-1}w_Bw_1) = \varepsilon^t(w_1^{-1}w_Bw_1) = 1 .$$

This establishes the claim.

Since $W(S_t)$ is the centralizer of t in W^0, the right hand side of (8.4) becomes

$$|W^0|^{-1}\sum_{t\in T}\ \sum_{w\in W(S^t)_{\mathrm{reg}}} \varepsilon(w)|\det(w-1)|^{-1} .$$

The set $W = W(S)$ operates on T, and

$$W(S^t)_{\mathrm{reg}} = \{w \in W_{\mathrm{reg}} : \ w(t) = t\} .$$

The last expression can then be written

$$\begin{aligned}
&|W^0|^{-1} \sum_{\{(t,w)\in T\times W_{\mathrm{reg}}:w(t)=t\}} \varepsilon(w)|\det(w-1)|^{-1} \\
&= |W^0|^{-1} \sum_{w\in W_{\mathrm{reg}}}\ \sum_{t\in T^w} \varepsilon(w)|T^w|^{-1} \\
&= |W^0|^{-1} \sum_{s\in W_{\mathrm{reg}}} \varepsilon(w) \\
&= \Delta(W,\varepsilon) ,
\end{aligned}$$

since $|\det(w-1)|$ equals the order of the fixed point set T^w of w in T. The lemma is proved. $\square$

LEMMA 8.6. $\Delta(W', \varepsilon') = \Delta(W, \varepsilon)$.

PROOF: The numbers $\Delta(W', \varepsilon')$ and $\Delta(W, \varepsilon)$ depend only on the Weyl set W. They are independent of the isogeny class of the underlying component S. We shall assume inductively that the required formula holds if S is replaced by a component of strictly smaller dimension.

We have the fixed Borel subgroups B and B' of S^0 and S', so we can speak of standard parabolic subgroups. Suppose that A is a standard torus in T'. In other words, A is the split component of a parabolic subgroup P' of S' which contains B'. Let M' be the Levi component of P' which contains T'. Then A equals $A_{M'} = Z(M')^0$, the connected component of the center of M'. Write

$$W'_A = W(M'/A)$$

for the Weyl group of M'/A, acting on T'/A. We can also take the centralizer M of A in S. Then M^0 is the Levi component of a standard parabolic subgroup of S^0. Write

$$W_A = W(M/A) ,$$

for the Weyl set of the component M, acting on T/A. The element w_B obviously embeds into W_A, and W'_A is just the centralizer of w_B in $W^0_A = W(M^0/A)$. If A is nontrivial, our induction hypothesis tells us that

$$\Delta(W'_A, \varepsilon'_A) = \Delta(W_A, \varepsilon_A) , \tag{8.8}$$

where ε'_A and ε_A are the sign characters on W'_A and W_A.

Suppose that w is an arbitrary element in W'. The identity component of the fixed point set $(T')^w$ is a torus in T', and equals a W'-translate

$$w_1^{-1}(A) , \qquad\qquad w_1 \in W' ,$$

of a standard torus A in T'. The element $w_1 w w_1^{-1}$ then lies in $W'_{A,\mathrm{reg}}$. It is also clear that

$$\varepsilon'(w) = \varepsilon'_A(w_1 w w_1^{-1}) .$$

Now the pair (A, w_1) is not uniquely determined by w. The number of such pairs actually equals

$$n(A)|W'_A| ,$$

where $n(A)$ is the number of chambers in $\mathrm{Hom}(X(A), \mathbf{R})$ cut out by the hyperplanes orthogonal to the roots of the corresponding standard parabolic subgroup. The elements in W' can be enumerated up to this ambiguity, however, as conjugates

$$w_1^{-1} w_A w_1 , \qquad w_A \in W'_{A,\mathrm{reg}} , \ w_1 \in W' .$$

We obtain

$$\begin{aligned} &|W'|^{-1} \sum_{w \in W'} \varepsilon'(w) \\ &= \sum_A n(A)^{-1} |W'_A|^{-1} \sum_{w_A \in W'_{A,\mathrm{reg}}} \varepsilon'_A(w_A) \\ &= \sum_A n(A)^{-1} \Delta(W'_A, \varepsilon'_A) . \end{aligned}$$

If W' is not equal to $\{1\}$, the sign character ε' is nontrivial, and the left hand side of the equation equals 0. Applying (8.8) to the right hand side, we conclude that the expression

$$\Delta(W', \varepsilon') + \sum_{A \neq \{1\}} n(A)^{-1} \Delta(W_A, \varepsilon_A) \tag{8.9}$$

vanishes if $W' \neq \{1\}$.

Now suppose that w is an arbitrary element in W. The identity component $(T^w)^0$ of the set of fixed points of w in T is a torus which commutes with any representative in S of the Weyl element w. Copying an argument from the proof of Lemma 8.5, we see that

$$(T^w)^0 = w_1^{-1}(A) ,$$

where w_1 belongs to W^0 and A is a torus in T'. In fact, we can assume that the centralizer M^0 of A in S^0 is the Levi component of a standard parabolic subgroup of S^0. This implies that A is standard torus in T'. The element $w_1 w w_1^{-1}$ then lies in $W_{A,\mathrm{reg}}$, and

$$\varepsilon(w) = \varepsilon_A(w_1 w w_1^{-1}) .$$

For a given w, how many such pairs (A, w_1) are there? We can certainly replace w_1 by a product

$$w' w_M w_1 , \qquad\qquad w' \in W', \ w_M \in W(M^0),$$

in which w' maps A to another standard torus in T'. However, this is the only possible ambiguity, so the number of pairs equals

$$n(A)|W(M^0)| \ = \ n(A)|W_A| \ .$$

We obtain

$$\begin{aligned} &|W^0|^{-1} \sum_{w \in W} \varepsilon(w) \\ &= \sum_A n(A)^{-1} |W_A|^{-1} \sum_{w_A \in W_{A,\mathrm{reg}}} \varepsilon_A(w_A) \\ &= \sum_A n(A)^{-1} \Delta(W_A, \varepsilon_A) \ . \end{aligned}$$

If W contains more than just the one element w_B, the left hand side of the equation equals 0. Therefore, the expression

$$\Delta(W, \varepsilon) + \sum_{A \neq \{1\}} n(A)^{-1} \Delta(W_A, \varepsilon_A) \tag{8.10}$$

vanishes if $W \neq \{w_B\}$.

The simple reflections in W' correspond to the orbits of simple roots of (B, T) under powers of $\mathrm{ad}(w_B)$. It follows that $W' = \{1\}$ if and only if $W = \{w_B\}$. In this case, both (8.9) and (8.10) are trivially equal to 1. We can therefore conclude that the expressions (8.9) and (8.10) are equal. The equality of $\Delta(W', \varepsilon')$ and $\Delta(W, \varepsilon)$ then follows. □

We have reached the end of the lemmas that make up the proof of Theorem 8.1. We obtain the general inequality of $i(S)$ with $e(S)$ immediately by combining Lemmas 8.3, 8.4, 8.5 and 8.6. The proof of Theorem 8.1 is now complete. □

§9. Concluding remarks

Theorem 8.1 tells us that the coefficients $i(x)$ and $e(x)$ in (5.10) and (7.15) are always equal. It follows that there is a term by term identification of the expansions for $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$. We conclude that Hypothesis 3.1 is a consequence of Hypothesis 4.1 (together with the local assumptions of §3, §4 and **[3, §7]**). This was the task we originally set for ourselves.

We have in fact shown that the contributions to $I_{\mathrm{disc},t}(f)$ and $E_{\mathrm{disc},t}(f)$ of each parameter $\psi \in \Psi(G,\chi,t)$ are equal. Now there are some parameters for which the representation theoretic hypotheses are known. Consider the special case that G is a connected quasi-split group. Suppose that ψ is the image of a parameter $\psi_0 \in \Psi(M_0)$ for a minimal Levi subgroup M_0 of G. Since M_0 is a maximal torus in this case, ψ_0 is trivial on $SL(2,\mathbb{C})$, and is the parameter of a unitary character on $M_0(F)\backslash M_0(\mathbb{A})$. We can take Π_ψ to be the set of irreducible constituents of the corresponding induced representation of $G(\mathbb{A})$. The parameter ψ_0 factors through the quotient W_F of L_F, so there is no problem with the hypothetical Langlands group. In particular, S_ψ equals the centralizer in $\hat{G}$ of the image of W_F, and the quotient $\mathcal{S}_\psi$ is just the R-group R_ψ. The pairing on $\mathcal{S}_\psi \times \Pi_\psi$ is then determined by the global normalized intertwining operators. In fact, Conjecture 7.1 of **[3]**, which we assumed in §5, is already known in this case thanks to Keys and Shahidi **[10, Theorem 5.1]**. If H_1 is associated to a point in a component $x \in \mathcal{S}_\psi$, we could just define the distribution $f \to f^{H_1}(\psi_1)$ by

$$f^{H_1}(\psi_1) = \sum_{\pi\in\Pi_\psi} < x,\pi > f_G(\pi) .$$

Then with these interpretations, the notions that went into the discussion in §5–§8 are all understood. The reader who dislikes arguments based on unproven conjectures can regard the earlier discussion as pertaining only to the parameters just described. It establishes that the contribution of these parameters to

$$E_{\mathrm{disc},t}(f) - I_{\mathrm{disc},t}(f) \tag{9.1}$$

vanishes.

This paper has concerned the conjectures in **[3]** on unipotent (and more general) automorphic representations. The long term goal is to prove them, at least in part, with the help of endoscopy and the trace formula. A first step towards the creation of a logical structure for the argument is to verify the compatibility of the notions involved and to analyze the reasons for it. This has been our emphasis, and we continue with some informal comments on the proof envisaged.

In general, Hypothesis 3.1 asserts the vanishing of the distributions (9.1). As we mentioned earlier, one should first try to deduce this from the trace formula. One would then use (9.1) to establish some version of the multiplicity formula (4.7). The formula could be assumed inductively for any proper Levi-subgroup. This would permit the application of the arguments in §5–§8 to any parameter $\psi \in \Psi(G,\chi,t)$ which is not the image of an *elliptic* parameter for an *elliptic* endoscopic datum. The contribution to (9.1) of all such parameters could then be shown to vanish. The only remaining contribution to (9.1) would come from parameters ψ such that $\bar{S}_{\psi,s}$ is finite for some element s in $\bar{S}_\psi = S_\psi/Z(\hat{G}^0)$. It is from this that we would hope to deduce some form of (4.7), again using arguments of Sections 5, 6 and 7. The sign characters ε_ψ^G would be forced on us at this stage, essentially because of Proposition 5.1.

Of course, it would not be feasible to apply the arguments of §5–§8 in precisely the way they were presented here. The correspondence from maps $W_F \to {}^LG$ to automorphic representations is much deeper than multiplicity formulas such as (4.7), and in any case, we would certainly not want to assume the existence of the Langlands group L_F. We would instead have to replace the parameters ψ by the families $\sigma = \{\sigma_v : v \notin S\}$ of conjugacy classes in ${}^LG$ attached to automorphic representations. (See [**3**, §**1**, §**8**].) For many G we can expect a bijection from $\Psi(G)$ onto the set $\Sigma(G)$ of such families. In these cases, the idea would be to define the centralizer S_ψ in terms of σ. This could probably be done by considering the set of endoscopic groups H for which σ lies in the image of the map $\Sigma(H) \to \Sigma(G)$. It is of course necessary to determine S_ψ in order to state the multiplicity formula (4.7). By definition, a parameter ψ has a Jordan decomposition $(\psi_{ss}, \psi_{\mathrm{unip}})$, where

$$\psi_{ss} : L_F \longrightarrow {}^LG$$

and

$$\psi_{\mathrm{unip}} : SL(2,\mathbb{C}) \longrightarrow S^0_{\psi_{ss}} .$$

We would describe the Jordan decomposition in terms of σ by first determining the family σ_{ss} attached to ψ_{ss}, and then describing the group $S^0_{\psi_{ss}}$ in terms of σ_{ss}. In the case of a general G, some understanding of the fibres of the map $\Psi(G) \to \Sigma(G)$ will probably be needed.

We have not said much about the local side of the conjectures. This includes the definition of the stable distributions $f_1 \to f_1^{G_1}(\psi_1)$, the construction of the packets Π_ψ and the pairing $< \cdot, \cdot >$, and the proof of the local character identity (4.4). Once the stable distributions have been defined, the packets and the pairing are determined by (4.4) (together with the maps $f \to f^{H_1}$). The essential part of the local conjecture is then the assertion that for a given ψ, certain linear combinations of the distributions

$$f \longrightarrow f^{H_1}(\psi_1), \qquad f \in C_c^\infty(G(\mathbb{A}), \chi),$$

are actually characters, as opposed to more general invariant distributions. Ideally, it would be best to deduce this locally. However, the global Hypothesis 3.1 itself carries some local information. For it ultimately implies some version of (4.7), and any such multiplicity formula tells us that certain distributions are in fact characters. I do not know how far this can be pushed. It is perhaps best to wait until Hypothesis 3.1 has actually been established.

The case in which Hypothesis 3.1 will lead to the most complete results is the example of outer twisting of $GL(n)$. The hypotheses of §4 (interpreted without reliance on the parameters $\psi \in \Psi(G)$) are now known for $GL(n)$. Moeglin and Waldspurger [**21**] have recently characterized the residual discrete spectrum for $GL(n)$ in terms of the cuspidal spectrum, and it is clear how to interpret this in terms of the Jordan decomposition [**3**, §**2**]. On the other hand, the twisted endoscopic groups for $GL(n)$ include all of the quasi-split classical groups of type B, C and D (up to isogeny). One should try to deduce the conjectural properties of the spectra of these classical groups from what is known for $GL(n)$. We will conclude with a very brief discussion of this example.

Set $G^0 = GL(n)$. If

$$J_n = \left.\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}\right\} n,$$

then

$$\theta_n(g) = J_n^{-1}({}^t g^{-1}) J_n, \qquad g \in G^0,$$

is an outer automorphism of G^0 which leaves invariant the standard Borel subgroup. Set

$$G = G^0 \rtimes \theta_n .$$

If $\hat{\theta}_n$ denotes the same outer automorphism of $\hat{G}^0 = GL(n, \mathbb{C})$, then

$$\hat{G} = \hat{G}^0 \rtimes \hat{\theta}_n .$$

It is easy to describe the elliptic endoscopic data for G. For each integer r, with $1 \leq r \leq \frac{n}{2}$, set

$$s_r = \begin{array}{c} r\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} -1 & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \rtimes \hat{\theta}_n ,$$

regarded as a semisimple coset in $\hat{G}/Z(\hat{G}^0)$. Then

$$\mathrm{Cent}(s_r, \hat{G}^0) \cong \mathrm{Sp}(2r, \mathbb{C}) \times O(n-2r, \mathbb{C}) .$$

Define

$$\hat{H}_r = \mathrm{Cent}(s_r, \hat{G}^0)^0 \cong \mathrm{Sp}(2r, \mathbb{C}) \times SO(n-2r, \mathbb{C}) .$$

Let ξ_r be any L-homomorphism

$$W_F \longrightarrow \mathrm{Cent}(s_r, \hat{G}^0) \times W_F \subseteq \hat{G}^0 \times W_F .$$

This determines an endoscopic datum $(H_r, \mathcal{H}_r, s_r, \xi_r)$ whose equivalence class depends only on the map

$$\xi_r^* : \mathrm{Gal}(\bar{F}/F) \longrightarrow \pi_0(\mathrm{Cent}(s_r, \hat{G}^0)) .$$

Thus,

$$H_r \cong \begin{cases} SO(2r+1) \times SO^*(n-2r), & n \text{ even}, \\ SO(2r+1) \times \mathrm{Sp}(n-1-2r), & n \text{ odd}, \end{cases}$$

where SO^* stands for the quasi-split orthogonal group determined by ξ_r^*. When $n - 2r = 2$ and ξ_r^* is trivial, $Z(\hat{H}_r)^\Gamma$ is infinite, and the endoscopic datum is not elliptic. If we rule out this exceptional case, however, we obtain a set of representatives of elliptic endoscopic data. Observe that $\mathcal{H}_r$ can be identified with ${}^L H_r$ in each case, so there is no need to introduce the extensions that were denoted by H_1 in §2.

Suppose that

$$\psi : \; L_F \times SL(2, \mathbb{C}) \longrightarrow {}^L G^0$$

is a parameter in $\Psi(G^0)$. We shall identify ψ with an n-dimensional representation of the group $L_F \times SL(2, \mathbb{C})$, which can then be decomposed into a direct sum

$$\psi = \bigoplus_{k=1}^{\ell} \psi_k$$

of irreducible representations. The centralizer in $\hat{G}^0$ of the image of ψ is the group of intertwining operators. That is,

$$S_\psi(G^0) = S_\psi^0 \cong \prod_j GL(m_j, \mathbb{C}) \, .$$

The parameter ψ belongs to $\Psi(G)$ if and only if it is self-contragredient as a representation of $L_F \times SL(2, \mathbb{C})$. In other words, the contragredient operation acts as a permutation of order two on the irreducible constituents ψ_k. Suppose that this is the case. Then $S_\psi = S_\psi(G)$ is isomorphic to a product of components of the form

$$GL(m, \mathbb{C}) \rtimes \hat{\theta}_m$$

or

$$\big(GL(m, \mathbb{C}) \rtimes GL(m, \mathbb{C})\big) \times \hat{\tau}_m \, ,$$

with

$$\hat{\tau}_m(g_1, g_2) = \big(\hat{\theta}_m(g_2), \hat{\theta}_m(g_1)\big) \, , \qquad g_1, g_2 \in GL(m, \mathbb{C}) \, .$$

We are especially interested in the parameters $\psi \in \Psi(G)$ which are the images of elliptic parameters $\psi_r \in \Psi_0(H_r)$, for elliptic endoscopic data H_r. Since $A_{\hat{G}} \cong \mathbb{Z}/2\mathbb{Z}$ is finite, this means that there is an element $s \in S_\psi$ such that $S_{\psi,s}$ is finite. The condition is equivalent to

$$(9.2) \qquad S_\psi \cong (\mathbb{C}^* \rtimes \hat{\theta}_1)^\ell \, ,$$

which is to say that the irreducible constituents ψ_k of ψ are self-contragredient and mutually inequivalent. Since

$$\hat{\theta}_1(z) = z^{-1}, \qquad z \in \mathbb{C}^*,$$

we see immediately from (9.2) that there is only an orbit of S_ψ^0 in S_ψ. Therefore, ψ factors through only the one endoscopic datum H_r.

Fix an elliptic endoscopic datum H_r, and let $\psi_r \in \Psi_0(H_r)$ be a fixed elliptic parameter. The image ψ of ψ_r in $\Psi(G)$ then satisfies (9.2). For reasons of induction it is not necessary to consider a product of two classical groups, so we may assume that r equals 0 or $\frac{n}{2}$. Then H_r is either an orthogonal group or a symplectic group. To study the representations of $H_r(\mathbb{A})$ attached to ψ_r, it will be necessary to apply Hypothesis 3.1 to both G and H_r.

A missing ingredient from the local conjectures was a canonical definition of the stable distribution

$$f_r \longrightarrow f_r^{H_r}(\psi_r), \qquad f_r \in C_c^\infty(H_r(\mathbb{A})). \tag{9.3}$$

Such a definition will be provided, at least in some cases, by the connection with G. The packet Π_ψ consists of one orbit $\{\pi_\psi\} \subseteq \Pi(G(\mathbb{A})^+)$ under the group $\pi_0(G^+)^* \cong \mathbb{Z}/2\mathbb{Z}$, and we can choose π_ψ so that

$$< \bar{s}_\psi \bar{s}, \pi_\psi > = < \bar{s}, \pi_\psi > = 1, \qquad \bar{s} \in \mathcal{S}_\psi.$$

It follows from (4.4) that

$$f^{H_r}(\psi_r) = f_G(\pi_\psi), \qquad f \in C_c^\infty(G(\mathbb{A})).$$

A similar formula holds for the corresponding stable distributions on the local groups $G(F_v)$. However, this formula may not determine (9.3) completely. The problem is that the anticipated injection

$$\{f^{H_r} : f \in C_c^\infty(G(\mathbb{A}))\} \hookrightarrow \{f_r^{H_r} : f_r \in C_c^\infty(H(\mathbb{A}))\},$$

obtained by transfer of twisted orbital integrals, could be a strict inclusion. This difficulty is tied up with the question of how many local parameters

$$\psi_r' = \bigotimes_v \psi_{r,v}', \qquad \psi_{r,v}' \in \Psi(H/F_v),$$

lift to ψ. If $\hat{H}_r$ is symplectic or odd orthogonal, the only such parameter will be ψ_r itself. However, there can be a number of ψ'_r in the even orthogonal case, and the formula then determines only a sum of distributions (9.3).

Once the distribution (9.3) has been defined (for H_r and its endoscopic groups), the packet Π_{ψ_r} and the pairing on $\mathcal{S}_{\psi_r} \times \Pi_{\psi_r}$ will be uniquely determined. Leaving aside the question of whether the required local properties of these objects can be deduced from Hypothesis 3.1, let us simply assume that the local assumptions of §3, §4 and [**3**, §**7**] hold for H_r. The next problem is to determine the stable distribution

$$SI^{H_r}_{\psi_r}(f_r) = \sigma(H_r,\psi_r) f^{H_r}_r(\psi_r)\ . \tag{9.4}$$

(See the notation of §7.) The distribution $f^{H_r}_r(\psi_r)$ is a local object which we are assuming is known, so it is the global constant $\sigma(H_r,\psi_r)$ which must be found. According to Hypothesis 3.1, we should take the contribution of ψ to (9.1), and set it equal to 0. I have not thought through the details, but it should just be a question of running backwards over a couple of the more trivial arguments of §5 and §7. The result will be a special case

$$\sigma(H_r,\psi_r) = |\mathcal{S}_{\psi_r}|^{-1} \varepsilon^{H_r}_{\psi_r}(\bar{s}_{\psi_r}) \tag{9.5}$$

of the general formula (7.12) we determined was compatible with Hypothesis 4.1. Observe that the sign character $\varepsilon^{H_r}_{\psi_r}(\bar{s}_{\psi_r})$ appears. It originates, through [**3, Conjecture 7.1**] and Proposition 5.1, from the normalizing factors for (nontempered) intertwining operators for $GL(n)$.

Having determined the stable distributions (9.4) (for H_r and its endoscopic groups), we can apply Hypothesis 3.1 to H_r. The contribution of ψ_r to $E_{\mathrm{disc},t}(f_r)$ can be calculated as an easy special case of the arguments in §7, or it can simply be read off from the formula (7.15), (applied to H_r instead of G). It equals

$$\sum_{\pi\in\Pi_{\psi_r}} \Big(|\mathcal{S}_{\psi_r}|^{-1} \sum_{x\in\mathcal{S}_{\psi_r}} \varepsilon^{H_r}_{\psi_r}(x) < x,\pi > \Big) \mathrm{tr}\big(\pi(f_r)\big)\ .$$

On the other hand, the parameter $\psi_r \in \Psi_0(H_r)$ is elliptic. Its contribution to $I_{\mathrm{disc},t}(f_r)$ equals

$$\sum_{\pi\in\Pi_{\psi_r}} m_0(\pi) \mathrm{tr}\big(\pi(f_r)\big)\ .$$

Identifying the coefficients in these two linear combinations of irreducible characters, we obtain the multiplicity formula

$$m_0(\pi) = |\mathcal{S}_{\psi_r}|^{-1} \sum_{x \in \mathcal{S}_{\psi_r}} \varepsilon_{\psi_r}^{H_r}(x) < x, \pi > .$$

Notice that the only contribution to $m_0(\pi)$ should come from the parameter ψ_r. This suggests that the map $\Psi(H_r) \to \Sigma(H_r)$ is bijective, at least if H_r is not an even orthogonal group, the case we left ambiguous.

This discussion has been very sketchy. We have simply tried to indicate that since the spectrum of $GL(n)$ can be understood in terms of a Jordan decomposition, the same should be true for the spectrum of its endoscopic groups. The arguments of §5–§8 will be essential for this, in that they allow for the elimination of the irrelevant parameters from the study of (9.1).

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Motifs et Formes Automorphes : Applications du Principe de Fonctorialité

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Sommaire

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Automorphic Forms, Shimura Varieties, and L-Functions

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0. Introduction

Ne t'attarde pas à l'ornière des résultats.
Seule les traces font rêver.

—René Char

Les conjecture générales de Langlands amènent à subodorer une relation étroite entre la théorie des formes automorphes et la théorie arithmétique des variétés algébriques - plus précisément, entre représentations automorphes et motifs au sens de Grothendieck. Dans son article à Corvallis, Langlands avait esquissé l'idée d'une théorie "tannakienne" des formes automorphes: si l'on suppose que cette théorie existe, on est amené à conjecturer l'existence d'un "groupe de Galois automorphe", qui devrait être lié au "groupe de Galois motivique" postulé par Grothendieck.

La théorie moderne des formes automorphes, cependant, a conduit - depuis l'introduction par Maass des formes qui portent son nom - à l'étude de nombreux objets qui, à la différence des formes modulaires classiques, n'ont qu'une relation ténue avec des objets arithmétiques. C'est déjà vrai des "Grössencharakterer" de Hecke: dans ce cas, Weil avait été amené à définir les "Grössencharakterer de type A_0" - ou caractères de Hecke algébriques - qui sont ceux qui apparaissent, par exemple, dans la théorie de la multiplication complexe.

Le propos de cet article est de définir, dans le cas non abélien, la catégorie des représentations automorphes qui devraient de même être liées aux "motifs" - l'analogue non abélien des caractères de Hecke.

Une fois acquise la "classification de Langlands" pour les groupes réels, la définition d'une telle catégorie s'impose assez naturellement (Borel l'avait proposée dans [10b]). On s'aperçoit assez vite, cependant, que la définition "évidente" ne remplit pas deux des conditions minimales requises: être accessible au formalisme tannakien, et vérifier les propriétés de **rationalité** qui sont satisfaites par les motifs, et par les caractères de Hecke algébriques.

L'origine du problème est que la classification de Langlands, basée sur l'induction unitaire, n'est pas "algébrique": elle ne respecte pas la rationalité des représentations.

En prenant pour guides le formalisme tannakien et les propriétés de rationalité, nous proposons ici une définition de la catégorie des représentations automorphes $GL(n, \mathbb{A}_F)$, $\mathbb{A}$ désignant les adèles d'un corps de nombres F. Nous montrons alors, à l'aide de l'extension à $GL(n)$ de la théorie d'Eichler-Shimura, que ces représentations -

au moins quand leur type à l'infini est convenablement régulier - ont les propriétés de rationalité attendues: si $\pi = \otimes_v \pi_v$ est une représentation automorphe algébrique, supposée cuspidale, de $GL(n, \mathbb{A}_F)$, sa partie finie $\pi_f = \otimes_{v\text{finie}} \pi_v$ est définie sur un corps de nombres $E \subset \mathbb{C}$, et toutes les conjuguées de π_f par $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ sont encore parties finies de représentations cuspidales. De plus, si l'on admet le "principe de fonctorialité" de Langlands, la catégorie des représentations algébriques devrait être une catégorie tannakienne.

Nous précisons ensuite, de façon assez détaillée, les relations conjecturales entre représentations algébriques et motifs. Par construction, les représentations algébriques sont munies de structures qui devraient être reliées aux structures diverses dont sont munis les motifs: théorie de Hodge; représentations des groupes de Galois dans la cohomologie l-adique. Les propriétés de **pureté** des motifs se traduisent par la "Conjecture de Ramanujan".

Dans quelques cas, que nous passons en revue, on dispose déjà d'une interprétation "motivique" des représentations automorphes: il s'agit essentiellement de formes automorphes de type arithmétique sur $GL(2)$, ou des caractères de Hecke. Dans le dernier paragraphe, nous montrons comment associer, sous certaines hypothèses locales, des systèmes compatibles de représentations l-adiques à des représentations automorphes autoduales de $GL(n)$ sur un corps totalement réel. En particulier, nous démontrons la conjecture de Ramanujan (à presque toutes les places) pour de telles représentations.

Le lecteur de penchants classiques pourra s'étonner de nous voir étudier les formes automorphes sur $GL(n)$ plutôt que sur les groupes réductifs liés aux domaines hermitiens symétriques: les formes modulaires holomorphes sur ceux-ci admettent souvent une interprétation "motivique" évidente. L'importance des groupes linéaires tient au fait que, d'après le principe de fonctorialité, ils "reçoivent" toutes les représentations automorphes des groupes classiques. Ainsi, toutes les formes automorphes sur ceux-ci qui ont une interprétation motivique doivent être liées à des représentations algébriques. La réciproque n'est pas vraie: en fait, seuls des motifs satisfaisant des conditions très fortes d'autodualité apparaissent, semble-t-il, dans la cohomologie d'intersection des variétés de Shimura (cf. §5.2.4). Par ailleurs, le fait de considérer $GL(n)$ permet d'utiliser - au moins conjecturalement - le formalisme tannakien.

Voici maintenant une description plus détaillée des différents paragraphes. Le paragraphe 1 décrit le formalisme tannakien pour $GL(n)$.

Nous avons repris les constructions de Langlands [38 f] en les explicitant: noter que le résultat fondamental qui est à la base du formalisme (Thm. 1.1), et qui était supposé par Langlands, a été démontré par Jacquet et Shalika. Il donne un sens, en particulier, à la notion de représentation isobare. La catégorie abélienne des représentations isobares est décrite en 1.4.3. Les représentations algébriques sont définies en 1.2.3. On a décrit en détail comment elles se traduisaient dans les cas "classiques" (formes modulaires de Hilbert).

Le paragraphe 2, utilitaire, décrit certains cas du principe de fonctorialité - établis pour la plupart pendant les dix dernières années - qui nous seront utiles.

Le paragraphe 3 étudie le **corps de rationalité** des représentations automorphes de $GL(n)$. On y démontre tout d'abord que celles-ci sont définies sur leur corps de rationalité (Prop. 3.1). On explique ensuite comment transformer la paramétrisation de Langlands des représentations non ramifiées pour la rendre rationnelle. On formule ensuite les conjectures fondamentales (Conjectures 3.7, 3.7′ , 3.8) sur la rationalité des représentations algébriques (en particulier, la Conjecture 3.8 est l'analogue de la conjecture de Weil, démontrée par Waldschmidt, et caractérisant par leurs valeurs les caractères de Hecke algébriques). On vérifie que ces conjectures sont compatibles avec notre définition des représentations algébriques (§3.4.3) ainsi qu'au changement de base (§3.4.4).

On démontre ensuite la Conjecture 3.7 pour les représentations **régulières** (Thm. 3.13). La démonstration repose sur la théorie d'Eichler-Shimura, reformulée dans le cadre adélique par Harder. Le point essentiel consiste à montrer que la cohomologie parabolique, considérée commes sous-espace de la cohomologie à support compact des groupes arithmétiques, est stable par automorphismes de $\mathbb{C}$. On le déduit des propriétés d'autodualité de la cohomologie parabolique: le résultat crucial (Prop. 3.18) a été démontré aussi, de façon différente, par Harder [30 a]. Il est ici déduit d'un lemme simple mais fondamental concernant l'image de la cohomogie à support compact dans la cohomogie L^2 (Lemme 3.17). Une conséquence importante de la démonstration est que la cohomologie parabolique (dans le cas des coefficients constants) est définie sur $\mathbb{Q}$: c'est le Théorème 3.19, qui étend à $GL(n)$ un résultat classique d'Eichler et Shimura.

Le paragraphe 4 est consacré au "dictionnaire" reliant motifs et représentations algébriques. On y définit les représentations algébriques rationnelles (§4.2), analogue des motifs "absolus" (sans coefficients).

Après avoir introduit le groupe de Galois (conjectural) associé aux représentations algébriques, et discuté ses relations avec le "groupe de Galois automorphe" de Langlands, on donne la conjecture fondamentale concernant la relation entre motifs et représentations algébriques (4.3.2, Conjecture 4.5). Elle est précisée en 4.3.3.

On donne quelques vérifications concernant la rationalité et le comportement fonctoriel en 4.3.4, 4.3.4. En particulier, on démontre le fondamental Lemme 4.9, concernant la pureté des représentations algébriques à l'infini; il est utilisé dans le paragraphe 3 et pour définir **a priori** la notion de **poids** pour les représentations algébriques. Dans le paragraphe 4.5, on formule un problème concernant la "théorie réciproque" pour les représentations algébriques.

Enfin, le paragraphe 5 décrit comment l'on peut associer, sinon des motifs, au moins des systèmes compatibles de représentations l-adiques, aux représentations automorphes de $GL(n, \mathbb{A}_F)$ (F totalement réel) qui sont régulières et autoduales. Les démonstrations reposent sur des résultats récents de Kottwitz.

Pendant la rédaction de cet article, j'ai bénéficié de discussions avec J.-L. Waldspurger sur les questions de rationalité du paragraphe 3. J'ai aussi eu accès à un manuscrit de Harder [30 a] concernant la cohomologie des groupes arithmétiques. Lors de la conférence, et ensuite, j'ai profité des explications de D. Blasius et D. Ramakrishnan sur les propriétés de la catégorie des motifs. Si le paragraphe 4 ne contient pas d'erreur grossière, cela leur est dû. J. Tits m'a décrit la cohomologie galoisienne des groupes unitaires. Enfin, R. Kottwitz m'a, à plusieurs reprises, expliqué les propriétés miraculeuses des groupes décrits au paragraphe 5 et la stabilisation de la formule des traces tordue. Je lui suis tout particulièrement reconnaissant de m'avoir communiqué avant rédaction les résultats essentiels de [36]. Qu'ils soient tous ici remerciés.

1. Catégories de représentations automorphes

Dans ce chapitre on décrit certaines catégories de représentations automorphes des groupes linéaires qui devraient être passibles d'un formalisme tannakien.

1.1. Principe de fonctorialité; opérations tannakiennes. Dans tout ce qui suit, on s'intéresse aux **représentations automorphes** de $GL(n, \mathbb{A}_F)$ où F est un corps de nombres et $\mathbb{A} = \mathbb{A}_F$. Rappelons [12, §4.6] qu'une représentation irréductible de $G(\mathbb{A}) = GL(L, \mathbb{A})$

est automorphe si elle est **isomorphe** à un sous-quotient irréductible de la représentation de $G(\mathbb{A})$ sur l'espace des formes automorphes (pour les problèmes que cela pose du point de vue tannakien, cf.§1.3). On notera Aut (n) la classe des représentations automorphes de $GL(n, \mathbb{A}_F)$, Aut $= \coprod_n$ Aut (n).

Rappelons que les représentations **cuspidales** sont celles qui interviennent dans l'espace des formes paraboliques $\mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$ [12]. La théorie des séries d'Eisenstein montre (Langlands [38 e]) que toute représentation automorphe est isomorphe à un sous-quotient d'une induite

$$\rho = \operatorname{Ind}_{P(\mathbb{A})=M(\mathbb{A})N(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes 1) \tag{1.1}$$

où l'on désigne par Ind l'induction **unitaire** à partir d'un parabolique $P = MN$ de G, et σ une représentation **cuspidale** de $M(\mathbb{A})$.

Dans tout ce qui suit, on considérera uniquement, sauf mention explicite, les paraboliques standard de $GL(n)$ formés de matrices triangulaires supérieures par blocs. Si P est de type $(n_1, \dots, n_r)$ où $n_1 + \dots + n_r = n$, σ peut s'écrire sous la forme $\sigma_1 \otimes \dots \otimes \sigma_r$ où σ_i est une représentation cuspidale de $GL(n_i, \mathbb{A})$. Le résultat suivant est dû à Jacquet et Shalika.

THÉORÈME 1.1 [33 a,b]. *Soient $n = n_1 + \dots + n_r = m_1 + \dots + m_s$ deux partitions de n, S un ensemble fini de places de F, contenant les places infinies. Pour tout $i = 1, \dots, r$, soit σ_i une représentation de $G_{n_i}(\mathbb{A})$ sur un sous-espace irréductible V_i de $\mathcal{A}_{\text{cusp}}(G_{n_i}(F)\backslash G_{n_i}(\mathbb{A}))$; de même, soit (τ_j, W_j) une représentation de $G_{m_i}(\mathbb{A})$ $(j = 1, \dots, s)$. On suppose les σ_i et les τ_j nonramifiées en dehors de S.*

Supposons que pour $v \notin S$, les matrices diagonales

$$\begin{pmatrix} t_{\sigma_1,v} & & \\ & \ddots & \\ & & t_{\sigma_r,v} \end{pmatrix}, \qquad \begin{pmatrix} t_{\tau_1,v} & & \\ & \ddots & \\ & & t_{\tau_s,v} \end{pmatrix}$$

où les $t_{\sigma_i,v}$, (resp. $t_{\tau_j,v}$) désignent les matrices de Hecke, sont égales modulo $\mathfrak{S}_n$. Alors $r = s$, et il existe une permutation $\alpha \in \mathfrak{S}_r$ telle que $m_{\alpha(i)} = n_i$ et les deux sous-modules V_i et $W_{\alpha(i)}$ de $\mathcal{A}_{\text{cusp}}(G_{n_i}(F)\backslash G_{n_i}(\mathbb{A}))$ coïncident.

L'analogue local de ce résultat est la classification de Langlands. Supposons pour un instant F **local**. Soit $P : n = n_1 + \dots + n_r$ une partition de n, et σ_i des représentations de carré intégrable mpodulo le centre de $GL(n_i, F)$.

Soit χ_i le caractère central de σ_i sur $F^\times$. On a $\chi_i = |\ |^{s_i}$, $s_i \in \mathbb{R}$. A permutation près, on peut supposer $s_1 \geq s_2 \geq \cdots \geq s_r$.

La représentation

$$\rho = \mathrm{Ind}_{P(F)}^{G(F)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1) = \rho(\sigma_1, \ldots, \sigma_r) \tag{1.2}$$

a alors un unique quotient irréductible, qu'on appelle son quotient de Langlands; il apparaît avec multiplicité 1 dans ρ et donc dans chacune des représentations $\rho(\sigma_{s.1}, \ldots, \sigma_{s.r})$ où $s \in \mathfrak{S}_r$. On l'appelle encore **sous-quotient de Langlands** de chacune de ces représentations (cf.[14, 50]). On notera que la définition habituelle du quotient de Langlands fait appel à l'induction à partir de représentations tempérées. Pour traduire ceci en notre définition, il faut savoir que pour $GL(n)$ les induites paraboliques à partir de représentations de carré intégrable unitaires sont irréductibles: [5, 4 ; 51 a].

D'après Langlands, on sait que toute représentation irréductible de $GL(n, F)$ est isomorphe au sous quotient de Langlands d'une $\rho(\sigma_1, \ldots, \sigma_r)$ et que les σ_i sont alors bien déterminées à permutation près.

Pour étendre la définition du quotient de Langlands aux composantes locales de (A.1), il faut introduire les représentations génériques. Soit

$$N_0 = \left\{ n = \begin{pmatrix} 1 & x_1 & * \\ \ddots & \ddots & x_{n-1} \\ 0 & & 1 \end{pmatrix} \right\}$$

ψ un caractère additif non nul de F, $\theta(n) = \psi(x_1 + \cdots + x_{n-1})$ le caractère non dégénéré de $N_0(F)$ associé. Rappelons qu'une représentation (π, V) de $GL(n, F)$ (F local) est **générique** s'il y a un vecteur distribution dans son dual se transformant sous $N_0(F)$ selon θ . On sait que les composantes locales d'une représentation cuspidale sont génériques [40, 49]. D'après des résultats de Kostant et Vogan dans le cas réel, de Bernstein et Zelevinsky dans le cas p-adique, on sait que toute représentation irréductible **générique** s'écrit comme l'induite totale $\rho(\sigma_1, \ldots, \sigma_r)$ (1.2) pour des représentations de carré intégrable $\sigma_1, \ldots, \sigma_r$.

Considérons alors les composantes locales de la représentation ρ de (1.1), où $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ est un produit tensoriel de représentations cuspidales de $GL(n, \mathbb{A}_F)$. La composante à la place v s'écrit

$$\rho_v = \mathrm{Ind}_{P(F_v)}^{G(F_v)}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{r,v} \otimes 1). \tag{1.3}$$

Puisque les $\sigma_{i,v}$ sont génériques, elles s'écrivent à leur tour comme des induites, à la (1.3), à partir de représentations de carré intégrable:

$$\rho_{i,v} = \mathrm{Ind}_{P_i(F_v)}^{G_{n_i}(F_v)}(\tau_{i,v}^1 \otimes \cdots \otimes \tau_{i,v}^{r_i} \otimes 1) \tag{1.4}$$

(**induite totale**). On en déduit, par induction par étapes, que ρ_v est l'induite totale à partir du produit tensoriel des τ_i et contient donc, avec multiplicité 1, le sous-quotient de Langlands correspondant.

Definition 1.2 (Langlands). Soit $\pi = \otimes \pi_v$ une représentation automorphe irréductible de $G(\mathbb{A})$. On sait que π est un sous-quotient d'une représentation (1.1) $\rho = \otimes \rho_v$. On dit que π est <u>isobare</u> si π_v est, à toute place, le sous-quotient de Langlands de ρ_v.

Remarquons tout d'abord que, si ρ_v est non-ramifiée, son quotient de Langlands est la représentation sphérique associée [31]. En particulier la représentation isobare associée à ρ a bien la propriété de continuité exigée des représentations automorphes.

Si $\sigma_1, \ldots, \sigma_r$ sont des représentations cuspidales de $G_{n_i}(A_F)$, on notera

$$\sigma_1 \boxplus \cdots \boxplus \sigma_r$$

l'unique sous-quotient isobare de la représentation (1.1).

Par construction, il ne dépend pas, à isomorphisme près, de l'ordre des σ_i. Par définition, toute représentation isobare est ainsi obtenue; enfin, d'après le Théorème 1.1, deux représentations $\sigma_1 \boxplus \cdots \boxplus \sigma_r$ et $\sigma_1' \boxplus \cdots \boxplus \sigma_t'$ sont isomorphes si et seulement si les σ_i sont obtenues par permutation des σ_i'.

Exemples: Dans (1.1), prenons P égal au sous-groupe de Borel, et

$$\rho = \mathrm{Ind}\left(|\ |^{\frac{-n+1}{2}}, |\ |^{\frac{-n+3}{2}}, \ldots, |\ |^{\frac{n-1}{2}}\right)$$

où $|\ |$ désigne la norme d'idèle. La représentation isobare associée est la représentation triviale.

Plus généralement, soit P homogène de type $(m,\ldots,m)$ $(n = am)$ et ω une représentation cuspidale de $GL(m,\mathbb{A})$. La représentation π telle que, à toute place v, π_v est l'unique sous-quotient irréductible de

$$\mathrm{Ind}_{P_v}^{G_v}(\omega_v, \omega_v|\,|, \ldots, \omega_v|\,|^{a-1}) \tag{1.5}$$

est isobare et intervient dans le spectre discret [51 b]. Moeglin et Waldspurger [39] ont montré récemment que ces représentations forment tout le spectre discret. On notera l'unique quotient irréductible de (1.5) $J(\omega_v, a)$ et le produit global $J(\omega, a)$.

Rappelons maintenant comment le principe de fonctorialité s'illustre dans le cas de $GL(n)$:

• **Sommes** : Soit $n = n_1 + n_2 + \cdots + n_r$, π_i des représentations automorphes de $GL(n_i, \mathbb{A})$. On peut considérer la représentation induite

$$\rho = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi_1 \otimes \cdots \otimes \pi_r \otimes 1.$$

Elle a pour matrices de Hecke, aux places non-ramifiées,

$$t_{\rho,v} = \begin{pmatrix} t_{\pi,v} & & \\ & \ddots & \\ & & t_{\pi_r,v} \end{pmatrix}.$$

Autrement dit, les matrices de Hecke sont dé terminées par l'homomorphisme de L-groupes

$${}^LG^0_{n_1} \times \cdots \times {}^LG^0_{n_r} \to {}^LG^0_n$$

envoyant $GL(n_1) \times \cdots \times GL(n_r)$ vers $GL(n)$ par blocs.

Si les représentations π_i sont **isobares**, on peut considérer l'unique sous-quotient isobare de ρ (il existe d'après les mêmes arguments utilisés pour π_i cuspidales). On le note

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_r.$$

• **Produits.** Soit $n = ab$, π_1 une représentation automorphe de $GL(a, \mathbb{A})$, π_2 une représentation automorphe de $GL(b, \mathbb{A})$.

CONJECTURE 1.3. *Il existe une représentation automorphe π de $GL(ab, A)$ telle que, pour v en dehors d'un ensemble fini S de places:*

$$t_{\pi,v} = t_{\pi,v} \otimes t_{\pi_2,v} \tag{1.6}$$

(le produit tensoriel de deux matrices diagonales définit une matrice diagonale, qui ne dépend que de π_1 et π_2 modulo $\mathfrak{S}_n$).

Cette conjecture mérite d'être appelée la conjecture principale pour $GL(n)$. Notons qu'il suffirait de la démontrer pour des représentations cuspidales. On peut la raffiner pour des représentations isobares: soient π_1, π_2 isobares. Si la conjecture est vraie pour π_1 et π_2, il existe une unique représentation **isobare** de $GL(n, A_F)$ vérifiant (1.6).

Definition 1.4. Si la représentation isobare de $GL(n, A_F)$ vérifiant (1.6) existe, on l'appelle produit tensoriel extérieur de π_1 et π_2; on la note $\pi_1 \boxtimes \pi_2$.

Du côté du L-groupe, cette opération correspond bien sûr à l'homomorphisme

$$^LG_a^0 \times {}^LG_b^0 \to {}^LG_n^0$$

donné par le produit tensoriel: $GL(a) \times GL(b) \to GL(n)$.

• **Puissances.** Plus généralement, soit r une représentation rationnelle de $GL(n)$, de degré N. Si $G = GL(n)$ et $H = GL(N)$, on considère r comme un homomorphisme $^LG^0 \to {}^LH^0$. On l'étend aux L-groupes, trivialement sur les groupes de Galois. D'après le principe de fonctorialité on en déduit de même une application conjecturale, envoyant les représentations isobares τ_n de $GL(n, A)$ vers celles, π_N, de $GL(N, A)$, et déterminée par le fait que

$$t_{\pi_N,v} = r\left(t_{\pi_n,v}\right)$$

comme classe de conjugaison dans $GL(N, \mathbb{C})$.

Des cas particuliers importants sont les puissances **symétriques** S^i, **extérieures** Λ^i de la représentation standard de $GL(n)$.

Plus généralement, on peut considérer des représentations rationnelles de $G = GL(n_1) \times \cdots \times GL(n_r)$. On retrouve ainsi le cas des produits.

• **Changement de base.** Soit E/F une extension de corps de nombres. On considère $GL(n)/E$ comme un F-groupe par restriction des scalaires. Son L-groupe est alors [10a,§5]

$$^LH = GL(n, \mathbb{C}) \times \cdots \times GL(n, \mathbb{C}) \times \mathfrak{g}_F;$$

il y a une copie de $GL(n, \mathbb{C})$ pour chaque plongement $E \to \bar{F}$ sur F, et $\mathfrak{g}_F$ opère par permutation. On en déduit deux opérations fonctorielles:

(a) **Restriction.** Posant $G = GL(n)/F$, on a un plongement diagonal ${}^LG \to {}^LH$ qui envoie ${}^LG^0$ diagonalement dans ${}^LH^0$. Dualement, on en déduit une opération de restriction $\rho_{F/E}$ envoyant les représentations isobares de $G(\mathbb{A}_F)$ vers celles de $G(\mathbb{A}_E)$.
(b) **Induction**. Soit $d = [E : F]$, $G_{dn} = GL(dn)/F$. On a un plongement

$$ {}^LH \to {}^LG_{dn} $$

envoyant les matrices de ${}^LH^0$, diagonalement par blocs, vers des matrices de $GL(dn, \mathbb{C})$. On l'étend à ${}^LH^0$ en faisant opérer les éléments de $\mathfrak{g}_F$ par des matrices de permutation. On en déduit, dualement, une opération (conjecturale) d'**induction automorphe** ι_E^F envoyant les représentations isobares de $GL(n, \mathbb{A}_E)$ vers celles de $GL(dn, \mathbb{A}_F)$.

Pour la description, dans le cas où E/F est cyclique d'ordre premier, de $\rho_{F/E}$ et ι_E^F en termes de matrices de Hecke, voir [18 d].

Le changement de base commute avec les opérations "géométriques" $\boxplus$ et $\boxtimes$ (les représentations étant considérées à isomorphisme près).

Dans le chapitre 2, on passera en revue certains cas où le principe de fonctorialité a été établi. Il sera commode de décrire tout d'abord certaines catégories de représentations.

1.2. Catégories de représentations automorphes et admissibles.

Il s'agit maintenant de définir certaines catégories de représentations automorphes, qui puissent être accessibles au formalisme tannakien (cf. §1.3). On se contente ici de définir les **objets**. Il faut que les ensembles de représentations automorphes considérés soient stables par les opérations tannakiennes $\boxtimes$ et $\boxplus$, ainsi que par l'adjonction qui à π associe sa contragrédiente $\tilde{\pi}$. Pour que celles-ci soient bien définies, on se limite dès l'abord aux représentations **isobares.**

1.2.1. La catégorie *Isob*. Ses objets sont les représentations isobares. La somme directe $\pi \boxplus \pi'$ y est bien définie; le produit tensoriel l'est (à isomorphisme près) modulo la Conjecture principale.

On notera $Isob(n)$ l'ensemble des objets de degré n (représentation de $GL(n, \mathbb{A})$); de même pour d'autres catégories.

1.2.2. Les catégories *Temp* **et** *IC*..

Les catégories *Temp* sont les représentations automorphes irréductibles de $GL(n, \mathbb{A})$- pour les différents n- dont les composantes à toutes les places sont **tempérées.**

LEMME 1.5. *Si $\pi \in Temp(n)$, π est isobare.*

En tant que représentation automorphe, π est en effet un sous-quotient de $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$ pour des représentations cuspidales σ_i de $GL(n_i, \mathbb{A})$. A presque toutes les places, π_v est non-ramifiée. Les représentations $(\sigma_i)_v$ le sont donc aussi, et π_v est, par induction par étages, le sous-quotient de Langlands de $\mathrm{Ind}_{B(F_v)}^{G(F_v)}(\chi_1, \ldots, \chi_n)$ où B est le sous-groupe de Borel, et les χ_i sont non-ramifiés. Comme π_v est tempérée, l'unicité dans la classification de Langlands implique que les χ_i sont unitaires et l'induite irréductible. On en déduit que les σ_i doivent être **unitaires** (à presque toutes les places, et donc partout); donc $\pi \cong \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$ est isobare.

Rappelons la:

CONJECTURE 1.6 ("CONJECTURE DE RAMANUJAN"). *Si π est une représentation cuspidale unitaire de $GL(n, A_F)$, les composantes locales de π sont tempérées.*

Il est clair que la somme directe de deux représentations de *Temp* est encore tempérée. Le Lemme 1.5 montre en fait que si π est une représentation automorphe dont presque toutes les composantes locales sont tempérées, elle est somme (pour $\boxplus$) de représentations cuspidales unitaires. Si l'on admet la Conjecture 1.6, et bien sûr la Conjecture 1.3, on voit que *Temp* est préservée par le produit tensoriel.

Soit *IC* la catégorie de représentations unitairement induites de cuspidales unitaires. La conjecture de Ramanujan impliquerait que *IC* est incluse dans *Temp.* Réciproquement, la démonstration du Lemme 1.5 montre que *Temp* est incluse dans *IC.* Il est souvent utile (cf. [2, ch. III]) de considérer directement la catégorie *IC*, qui se décrit simplement sans attendre la solution de la conjecture de Ramanujan. Elle est formée de représentations isobares, et est stable par $\boxplus$.

La conjecture de Ramanujan a une conséquence intéressante pour les représentations de *Isob*:

LEMME (QUOTIENT DE LANGLANDS GLOBAL). Supposons la Conjecture 1.6. *Alors, si $\pi \in Isob(n)$, π est l'unique quotient irréductible d'une représentation.*

$$Ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

où les σ_i *sont des représentations cuspidales de* $G_{n_i}(\mathbb{A})$, P *est le parabolique de type* $(n_1, \ldots, n_r)$, $\sigma_i \cong \sigma_i^0 \mid \mid^{s_i}$ *où* σ_i^0 *est unitaire et* $s_i \in \mathbb{R}$, $s_1 \geq s_2 \geq \cdots \geq s_r$.

Cela résulte immédiatement des formules (1.3) et (1.4) puisque les σ_i^0 sont alors tempérés.

1.2.3. La catégorie *Alg*..

La théorie des formes automorphes sur $GL(1)$ - i.e., des caractères du groupe des classes d'idèles - montre que les catégories introduites jusqu'ici sont trop vastes pour être liées à des objets arithmétiques. On veut maintenant définir la catégorie de représentations de $GL(n)$ qui joue un rôle analogue à celui des Grössencharakterer de type *Ao*, ou caractères de Hecke algébriques, de Weil. Comme on va le voir (cf. aussi §3), cela amène à modifier la définition de $\boxplus$.

Rappelons que d'après Langlands [38 d, 51] il y a une bijection canonique entre représentations admissibles irréductibles de $GL(n, F)$ ($F = \mathbb{R}$ ou $\mathbb{C}$), modulo équivalence, et représentations complexes de degré n, à image semi-simple, à équivalence près, du groupe de Weil W_F. On a une suite exacte

$$1 \longrightarrow W_{\mathbb{C}} \longrightarrow W_{\mathbb{R}} \longrightarrow \mathfrak{g}_{\mathbb{C}/\mathbb{R}} \longrightarrow 1$$

et $W_{\mathbb{C}}$ est canoniquement isomorphe à $\mathbb{C}^\times$ [53]. Il est parfois commode de considérer $\mathbb{C}^\times$ comme l'ensemble des points réels de $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(Gm)$.

On notera r les représentations du groupe de Weil réel ou complexe. Insistons sur le fait que la paramétrisation que nous utilisons est celle de Langlands. En particulier, elle envoie les représentations **tempérées** de $GL(n, F)$ sur les représentations à image bornée de W_F.

Soit $\pi = \otimes_v \pi_v$ une représentation irréductible de $GL(n, \mathbb{A})$. Soit v une place infinie de F, et fixons un isomorphisme de F_v avec $\mathbb{R}$ ou $\mathbb{C}$. La classification de Langlands associe à v une représentation r_v de W_{F_v}. Par restriction dans le cas réel, on en déduit une représentation - encore notée r_v- de $\mathbb{C}^\times$; dans le cas complexe, r_v est la représentation de $\mathbb{C}^\times$ définie par le choix d'isomorphisme.

Enfin, soit $\mid \, \mid_{\mathbb{C}}$ la valeur absolue complexe sur $\mathbb{C}^\times$.

Definition 1.8. *Alg*(n) est l'ensemble des représentations isobares de $GL(n, \mathbb{A})$, $\pi = \otimes \pi_v$, telles que, pour toute place infinie v, r_v soit égale à $\chi_1 \otimes \cdots \otimes \chi_n$ où χ_i est un caractère de $S(\mathbb{R}) = \mathbb{C}^\times$ tel que $\chi_i \mid |_{\mathbb{C}}^{\frac{1-n}{2}}$ soit un caractère algébrique.

Autrement dit, $\chi_i(z) = z^{p_i + \frac{n-1}{2}} (\bar{z})^{q_i + \frac{n-1}{2}}$, avec p_i, $q_i \in \mathbb{Z}$.

Remarque. Soit v une place infinie de F. Soit H le tore maximal (diagonal) de G. Si v est complexe, soit $\mathfrak{h}_0 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ (Lie H); si v est réelle, $\mathfrak{h}_0 = \mathrm{Lie}\, H$. Soit $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$. On a $\mathfrak{h} \cong \mathbb{C}^n \times \mathbb{C}^n$ (cas complexe), $\mathfrak{h} \cong \mathbb{C}^n$ (cas réel), le groupe de Weyl complexe W opérant comme $\mathfrak{S}_n \times \mathfrak{S}_n$ (resp. $\mathfrak{S}_n$). D'après Harish-Chandra, le centre $\mathfrak{Z}$ de $U(\mathfrak{g})$, où $\mathfrak{g}_0 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{Lie}\, G$ ou $\mathrm{Lie}\, G$ suivant le cas, et $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, s'identifie à $S(\mathfrak{h})^W$.

Le caractère infinitésimal de π_v s'identifie alors à l'élément $(p, q) \in \mathbb{C}^{2n}$ (cas complexe), ou à $(p_1, \ldots p_{2r}, p'_{2r+1}, \cdots, p'_n) \in \mathbb{C}^n$ si r_v est de la forme $(z^{p_1} \bar{z}^{p_2}, z^{p_2} \bar{z}^{p_1}, \ldots, (z\bar{z})^{p'_{2r+1}}, \ldots, (z\bar{z})^{p'_n})$ avec $p_i \neq p_j$ (cas réel). Laumon m'a fait remarquer que - modulo le centre - les représentations de Alg sont donc associées à des formes automorphes satisfaisant des équations différentielles associées à un caractère de $\mathfrak{Z}$ qui est **entier** pour la structure entière sur $\mathfrak{h}$ déduite d'une base de Chevalley.

Exemple. $GL(2)$ sur un corps réel.

Il sera utile de décrire ici comment la définition de *Alg* se traduit dans les cas *classiques*.

Commençons par le cas de $GL(2)$ sur $\mathbb{Q}$. Soit f une forme parabolique de poids $k \in \mathbb{Z}$ sur le demi-plan de Poincaré pour le groupe $\Gamma_0(n)$, associée à un caractère ψ de $(\mathbb{Z}/N\mathbb{Z})^\times$:

$$f\left(\frac{az+b}{cz+d}\right) = \psi(a)^{-1} j(\gamma, z)^k f(z), \qquad \gamma \in \Gamma_0(N),$$

où $j(\gamma, z) = cz+d$. Le relèvement de poids k [26] lui associe une forme parabolique φ_f sur $GL(2, \mathbb{Q}) \backslash GL(2, \mathbb{A})$, dont le caractère central est le caractère d'Artin ω de $\mathbb{A}^\times$ associé à ψ. On suppose que f est fonction propre des opérateurs de Hecke pour $p|N$, et est holomorphe si $k > 0$, antiholomorphe pour $k < 0$, et fonction propre du laplacien hyperbolique:

$$\Delta f = \lambda f, \qquad \Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

pour $k = 0$; enfin on peut supposer que f est une newform. Plus généralement, on peut tordre le relèvement de poids k par un caractère $|\det g|^s$ du déterminant, remplaçant ainsi ω par $\omega|\ |^{2s}$.

• **Cas holomorphe** $k \geq 2$. Si f est holomorphe, φ_f engendre alors une représentation $\pi = \pi_\infty \otimes \pi_f$ de $GL(2, \mathbb{A}_\mathbb{Q})$ telle que π_∞ est associée, par la correspondance de Langlands, à un couple de caractères $(z^p \bar{z}^q, z^q \bar{z}^p)$ $(p \neq q)$ de $\mathbb{C}^\times$. Elle est algébrique si $p = p_0 + \frac{1}{2}$, $q = q_0 + \frac{1}{2}$ avec $p_0, q_0 \in \mathbb{Z}$; son caractère central sur $\mathbb{R}_+^\times$ est $x \to x^m$, avec $m = p + q = p_0 + q_0 + 1 \equiv p_0 - q_0 + 1 \pmod 2$; le poids de f est $k = |p - q| + 1 = |p_0 - q_0| + 1$. Donc:

Les relèvements π de f qui sont algébriques sont ceux de caractère central $x \to x^m (\operatorname{sgn} x)^\epsilon$ avec $m \equiv k \pmod 2$.

• **Le cas antiholomorphe** $(k \leq -2)$ se traite de même.

• **Formes de poids 0 ou 1.** Si f est de poids 0 ou ± 1, la représentation π associée est telle que $\pi_\infty = \xi_1 \boxplus \xi_2$, où $\xi_i = |x|^{s_i} (\operatorname{sgn} x)^{\epsilon_i}$ sont des caractères de $\mathbb{R}^\times$. Elle est algébrique si $s_i \in \frac{1}{2} + \mathbb{Z}$. Comme elle est unitaire mpodulo le centre, on a $s_1 - s_2 \in i\mathbb{R} \cup]-1, +1[$. On doit donc avoir $s_1 = s_2 \in \frac{1}{2} + \mathbb{Z}$. Si $\epsilon_1 = \epsilon_2$, f est une forme de poids 0, de valeur propre du laplacien $\lambda = -(s - \frac{1}{2})(s + \frac{1}{2})$ où $s = s_1 - s_2$. On a donc $\lambda = \frac{1}{4}$. Enfin, le caractère central de π_∞ sur $\mathbb{R}_+^\times$, $x \to |x|^{s_1 + s_2}$, est de la forme x^m avec $m \in 1 + 2\mathbb{Z}$:

Si f est une forme de Maass, la représentation π associée ne peut être algébrique que si $\lambda = \frac{1}{4}$. Si f est de poids 0, -1, **ou** 1, **les relèvements algébriques sont les représentations π de caractère central** $x \to x^m (\operatorname{sgn} x)^\epsilon$ **avec** $m \equiv 1 \pmod 2$.

Supposons maintenant F **totalement réel** de degré r et soit f une forme parabolique de poids $k_1, \ldots, k_r)$ $(k_i \in \mathbb{Z})$. On dispose également du relèvement de poids $k = (k_1, \ldots, k_r)$ de f; il engendre une représentation cuspidale irréductible π de $GL(2, \mathbb{A}_F)$ sous les hypothèses précédentes sur f. On peut le tordre par des caractères de la forme $|\det g|^s$ (rappelons par ailleurs que les caractères de Hecke algébriques de $\mathbb{A}_F^\times$ sont de la forme $\omega(x)\,|x|^m$, $m \in \mathbb{Z}$, ω un caractère d'Artin).

Cela étant, si $\pi_\infty = \bigotimes_{v_i \text{ infinie}} \pi_i$ est algébrique, (π_i doit être associée à $z^{p_i} \bar{z}^{q_i}, z^{q_i} \bar{z}^{p_i}$) avec $p_i = p_{i,o} + \frac{1}{2}$, $q_i = q_{i,o} + \frac{1}{2}$, $p_{i,o}$, $q_{i,o} \in \mathbb{Z}$. Comme de plus π, à torsion près par un caractère, est **unitaire** , on voit de plus que $p_{i,o} + q_{i,o} = w$ est indépendant de i; comme $|k_i| = |p_{i,o} - q_{i,o}| + 1$ si $p_{i,o} \neq q_{i,o}$, $k_i = 0, 1, -1$ sinon, on a le résultat suivant.

Posons $\ell_i = k_i$ si $|k_1| \geq 2$, $\ell_i = 1$ si $|k_i| \leq 1$.

La représentation π associée à f ne peut être algébrique

que si $\ell_i \equiv \ell_j \pmod 2$ pour tous i, j. Si cette condition est vérifiée, les relèvements algébriques π de f sont ceux de caractère central $\omega(x)\,|x|^m$, ω étant un caractère d'Artin et $m \equiv \ell_i$ $\pmod 2$.

Pour l'instant, *Alg* n'est pas stable par la somme. On définit une nouvelle opération $\boxplus$ à l'aide d'un twist à la Tate. Si $\pi \in Isob(n)$, soit $\pi|\ |^s$ la représentation $\pi(g)\,|\det g|^s$, où $|\ |$ désigne la norme d'idèle.

Definition 1.9. Soit $\pi_1 \in Isob(n_i)$, $\pi_2 \in Isob(n)$. On définit

$$\pi_1 \overset{T}{\boxplus} \pi_2 = \left(\pi_1 |\ |^{\frac{1-n_1}{2}} \boxplus \pi_2 |\ |^{\frac{1-n_2}{2}}\right) |\ |^{\left(\frac{n_1+n_2-1}{2}\right)}$$

Par construction, l'opération $\overset{T}{\boxplus}$ conserve la catégorie $Alg = \coprod Alg(n)$ puisque, aux places infinies, $\boxplus$ se traduit par la somme de représentations du groupe de Weil.

Modulo la Conjecture 1.3, on a de même:

Definition 1.10. On pose

$$\pi_1 \overset{T}{\boxtimes} \pi_2 = \left(\pi_1 |\ |^{\frac{1-n_1}{2}} \boxtimes \pi_2 |\ |^{\frac{1-n_2}{2}}\right) |\ |^{\left(\frac{n_1 n_2-1}{2}\right)}.$$

Elle conserve aussi la catégorie *Alg*. De plus les propriétés de $\boxplus$ et $\boxtimes$ impliquées par le formalisme tannakien (§1.3) sont conservées: par exemple, $(\pi_i \overset{T}{\boxplus} \pi_2) \overset{T}{\boxplus} \pi_3$ est isomorphe (en fait canoniquement isomorphe, par induction par étages) à $\pi_1 \overset{T}{\boxplus} (\pi_2 \overset{T}{\boxplus} \pi_3)$, $\overset{T}{\boxplus}$ et $\overset{T}{\boxtimes}$ satisfont (conjecturalement et à isomorphisme près) les compatibilités habituelles...

Par contraste avec la somme de Langlands, l'opération $\overset{T}{\boxplus}$ ne commute pas au passage à la contragrédiente.

Definition 1.11. Si $\pi \in$ Aut (n), son dual de Tate $\check{\pi}$ est la représentation $\tilde{\pi}|\ |^{n-1}$.

Il est clair que $\pi \to \check{\pi}$ est une involution. On vérifie que $(\pi_1 \overset{T}{\boxplus} \pi_2)\check{} = \check{\pi}_1 \overset{T}{\boxplus} \check{\pi}_2$ si π_1, $\pi_2 \in Isob(n_1)$, $Isob(n_2)$.

La décomposition des représentations isobares prend une forme nouvelle dans *Alg*:

LEMME 1.12. *Si $\pi \in Alg(n)$, ou peut écrire*

$$\pi \cong \pi_1 \overset{T}{\boxplus} \pi_2 \ldots \overset{T}{\boxplus} \pi_r,$$

où $\pi_i \in Alg^0(n_i)$ (l'ensemble des représentations *cuspidales* *de $Alg(n_i)$); les (π_i, n_i) sont uniques à permutations près.*

Ceci résulte directement des définitions.

1.2.4. Les catégories *Tan* et *Art*.

La catégorie *Tan* est formée des représentations de *Alg* qui, par restriction automorphe (§1.1) à une extension finie E/F, deviennent des sommes de caractères de Hecke (algébriques) de $GL(1, \mathbb{A}_E)$:

$$\rho_{F/E}\pi \cong \chi_1 \overset{T}{\boxplus} \chi_2 \overset{T}{\boxplus} \ldots \overset{T}{\boxplus} \chi_n,$$

$\chi_i \in Alg(1, E)$.

La catégorie *Art* est formée des représentations de *Alg* telles qu'on ait, aux places infinies, avec les notations 1.2.3, $r_v = |\ |_{\mathbb{C}}^{\frac{n-1}{2}} \oplus \cdots \oplus |\ |_{C}^{\frac{n-1}{2}}$.

Ces catégories sont stables par $\overset{T}{\boxplus}$ et, conjecturalement, par $\overset{T}{\boxtimes}$.

1.2.5. Sous-catégories self-duales.

Dans chacune des catégories considérées, on peut considérer l'ensemble des représentations stables par la dualité ($\pi \mapsto \tilde{\pi}$ pour *Isob* ou *Temp*, $\pi \mapsto \breve{\pi}$ pour les sous-catégories de *Alg*). On obtient alors des sous-catégories (stables par $\boxplus$ ou $\overset{T}{\boxplus}$ et, conjecturalement, par $\boxtimes$) de représentations. On verra (§5) que pour celles-ci la correspondance entre motifs et représentations automorphes paraît plus accessible.

1.3. Formalisme, et problèmes, tannakiens.

1.3.1.- On suppose le lecteur quelque peu familier avec le formalisme des *catégories tannakiennes* dû à Grothendieck et Saavedra Rivano (cf. [45, 21]).

Rappelons simplement qu'une **catégorie tannakienne neutre** est une catégorie abélienne k -linéaire (pour un corps k) $\mathcal{C}$ munie d'un foncteur $\otimes : \mathcal{C} \to \mathcal{C}$, $(X,Y) \to X \otimes Y$, de certaines données (contraintes d'associativité et de commutativité) exprimant les propriétés habituelles du produit tensoriel, et d'un foncteur k-linéaire (exact, fidèle, compatible avec $\otimes$) $\mathcal{C} \to \mathrm{Vec}_k$, où Vec_k est la catégorie tensorielle des espaces vectoriels sur k.

Dans [38 f], Langlands exprime l'espoir qu'on puisse munir l'ensemble *Isob* des représentations isobares d'une structure de catégorie tannakienne. Au niveau des **objets**, à isomorphisme près, la somme directe devrait être donnée par l'opération $(\pi, \pi') \to \pi \boxplus \pi'$ et le produit tensoriel par $(\pi, \pi') \to \pi \boxtimes \pi'$ (rappelons que l'existence de celui-ci est conjecturale!). Le corps k devrait être ici le corps des complexes.

Comme le remarque Langlands "the attempt [to define Tannakian categories of automorphic representations] may be vain but the prize to be won is so great that one cannot refuse to hazard it". En effet, même en supposant démontrée la Conjecture principale (§1.1), les problèmes à résoudre semblent formidables.

Le premier problème à résoudre consiste à munir la catégorie *Isob* - ou les autres catégories du §1.2 - d'une structure de catégorie **abélienne** $\mathbb{C}$-linéaire. Dans le §1.4, on propose une définition de $\mathrm{Hom}_{Isob}(\pi, \pi')$ pour π ,π' dans un sous-ensemble convenable de *Isob*. On verra que, même pour définir $\mathrm{Hom}_{Isob}(\pi, \pi')$, on ne peut pas travailler avec les représentations **isomorphes** à des sous-représentations (isobares) de $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$. On est contraint à se limiter à un sous-ensemble, contenant à $G(\mathbb{A})$-isomorphisme près toutes les représentations de *Isob* (on vérifie alors qu'on obtient bien une catégorie abélienne).

Le problème consistant à définir un sous-ensemble convenable de *Isob* (ou des autres catégories de 1.2) devient encore plus crucial si l'on considère le produit tensoriel. La définition des catégories tannakiennes impose qu'on puisse exhiber, pour π , $\pi' \in Isob$, leur produit tensoriel $\pi \boxtimes \pi'$, un objet bien défini de *Isob*. Même en supposant la Conjecture 1.3, il paraît peu plausible que l'on puisse définir naturellement un objet $\pi \boxtimes \pi'$ si π, π' sont seulement **isomorphes** à des représentations apparaissant dans $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$.

En revanche, on peut espérer qu'un *théoréme de multiplicité 1* convenable (cf. Th. 1.1) implique par exemple, l'existence d'une unique représentation (isomorphe à) $\pi \boxtimes \pi'$ et apparaissant comme sous-module de $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$. On voit que cela amène à chercher des **modèles** des représentations automorphes, i.e., des représentants bien déterminées des classes d'isomorphisme de représentations automorphes (isobares).

1.3.2. Modèles de Whittaker.

Soit ψ un caractère non trivial de $F\backslash \mathbb{A}_F$, θ le caractère de

$$N_0(\mathbb{A}) = \left\{ \begin{pmatrix} 1 & x_1 & & * \\ & & \ddots & \\ & & & x_{n-1} \\ & & & 1 \end{pmatrix} \right\}$$

défini par $\theta(n) = \psi(x_1 + \cdots + x_{n-1})$. Si $\pi = \otimes \pi_v$ est **générique**, π a un unique représentant $W(\pi)$ dans l'espace $W(\theta)$ des fonctions sur $G(\mathbb{A})$ se transformant à gauche suivant θ [40, 49]. Ceci s'applique en particulier aux représentations de *Temp* (ou *IC*).

Notons l'analogue local: on suppose F local non-archimédien; ψ est alors un caractère additif non-trivial de F, θ le caractère associé de $N_0(F)$. Par définition, une représentation générique π de $G(F)$ a un représentant -alors unique- dans l'espace W_θ des fonctions sur $G(F)$ telles que $f(ng) = \theta(n)f(g)$ $(n \in N(F))$ [28, 33 a] (cf. §1.1).

1.3.3. Modèles automorphes.

Soit $\pi \in \mathrm{Aut}^0(n)$. D'après le Théorème 1.1, il existe un unique sous-module de $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$ isomorphe à π. On peut se demander s'il existe un résultat analogue pour les représentations de *Isob* ou *Temp*. Le rédacteur ne le sait pas, mais on peut remarquer les faits suivants:

(1) La théorie des séries d'Eisenstein [38 a] montre que les représentations de *IC* (de *Temp*, modulo la conjecture de Ramanujan) interviennent avec multiplicité 1 dans la décomposition hilbertienne de $L^2(G(F)\backslash G(\mathbb{A}))$.

(2) Pour les représentations (isobares) de Speh introduites dans le §1.1, on a un théorème, dû à Moeglin et Waldspurger. Fixons un caractère central $\chi : F^\times\backslash \mathbb{A}^\times \to \mathbb{C}^\times$. Soit $L^2_{\mathrm{dis}}(G(F)\backslash G(\mathbb{A}), \chi)$ la partie discrète de l'espace des fonctions sur $G(F)\backslash G(\mathbb{A})$, se transformant suivant χ sous $Z(\mathbb{A}) \cong \mathbb{A}^\times$, et de carré intégrable modulo le centre.

THÉORÈME 1.13. *(cf. [39]). Si $\pi \cong J(\omega, a)$ est un module de Speh* (notations 1.1) *de caractère central χ, π intervient dans $L^2(G(F)\backslash G(\mathbb{A}), \chi)$ avec multiplicité 1.*

1.3.4. Modèles principaux.

Soit π une représentation isobare de $GL(n, \mathbb{A})$. On peut l'écrire sous la forme $\sigma_1 \boxplus \cdots \boxplus \sigma_r$, où les σ_i sont des représentations cuspidales de $GL(n_1, \mathbb{A})$ (§1.1). Puisque toute représentation σ_i a un unique modèle dans $\mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$, on obtient - pour tout choix de l'ordre des σ_i - un unique modèle de π, l'unique sous quotient de

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

de type π . Il y a autant de "modèles" (sic) que de permutations possibles des σ_i. Ces modèles seront appelés **modèles principaux.**

1.3.5. Terminons ce paragraphe en rappelant que les représentations admissibles irréductibles de $GL(n,F)$, où F est **local** devraient être paramétrées par les représentations complexes de degré n du groupe de Weil-Deligne WD_F [53, 38 f]. En particulier, les opérations $\boxplus$ et $\boxtimes$ devraient être définies (au moins à isomorphisme près) pour les représentations locales. Dans le cas archimédien, où la classification est connue, cela suggère le problème suivant:

PROBLÈME 1.14 ($F = \mathbb{R}$ OU $\mathbb{C}$).

Définir une sous-catégorie convenable de la catégorie des représentations admissibles irréductibles de $GL(n,F)$ et la munir d'une structure de catégorie tannakienne (pour une description d'une petite sous-catégorie dont on peut faire au moins une catégorie **abélienne**, *cf. 1.4.2).*

1.4. Catégories abéliennes de représentations isobares.

1.4.0.- Dans ce paragraphe, on construit une (petite) catégorie abélienne de représentations isobares, où l'on décrit les espaces Hom qui devraient être ceux de la catégorie tannakienne. Elle contient un objet isomorphe à toute représentation isobare; elle se décrit simplement à l'aide des modèles principaux. Il y a une construction locale analogue.

Cette catégorie a le défaut que l'application $\boxtimes$ (ou niveau des objets) ou la restriction global - local ne peuvent y être définies. Dans le dernier paragraphe, on indique brièvement la construction d'une autre catégorie, équivalente, où ces applications sont définies.

Il est clair que ces constructions sont insuffisantes au regard des espoirs du §1.3: en particulier, on n'a pas montré ici comment rendre l'opération $\boxtimes$, ou même la restriction global-local, fonctorielles.

1.4.1. On appelle $Isob_1 = Isob_1(F)$ la catégorie dont les objets sont les modèles principaux (1.3.4). Un objet $\pi \in Isob_1$ s'identifie à la donnée d'une partition $n = n_1 + \cdots + n_r$ et de r représentations cuspidales σ_i de $G_{n_i}(\mathbb{A})$. On appellera ici **représentation cuspidale**

une sous-représentation de $\mathcal{A}_{\text{cusp}}(G_{n_i}(F)\backslash G_{n_i}(\mathbb{A}))$: elle est uniquement définie par sa classe d'isomorphie.

On notera $\mathrm{Hom}_{\mathcal{G}}(\pi,\pi')$ l'espace des homomorphismes de π vers π' dans $Isob_1$ (soit §4 pour la notation).

Definition 1.15. (1) Si σ, τ sont cuspidales, on pose

$$\mathrm{Hom}_{\mathcal{G}}(\sigma,\tau)=0 \qquad \textbf{si} \quad \sigma\neq\tau$$
$$\mathrm{Hom}_{\mathcal{G}}(\sigma,\sigma)=\mathrm{Hom}_{G(\mathbb{A})}(\sigma,\sigma)=\mathbb{C}$$

(2) On pose $\mathrm{Hom}_{\mathcal{G}}(\sigma_1\boxplus\cdots\boxplus\sigma_r,\tau_1\boxplus\cdots\boxplus\tau_s)=\oplus_{i,j}\mathrm{Hom}_{\mathcal{G}}(\sigma_i,\tau_i)$.

Dans la partie (1), noter que $\mathrm{Hom}_{G(\mathbb{A})}(\sigma,\sigma)$ est canoniquement isomorphe à $\mathbb{C}$!

Composition. Les morphismes se composent de la manière évidente, comme des matrices.

On définit l'objet nul comme l'unique représentation cuspidale de $GL(0,\mathbb{A}_F)$.

Soit $\pi=\sigma_1\boxplus\cdots\boxplus\sigma_r$, $\pi'=\tau_1\boxplus\cdots\boxplus\tau_s$ deux éléments de $Isob_1$, avec σ_i, τ_j cuspidales. Comme $\pi\boxplus\pi'$ est, par définition, **égal** à $\sigma_1\boxplus\cdots\boxplus\sigma_r\boxplus\tau_1\boxplus\cdots\boxplus\tau_s$, on vérifie aisément que c'est à la fois une somme directe et un produit direct de π et π'. La notation $\boxplus$ est donc justifiée.

On vérifie aisément l'existence de noyaux et conoyaux (il est commode de remarquer que tout $\pi\in Isob_1$ s'écrit de façon unique comme somme directe dans $Isob_1$ de représentations isotypiques, i.e. de la forme $\sigma\boxplus\ldots\boxplus\sigma$; on est ramené à l'existence de noyaux et conoyaux dans la catégorie des espaces vectoriels munis d'une base). On voit de même que la catégorie ainsi définie est abélienne.

On remarquera que, même en supposant la Conjecture 1.3, le produit tensoriel n'est pas défini dans $Isob_1$. Dans une catégorie munie de produit tensoriels, on définit le dual d'un objet à l'aide du Hom interne $\mathrm{Hom}(X,Y)$ ([21 , p. 110]). Dans le cas de $Isob_1$, on peut définir le dual de $\pi=\sigma_1\boxplus\cdots\boxplus\sigma_r$ comme l'objet $\tilde\sigma_1\boxplus\cdots\boxplus\tilde\sigma_r$; notons-le $\tilde\pi$. On vérifie aisément qu'on obtient ainsi une involution contravariante de $Isob_1$.

1.4.2. Si F est un corps **local** , l'analogue local de $Isob_1$ se définit ainsi. Tout d'abord, il n'y a pas à imposer des restrictions aux représentations locales: on considère toutes les représentations irréductibles admissibles. Une telle représentation de $G_n(F)$ est isomorphe à l'unique sous-quotient de Langlands d'une induite

$$\mathrm{Ind}_{P(F)}^{G(F)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

(formule 1.2) où les σ_i sont de carré intégrable. On note ce sous-quotient $\sigma_1 \boxplus \cdots \boxplus \sigma_r$.

Pour avoir l'analogue des modèles principaux (1.3.4), il faut fixer des modèles des représentations de carré intégrable. On le fait à l'aide du modèle de Whittaker local (cf. 1.3.2, [28]). On a alors un modèle unique pour chaque représentation de carré intégrable; on définit alors la catégorie abélienne $Isob_1(F)$, comme en 1.4.1, à l'aide des modèles principaux (remarquons qu'on aurait aussi pu utiliser les modèles de Whittaker 1.3.2 au lieu des modèles automorphes 1.3.3 des représentations cuspidales pour définir $Isob_1(F)$ pour F global).

Le lecteur se convaincra qu'il n'y a pas d'application naturelle $Isob_1(F) \to Isob_1(F_v)$ si v est une place du corps global F.

1.4.3. Indiquons brièvement la construction d'une nouvelle catégorie $Isob_2$, équivalente à $Isob_1$, et où le produit tensoriel est bien défini modulo la Conjecture principale. Ses objets sont les classes d'équivalence de représentations isobares: on peut les identifier aux données $\sigma_1 \boxplus \cdots \boxplus \sigma_r$ de $Isob_1$, prises modulo permutation.

Chaque objet de $Isob_2$ peut être identifié à une somme formelle $\oplus_\sigma n_\sigma \sigma$ (somme finie), où σ décrit $\coprod_{n=1}^{\infty} \mathrm{Aut}^{\,0}(n)$. On définit

$$\mathrm{Hom}_{\mathcal{G}}(\oplus n_\sigma \sigma, \oplus m_\sigma \sigma) = \oplus_\sigma \mathrm{Hom}(\mathbb{C}^{n_\sigma}, \mathbb{C}^{m_\sigma}).$$

On vérifie aisément qu'on a défini une catégorie abélienne, avec la composition évidente des homomorphismes. Il y a une application évidente $Isob_1 \xrightarrow{\Phi} Isob_2$; utilisant le fait, remarqué en 1.4.1, que $\mathrm{Hom}_{\mathcal{G}}(\pi, \pi')$ pour $\pi, \pi' \in Isob_1$ se décompose suivant les composantes isotypiques, on voit que c'est un foncteur, et que $\mathrm{Hom}_{\mathcal{G}}(\Phi\pi, \Phi\pi') = \mathrm{Hom}_{\mathcal{G}}(\pi, \pi')$ pour $\pi, \pi' \in Isob_1$. Les deux catégories sont donc équivalentes (cf. e.g. [16, Prop. 1.19]).

On construit de manière analogue la catégorie $Isob_2(F)$ pour F local, à partir de 1.4.2. On remarque alors que:

(1) Modulo la Conjecture principale, $\boxtimes$ est bien défini comme application $(Isob_2)^2 \to Isob_2$ (F global).

(2) Si v est une place d'un corps global F, il y a une application de restriction $Isob_2(F) \to Isob_2(F_v)$.

1.4.4. Les constructions des paragraphes 1.4.1 et 1.4.3 peuvent être transposées dans la catégorie *Alg*, en remplaçant bien sûr la somme directe $\boxplus$ par l'opération $\overset{T}{\boxplus}$. En particulier, on munit ainsi des sous-catégories convenables Alg_1 ou Alg_2 de Alg d'une structure de catégorie abélienne.

Par restriction, on a donc des structures de catégories abéliennes sur toutes les catégories de 1.2.

2. Certains cas connus du principe de fonctorialité

Dans ce paragraphe on passe en revue certains cas où le principe de fonctorialité a été établi. On s'intéresse surtout aux groupes liés à $GL(n)$ et aux exemples décrits en 1.1.

Rappelons tout d'abord l'énoncé du principe de fonctorialité. Soit H un groupe réductif connexe sur F , ${}^LH = {}^LH^0 \times W_F$ la *forme de Weil* de son L-groupe. Si $\pi = \otimes\pi_v$ est une représentation automorphe de $H(\mathbb{A}_F)$, non ramifiée en-dehors d'un ensemble fini de places S (contenant les places infinies), on lui associe, pour $v \notin S$, sa *matrice de Hecke* : c'est une classe de conjugaison (sous ${}^LH^0$) dans ${}^LH$, s'envoyant par la projection ${}^LH \to W_F$ vers un élément de Frobenius pour F_v .

Soit alors H_1, H_2 deux groupes, $\varphi : {}^LH_1 \to {}^LH_2$ un homomorphisme de L-groupes [10 a]. Si π_1 est une représentation automorphe de $H_1(\mathbb{A}_F)$, le principe de fonctorialité affirme l'existence d'une représentation π_2 , automorphe, de $H_2(A)$ telle que, en presque toute place, $t_{\pi_2,v} = \varphi(t_{\pi_1,v})$. En l'absence de "théorèmes de multiplicité 1" (cf. Thm. 1.1), elle n'est pas nécessairement unique.

Rappelons quelques illustrations du principe qui ont été démontrées pour $GL(n)$ et certains groupes classiques.

2.1. Induction et restriction automorphes.

Les homomorphismes de L-groupes sont décrits en 1.1. Dans le cas d'une extension **cyclique** , le théorème suivant a été démontré par Arthur et l'auteur:

Théorème 2.1. *Soit F_1/F une extension cyclique de degré d de corps de nombres. Alors:*

(i) Si π_F est une représentation induite de cuspidale (catégorie ICde 1.2.2) de $GL(n,\mathbb{A}_{F_1})$, la représentation $\iota_{F_1}^F\pi_{F_1}$ de $GL(nd,\mathbb{A}_F)$ existe, et est uniquement déterminée; elle appartient à IC.

(ii) Si π_F est une représentation de $GL(n, \mathbb{A}_F)$ appartenant à IC, la représentation $\rho_{F/F_1}\pi_F$ de $GL(n, \mathbb{A}_{F_1})$ existe; elle est uniquement déterminée et appartient à IC.

Dans le cas où $n = 1$, la partie (i) est due à Kazhdan et Flicker [34, 25 a]: c'est la fonctorialité "de Hecke" envoyant Grössencharakterer sur F_1 vers représentations de $GL(d, \mathbb{A}_F)$.

Pour une description plus détaillée des résultats, et de l'image de ρ_{F/F_1} dans le cas cyclique, on renvoie le lecteur à [18 d, e].

2.2. Carré symétrique.

Il s'agit de la fonctorialité entre $GL(2)$ et $GL(3)$ déduite de l'homomorphisme de L-groupes $GL(2, \mathbb{C}) \to GL(3, \mathbb{C})$ donné par le carré symétrique. Elle a été démontrée par Gelbart et Jacquet [27]. Flicker, à la suite de travaux antérieurs de Jacquet et Langlands, a annoncé une démonstration différente, à l'aide de la formule des traces, qui permettrait d'en caractériser l'image.

2.3. Fonctorialité associée au carré extérieur.

Soit $GSp(4)$ le groupe des similitudes symplectiques sur un espace de dimension 4. C'est un groupe déployé, dont le L-groupe est $GSp(4, \mathbb{C}) \times W_F$. La fonctorialité "évidente" associée au morphisme $GSp(4, \mathbb{C}) \to GL(4, \mathbb{C})$ devrait avoir pour image les représentations π de $GL(4, \mathbb{C})$ telle que la fonction L, $L(s, \pi, \Lambda_2)$, associée au carré extérieur, a un pôle. Ceci a été démontré par Jacquet et Shalika (voir l'exposé de Jacquet dans ce volume, et [6 a] pour un énoncé).

2.4. Groupes unitaires.

Soit F_1/F une extension quadratique de corps de nombres. Soit G un groupe unitaire sur F, associé à une forme F_1/F -hermitienne sur un espace de dimension n sur F_1. Son L -groupe est $GL(n, \mathbb{C}) \times W_F$, W_F opérant via $\mathrm{Gal}(F_1/F)$, dont l'élément non trivial σ agit sur $GL(n, \mathbb{C})$ par l'automorphisme extérieur d'ordre 2. On a $G \times_F F_1 \cong GL(n)/F_1$; le changement de base devrait relier représentations automorphes de $GL(n, \mathbb{A}_{F_1})$ fixées par l'automorphisme $g \mapsto^\sigma ({}^t g^{-1})$ et représentations automorphes de $GL(\mathbb{A}_F)$.

En général, le problème est compliqué par les phénomènes de L -indiscernabilité: il est résolu, pour $n = 3$, par Rogawski [43 b].

Rapoport, Zink et Kottwitz ont découvert que la L -indiscernabilité était mise en échec pour certains groupes unitaires venant d'algèbres à division sur F_1 munies d'une involution de seconde espèce (ce cas a aussi étudié, sous certaines hypothèses locales et en utilisant un

"Lemme fondamental" apparemment non démontré, par Flicker [25 b]). On utilisera les résultats de Kottwitz dans le paragraphe 5.

3. Questions de rationalité

Dans ce chapitre, on étudie systématiquement le *corps de définition* des représentations automorphes; c'est la généralisation du corps engendré par les valeurs d'un caractère de Hecke, ou par les coefficients de Fourier d'une forme modulaire. On formule une conjecture, suivant laquelle les représentations algébriques sont caractérisées par leurs propriétés de rationalité. On montre que les représentations algébriques *régulières* cuspidales sont définies sur des corps de nombres, et que leurs conjuguées par Aut ($\mathbb{C}$) sont automorphes. On en déduit la généralisation à $GL(n)$ d'un théorème d'Eichler-Shimura (Thm. 3.19).

3.1. Le corps de rationalité d'une représentation automorphe.

Les définitions de ce paragraphe sont celles de Waldspurger [58], et j'ai bénéficié de nombreuses discussions avec lui durant la rédaction.

Soit $\pi = \otimes \pi_v$ une représentation admissible irréductible de $GL(n, \mathbb{A})$. On note $\pi_f = \underset{v \nmid \infty}{\otimes} \pi_v$ sa partie finie: c'est une représentation admissible irréductible de $GL(n, \mathbb{A}_f)$, définie à isomorphisme près par π.

Puisque $G(\mathbb{A}_f) = GL(n, \mathbb{A}_f)$ est un groupe totalement discontinu, on peut considérer ses représentations lisses sur un corps arbitraire (de caractéristique nulle).

Pour $\sigma \in$ Aut ($\mathbb{C}$), soit ${}^\sigma\pi_f$ la représentation définie de la façon suivante: si V est l'espace de π_f, on pose ${}^\sigma V = V \underset{\mathbb{C}, \sigma^{-1}}{\otimes} \mathbb{C}$ (le produit tensoriel de V avec $\mathbb{C}$ sur $\mathbb{C}$, l'homomorphisme $\mathbb{C} \to \mathbb{C}$ étant σ^{-1}); ${}^\sigma V$ est un $G(\mathbb{A}_f)$-module par l'action sur le premier facteur.

Il est clair que ${}^\sigma\pi_f$ est encore irréductible admissible; on a ${}^{\tau\sigma}\pi_f \cong {}^\tau({}^\sigma\pi_f)$. Le *corps de rationalité* de la représentation π_f est le sous-corps de $\mathbb{C}$ fixé par les automorphismes σ tels que ${}^\sigma\pi_f \cong \pi_f$. On le note $\mathbb{Q}(\pi_f)$.

Si F est un corps local non-archimédien, on définit de même le corps de rationalité $\mathbb{Q}(\pi)$ d'une représentation irréductible admissible π de $G(F)$. Il est clair que le facteur, à la place finie v, de ${}^\sigma\pi_f$, est isomorphe à ${}^\sigma\pi_v$. On en déduit que la stabilisateur de π_f dans Aut ($\mathbb{C}$) est l'intersection des stabilisateurs des π_v.

En général, une représentation, même d'un groupe *fini*, n'est pas définie sur son corps de rationalité. Pour $GL(n)$, en revanche, on a:

PROPOSITION 3.1. *Soit $(\pi, V)(\pi = \otimes\pi_v)$ une représentation irréductible admissible complexe de $G(\mathbf{A}_f)$. Soit $E = \mathbf{Q}(\pi)$.*

(i) Il existe un sous-espace vectoriel sous E, V_E, de V tel que $V_E \otimes_E \mathbf{C} = V$, et stable par $G(\mathbf{A}_f)$. En particulier, π est obtenue par extension des scalaires à partir d'une représentation sur E.

(ii) L'espace V_E est unique à homothéties complexes près.

(iii) Le corps $\mathbf{Q}(\pi\ )$ est le composé des $\mathbf{Q}(\pi_v)$.

On a aussi l'analogue local:

PROPOSITION 3.2. *Soit (π, V) une représentation irréductible admissible complexe de $G(F)$ pour F local non-archimédien, $E = \mathbf{Q}(\pi)$. Alors les propriétés (i), (ii) de la Proposition 3.1 sont vraies pour (π, V).*

Remarquons tout d'abord que la Proposition 3.1 résulte de l'assertion locale 3.2. Si en effet π_v est obtenue par extension des scalaires à partir de la représentation $\pi_v^{E_v}$ (où $E_v = \mathbf{Q}(\pi_v)$), on construit le produit tensoriel $\otimes_v\pi_v$, une représentation de $G(\mathbf{A}_f)$ évidemment définie sur le composé E' des $\mathbf{Q}(\pi_v)$. Etendant les scalaires à $\mathbf{C}$, on obtient π ; π peut donc être définie sur E'; par ailleurs, $E' \subset \mathbf{Q}(\pi)$ puisqu'un automorphisme de $\mathbf{C}$ fixant π fixe les $\mathbf{Q}(\ \pi_v)$, donc E'. Donc π peut être définie sur $\mathbf{Q}(\pi\ )$ et $E' = \mathbf{Q}(\pi)$ d'où (i) et (iii). L'assertion (ii) se dé montre de la façon suivante [58, Lemme I.1]: Soit V_1, V_2 deux sous-E-espaces vectoriels de V tels que $V_i \otimes_E \mathbf{C} = V$. Si $\sigma \in \mathrm{Aut}\ (\mathbf{C}/E)$, notons σ_1, σ_2 les automorphismes σ-linéaires de V déduits de V_1, V_2.

L'automorphisme $\lambda_\sigma = \sigma_1^{-1}\sigma_2$ de V est alors linéaire, et il commute avec l'action de $G(\mathbf{A}_f)$. D'après le lemme de Schur (qui est vrai pour les représentations admissibles irréductibles), λ_σ est scalaire: on l'identifie à un nombre complexe. Soit alors $0 \neq f \in V_2$. On peut écrire $f = \sum e_i \otimes \lambda_i$, où (e_i) est une base de V_1 sur E et $\lambda_i \in \mathbf{C}$.

On a alors

$$\sigma_2 f = f = \sum e_i \otimes \lambda_i,$$
$$\sigma_2 f = \lambda_\sigma \sigma_1 f = \lambda_\sigma \sum e_i \otimes \sigma(\lambda_i).$$

On en déduit que $\lambda_\sigma = \lambda_i\sigma(\lambda_i)^{-1}$ pour quelque i tel que $\lambda_i \neq 0$, et pour tout σ. Ceci implique que V_1 et V_2 sont homothétiques de rapport λ_i.

Il reste à démontrer la Proposition 3.2*. Supposons tout d'abord π *générique.* Alors, d'après Jacquet, Piatetski-Shapiro et Shalika [32 b], il existe un sous-groupe compact ouvert $K \subset G(F)$, tel que la représentation triviale de K intervienne avec multiplicité 1 dans π. La Proposition résulte alors du Lemme I.1 de Waldspurger [58].

Dans le cas général, on va se ramener au cas générique à l'aide de la classification de Zelevinsky. Comme cela nous sera utile plus tard, nous préférons utiliser la classification de Langlands reformulée dans le cadre de la théorie de Zelevinsky [58] par Rodier [42].

Si τ est une représentation supercuspidale de $GL(a, F_v)$ et $b = [b', \ldots, b'']$ est une suite d'entiers consécutifs de longueur $|b| = b'' - b' + 1$, soit $L(b, \tau)$ l'unique quotient irréductible de

$$(3.1) \qquad \operatorname{Ind}_{P(F)}^{GL(a|b|,F)}(\tau \mid \,|^{b'} \otimes \cdots \otimes \tau \mid \,|^{b''}),$$

où P est le parabolique homogène de type $a, \ldots, a$. Une suite de représentation supercuspidales de la forme $(\tau \mid \,|^{b'}, \ldots, \otimes \tau \mid \,|^{b''})$ est un **segment**. On le note $\Delta(b, \tau)$. On écrit $\Delta_1 \to \Delta_2$ (Δ_1 précède Δ_2) si $\Delta_1 \neq \Delta_2$, $b_1' \leq b_2'$ et $b_1'' \leq b_2''$. Notons que $L(b, \tau)$ est de carré intégrable.

On sait alors que toute repré sentation irréductible π de $GL(n, F)$ s'écrit comme induite totale

$$(3.2) \qquad \operatorname{Ind}_{Q(F)}^{G(F)}(\pi_1 \otimes \cdots \otimes \pi_r)$$

où Q est le parabolique de type $(n_1, \ldots, n_r)$ et les π_i sont du type suivant: π_i est l'unique quotient irréductible de

$$(3.3) \qquad \operatorname{Ind}_{P_i(F)}^{G_{n_i}(F)}(L(b_1^i, \tau_i) \otimes \cdots \otimes L(b_{s_i}^i, \tau_i)),$$

τ_i étant une représentation supercuspidale de $GL(a_i, F)$ et les segments $\Delta(b_j^i, \tau_i)$ satisfaisant la condition

$$(3.4) \qquad \Delta(b_j^i, \tau_i) \to \Delta(b_k^i, \tau_i) \Rightarrow j > k.$$

De plus, les τ_i sont alors uniquement déterminées si on les suppose distinctes modulo torsion par des puissances entières de $|\det|$. La représentation (3.2) est unique à permutation près des τ_i ou des π_i.

*Waldspurger m'a indiqué une démonstration différente de la Proposition, utilisant directement les modèles de Whittaker dégénérés.

Pour l'instant, la classification n'est pas donnée en termes rationnels car l'induction unitaire ne respecte pas la rationalité des représentations. Si P est un parabolique de G, de type $(n_1, \ldots, n_r)$, on vérifiera plus tard (Lemme 3.9) que l'opération d'induction utilisée dans la Définition 1.9, qui à $(\pi_1, \ldots, \pi_r)$ associe

$$\mathrm{Ind}_{P(F)}^{G(F)}(\pi \mid \mid^{\frac{1-n_1}{2}} \otimes \cdots \otimes \pi_r \mid \mid^{\frac{1-n_r}{2}}) \mid \mid^{\frac{n-1}{2}},$$

qu'on notera $\mathrm{Ind}_{P(F)}^{G(F)}(\pi_1 \otimes \cdots \otimes \pi_r)$, conserve le corps de définition.

On reformule donc la classification décrite en remplaçant à chaque étape (3.1, 3.2, 3.3) l'induction unitaire par l'induction tordue. La classification obtenue, appelée *classification rationnelle* , a les propriétés mentionnées précédemment.

Supposons alors π associée, par la classification *rationnelle*, à une famille de segments $\Delta(b_j^i, \tau_i)$ satisfaisant (3.4). D'après l'unicité, le groupe $\Sigma = \mathrm{Aut}\ (\mathbb{C}, E)$, où $E = \mathbb{Q}(\pi)$, doit permuter les τ_i. De plus, si i, k appartiennent à la même orbite I de Σ, on doit avoir $s_i = s_k = s$ et $b_j^i = b_j^k$ pour tout $j = 1, \ldots, s$. Considérons alors, pour tout j, la représentation

$$\mathrm{Ind}_{R(F)}^{G_{ta|b|}}(\mathcal{L}(b, \tau_{i_1}) \otimes \cdots \otimes \mathcal{L}(b, \tau_{i_t}))$$

(induction tordue), où $I = \{i_1, \ldots, i_t\}$, $b = b_j$, τ_i est une représentation de $GL(a)$ $(i \in I)$, et R est le parabolique évident; on note $\mathcal{L}$ le quotient de Langlands (3.1) pour l'induction rationnelle. Cette représentation est irréductible et générique [58]. Puisque Σ permute les τ_i, elle est fixée par Σ et donc, d'après le résultat déjà démontré, peut être définie sur E. En considérant ces représentations pour toutes les orbites et tous les segments, on en déduit alors, par induction par étages, que π peut être réalisée comme l'unique quotient irréductible d'une induite (rationnelle !) définie sur E.

Il reste à démontrer que le quotient est défini sur E. C'est l'objet du lemme bien connu suivant:

LEMME 3.2.1. *Soit $X = X_E \otimes_E \mathbb{C}$ un espace vectoriel défini sur E, $W \subset X$ un sous-espace complexe de X stable par $\Sigma = \mathrm{Aut}\ (\mathbb{C}/E)$. Alors $W = W_E \otimes_E \mathbb{C}$, où $W_E = W \cap X_E$.*

On applique le lemme au noyau W de l'application $X \to V$, où X est l'espace de la représentation induite, définie sur E, dont V est l'unique quotient irréductible. Il est clair que W est stable par Σ, et ceci démontre la Proposition 3.2.

Démonstration du Lemme: Soit (e_i) une base de X_E sur E. Soit $v = e_1\lambda_1 + \cdots + e_n\lambda_n \in W$, et supposons que tout vecteur de W admettant une telle expression de longueur inférieure à n appartienne à $W_E \otimes \mathbb{C}$. On peut donc supposer $\lambda_i \neq 0$ pour tout i; soit de plus $\lambda_1 = 1$. Si $\lambda_i \notin E$, il existe $\sigma \in \Sigma$ tel que $\sigma\lambda_i \neq \lambda_i$. Considérons alors

$$v' = \sigma v - v = e_2(\sigma\lambda_2 - \lambda_2) + \cdots + e_n(\sigma\lambda_n - \lambda_n).$$

Il appartient à W; d'après l'hypothèse de récurrence, on a donc $v' \in W_E \otimes \mathbb{C}$. Soit alors $\xi_i = \sigma\lambda_i - \lambda_i \neq 0$. Alors $v'' = v\xi_i - v'\lambda_i$ a une composante nulle suivant e_i; donc, par récurrence, $v'' \in W_E \otimes \mathbb{C}$: on a donc enfin $v \in W_E \otimes \mathbb{C}$.

3.2. Représentations non-ramifiées: paramétrisation rationnelle. La paramétrisation de Langlands des représentations non-ramifiées n'est pas rationnelle. On doit donc la transformer par une torsion à la Tate.

Dans ce paragraphe, F est local non-archimédien. On écrit G pour $G(F)$, K pour $G(\mathcal{O}_F)$. Soit $T \subset G$ le tore diagonal, $K_T = T(\mathcal{O})$. Les algèbres de Hecke $\mathcal{K}(G, K)$ et $\mathcal{K}(T, K_T)$ sont définies sur $\mathbb{Q}$ (on écrit $\mathcal{H} = \mathcal{H}(G, K)$, $\mathcal{H}_T = \mathcal{H}(T, K_T)$).

3.2.1. Transformée de Satake..

Rappelons que la transformée de Satake de $f \in \mathcal{H}$ est définie par $Sf(h) = \delta_{P_0}(h)^{\frac{1}{2}} \int_{N_0} f(hn)dn$, $h \in T$ (notations 1.1; δ_{P_0} est le module de $P_0 = TN_0$.

Definition 3.3. Si $f \in \mathcal{H}$, on définit

$$S^T f(h) = |\det h|^{\frac{n-1}{2}}\, Sf(h) \qquad (h \in T).$$

On vérifie que S^T est un isomorphisme défini sur $\mathbb{Q}$ de $\mathcal{H}$ sur $\mathcal{H}_T^W$. On identifie $\mathcal{H}_T(\mathbb{Q})$ à $\mathbb{Q}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ en envoyant la fonction caractéristique de $\bar{\omega}^\lambda$ sur z^λ ($\lambda \in \mathbb{Z}$), où $\bar{\omega}^\lambda$ est la matrice diagonale $(\bar{\omega}^{\lambda_1}, \ldots, \bar{\omega}^{\lambda_n})$. Le tore dual ${}^LT^0$ est ainsi muni canoniquement d'une structure rationnelle (on notera $\hat{f}^T$, pour $f \in \mathcal{H}$, l'élément de $\mathbb{C}({}^LT^0)^W$ ainsi défini).

3.2.2. Représentations non-ramifiées.

Definition 3.4. Si $\chi = (\chi_1, \dots, \chi_n)$, où les χ_i sont des caractères non-ramifiés de $F^\times$, on pose

$$\pi^T(\chi_1, \dots, \chi_n) = \pi(\chi_1 | \, |^{\frac{n-1}{2}}, \dots, \chi_n | \, |^{\frac{n-1}{2}}).$$

Dans le membre de droite, $\pi(\ )$ désigne le sous-quotient de Langlands de l'induite; il est non-ramifié. On pourrait écrire, utilisant l'analogue local de la Définition 1.10:

$$\pi^T(\chi_1, \dots, \chi_n) = \chi_1 \overset{T}{\boxplus} \chi_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \chi_n.$$

On vérifie alors:

LEMME 3.5. *(i) Si $f \in \mathcal{H}$, f opère sur le vecteur sphérique de $\pi^T(\chi)$ par $\hat{f}^T(t_\chi)$, où $t_\chi = (\chi_1(\bar{\omega}), \dots, \chi_n(\bar{\omega})) \in {}^LT^0$.*

(ii) Si $\sigma \in Aut$ $(\mathbb{C})$, on a ${}^\sigma(\pi^T(\chi)) \cong \pi^T({}^\sigma\chi)$.

(iii) En particulier, le corps de rationalité de $\pi^T(\chi)$ est celui de l'orbite de $t_\chi \in {}^LT^0(\mathbb{C})$ sous W.

La partie (i) est évidente. La partie (ii) en résulte puisque l'application $f \mapsto \hat{f}^T$ est définie sur $\mathbb{Q}$, et les représentations non ramifiées sont classifiées par les caractères de l'algèbre de Hecke. Comme ceux-ci correspondent à ${}^LT^0/W$, (iii) résulte de (ii).

Remarque. Si π est une représentation non-ramifiée de $G(F)$, on notera t_π^T sa matrice de Hecke pour la paramétrisation donnée par la Définition 3.4; c'est t_π multipliée par la matrice diagonale $q_v^{\frac{1-n}{2}}$.

3.3.- L'action de Aut $(\mathbb{C})$ sur les types à l'infini des représentations algébriques..

On va commencer par reformuler la construction utilisée dans le paragraphe 1.2.3 pour définir la catégorie Alg. Notons I l'ensemble des plongements $\iota : F \to \mathbb{C}$. L'ensemble des places infinies de F s'identifie donc à I modulo l'action de la conjugaison complexe.

A toute représentation $\pi \in Alg(n)$, on associe son *type à l'infini*, un élément de $(\mathbb{Z}^n/\mathfrak{S}_n)^I$, de la façon suivante: soit $\pi' = \pi | \, |^{\frac{1-n}{2}}$; la paramétrisation de Langlands lui associe, par hypothèse, des caractères algébriques de $\mathbb{C}^\times$.

Si ι est un plongement réel, π'_ι est associée à une représentation de $W_\mathbb{C} = \mathbb{C}^\times$ de la forme $(z^{p_1}\bar{z}^{p_2}, z^{p_2}\bar{z}^{p_1}, \dots, (z\bar{z})^{p'_{2r+1}}, \dots, (z\bar{z})^{p'_n})$ (cf.

après la Définition 1.8). On pose $p_\iota = (p_1, p_2, \ldots, p_{2r}, p'_{2r+1}, \ldots, p'_n)$ considéré comme élément de $\mathbf{Z}^n/\mathfrak{S}_n$.

Si ι est un plongement complexe, il définit une identification $F_v = \mathbb{C}$ où v est la place déduite de ι. On a $W_{\mathbb{C}} = \mathbb{C}^\times$ (canoniquement), et π'_v est la représentation de $GL(n, \mathbb{C})$ associée aux caractères $(z_1^p \bar{z}^{q_1}, \ldots, z^{p_n}\bar{z}^{q_n})$. On pose $p_\iota = (p_1, \ldots, p_n)$ (modulo $\mathfrak{S}_n$. Il est clair qu'alors $p_{\bar\iota} = (q_1, \ldots, q_n)$ où désigne la conjugaison complexe.

Definition 3.6. Soit $\pi \in Alg(n)$, et soit $p(\pi) = p = (p_\iota)_{\iota \in I}$ son type à l'infini. Si $\sigma \in$ Aut $(\mathbb{C})$, on définit ${}^\sigma p$ par $({}^\sigma p)_\iota = p_{\sigma^{-1}\iota}$ $(\iota \in I)$.

Dans le cas de $Alg(1)$ -c'est-à-dire des caractères de Hecke algébriques - cela revient à la définition habituelle de l'action de Aut $(\mathbb{C})$ sur les types: si $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ est un caractère de Hecke algébrique, on a, à un caractère d'Artin près, $\chi_\iota(x) = x^{p_\iota}$ (ι réelle) et $\chi_v(z) = z^{p_\iota}\bar{z}^{q_\iota}$ (ι complexe, F_v identifié à $\mathbb{C}$ via ι). Si $\sigma \in$ Aut $(\mathbb{C})$, on vérifie que le type de ${}^\sigma\chi$ (σ agissant sur les valeurs de χ_f sur $\mathbb{A}_f^\times$) est bien celui prescrit par la Définition 3.2, à l'aide de la formule $\chi_\infty(x)\chi_f(x) = 1$ $(x \in F^\times)$.

Remarque. Le *type à l'infini* de π - qu'on devrait plutôt appler son type à l'infini algébrique - ne détermine par π_∞ ; c'est déjà le cas pour $n = 1$: pour un Grössencharakter, le type algébrique ne définit χ_∞ que sur $(F_\infty^\times)^+$. Néanmoins, pour π **cuspidale** , le Lemme de pureté 4.9 montre que $p(\pi)$ détermine le caractère de π sur l'ensemble des éléments de $G(F_\infty)$ qui sont des carrés: cf. Corollaire 4.11. Cela est faux si π n'est pas cuspidale.

3.4.- Conjectures, et vérifications.

3.4.1. Conjectures.

CONJECTURE 3.7. *Soit $\pi \in Alg(n)$, $p = (p_\iota)_{\iota \in I}$ son type à l'infini. Alors*

(i) π_f est définie sur un sous-corps $E \subset \mathbb{C}$ qui est un corps de nombres.

(ii) Pour tout $\sigma \in$ Aut $(\mathbb{C})$, ${}^\sigma\pi_f$ est la partie finie d'une représentation ${}^\sigma\pi \in Alg(n)$ dont le type à l'infini est ${}^\sigma p$.

La conjecture 3.7 pourrait se formuler à l'aide de la fonction L de π; elle implique que $L_f(\pi, s + \frac{1-n}{2})$ est donnée par une série de

Dirichlet à coefficients dans E (cf. Lemme 4.7) (L_f est la partie non-archimédienne de la fonction L; la fonction $L(\pi)$ est celle de Langlands: équation fonctionnelle $s \mapsto 1-s$, échangeant π et $\tilde{\pi}$). Elle implique, en particulier, qu'aux places non-ramifiées,

$$t^T_{\pi,v} \in ({}^LT^0/W)(E). \tag{3.5}$$

Pour la sous-catégorie des représentations d'Artin, on est amené à conjecturer de plus:

CONJECTURE 3.7′. *Si $\pi \in Art(n)$, les valeurs propres de $t^T_{\pi,v}$ (v non-ramifiés) sont des racines de l'unité.*

Comme Weil l'avait fait pour les caractères de Hecke algébriques, il paraît naturel de formuler une conjecture réciproque, qui est une question de théorie de la transcendance. Sous la forme la plus forte:

CONJECTURE 3.8. *Soit $\pi \in Aut\ (n)$, et supposons qu'à toutes les places non-ramifiées pour π, sauf un nombre fini:*

$$t^T_{\pi,v} \in ({}^LT^0/W)(E), \qquad E \subset \mathbf{C}$$

où E est un corps de nombres. Alors $\pi \in Alg(n)$.

On vérifie maintenant la compatibilité de la conjecture avec quelques opérations fonctorielles.

3.4.2. Caractères centraux.

Si $\pi \in Alg(n)$, son caractère central ω_π est un caractère de Hecke algébrique. En particulier, les conséquences évidentes de la Conjecture 3.7 pour les caractères centraux sont vraies (rappelons que la Conjecture 3.8 pour Aut (1) a été démontrée par Waldschmidt [57]).

3.4.3. Induction parabolique.

La conjecture est compatible avec l'opération $\overset{T}{\boxplus}$:

LEMME 3.9. *Soit $\pi = \pi' \overset{T}{\boxplus} \pi''$, π', $\pi'' \in Alg$.*

(i) Si π'_f, π''_f sont définies sur E, π l'est aussi.

(ii) Soit p, p', p'' les types à l'infini de π, π' , π''. Alors $p = p' \oplus p''$ dans un sens évident.

L'assertion (ii) est évidente. Pour (i), il faut d'abord vérifier que l'induction (re)-normalisée associant à π'_v et π''_v la représentation

$$\rho_v = \mathrm{Ind}_{P(F_v)}^{G(F_v)}(\pi'_v\,|\ |^{\frac{1-n'}{2}} \otimes \pi''_v\,|\ |^{\frac{1-n''}{2}} \otimes 1) \otimes |\ |^{\frac{n'+n''-1}{2}}$$

est *définie sur* $\mathbb{Q}$, i.e., conserve le corps de rationalité des représentations. Le module parasite, qui introduit des irrationnelles quadratiques dans la définition de l'induite unitaire Ind_P^G, est donné par $|\det m'|^{\frac{n''}{2}} |\det m''|^{\frac{-n'}{2}} = \delta_P^{\frac{1}{2}}(m)$,

$$m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix} \in M.$$

Les twists indiqués le remplacent par $|\det m'|^{n''}$, un nombre rationnel.

Il faut ensuite vérifier que le sous-quotient de Langlands de ρ_v est défini par des conditions rationnelles (ce n'est pas clair d'après la définition de Langlands, qui s'exprime par une condition de positivité). Pour cela, on utilise la *classification de Langlands rationnelle* décrite dans la démonstration de la Proposition 3.2. Elle réalise toute représentation π de $GL(n, F_v)$ comme induite

$$\mathcal{I}\mathrm{nd}_{Q(F)}^{G(F)}(\pi_1 \otimes \cdots \otimes \pi_r) \tag{3.6}$$

(notations 3.1: $\mathcal{I}\mathrm{nd}$ désigne l'induite rationnelle) où π_i est l'unique quotient irréductible de

$$\rho_i = \mathcal{I}\mathrm{nd}_{P_i(F)}^{G_{n_i}(F)}(\mathcal{L}(b_1^i, \tau_i) \otimes \cdots \otimes \mathcal{L}(b_{s_i}^i, \tau_i)) \tag{3.7}$$

(rappelons que $\mathcal{L}$ est le quotient de Langlands (3.1) pour l'induction rationnelle). La représentation générique dont π est le quotient de Langlands (et uniquement définie par cette condition) est alors

$$\rho = \mathcal{I}\mathrm{nd}_{Q(F)}^{G(F)}(\rho_1 \otimes \cdots \otimes \rho_r)$$

Il est clair, puisque l'induction rationnelle commute avec l'action de Aut ($\mathbb{C}$), que le quotient de Langlands de ${}^\sigma\rho_v$ est alors ${}^\sigma\pi_v$, si π_v est celui de ρ_v. En particulier, le corps de définition de π_v est contenu dans celui de ρ_v, d'où le Lemme 3.9.

Remarques. (a) Le lecteur pourra s'inquiéter de ce qu'on a utilisé le Lemme 3.9 pour démontrer la Proposition 3.2. On n'avait utilisé, évidemment, que la propriété de rationalité de l'induite totale ρ.

(b) Le corps de rationalité de $\pi' \overset{T}{\boxplus} \pi''$ peut bien sûr être plus petit que celui de π' et π'': penser par exemple au cas de deux caractères d'Artin $\chi' = \mathbb{A}_f^\times \to E^\times$, où $E/\mathbb{Q}$ est quadratique, et $\chi'' = (\chi')^\sigma$ où σ engendre $\mathrm{Gal}(E/\mathbb{Q})$.

(c) Dans le cas où π_v'' st π_v'' sont **non-ramifiées** , on sait que le sous-quotient de Langlands et l'unique sous-quotient non-ramifié de l'induite: il est clair que c'est une condition rationnelle, et ceci donne une autre démonstration de (i) à ces places. On aimerait savoir que, dans tous les cas, le sous-quotient de Langlands est défini par des conditions de ramification (K- type minimal). Cela pourrait résulter de travaux récents de Howe, Moy et Bushnell.

Le Lemme 3.9 ramène la vérification de la Conjecture 3.7 au cas des représentations cuspidales (cf. Lemme 1.12).

3.4.4. Changement de base. Soit K/F une extension de corps globaux.

Soit d'abord $\pi \in Alg^0(n,F)$ une représentation cuspidale. Soit $\Pi \in Alg(n,K)$ la représentation de $GL(n,\mathbb{A}_k)$ déduite (conjecturalement) de Π par changement de base (§1.1,2). On vérifie alors facilement que, aux places non-ramifiées, l'équation (3.1) pour π implique l'équation (3.1) pour Π . En particulier, si π_v est définie sur $E \subset \mathbb{C}$, Π_w l'est $(w|v)$ aux places non-ramifiées de π.

Soit ι un plongement $F \to \bar{\mathbb{Q}}$, η un plongement de K étendant ι . Les propriétés conjecturales du changement de base aux places archimédiennes impliquent que $p(\Pi,\eta) = p(\pi,\iota)$ (notations 3.3). Comme $\sigma^{-1}\eta$ étend $\sigma^{-1}\iota$ ($\sigma \in$ Aut $(\mathbb{C})$), on voit donc que la Conjecture 3.4 pour π l'implique pour Π (au moins aux places non-ramifiées de π , ou archimédiennes). Dans le cas résoluble, on a davantage; notons que Π existe d'après les résultats de [2].

Lemme 3.10 (K/F résoluble). *Soit $\pi \in Alg^0(n,F)$, $\Pi \in Alg(n,K)$ la représentation déduite de π par changement de base. Alors la Conjecture 3.4 pour π l'implique pour Π.*

Démonstration. On se ramène à K/F abélien (aux étapes intermédiaires, on n'a plus $\pi \in Alg^0$, mais $\pi \in Alg \cap IC$; on laisse les arguments ancillaires au lecteur). Il reste à considérer les places ramifiées. Dans ce cas, on a une identité de caractères [2, ch. I], quand $w|v$ sont des places finies:

$$\mathrm{trace}(\Pi_w(g)I_\sigma) = \mathrm{trace}\,\pi_v(\mathcal{N}g),$$

$g \in GL(n, E)$ de norme régulière, σ un générateur de $\mathrm{Gal}(K/F)$. On en déduit que le caractère tordu de Π prend ses valeurs dans E. Comme les valeurs du caractère tordu sur les éléments réguliers déterminent la représentation [2, 18 b], ceci implique que Π est rationnelle, et donc définie, sur E.

On a des résultats analogues dans le cas de l'induction automorphe. Soit $\pi \in Alg^0(n, K)$, $\pi_F \in Alg(nd, F)$ conjecturalement déduite de π (§1.1) par induction automorphe. On vérifie alors qu'aux places non-ramifiées, $(\pi_F)_v$ est définie sur E si π_w l'est pour tout $w|v$. Aux places archimédiennes, on a

$$p(\pi_F, \iota) = \otimes_\eta p(\pi, \eta)$$

où la somme porte sur les plongements $K \to \bar{\mathbb{Q}}$ étendant $\iota : F \to \bar{\mathbb{Q}}$. On en déduit encore la compatibilité avec la Conjecture 3.4 aux places non-ramifiées ou archimédiennes. Dans le cas résoluble, on pourrait démontrer l'analogue du Lemme 3.10 modulo la conjecture suivante, que je n'ai pu démontrer:

CONJECTURE 3.11. *Soit K/F une extension cyclique de corps locaux, de degré premier ℓ. Soit σ un générateur de* Gal(K/F). *Soit π une représentation supercuspidale de $GL(n, K)$ telle que $\pi \not\cong \pi \circ \sigma$. Soit π_F l'unique représentation de $GL(n\ell, F)$ automorphiquement induite de π ([2, Prop. 6.6] ; [18 d]). Alors, si π est définie sur $E \subset \mathbb{C}$, π_F est définie sur E.*

(on sait que la restriction de π_F au noyau de det: $GL(n\ell, F) \to F^\times / N_{K/F} K^\times$ est définie sur E).(*)

3.5. Cas des représentation régulières.

Definition 3.12. Soit $\pi \in Alg(n)$, $p = (p_\iota)_{\iota \in I}$ son type à l'infini. On dit que π est <u>régulière</u> si, pour tout $\iota \in I$, $p_\iota = (p_1, \ldots, p_n)$ avec $p_i \neq p_j$ $(i \neq j)$.

Cela revient à supposer son caractère infinitésimal **régulier** (cf. la Remarque après la Définition 1.8).

Dans ce paragraphe, on vérifie la Conjecture 3.7 pour les représentations régulières:

(*)**Ajouté sur épreuves:** Comme me l'a fait remarques G. Henniart, la Conjecture 3.11 résulte facilement du fait que π_F est l'unique représentation de $G_{nl}(F)$ dont le relèvement à K est égal à $\pi \boxplus (\pi \circ \sigma) \boxplus \cdots \boxplus \pi \circ \sigma^{l-1}$.

THÉORÈME 3.13. *Soit $\pi \in Alg(n)$, p son type à l'infini. Supposons π régulière. Alors π_f est définie sur un corps de nombres $E \subset \mathbb{C}$; pour tout $\sigma \in Aut\,(\mathbb{C})$, ${}^\sigma\pi_f$ est la partie finie d'une représentation ${}^\sigma\pi \in Alg^0(n)$, de type à l'infini ${}^\sigma\pi$.*

Remarque. La représentation ${}^\sigma\pi$ est alors unique (Thm. 1.1).

DÉMONSTRATION: Le fait que π_f est définie sur un corps de nombres résulte des relations entre formes automorphes et cohomologie des quotients arithmétiques de $G(F_\infty)/K_\infty$ (pour $GL(2)$ sur $\mathbb{Q}$, c'est l'isomorphisme d'Eichler-Shimura). On décrit succintement l'argument: cf. Harder [30 b] et Walspurger [58] pour $GL(2)$. On va utiliser le résultat suivant, qui serad démontré dans le paragraphe 4.4.

LEMME DE PURETÉ 4.9. - *Soit $\pi \in Alg^0(n)$. Alors il existe $w \in \mathbb{Z}$ tel que, pour toute place infinie v, les caractéres algébriques de $\mathbb{C}^\times$ associés à $\pi_v|\;|_v^{\frac{n-1}{2}}$ soient de la forme $z^p(\bar{z})^q$ avec $p+q=w$.*

En d'autres termes, $\pi|\;|^{\frac{n-1-w}{2}}$ est tempérée aux places infinies.

Soit $L = \mathrm{Res}_{F/\mathbb{Q}}\,GL(n)$. On associe tout d'abord au type à l'infini de π une représentation rationnelle, définie sur $\bar{\mathbb{Q}}$, de L. On la note τ, et V son espace. On a $\tau = \otimes_{\iota \in I}\tau_\iota$.

Si ι est une plongement complexe, soit $\bar{\iota}$ son conjugué et v la place associée. Identifions F_v à $\mathbb{C}$ via ι (cf. §3.3); soit $\pi_v = \pi_{\mathbb{C}}(z^{a_1}\bar{z}^{b_1},\ldots,z^{a_n},\bar{z}^{b_n}$ l'expression de π_v dans la classification de Langlands; on a

$$a_i = p_i + \frac{n-1}{2}, \quad b_i = q_i + \frac{n-1}{2}$$

$p_i,\ q_i \in \mathbb{Z}$ et $(p_i) = p_\iota$, $(q_i) = p_{\bar{\iota}}$. On peut ordonner les indices de façon que

$$a_1 > a_2 > \cdots > a_n$$

d'où, d'après le Lemme de pureté :

$$b_1 < b_2 < \cdots < b_n.$$

On pose

$$m_\iota = (p_1, p_2+1, p_3+2, \ldots, p_n+(n-1)) \tag{3.5}$$

soit

$$m_\iota = a - \delta \in \mathbb{C}^n, \quad a = (a_1, \ldots, a_n)$$

où δ est la demi-somme de racines de $\mathfrak{gl}(n, C)$ dans $\mathfrak{h}^*$ (cf. Remarque après la Définition 1.8):

$$\delta = \left(\frac{n-1}{2}, \frac{m-3}{2}, \ldots, -\frac{n-1}{2}\right) \in \mathbb{C}^n.$$

La définition appliquée à $\bar{\iota}$ donne

$$m_{\bar{\iota}} = (q_n, q_{n-1} + 1, \ldots, q_1 + (n-1));$$

m_ι et $m_{\bar{\iota}}$ s'identifient à des poids dominants pour $GL(n)$ par rapport au tore diagonal, car ce sont des suites d'entiers $(x_1, \ldots, x_n)$ avec $x_1 \geq x_2 \geq \cdots \geq x_n$. On définit τ_ι comme la représentation rationnelle de plus haut poids m_ι par rapport au système de racines sur $\mathfrak{h} \cong \mathbb{C}^n$ associé au parabolique minimal standard.

Le choix de ι donne un isomorphisme de $GL(n, F_v)$ sur $GL(n, \mathbb{C})$, d'où $GL(n, F_v) \times_{\mathbb{R}} \mathbb{C} \cong GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$. Pour cette identification, le caractère infinitésimal de la représentation $\tau_\iota \otimes \tau_{\bar{\iota}}$ s'identifie à

$$(m_\iota + \delta, m_{\bar{\iota}} + \delta) \in \mathbb{C}^n \times \mathbb{C}^n.$$

On vérifie que $m_{\bar{\iota}} + \delta = w_0 b$, où $w_0 \in \mathfrak{S}_n$ est l'élément de plus grande longueur pour le choix de racines; modulo le groupe de Weyl complexe, le caractère infinitésimal de $\tau_\iota \otimes \tau_{\bar{\iota}}$ est donc égal à (a, b), c'est-à-dire à celui de π_v.

Soit ι un plongement réel, et v la place associée. Si $p_\iota = (p_1, \ldots, p_n)$, où l'on suppose $p_1 > p_2 > \cdots > p_n$ on définit encore m_ι par la formule (3.5), et τ_ι comme précédemment. Des calculs bien connus (cf. e.g. [51 b]; [18 c, p. 482]) montrent encore que π_v a le même caractère infinitésimal que $\tau_v = \tau_\iota$. On définit enfin $\tau = \otimes_{\iota \in I} \tau_\iota$. Remarquons que le caractère central de τ - un caractère algébrique de $\mathrm{Res}_{F/\mathbb{Q}}\, GL(1)$ - est égal au type à l'infini du caractère central de π.

Si v est une place infinie, soit $\mathfrak{g}_v$ l'algèbre de Lie complexe de $GL(n, F_v)$.

Soit A_v la composante neutre topologique du tore réel déployé maximal de $GL(n, F_v)$ (considéré comme groupe réel), $\mathfrak{a}_v = \mathrm{Lie}(A_v) \otimes_{\mathbb{R}} \mathbb{C}$. Soit $\mathfrak{g}_v = \mathfrak{a}_v \times \tilde{\mathfrak{g}}_v$ la décomposition de Langlands de $\mathfrak{g}_v$: $\mathfrak{g}_v$ est donc l'ensemble des matrices de $\mathfrak{g}_v$ dont la trace est imaginaire, et est invariante par le sous-groupe compact maximal standard K_v, égal à

$O(n)$ ou U_n. On note $H^\bullet(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \tau_v$ les espaces de $(\tilde{\mathfrak{g}}_v, K_v)$-cohomologie [14]. Si $\mathfrak{a}$ est un espace vectoriel, on note $\Lambda^\bullet\mathfrak{a}$ l'espace vectoriel gradué égal à $\Lambda^i\mathfrak{a}$ $(0 \leq i \leq \dim \mathfrak{a})$, à 0 ailleurs; si $\mathfrak{a} = \{0\}$, $\Lambda^\bullet\mathfrak{a} = \mathbb{C}$ en dimension 0.

LEMME 3.14. *Soit v une place infinie. Alors il existe un caractére d'ordre 2, ϵ_v, de $F_v^\times$ tel que, notant $\pi_v \otimes \epsilon_v$ la représentation π_v tordue par $\epsilon_v \circ \det$:*

$$\begin{aligned} H^\bullet(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \epsilon_v \otimes \tilde{\tau}_v) &\cong \Lambda^{\bullet - \frac{n(n-1)}{2}}\mathbb{C}^{n-1} && (v \textit{ complexe}) \\ &\cong \Lambda^{\bullet - m^2}\mathbb{C}^{m-1} && v \textit{ réelle, } n = 2m) \\ &\cong \Lambda^{\bullet - m(m+1)}\mathbb{C}^{m} && (v \textit{ réelle, } n = 2m+1). \end{aligned}$$

DÉMONSTRATION: D'après le Lemme 4.9, la restriction de π_v au groupe $\tilde{G}_v = \{g \in G_v : |\det g| = 1$ est tempérée. Considérons d'abord le cas où $F_v \cong \mathbb{C}$. Dans ce cas, $\epsilon_v = 1$. La représentation τ_v de $GL(n, F_v)$ a pour complexification $\tau_\iota \otimes \tau_{\bar{\iota}}$, représentation de $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$.

Les représentations τ_v et π_v ont même caractère central et même caractère infinitésimal; $\pi_v|_{\tilde{G}_v}$ étant tempérée, le Lemme 3.4 résulte aisément du calcul, standard, de la cohomologie des groupes complexe [24].

Considérons maintenant F_v réel, et supposons d'abord $n = 2m$ pair. Soit r_v le paramètre de Langlands de π_v restreint à $\mathbb{C}^\times$; il s'écrit

$$(z^{a_1}(\bar{z})^{a_2}, z^{a_2}\bar{z}^{a_1}, \ldots, z^{a_{2r}}\bar{z}^{a_{2r-1}}, (z\bar{z})^{a_{2r+1}}, \ldots, (z\bar{z})^{a_n})$$

avec $a_i = p_i + \frac{n-1}{2}$, $p_i \in \mathbb{Z}$, $a_1 \neq a_2, \ldots, a_{2r} \neq a_{2r-1}$. Par régularité, on a en fait $a_i \neq a_j$ $\forall i \neq j$. D'après le Lemme de Pureté, on a de plus

$$w - n - 1 = a_1 + a_2 = a_3 + a_4 = \cdots = 2a_{r+1} = \cdots = 2a_n.$$

Il ne peut donc y avoir qu'un terme en $(z\bar{z})^{a_j}$; n étant pair, il n'y en a aucun; r_v s'écrit donc

$$(z^{a_1}\bar{z}^{a_2}, z^{a_2}\bar{z}^{a_1}, \ldots, z^{a_{2m}}\bar{z}^{a_{2m-1}}).$$

On supposera que $a_1 < a_2$, $a_3 < a_4$, $\ldots$, $a_{2m-1} < a_{2nm}$. On peut supposer de plus $a_1 < a_3 < \cdots < a_{2m-1}$, d'où d'aprés les relations de complémentarité dues au Lemme de Pureté(*):

$$a_2 > a_4 > \cdots > a_{2m} > a_{2m-1} > \cdots > a_3 > a_1.$$

On voit donc que les paramètres $p = p_\iota$, $m = m_\iota$ associés à ω_v s'écrivent

$$\begin{aligned} p &= (a_2 - \frac{n-1}{2}, a_4 - \frac{n-1}{2}, \ldots, a_1 - \frac{n-1}{2}) \\ m &= (a_2 - \frac{n-1}{2}, a_4 - \frac{n-3}{2}, \ldots, a_1 + \frac{n-1}{2}). \end{aligned}$$

Considérons d'abord le cas $n = 2$. La représentation π_v de la série discrète de $GL(2, \mathbf{R})$ associée par la classification de Langlands à $(z^{a_1}\bar{z}^{a_2}, z^{a_2}\bar{z}^{a_1})$ s'inscrit dans une suite exacte non scindée:

$$0 \longrightarrow \pi_v \longrightarrow \mathrm{Ind}(\nu_2, \nu_1) \longrightarrow F(\nu_2, \nu_1) \longrightarrow 0$$

où la représentation centrale est induite, à partir du sous-groupe de Borel, des caractères

$$\begin{aligned} \nu_2(x) &= |x|^{a_2} (\operatorname{sgn} x)^{\epsilon_2} \\ \nu_1(x) &= |x|^{a_1} (\operatorname{sgn} x)^{\epsilon_1} \end{aligned}$$

avec

$$\epsilon_1 + \epsilon_2 \equiv a_1 - a_2 - 1 \pmod 2,$$

ϵ_1 et ϵ_2 étant arbitraires à cette condition près (rappelons que $a_1 < a_2$). Le quotient $F(\nu_2, \nu_1)$, de dimension finie, a pour plus haut poids

$$\begin{pmatrix} x & \\ & y \end{pmatrix} \mapsto \nu_2(x)\,|x|^{-\frac{1}{2}}\,\nu_1(y)\,|y|^{\frac{1}{2}}\,.$$

Puisque $a_i = p_i + \frac{1}{2}$, $F(\nu_2, \nu_1)$ est rationnelle si $\epsilon_1 \equiv p_1 + 1 \pmod 2$, $\epsilon_2 \equiv p_2 \pmod 2$, ce qui est compatible avec les relations entre les ϵ; c'est la représentation $\tau_v = \tau_\iota$ définie précédemment. La suite exacte ci-dessus montre que $\mathrm{Ext}^1_{\tilde{\mathfrak{g}}_v, K_v}(\pi_v, \tau_v) = H^1(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \tilde{\tau}_v)$ est non-nul; il est bien connu que dans ce cas

$$H^\bullet(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \tilde{\tau}_v) \cong \Lambda^{\bullet - 1}\mathbf{C}^0.$$

(*)Attention: **cette notation n'est pas conforme à celle utilisée avant (3.5): l'ordre des a_i est différent.**

Passons maintenant au cas général où n est pair; soit $n = 2m$. Ecrivons (oubliant pour simplifier les indices v dans le reste de la démonstration)

$$\pi_v = \pi = \sigma_1 \boxplus \cdots \boxplus \sigma_m$$

σ_i étant la représentation de carré intégrable de $GL(2,\mathbf{R})$ associée à $(z^{a_{2i-1}}\bar{z}^{a_{2i}}, z^{a_{2i}}\bar{z}^{a_{2i-1}})$.

On va utiliser le Théorème 3.3 de Borel-Wallach [14, Ch. III]. Soit $P = MN$ le parabolique, de type (2,2,...,2), associé à π. Soit A_P la composante déployée connexe (pour la topologie réelle) de M : $A_P \cong (\mathbf{R}_+^\times)^m$. On a la décomposition de Langlands $M = {}^0MA_P$, avec ${}^0M \cong (SL_2^\pm)^m$, où $SL_2^\pm = \{g \in GL(2,\mathbf{R}) : \det g = \pm 1\}$. On identifie $\mathrm{Lie}(A_P)\otimes\mathbf{C} = \mathfrak{a}_P$ à $\mathbf{C}^m$. Pour nous conformer aux notations de Borel-Wallach, posons:

σ = la représentation de ${}^0M$ obtenue par restriction à partir de $\sigma_1 \otimes \cdots \otimes \sigma_m$.

ν = le caractère de A_P obtenu en restreignant le caractère central de $\sigma_1 \otimes \cdots \otimes \sigma_m$.

Le couple (σ, ν) détermine une représentation de M, et

$$\pi = \mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1).$$

Soit $\mathfrak{t}$ une sous-algèbre de Cartan de ${}^0\mathfrak{m}\otimes\mathbf{C}$; $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}_p$ est alors une sous-algèbre de Cartan de $\mathfrak{g}$ (on prendra pour $\mathfrak{t}$ la sous-algèbre diagonale, de sorte que $\mathfrak{h}$ est la sous-algèbre diagonale de $\mathfrak{g}$). Enfin, on notera λ le plus haut poids (pour l'ordre des racines sur $\mathfrak{h}$ déjà utilisé) d'une représentation rationnelle de G. On note celle-ci V^λ.

Soit $\lambda_\sigma \in \mathfrak{t}^*$ le paramètre d'Harish-Chandra de σ : il est défini modulo le groupe de Weyl complexe de ${}^0\mathfrak{m}$. On considère λ_σ tel que $\langle\lambda_\sigma, \alpha\rangle < 0$ pour toute racine positive α de ${}^0\mathfrak{m}$ par rapport à $\mathfrak{t}$, pour l'ordre induit par celui fixé sur les racines de $(\mathfrak{g}, h)$.

Soit $W = W(\mathfrak{g},\mathfrak{h})$, $W_M = W({}^0\mathfrak{m},\mathfrak{t}) = W(\mathfrak{m},\mathfrak{h})$. On a la décomposition de Kostant

$$W = W_M W^P$$

où $W^P = \{s \in W : s^{-1}\alpha > 0, \forall\alpha \text{ racine simple de } (\mathfrak{m},\mathfrak{h})\}$.

Cela dit, d'après [14, Thm. 3.3, Ch. III],

(a) Si $H^\bullet(\mathfrak{g}, K, \pi \otimes V^\lambda) \neq 0$, il existe un unique $s \in W^P$ tel que

$$-(\delta + \lambda) = s^{-1}(\lambda_\sigma + \nu)$$

(b) Si l'élément s de W^P satisfait (a),

$$H^{\bullet+\ell(s)}(\mathfrak{g}, K; \pi \otimes V^\lambda) \cong H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}) \otimes \Lambda^\bullet \mathfrak{a}_p^*.$$

Dans le membre de droite, $s(\lambda+\delta)-\delta$ est le plus haut poids d'une représentation rationnelle $V^{s(\lambda+\delta)-\delta}$.

Nous appliquons ceci à la cohomologie de $\pi \otimes \tilde{V}^m$; on a donc $\tilde{V}^m = V^\lambda$, avec $\lambda = -w_0 m$, w_0 étant l'élément de plus grande longueur de W. On a:

$$\lambda_\sigma + \nu = (a_1, a_2, a_3, a_4, \ldots, a_{2m});$$

$$\begin{aligned} -(\delta + \lambda &= -(\delta - w_0 m) = w_0(\delta + m) \\ &= w_0(a_2, a_4, \ldots, a_{2m}, a_{2m-1}, \ldots, a_1) \\ &= (a_1, a_3, \ldots, a_{2m-1}, a_{2m}, \ldots, a_2). \end{aligned}$$

On a donc bien

$$\lambda_\sigma + \nu = -s(\delta + \lambda)$$

avec

$$s^{-1} = \begin{pmatrix} 1 & 3 & 5 & \ldots & 2m-1, 2m, & \ldots, & 2 \\ 1 & 2 & 3 & \ldots & & & 2m \end{pmatrix},$$

soit

$$s^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & 2m-1, & 2m \\ 1 & 2m & 2 & 2m-2 & 3 & \ldots & m, & m+1 \end{pmatrix}$$

(noter que $s \in W^P$). La condition (a) est donc satisfaite, et d'après (b):

$$H^\bullet(\mathfrak{g}, K; \pi \otimes \tilde{V}^m) = H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}) \otimes \Lambda^\bullet \mathfrak{a}_p^*.$$

On a $s(\lambda+\delta)-\delta = -\lambda_\sigma - \nu - \delta$, soit

$$s(\lambda+\delta)-\delta = (-a_1 - \frac{n-1}{2}, -a_2 - \frac{n-3}{2}, \ldots, -a_n + \frac{n-1}{2}).$$

D'après le formule de Künneth appliquée à ${}^0M = (SL(2)^\pm)^m$, on doit calculer les termes

$$H^\bullet(s\ell(2), 0(2); \sigma_i \otimes V_i)$$

où V_i est la représentation rationnelle de $GL(2)$ de plus haut poids $(-a_{2i-1} - \frac{n-1}{2} + 2i - 2, -a_{2i} - \frac{n-1}{2} + 2i - 1)$.

On sait, d'après le calcul pour $GL(2)$, que

$$H^\bullet(s\ell(2), 0(2); \sigma_i \otimes (V^{a_{2i}-\frac{1}{2}, a_{2i-1}+\frac{1}{2}})^\sim) \cong \Lambda^{\bullet-1}\mathbf{C}^0,$$

et V_i est obtenu à partir de la représentation indiquée en tordant par le caractère

$$(\det x)^{a_{2i}+a_{2i-1}+\frac{n}{2}-1-2i+2}$$

(noter que $\frac{n}{2} = m \in \mathbf{Z}$, $a_{2i} + a_{2i-1} \in \mathbf{Z}$). Comme la représentation σ_i, restreinte à $SL(2)^\pm$, est stable par torsion par $\det x$ (puisque σ_i appartient à la série discrète), la cohomologie ne change pas. On obtient donc

$$H^{\bullet+\ell(s)}(\mathfrak{g}, K; \pi \otimes \tilde{V}^m) \cong \otimes^m \mathbf{C}^{\bullet-1}\mathbf{C}^0 \otimes \Lambda^\bullet \mathbf{C}^m \cong \Lambda^{\bullet-m}\mathbf{C}^m.$$

On a $\tilde{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{a}$; le caractère central de $\pi \otimes \tilde{V}^m$, restreint à $A = A_G$, est trivial; la formule de Künneth pour $\tilde{\mathfrak{g}} \times \mathfrak{a}$ donne alors le résultat dans le cas pair, au décalage près; d'après Borel-Wallach [14, Ch. IV, Thm. 5.1], on a $\ell(s) = m(m-1)$, d'où le Lemme 3.14 dans ce cas.

Considérons maintenant le cas où $n = 2m + 1$ est **impair**. La représentation π est donc associée aux caractères

$$(z^{a_1}\bar{z}^{a_2}, z^{a_2}\bar{z}^{a_1}, \ldots, z^{a_{2m}}\bar{z}^{a_{2m-1}}, (z\bar{z})^{a_{2m+1}})$$

avec

$$a_1 < a_2, \quad a_3 < a_4, \quad \ldots,$$

$$a_1 + a_2 = \cdots = 2a_{2m+1} = w + 1 - n.$$

On suppose encore $a_1 < a_3 < \ldots$, d'où enfin:

$$a_2 > a_4 > \cdots > a_{2m} > a_{2m+1} > a_{2m-1} > \cdots > a_1.$$

La représentation π est induite, à partir du parabolique $P = MN$ de type (2,2,...,2,1), de la représentation

$$\sigma = \sigma_1 \otimes \cdots \otimes \sigma_m \otimes \chi$$

où σ_i est la série discrète évidente, et χ un caractère de $\mathbf{R}$ de la forme $x \mapsto |x|^{a_{2m+1}} (\operatorname{sgn} x)^\epsilon$. On utilise des notations analogues au cas pair: ainsi $M = {}^0MA_M$, $\mathfrak{a}_M \cong \mathbf{C}^{m+1}$, ${}^0M \cong (SL_2^\pm)^m \times (\pm 1)$.

On a, toujours avec les notations introduites dans le cas pair:

$$\begin{aligned} p &= (a_2 - \frac{n-1}{2}, a_4 - \frac{n-1}{2}, \dots, a_1 - \frac{n-1}{2}) \\ m &= (a_2 - \frac{n-1}{2}, a_4 - \frac{n-3}{2}, \dots, a_1 + \frac{n-1}{2}) \\ \lambda &= -w_0 m = (-a_1 - \frac{n-1}{2}, -a_3 - \frac{n-3}{2}, \dots, -a_2 + \frac{n-1}{2}) \\ \lambda_\sigma + \nu &= (a_1, a_2, \dots, a_{2m+1}) \\ \delta + \lambda &= (-a_1, -a_3, \dots, -a_2). \end{aligned}$$

On doit écrire $-(\delta + \lambda) = s^{-1}(\lambda_\sigma + \nu)$, d'où

$$s^{-1} = \begin{pmatrix} 1 & 3 & 5 & 7 \dots & 2m-1, & 2m+1, & 2m, & \dots, & 4 & 2 \\ 1 & 2 & 3 & \dots & m, & m+1, & m+2, & \dots, & 2m & 2m+1 \end{pmatrix}$$

soit

$$s^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2m-1 & 2m & 2m+1 \\ 1 & 2m+1 & 2 & 2m & \dots & m & m+2 & m+1 \end{pmatrix}$$

(noter que $s \in W^P$). Utilisant de nouveau la partie (b) du théorème de Borel-Wallach, on est amené à calculer

$$H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}).$$

On a

$$\begin{aligned} s(\lambda + \delta) - \delta &= -\lambda_\sigma - \nu - \delta \\ &= (-a_1 - \frac{n-1}{2}, -a_2 - \frac{n-3}{2}, \dots, -a_{2m+1} + \frac{n-1}{2}). \end{aligned}$$

Utilisant la formule de Künneth pour ${}^0M$, on est amené à calculer d'abord les termes en

$$H^\bullet(s\ell(2), 0(2); \sigma_i \otimes V_i)$$

où V_i est décrit dans la démonstration du cas pair; on sait que chacun de ces facteurs est égal à $\Lambda^{\bullet-1}\mathbb{C}^0$.

Il reste à calculer

$$H^\bullet(0, \pm 1; \chi \otimes x^{-a_{2m+1}+\frac{n-1}{2}})$$

(**sic**), où, rappelons-le, $\chi(x) = |x|^{a_{m+1}} (\operatorname{sgn} x)^\epsilon$; on tord par le caractère rationnel de $GL(1)$:

$$x \longmapsto x^{-a_{2m+1}+\frac{n-1}{2}}$$

(noter que $a_{2m+1} + \frac{n-1}{2} \in \mathbb{Z}$); on considère ici la $(\mathfrak{g}, K)$-cohomologie pour $\mathfrak{g} = \{0\}$, $K = \{\pm 1\} \subset \mathbb{R}^\times$. Cet espace est égal à $\mathbb{C}$ en degré zéro si $\chi(x)x^{-a_{2m+1}+\frac{n-1}{2}}$ est un caractère **pair** de $\mathbb{R}^\times$, à zéro sinon.

Dans le premier cas, on obtient donc un terme en $\Lambda^\bullet\mathbb{C}^0$; multipliant par les termes déjà obtenus, on a

$$\begin{aligned} H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}) &\cong \otimes^m \Lambda^{\bullet-1}\mathbb{C}^0 \otimes \Lambda^\bullet\mathbb{C}^0 \otimes \Lambda^\bullet\mathbb{C}^{m+1} \\ &\cong \Lambda^{\bullet-m}\mathbb{C}^{m+1}. \end{aligned}$$

D'après Borel-Wallach [14, Thm. 5.1 p. 101], on a $\ell(s) = m^2$, d'où

$$H^\bullet(\mathfrak{g}, K; \pi \otimes \tilde{V}^m) \cong \Lambda^{\bullet-m(m+1)}\mathbb{C}^{m+1}.$$

Divisant, comme dans le cas pair, par le terme $\Lambda^\bullet\mathfrak{a} = H^\bullet(\mathfrak{a}, \mathbb{C})$ associé à la décomposition $\mathfrak{g} = \tilde{\mathfrak{g}} \times \mathfrak{a}$, on en déduit le résultat cherché.

Supposons maintenant que $\chi(x)x^{-a_{2m+1}+\frac{n-1}{2}}$ est un caractère impair. Soit ϵ le caractère d'ordre 2 non trivial de $\mathbb{R}^\times$. On a alors

$$H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes \epsilon \otimes V^{s(\lambda+\delta)-\delta}) \cong \Lambda^{\bullet-m}\mathbb{C}^{m+1},$$

$\sigma \otimes \epsilon$ désignant la représentation $(\sigma_1 \otimes \epsilon, \ldots, \sigma_m \otimes \epsilon, \chi_\epsilon)$ de M (ou ${}^0M$, où l'on tord tous les facteurs par le caractère ϵ du déterminant (noter que $\sigma_i \cong \sigma_1 \otimes \epsilon$). Cette représentation s'induit en la représentation $\pi \otimes \epsilon$ de $GL(n, \mathbb{R})$. On a donc enfin $H^\bullet(\mathfrak{g}, K, \pi \otimes \epsilon \otimes \tilde{V}^m) \cong \Lambda^{\bullet-m(m+1)}\mathbb{C}^{m+1}$, ce qui termine la démonstration.

Remarque. Si n est **pair**, on voit donc qu'il n'y a pas lieu de tordre par un caractère.

Pour démontrer le Théorème 3.13, on considère maintenant, suivant Harder [30 a, c] les quotients arithmétiques de $X = G(F_\infty)/K_\infty$, où

$K_\infty = \prod_{v \text{ infinie}} K_v$, par des sous-groupes de congruence Γ. Pour tout sous-groupe compact-ouvert $K \subset G(\mathbb{A}_f)$, on a

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K \cong \coprod_i \Gamma_i \backslash X$$

(union finie, les Γ_i étant des sous-groupes de congruences) . On pose $S_K = \coprod_i \Gamma_i \backslash X$; on a

$$\tilde{S} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty = \varprojlim_K S_K$$

avec les morphismes évidents; on **définit**

$$H^\bullet(\tilde{S}, \mathbb{C}) = \varinjlim H^\bullet(S_K, \mathbb{C}).$$

Plus généralement, une représentation rationnelle (τ, V) de $L = \mathrm{Res}_{F/\mathbb{Q}} GL(n)$ définit un système de coefficients $\mathcal{V}$ sur S, et des systèmes de coefficients, aussi notés $\mathcal{V}$, sur les S_K. On **définit**

$$H^\bullet(\tilde{S}, \mathcal{V}) = \varinjlim H^\bullet(S_K, \mathcal{V}).$$

Le groupe $G(\mathbb{A}_f)$ opère naturellement sur $H^\bullet(\tilde{S}, \mathcal{V})$. Pour tout niveau fini K, soit

$$H^\bullet_{\mathrm{cusp}}(S_K, \mathcal{V}) = \oplus H^\bullet_{\mathrm{cusp}}(\Gamma_i \backslash X, \mathcal{V}).$$

la cohomologie parabolique [10 c]. On pose, **par définition** :

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) = \varinjlim H^\bullet_{\mathrm{cusp}}(S_K, \mathcal{V}).$$

D'après un résultat fondamental de Borel [10 c], la cohomologie parabolique, aux niveaux finis, s'injecte dans $H^\bullet(S_K, \mathcal{V})$. Passant à la limite inductive, on en déduit:

Lemme 3.15.
(i) On a $H^\bullet_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) \hookrightarrow H^\bullet(\tilde{S}, \mathcal{V})$.

(ii) $H^\bullet_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) = \bigoplus_\pi H^\bullet(\tilde{\mathfrak{g}}_\infty, K_\infty; \pi_\infty \otimes \mathcal{V}) \otimes \pi_f$; la somme porte sur les sous-modules π de $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}_F))$ dont le caractère central coïncide, sur la composante neutre $Z(F_\infty)^+$, avec celui de $\tilde{\mathcal{V}}$

(on a posé $\mathfrak{g}_\infty = \bigotimes_{v \text{ infinie}} \mathfrak{g}_v$*).*

Soit E le corps de rationalité de la représentation (τ, V). Remarquons que (τ, V) est en fait définie sur E. En effet, τ est définie par ses poids dominants $(m_\iota)_{\iota \in I}$; on a donc $m_\iota = (m_\iota^1, \ldots, m_\iota^n)$, $m^1 \geq \ldots, m^n$. Le corps E est l'extension de $\mathbb{Q}$ fixée par le sous-groupe Γ de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ défini par les équations $m_{\sigma\iota} = m_\iota$ ($\sigma \in \Gamma$, $\iota \in I$). En particulier, $m_{\sigma\iota}^i = m_\iota^i$ et le caractère de $GL(1, F)$ défini par $x \mapsto \Pi(x_\iota)^{m_\iota^i}$ est ainsi défini sur E, et donc à valeurs dans E. Il en est de même du caractère du tore maximal $GL(1, F)^n$ de $GL(n, F)$ défini par (m_ι). Puisque ce caractère apparaît avec multiplicité 1 dans V, τ est définie sur E d'après le Lemme 1.1 de Waldspurger [58].

Par conséquent, les morphismes de transition du système de coefficients $\mathcal{V}$ sont donnés par des matrices à coefficients dans E. On en déduit que la cohomologie de Betti $H_B^\bullet(S_K, \mathcal{V})$, et sa limite projective $H_B^\bullet(\tilde{S}, \mathcal{V})$ sont de façon naturelle définis sur E. On note $H_B^\bullet(\tilde{S}, \mathcal{V}_E)$ cet espace de cohomologie; on a

$$H^\bullet(\tilde{S}, \mathcal{V}) = H_B^\bullet(\tilde{S}, \mathcal{V}_E) \otimes_E \mathbb{C}.$$

L'action de $G(\mathbb{A}_f)$ sur $H^\bullet(\tilde{S}, \mathcal{V})$ par les correspondances de Hecke préserve cette E-structure. Pour l'instant, nous ne savons pas si le sous-espace $H_{\text{cusp}}^\bullet \subset H^\bullet$ est défini sur E. Néanmoins:

Proposition 3.16 (Drinfeld-Manin). *Soit W un sous-quotient irréductible de $H^\bullet(\tilde{S}, \mathcal{V})$ pour l'action de $G(\mathbb{A}_f)$. Alors il existe une extension finie E' de E telle que W soit rationnel sur E'.*

Remarquons tout de suite que la Proposition implique la première partie du théorème: si π est régulière, on a construit un système de coefficients τ tel que $H^\bullet(\mathfrak{g}_\infty, K_\infty; \pi_\infty \otimes \epsilon_\infty \otimes \tau_\infty) \neq 0$ (Lemme 3.14), ϵ_∞ étant un caractère d'ordre 2 de $F_\infty^\times$, qu'on étend en un caractère d'Artin d'ordre 2, ϵ, de $\mathbb{A}_F^\times$ [3]. Le Lemme 3.15 produit un sous-espace irréductible de $H^\bullet(\tilde{S}, \mathcal{V})$ isomorphe à $\pi_f \otimes \epsilon$ comme représentation de $G(\mathbb{A}_f)$, d'où le résultat.

Démonstration de la Proposition 3.16 : On remarque que $H^\bullet(\tilde{S}, \mathcal{V})$ est un $G(\mathbb{A}_f)$-module lisse et admissible. En particulier, W est une représentation irréductible admissible de $G(\mathbb{A}_f)$. Soit $K \subset G(\mathbb{A}_f)$ tel que $W^K \neq \{0\}$. On peut supposer de plus que

$K = \prod_{v \text{ finie}} K_v$. Soit $\mathcal{H} = C_c^\infty(G(\mathbb{A}_f)//K)$. On sait (cf. [5]) que $\mathcal{H}$ opère de façon irréductible sur W^K, et que cette action détermine la classe d'isomorphisme de W. L'algèbre $\mathcal{H}$ est définie sur $\mathbb{Q}$; d'après le paragraphe 3.1, le corps de rationalité de W est alors celui de cette représentation de $\mathcal{H}$. Considérons l'espace $T_E = H_B^\bullet(\tilde{S}, \mathcal{V}_E)^K$; c'est l'espace d'une représentation de $\mathcal{H}$, définie sur E. Soit $T_{\mathbb{C}} = T_E \otimes_E \mathbb{C}$. D'après le théorème de Burnside, les sous-quotients irréductibles de $T_{\mathbb{C}}$ (comme $\mathcal{H}(\mathbb{C})$-module) ou de $T_{\bar{\mathbb{Q}}} = T_E \otimes_E \bar{\mathbb{Q}}$ (comme $\mathcal{H}(\bar{\mathbb{Q}})$-module) coïncident. Par conséquent, W^K est obtenu par extension des scalaires à partir d'un des sous-quotients irréductibles de $T_{\bar{\mathbb{Q}}}$, et celui-ci peut être défini sur une extension finie E'.

Remarque. Même si W est un sous-module, cela n'implique pas qu'il est de la forme $W_{E'} \otimes_{E'} \mathbb{C}$ avec $W_{E'} \in H_B^\bullet(\tilde{S}, \mathcal{V}_E) \otimes E'$; c'est faux en général, cf. [30 b].

Le reste du Théorème 3.13 est plus difficile à démontrer. Pour simplifier, supposons d'abord $F = \mathbb{Q}$. La représentation (τ, V) peut être alors choisie définie sur $\mathbb{Q}$.

Soit $H_c^\bullet(S_K, \mathcal{V})$ l'espace de cohomologie à supports compacts de S_K dans $\mathcal{V}$. Si $\overline{S_K}$ est le compactifié de Borel-Serre [13, 47] de S_K, ∂S_K son bord, on a un triangle exact:

$$\begin{array}{ccc} H_c^i(S_K, \mathcal{V}) & \longrightarrow & H^i(\bar{S}_K, \mathcal{V}) = H^i(S_K, \mathcal{V}) \\ \delta \nwarrow & & \swarrow \\ & H^i(\partial \bar{S}_K, \mathcal{V}) & \end{array}$$

l'homomorphisme δ étant de degré $+1$. En particulier, on voit que $H_c^i(S_K, \mathcal{V})$ est défini sur $\mathbb{Q}$ dans la cohomologie de Betti.

On sait que les classes de cohomologie cuspidales ont une restriction nulle à $\partial \bar{S}_K$ [47, Satz 1.10], d'où des injections naturelles

$$H^i_{\text{cusp}}(S_K, \mathcal{V}) \to H_c^i(S_K, \mathcal{V}).$$

Soit $d = \dim(A \backslash X)$, où A est la composante neutre (topologique) de $Z(\mathbb{R})$: $A \cong \mathbb{R}_+^\times$. L'application naturelle $\Gamma \backslash X \to A\Gamma \backslash X$ est une équivalence d'homotopie; pour Γ arithmétique, $A\Gamma \backslash X$ est de volume fini. Posons $X = A \backslash X$, $S_K^1 = AG(\mathbb{Q}) \backslash G(\mathbb{A})/K_\infty K$. Dans les descriptions qui précèdent, on peut remplacer partout S_K par S_K^1. Notons que $G(\mathbb{A}) = AG(\mathbb{A})^1$, où $G(\mathbb{A})^1$ est le sous-groupe de $G(\mathbb{A})$ annulé par les caractères de la forme $|\chi(g)|$ où $\chi : G \to G_m$ est un caractère rationnel. On a donc des décompositions directes $S_K = AS_K^1$, avec

$$S_K^1 = G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_\infty K.$$

Si (τ^*, V^*) est la représentation duale de (τ, V), $\mathcal{V}^*$ le système local associé, on a une dualité naturelle, non-dégénérée

$$H_c^i(S_K^1, \mathcal{V}) \times H^{d-i}(S_K^1, \mathcal{V}^*) \to \mathbb{C}.$$

Elle est définie sur $\mathbb{Q}$ (pour les structures de Betti). Si $\tilde{\omega}$, $\tilde{\eta}$ sont deux formes différentielles représentant, dans la cohomologie de De Rham, des classes $\omega \in H_c^i$, $\eta \in H^{d-i}$, la dualité est donnée par

$$\langle \omega, \eta \rangle = \int_{S_K^1} \langle \tilde{\omega}(x), \tilde{\eta}(x) \rangle .$$

Restreinte à $H_{\text{cusp}}^i \times H_{\text{cusp}}^{d-i} \subset H_c^i \times H^i$, la dualité est encore non-dégénérée. Du point de vue automorphe, elle s'exprime de la façon suivante: soit $\omega \in H_{\text{cusp}}^i(S_K, \mathcal{V})$, représentée par une forme différentielle

$$\tilde{\omega} \in \mathrm{Hom}_{K_\infty}(\Lambda^i \tilde{\mathfrak{p}}, \mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K) \otimes V);$$

on a posé $\mathfrak{p} = \mathfrak{g}_\infty/\mathfrak{k}_\infty$, $\tilde{\mathfrak{p}} = \mathfrak{p}/\mathfrak{a}$ où $\mathfrak{a} = \mathrm{Lie}(A)$. Soit de même $\tilde{\eta}$ représentant $\eta \in H_{\text{cusp}}^{d-i}(S_K, \mathcal{V}^*)$. Alors

$$\tilde{\omega} \wedge \tilde{\eta} \in \mathrm{Hom}_{K_\infty}(\Lambda^d \tilde{\mathfrak{p}}, C^\infty(S_K))$$

définit une forme différentielle à décroissance rapide sur S_K. Son intégrale sur S_K^1 est égale à $\langle \omega, \eta \rangle$.

On considère maintenant les espaces de L^2-cohomologie réduite [61] des variétés S_K^1. Ils sont définis de la façon suivante: soit $\Omega_{(2)}^i(S_K^1, \mathcal{V})$ l'espace des i-formes φ à valeurs dans $\mathcal{V}$ qui sont de carré intégrable, ainsi que leur dérivée $d\varphi$ (définie comme distribution), pour la mesure invariante sur S_K^1 (on a muni V et $\lambda^i \tilde{\mathfrak{p}}$ de produits scalaires admissibles, cf. [11, 61]); $H_{(2)}^\bullet(S_K^1, \mathcal{V})$ est la cohomologie de ce complexe, et

$\bar{H}_{(2)}^\bullet(S_K^1, \mathcal{V})$, la cohomologie réduite, est définie par

$$\bar{H}_{(2)}^i(S_K, \mathcal{V}) = \{\varphi \in \Omega_{(2)}^i : d\varphi = 0\}/\overline{d\Omega_{(2)}^i}$$

où la clôture est pour la topologie L^2 sur les i-formes. On a une application naturelle, en réalisant la cohomologie à supports en cohomologie de De Rham:

$$j : H_c^i(S_K^1, \mathcal{V}) \to H_{(2)}^i(S_K^1, \mathcal{V}).$$

Le lemme suivant, quoique simple, est fondamental:

LEMME 3.17. *Soit* $\alpha \in H^i_c(S^1_K, \mathcal{V})$ *et* $\beta \in H^{d-i}_c(S^1_K, \mathcal{V}^*)$ *telles que* $\langle\alpha,\beta\rangle \neq 0$. *Alors* $j\alpha \neq 0$, $j\beta \neq 0$.

Démonstration. On définit une forme linéaire sur $\bar{H}^i_{(2)}(S_K, \mathcal{V})$ par

$$\lambda_\beta : \alpha \mapsto \langle\alpha,\beta\rangle = \int_{S^1_K} (\alpha,\beta)$$

où (α,β) est la forme différentielle (à coefficients triviaux) obtenue à l'aide de la dualité entre V et V^*. Montrons que ceci est bien défini. Tout d'abord, si $\alpha \in \Omega^i_{(2)}$, (α,β) est à support compact et de carré intégrable, donc l'intégrale est bien définie. Supposons que $\alpha = d\eta$, avec $\eta \in \Omega^{i-1}_{(2)}$. On a la formule

$$d((\alpha,\beta)) = (d\alpha,\beta) + (-1)^{\deg\alpha}(\alpha, d\beta),$$

d'où

$$\langle d\eta,\beta\rangle = \int_{S^1_K} [d((\eta,\beta)) + (-1)^{\deg\eta+1}(\alpha, d\beta)].$$

Comme $d\beta = 0$, il suffit de montrer que la première intégrale s'annulle. Soit $\omega = (\eta,\beta)$. C'est une forme différentielle à support compact, de carré intégrable, et telle que $d\omega = (d\eta,\beta)$ est de carré intégrable puisque $\eta \in \Omega^{i-1}_{(2)}$. On doit montrer que $\int_{S^1_K} d\omega = 0$. Utilisant une partition de l'unité, on peut supposer ω à support dans un système de coordonnées locales $(x_1,\dots,x_d)$; on a donc

$$\omega = \sum \omega_i\, dx_1\wedge\cdots\wedge\hat{dx_i}\wedge\dots dx_d, \quad d\omega = \sum(-1)^i \frac{\partial\omega_i}{\partial x_i}\, dx_1\wedge\cdots\wedge dx_d,$$

et $\int d\omega = \sum(-1)^i \int \frac{\partial\omega_i}{\partial x_i} dx_1\dots dx_n$. Si F est une fonction C^∞ à support compact, égale à 1 sur le support de ω , chaque terme est alors égal à $(-1)^i(-1)\int\left(\frac{\partial F}{\partial x_i}\omega_i\right)$ par définition de la dérivée distribution $\frac{\partial\omega_i}{\partial x_i}$. Cette intégrale est nulle, d'où le résultat.

On a donc montré que λ_β définit une forme linéaire sur l'espace de cohomologie non réduit $H^i_{(2)}(S^1_K, \mathcal{V})$. Il reste à voir que λ_β passe au quotient pour définir une forme linéaire sur $\bar{H}^i_{(2)}$. Par définition de cet espace, il suffit de vérifier que λ_β est continue pour la topologie sur $\Omega^i_{(2)}$ donnée par le produit scalaire sur les i-formes. C'est clair: supposons $(\lambda^i\tilde{\mathfrak{p}})^*$, V munis de leurs produits scalaires admissibles;

soit (λ_a, e_b) des bases orthonormales, et (λ'_a, e'_b) des bases duales de $(\lambda^{d-i}\tilde{\mathfrak{p}})^*$, V.

Si $\alpha \in \Omega^i_{(2)}(S^1_K, \mathcal{V})$, on peut écrire

$$\alpha = \sum_{a,b} f_{ab}\lambda_a \otimes e_b \in \mathrm{Hom}_K(\Lambda^i\tilde{\mathfrak{p}}^*, L^2(S^1_K) \otimes V)$$

avec $f_{a,b} \in L^2(S^1_K)$. De même, $\beta = \sum_{a,b} g_{a,b}\lambda'_a \otimes e'_b$ avec $g_{ab} \in C^\infty_c(S^1_K)$.

On a

$$\langle \alpha, \beta \rangle = \sum_{a,b} \int_{S^1_K} f_{ab} g_{ab}\, dx$$

qui est bien sûr continu pour la structure L^2. La forme λ_β est donc bien définie sur $\bar{H}^i_{(2)}$. Si elle ne s'annulle pas sur α, on a donc $j\alpha \neq 0$, q.e.d.

Remarque. Notons $H^i_!$ l'image de H^i_c dans H^i. Puisque H^i et H^{d-i}_c sont en dualité non-dégénérée, $H^i_!$ est le quotient de H^i_c par le noyau E^i_c de la forme d'accouplement sur $H^i_c \times H^{d-i}_c$. Le Lemme 3.17 a donc la conséquence suivante; la démonstration s'applique à n'importe quel groupe réductif G défini sur un corps de nombres F et à tout système de coefficients $\mathcal{V}$ venant d'une représentation algébrique de G à coefficients complexes; on pose encore $S^1_K = G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_\infty K_f$:

PROPOSITION 3.18. *On a une injection canonique*

$$H^i_!(S^1_K, \mathcal{V}) \hookrightarrow \bar{H}^i_{(2)}(S^1_K, \mathcal{V}).$$

Ce théorème a été démontré, de façon un peu différente, par Harder [30 a]. De plus, puisque la dualité restreinte à H^i_{cusp} est non-dégénérée, on sait que la flèche naturelle

$$H^i_{\mathrm{cusp}}(S^1_K, \mathcal{V}) \hookrightarrow H^i_!(S^1_K, \mathcal{V})$$

est aussi une injection.

Soit alors π la représentation considérée dans le Théorème 3.13. Supposons d'abord le Lemme 3.14 vérifié avec $\epsilon_\infty = 1$. On a vu que π_f s'identifiait à un sous-module de $H^i_{\mathrm{cusp}}(\tilde{S}^1, \mathcal{V})$. Pour K assez profond, la représentation de l'algèbre de Hecke $\mathcal{H} = C^\infty_c(G(\mathbb{A}_f)//K)$ associée à π_f (cf. Prop. 3.16) apparaît donc dans $H^i_{\mathrm{cusp}}(S^1_K, \mathcal{V})$ et donc dans $H_!^i(S_K^1, \mathcal{V})$. Comme cet espace est défini sur $\mathbb{Q}$, la représentation ${}^\sigma\pi_f$, pour $\sigma \in \mathrm{Aut}\,(\mathbb{C})$ apparaît donc aussi comme sous-module de $H_!^i(S_K^1, \mathcal{V})$ et donc de $\bar{H}^i_{(2)}(S_K^1, \mathcal{V})$ d'après la Proposition 3.18.

D'après Borel et Casselman [11], on a

$$\bar{H}^i_{(2)}(S_K^1, \mathcal{V}) = \bigoplus_\xi H^i(\tilde{\mathfrak{g}}, K_\infty; \xi_\infty \otimes V) \otimes \xi_f^K \tag{3.6}$$

où ξ décrit un système complet de sous-modules du spectre discret $L^2_{\mathrm{dis}}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. Remarquons qu'une représentation irréductible de $G(\mathbb{A})$ s'identifie à une représentation de $G(\mathbb{A})^1$, une fois qu'on connaît son caractère central sur A; en particulier, si l'on fixe celui-ci égal au caractère central $\omega_\pi|_A$, une représentation ξ de $G(\mathbb{A})^1$ détermine une représentation ξ_f de $G(\mathbb{A}_f)$. C'est l'action de $G(\mathbb{A}_f)$ sur $\bar{H}^i_{(2)}(S_K^1, \mathcal{V})$ obtenue à l'aide de cette description.

Soit alors ξ une représentation apparaissant dans (3.6), dont le caractère central ω_ξ satisfait $\omega_\xi|_A = \omega_\pi|_A$, et telle que $\xi_f \cong {}^\sigma\pi_f$. Soit ξ_∞ sa composante à l'infini, et soit

$$(z^{r_1}(\bar{z})^{r_2}, z^{r_2}\bar{z}^{r_1}, \dots, z^{r_{2k}}\bar{z}^{r_{2k-1}}, (z\bar{z})^{r_{2k+1}}, \dots, (z\bar{z})^{r_n})$$

les caractères de $\mathbb{C}^\times$ associés à ξ_∞ par la classification de Langlands, avec $r_i \neq r_j$ $(i \leq 2k)$.

La condition $H^i(\tilde{\mathfrak{g}}, K_\infty; \xi_\infty \otimes V) \neq 0$ implique que $\xi_\infty|\tilde{\mathfrak{g}}$ a le même caractère infinitésimal que $\pi_\infty|\tilde{\mathfrak{g}}$, d'où

$$(r_1, \dots, r_n) = w(a_1 + s, \dots, a_n + s) \qquad (w \in \mathfrak{S}_n)$$

pour un $s \in \mathbb{C}$. D'après la condition sur le caractère central, on a de plus $\sum a_i = \sum r_i$, soit $s = 0$. Si r est la représentation de $W_\mathbb{C} = \mathbb{C}^\times$ associée à ξ_∞, on voit donc que $r \,|\,|_\mathbb{C}^{\frac{1-n}{2}}$ est somme de caractères algébriques (**Attention** : ceci n'implique pas que ξ_∞ et π_∞ sont isomorphes: penser aux représentations de $GL(2,\mathbb{R})$ associées aux caractères $(z\bar{z}\,|\,|^{\frac{1}{2}}, |\,|^{\frac{1}{2}})$ et $(z\,|\,|^{\frac{1}{2}}, \bar{z}\,|\,|^{\frac{1}{2}})$ de $\mathbb{C}^\times$).

La théorie des séries d'Eisenstein [38 a, f] implique que ξ est obtenue à l'aide de séries d'Eisenstein formées à partir d'une représentation cuspidale d'un parabolique P de type $(n_1, \dots, n_r)$; ξ est alors isomorphe à un sous-quotient d'une induite ρ qu'on peut normaliser à la Tate:

$$\rho = \left[\mathcal{I}\mathrm{nd}_{P(\mathbb{A})}^{G(\mathbb{A})}(\xi_1\,|\,|^{\frac{1-n_1}{2}} \otimes \cdots \otimes \xi_r\,|\,|^{\frac{1-n_r}{2}})\right] \cdot |\,|^{\frac{n-1}{2}}, \tag{3.7}$$

les ξ_i étant des représentations cuspidales de $GL(n_i, \mathbb{A})$. Si r_i est la représentation de $W_{\mathbb{C}}$ associée par Langlands à $(\xi_i)_\infty$, r' celle associée au sous-quotient de Langlands de ρ_∞, on a alors

$$r' \mid \, |_{\mathbb{C}}^{\frac{1-n}{2}} \cong \oplus_i r_i \mid \, |_{\mathbb{C}}^{\frac{1-n_i}{2}} .$$

Les considérations précédentes sur le caractère central et le caractère infinitésimal montrent que $r' \mid \, |_{\mathbb{C}}^{\frac{1-n}{2}}$, et donc les $r_i \mid \, |_{\mathbb{C}}^{\frac{1-n_i}{2}}$, sont algébriques. Par conséquent, $\xi_i \in Alg(n_i)^0$. De plus, ξ_i est régulière (caractère infinitésimal !).

Supposons alors $P \neq G$. Par récurrence, on peut appliquer le Théorème 3.13 aux ξ_i. Soit $\alpha = \sigma^{-1}$. Il existe donc des représentations cuspidales ${}^\alpha\xi_i$, dont la partie finie est ${}^\alpha\xi_{i,f}$. Comme l'induction (3.7) est, pour la partie finie des représentations, une opération rationnelle (Lemme 3.9), ${}^\alpha\xi_f = \pi_f$ serait alors un sous-quotient de ${}^\alpha\rho_f$. Ceci contredit le Théorème 1.1.

On voit donc que ξ est cuspidale. Les considérations sur le caractère infinitésimal et le caractère central impliquent que le type à l'infini de ξ est égal à celui de π, d'où le Théorème 3.13 pour π. Enfin, en général, si π vérifie le Lemme 3.14 avec un caractère ϵ_∞ non-trivial, on l'étend en un caractère d'Artin d'ordre 2, ϵ, de $\mathbb{A}^\times$: le Théorème 3.13 pour π se déduit de l'assertion pour $\pi \otimes \epsilon$.

Indiquons rapidement comment étendre l'argument à un corps quelconque F. Soit $\pi \in Alg^0(n, F)$ une représentation régulière, (τ, V) la représentation de $L = \mathrm{Res}_{F/\mathbb{Q}}\, GL(n)$ associée à π. On peut la supposer définie sur $\bar{\mathbb{Q}}$; on a $\tau = \otimes_{\iota \in I} \tau_\iota$. Soit $\sigma \in \mathrm{Aut}\, \mathbb{C}$; on note aussi σ son image dans $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ où l'on considère $\bar{\mathbb{Q}}$ comme plongé dans $\mathbb{C}$. On peut considérer la représentation abstraite $({}^\sigma\tau, {}^\sigma V)$ de $L(\mathbb{Q})$ sur $V \otimes_{\mathbb{C}, \sigma^{-1}} \mathbb{C}$ (cf. 3.1). Il est clair qu'elle est obtenue par extension des scalaires à $\mathbb{C}$ de la représentation rationnelle, encore notée ${}^\sigma\tau$, de $H(\mathbb{Q}) : {}^\sigma\tau = \otimes_\iota \tau_{\sigma^{-1}\iota}$.

La réalisation simpliciale de la cohomologie de Betti montre qu'il y a un isomorphisme σ-linéaire:

$$H^\bullet(S_K, \mathcal{V}) \to H^\bullet(S_K, {}^\sigma\mathcal{V})$$

où ${}^\sigma\mathcal{V}$ est le système de coefficients déduit de ${}^\sigma V$. On a des isomorphismes analogues pour les espaces $H_c^\bullet(S_K, \mathcal{V})$, $H_!^\bullet(S_K, \mathcal{V})$. Puisque π_f est un sous-module de $H_!^\bullet(S_K, \mathcal{V})$, ${}^\sigma\pi_f$ est donc un sous-module de $H_!^\bullet(S_K, {}^\sigma\mathcal{V})$. On procède ensuite comme dans le cas de $\mathbb{Q}$, en remplaçant S_K par $S_K^1 = AG(F)\backslash G(\mathbb{A})/K_\infty K \cong G(F)\backslash G(\mathbb{A})^1/K_\infty K$, où $A = \mathbb{R}_+^\times$ est plongé diagonalement dans $Z(F_\infty) = \prod_{v\in\infty} Z(F_v)$, le centre de $G(F_\infty)$. On obtient de même une représentation ξ de $G(\mathbb{A})$, intervenant dans le spectre discret, de même caractère central sous$A$ que π, et telle que $\xi_f = {}^\sigma\pi_f$. On démontre de même, en considérant séparément toutes les places infinies, que ξ est algébrique, puis cuspidale. Il reste à vérifier que le type à l'infini $p(\xi) = (p_\iota(\xi))_\iota$ est égal à $\sigma(p(\pi)) = (p_{\sigma^{-1}\iota}(\pi))_\iota$. C'est clair - considérer le caractère infinitésimal de ξ_∞ - puisque ξ_∞ a de la cohomologie à coefficients dans ${}^\sigma\tau = \otimes_\iota \tau_{\sigma^{-1}\iota}$.

Notons la conséquence suivante de la démonstration. Si $\tau = \otimes_\iota \tau$, est une représentation irréductible, rationnelle, de L, soit E son corps de rationalité (ou de définition). Le système de coefficients $\mathcal{V}$ est donc défini sur E, et l'on a:

THÉORÈME 3.19. *Le sous-espace $H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) \subset H^i_c(\tilde{S}, \mathcal{V})$ est défini sur E. Plus précisément, on a*

$$H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) = H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}_E) \otimes_E \mathbb{C}$$

avec

$$H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}_E) = H^i_c(\tilde{S}, \mathcal{V}) \cap H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}_E).$$

C'est clair: on a $H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) = \bigoplus_\pi H_i(\tilde{\mathfrak{g}}, K; \pi_\infty) \otimes \pi_f$, où π_f décrit l'ensemble des représentations cuspidales de type à l'infini prescrit par V. Rappelons les injections

$$H^i_{\mathrm{cusp}} \hookrightarrow H^i_! \hookrightarrow \bar{H}^i_{(2)}.$$

Soit $W = H^i(\tilde{\mathfrak{g}}, K, \pi_\infty) \otimes \pi_f$ le sous-module de type π_f de $H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V})$ où π_f (et donc π) est fixée. On a $W \subset H^i_!$; pour $\sigma \in \mathrm{Aut}\,(\mathbb{C}/E)$ soit ${}^\sigma W$ sont transformé. Il s'injecte donc dans $\bar{H}^i_{(2)}(\tilde{S}, \mathcal{V})$ d'après le Lemme 3.17; la démonstration précédente montre qu'il est en fait contenu dans l'image de H^i_{cusp} dans $\bar{H}^i_{(2)}$; on en déduit que l'image de H^i_{cusp} dans $H^i_!$ est invariante par Aut $(\mathbb{C}/E)$; le théorème résulte alors du Lemme 3.2.1.

Remarque. Dans le cas de $GL(2)$, ce théorème est dû à Eichler, Shimura, Waldspurger et Harder.

4. Motifs et représentations algébriques

Dans ce chapitre, on étudie le lien entre deux théories conjecturales: celle, due à Grothendieck, des motifs associés aux variétés algébriques sur F, et la théorie tannakienne des formes automorphes sur GL. Il paraît naturel de s'attendre à un isomorphisme entre les deux catégories, du moins si l'on se limite aux représentations algébriques. Néanmoins, des proprités délicates du comportement des motifs, en ce qui concerne la rationalité et la descente (§4.3) limitent le champ du possible(*). Les conjectures précises sont dans le paragraphe 4.3. On étudie ensuite leurs conséquences pour les représentations algébriques, et en partiuclier leur type à l'infini. On suggère enfin une question portant sur les fonctions L, et motivée par ces conjectures.

4.1.- Motifs. On suppose le lecteur quelque peu familier avec la théorie des motifs de Grothendieck, pour laquelle on renvoie à [21, 23]. On ne s'intéresse qu'aux motifs sur le corps de nombres F. Mentionnons seulement les points suivants:

4.1.1.- Il nous suffit de considérer les motifs associés à la catégorie des variétés projectives et lisses sur F. On construit une catégorie $\mathcal{M}(F)$, munie d'un foncteur contravariant $H^* : \mathcal{V}(F) \to \mathcal{M}(F)$, à l'aide d'une relation d'équivalence $\mathcal{R}$ sur les cycles algébriques de variétés sur F: l'équivalence numérique ou cohomologique [23]; [19 b, §0.6].

La catégorie $\mathcal{M}(F)$ est une catégorie abélienne $\mathbb{Q}$-linéaire munie d'un produit tensoriel. **Modulo les conjectures standard**, c'est une catégorie tannakienne semi-simple [45].

On peut aussi considérer la catégorie $\mathcal{M}_{HA}(F)$ dont les correspondances sont données par les cycles de Hodge absolus [21, II.6]. Elle est tannakienne et semi-simple [21, II Thm. 6.7].

4.1.2.- On dispose des réalisations suivantes de la cohomologie motivique: ce sont des foncteurs de $\mathcal{M}(F)$ dans des catégories d'espaces vectoriels munis de quelques structures:

. **Réalisation de Betti.** On se fixe un plongement $\iota : F \to \mathbb{C}$ (on considèrera dans ce qui suit $\bar{\mathbb{Q}}$ comme plongé dans $\mathbb{C}$). Si $M \in M(F)$, $H_\iota(M)$ est un espace vectoriel sur $\mathbb{Q}$.

. **Réalisation de De Rham.** $H_{DR}(M)$ est un espace vectoriel sur F.

(*)Je remercie D. Blasius et D. Ramakrishnan de m'avior expliqué ces phénomènes lors de la conference. Leurs objections ont conduit à la formulation présente, que j'espère correcte, du paragraphe 4.3

. **Réalisation de Hodge.** C'est un foncteur à valeurs dans les structures de Hodge rationnelles. On notera encore $H_\iota(M)$ l'espace $H_\iota(M)$ muni de sa structure de Hodge $H_\iota(M) \otimes_{\mathbf{Q}} \mathbf{C} = \oplus H_\iota(M)^{pq}$ (la réalisation de Betti est sous-jacente à la réalisation de Hodge). On a un isomorphisme $F_\infty : H_\iota(M) \cong H_{\bar\iota}(M)$, et $F_\infty \otimes c$ échange H_ι^{pq} et $H_{\bar\iota}^{qp}$, c étant la conjugaison complexe sur $\mathbf{C}$.
. **Réalisation ℓ-adique.** $H_\ell(M)$ est donné par la cohomologie ℓ-adique des variétés $V \otimes_F \bar{F}$, où $V \in \mathcal{M}(F)$. C'est un espace vectoriel sur $\mathbf{Q}_\ell$, muni d'une action de $\mathrm{Gal}(\bar{F}/F)$. Si ℓ varie, on obtient un système strictement compatible de représentations ℓ -adiques [21, 1.1]; [48] (**Attention** : on ne sait pas si la catégorie $\mathcal{M}_{HA}(F)$ donne naissance à des systèmes compatibles de représentations ℓ-adiques: [21, 0.11]).

Ces réalisations sont soumises aux compatibilités suivantes:
. **Théorème de comparaison** : pour tout plongement ι,

$$H_\iota(M) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \cong H_\ell(M).$$

. **Théorie de De Rham** : pour tout ι,

$$H_\iota(M) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \cong H_{DR}(M) \otimes_{F,\iota} \mathbf{C}.$$

Cet isomorphisme est compatible aux filtrations de Hodge [21, 0.2.6,0.4].
4.1.3.- Si l'on munit la catégorie $\mathcal{M}(F)$ d'un foncteur fibre à valeurs dans les espaces vectoriels sur $\mathbf{Q}$, on en déduit (conjecturalement pour $\mathcal{M}$, inconditionnellement pour $\mathcal{M}_{AH}$) un *groupe de Galois motivique* $\mathcal{G}_{\mathrm{Mot}}$. *On prend le foncteur fibre donné* par la cohomologie de Betti H_ι, d'où un groupe $\mathcal{G}_{\mathrm{Mot}}(\iota F)$: c'est un schéma en groupes affine, proréductif, sur $\mathbf{Q}$. Si $F = \mathbf{Q}$, on le note $\mathcal{G}_{\mathrm{Mot}}(\mathbf{Q})$.

4.2. Représentations algébriques rationnelles.

4.2.1.- Dans la théorie des motifs, on dispose des motifs *absolus* (= morceaux de la cohomologie des variétés algébriques), et des motifs à coefficients dans des corps de nombres E. Le but de ce paragraphe est de définir les notions analogues dans la catégorie $Alg(F)$. Pour éclairer cette construction, il est utile de garder à l'esprit le cas des caractères de Hecke algébriques d'un corps F de type CM: si χ est un tel caractère, de type (1,0) à toutes les places infinies, et E son corps des valeurs, de degré $2d$ sur $\mathbf{Q}$, il existe une variété abélienne A

de dimension d sur F, à multiplication complexe par E, telle que la fonction L du motif $H^1(A)$ s'écrive

$$L_1(s, A) = \prod_\iota L(s, {}^\iota\chi)$$

où ι décrit les plongements complexes de E. Le membre de droite est alors associé à une représentation de Alg qui est rationnelle sur $\mathbb{Q}$ - ainsi que le motif $H^1(A)$ - mais dont les composantes ${}^\iota\chi$ ne le sont pas.

DEFINITION 4.1. *Soit $\pi \in Alg(n)$. On dit que π est rationnelle si π_f est définie sur $\mathbb{Q}$.*

On notera $Alg_{\mathbb{Q}}(n)$ la classe des représentations rationnelles. Plus généralement:

DEFINITION 4.2. *Soit $E \subset \mathbb{Q}$ un corps de nombres. On dit que π est E-rationnelle si π_f est définie sur E.*

Notation : $Alg_E(n)$.

D'après 3.4, on a une action de Aut $(\mathbb{C})$ - ou plutôt de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$: les représentations de Alg devraient être définies sur $\bar{\mathbb{Q}}$ - sur $Alg(n)$: conjecturalement, et effectivement pour les représentations régulières (Thm. 3.13). La définition 4.2 est donc équivalente à

$$\pi \cong {}^\sigma\pi \quad (\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)) \tag{4.1}$$

Il est clair que la catégorie des représentations rationnelles est stable par $\overset{T}{\boxplus}$, et conjecturalement par $\overset{T}{\boxtimes}$. On va maintenant vérifier, les définitions données en 1.4 (**Caveat lector** : ces définitions sont tout-à-fait provisoires, cf. 1.4.0) que c'est une catégorie $\mathbb{Q}$-linéaire. On considère la catégorie $Alg_{\mathbb{Q},2}$ construite (dans le cas de $Isob_2$) dans le paragraphe 1.4.3. **On admet la conjecture 3.7.**

Soit $\pi \in Alg_{\mathbb{Q}}(n)$. On peut écrire (Lemme 1.12)

$$\pi \cong \pi_1 \overset{T}{\boxplus} \pi_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \pi_r,$$

$\pi_i \in Alg^0(n_i)$. Si $\sigma \in \mathfrak{g} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, on a alors

$${}^\sigma\pi \cong {}^\sigma\pi_1 \overset{T}{\boxplus} {}^\sigma\pi_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} {}^\sigma\pi_r.$$

On en déduit que $\mathfrak{g}$ permute les composantes; on peut donc écrire, de façon unique à permutation près:

$$\pi \cong \sigma_1 \overset{T}{\boxplus} \sigma_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \sigma_r$$

avec $\sigma_k \cong \pi_{i_1} \boxplus \dots \boxplus \pi_{i_k}$, σ_k rationnelle, et les π_{i_s} étant une orbite sous l'action de $\mathfrak{g}$.

Une telle représentation sera dite **$\mathbf{Q}$-irréductible**.

Soit maintenant $\pi \in Alg_{\mathbf{Q},2} = Alg_{\mathbf{Q}} \cap Alg_2$ (§1.4.3): π est donc une représentation de $Alg_{\mathbf{Q}}$ à isomorphisme près, qu'on identifie à une somme formelle

$$\oplus n_\tau \tau \qquad (\tau \in Alg^0(n));$$

la représentation associée à cette somme est

$$\tau_1 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \tau_1 \overset{T}{\boxplus} \tau_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \tau_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \tau_r$$

(n_{τ_i} facteurs pour chaque τ_i).

Par définition, si $\pi' = \otimes m_\tau \tau$:

$$\mathrm{Hom}_{\mathcal{G}}(\pi, \pi') = \oplus_\tau \mathrm{Hom}(\mathbf{C}^{n_\tau}, \mathbf{C}^{m_\tau}).$$

On définit alors $\mathrm{Hom}_{\mathcal{G},\mathbf{Q}}(\pi, \pi')$ comme l'ensemble des $h = (h_\tau : h_\tau \in M_{n_\tau m_\tau}(\mathbf{C})) \in \mathrm{Hom}_{\mathcal{G}}(\pi, \pi')$ tels que $h_\tau \in M_{n_\tau m_\tau}(\bar{\mathbf{Q}})$ et $h_{\sigma_\tau} = \sigma(h_\tau)$ pour tout $\sigma \in \mathfrak{g}$. On peut regrouper les τ par orbites sous l'action de $\mathfrak{g}$; il est clair que $\mathrm{Hom}_{\mathcal{G},\mathbf{Q}}(\pi, \pi')$ est une somme directe suivant les orbites.

Supposons donc que $\mathfrak{g}$ permute les représentations τ. Si $E(\tau)$ est le corps de rationalité de τ, on a ${}^\sigma\tau \cong \tau \Leftrightarrow \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/E(\tau))$; on en déduit que l'orbite de τ sous $\mathfrak{g}$ correspond bijectivement à l'ensemble des plongements de $E(\tau)$ dans $\bar{\mathbf{Q}} \subset \mathbf{C}$, le corps de définition de ${}^\sigma\tau$ étant $\sigma(E(\tau))$. Notons aussi que les coefficients n_τ et m_τ sont constants sur l'orbite. On les note n et m.

Fixons une représentation τ_0 dans l'orbite, et soit $E = E(\tau_0)$. On voit alors que $\mathrm{Hom}_{\mathcal{G},\mathbf{Q}}(\pi, \pi')$ s'identifie au groupe

$$\{h_\iota \in M_{nm}(\iota E) : \sigma h_\iota = h_{\sigma\iota} \quad (\sigma \in \mathfrak{g})\},$$

où ι décrit les plongements de E dans $\bar{\mathbf{Q}} \subset \mathbf{C}$. Autrement dit, $\mathrm{Hom}_{\mathcal{G},\mathbf{Q}}(\pi, \pi')$ est obtenu, par restriction des scalaires de E à $\mathbf{Q}$, à partir de l'espace vectoriel $M_{nm}(E)$ sur E. On a alors naturellement

$$\mathrm{Hom}_{\mathcal{G},\mathbf{Q}}(\pi, \pi') \otimes_{\mathbf{Q}} \mathbf{C} = M_{nm}(E) \otimes_{\mathbf{Q}} \mathbf{C} = M_{nm}(E \otimes_Q \mathbf{C}) = M_{mn}(\mathbf{C} \oplus \dots \oplus \mathbf{C}),$$

le nombre de facteurs étant égal à $[E : \mathbb{Q}]$, c'est-à-dire au cardinal de l'orbite. En résumé:

LEMME 4.3. (ON ADMET LA CONJECTURE 3.7).
(i) La catégorie $Alg_{\mathbb{Q},2}$ est $\mathbb{Q}$-linéaire.

(ii) Si $\tau \cong \tau_1 \overset{T}{\boxplus} \tau_2 \overset{T}{\boxplus} \ldots \overset{T}{\boxplus} \tau_k$ *est $\mathbb{Q}$-irréductible,* $\mathrm{Hom}_{\mathcal{G},\mathbb{Q}}(n\tau, m\tau) \cong M_{nm}(E)$, *où E est le corps de définition de l'une quelconque des τ_i.*

(iii) Les représentations $\mathbb{Q}$-irréductibles sont les objets simples de $Alg_{\mathbb{Q},2}$, et tout élément de $Alg_{\mathbb{Q},2}$ est somme directe d'objets simples.

(iv) $\mathrm{Hom}_{\mathcal{G}}(\pi, \pi') = \mathrm{Hom}_{\mathcal{G},\mathbb{Q}}(\pi, \pi') \otimes_{\mathbb{Q}} \mathbb{C}$ *pour tous $\pi, \pi' \in Alg_{\mathbb{Q},2}$.*

La partie (iii) est claire: il suffit de montrer que tout représentation $\mathbb{Q}$-irréductible est simple, et cela est évident puisque $\mathrm{Hom}_{\mathcal{G},\mathbb{Q}}(\tau, \tau)$ est un corps. D'après (iv), la catégorie obtenue à partir de $Alg_{\mathcal{G},\mathbb{Q}}$ par extension des scalaires à $\mathbb{C}$ est la sous-catégorie pleine de Alg_2 dont les objets sont les représentations rationnelles.

Si E est un corps de nombres, supposé plongé dans $\mathbb{C}$, on démontre facilement l'analogue du Lemme 4.3 pour la catégorie $Alg_{E,2}$. En particulier, elle est E-linéaire.

4.2.2.- Propriétés de $Alg_{\mathbb{Q}}(F)$: Conjectures..

Tout d'abord, le principe de fonctorialité amène à conjecturer que **la catégorie $Alg_{\mathbb{Q}}(F)$** - ou plutôt une sous-catégorie équivalente convenable - **peut être munie d'une structure de catégorie tannakienne $\mathbb{Q}$-linéaire.**

On conjecture aussi [38 f] qu'il existe un **foncteur fibre** de la catégorie $Alg_{\mathbb{Q}}(F)$, au moins dans la catégorie des espaces vectoriels complexes. Un objet de $Alg_{\mathbb{Q}}(n, F)$ devrait être associé à un espace vectoriel de dimension n. On n'a aucune idée de la façon de construire un tel foncteur fibre.

L'existence d'un tel foncteur fibre impliquerait alors l'existence d'un groupe $\mathcal{G}_{Alg,\mathbb{Q}}(F)$, groupe pro-algébrique complexe. On peut supposer de plus l'existence d'un foncteur fibre à valeurs dans $\mathbb{Q}$, qui devrait dépendre (cf. 4.13) d'un plongement $\iota : F \to \bar{\mathbb{Q}}$. On obtient alors un groupe $\mathcal{G}_{Alg,\mathbb{Q}}(\iota)$, forme sur $\mathbb{Q}$ du groupe complexe $\mathcal{G}_{Alg,\mathbb{Q}}(F)$.

Indiquons rapidement la relation de ces objets conjecturaux avec ceux considérés par Langlands [38 f, §2] (on ne considère ici que des groupes **complexes**).

Le groupe $\mathcal{G}_{Alg,\mathbb{Q}}(F)$ ne paramètre que les représentations algébriques **rationnelles**. Celles-ci, après extension des scalaires à $\mathbb{C}$, déterminent des représentations algébriques. On s'attend donc à un homomorphisme surjectif ([21: Prop. II 2.21 et Remark 2.29]).

$$\mathcal{G}_{Alg}(F) \to \mathcal{G}_{Alg,\mathbb{Q}}(F). \tag{4.2}$$

Plus généralement, on a des homomorphismes analogues en remplaçant $\mathbb{Q}$ par E, et on devrait avoir

$$\mathcal{G}_{Alg}(F) = \varinjlim_{E} \mathcal{G}_{Alg,E}(F) \tag{4.3}$$

les homomorphismes $\mathcal{G}_{Alg,E} \to \mathcal{G}_{Alg,E'}$ $(E' \subset E)$ étant évidents.

Le groupe de Galois automorphe considéré par Langlands dans [38 f] est associé à la catégorie *Isob*(F). Notons-le $\mathcal{G}_{Isob}(F)$ (notations de Langlands: $\mathcal{G}_{\Pi(F)}$). Il devrait être muni d'un homomorphisme surjectif ([21, loc. cit.]):

$$\mathcal{G}_{Isob}(F) \to \mathcal{G}_{Alg}(F). \tag{4.4}$$

Le noyau de (4.4) serait très gros, correspondant à toutes les représentations non-algébriques.

Pour l'instant, l'homomorphisme (4.4) n'a de sens que si les espaces $\mathrm{Hom}_{\mathcal{G}}$ ($\mathcal{G} = \mathcal{G}_{Isob}$ ou $\mathcal{G}_{Alg}$) sont définis de façon compatible, ce qui n'est pas le cas: l'un est défini par les sommes directes à la Langlands (Def. 1.15), l'autre par les sommes à la Tate (§4.2).

On définit une équivalence de catégories entre $(Isob, \overset{T}{\boxplus})$ et $(Isob, \boxplus)$ (i.e., *Isob* muni des homomorphismes exprimés à l'aide de $\overset{T}{\boxplus}$ et $\boxplus$ respectivement) par $\pi \mapsto \pi|\ |^{\frac{1-n}{2}}$ $(\pi \in Isob(n))$. On en déduit que $\mathcal{G}_{Isob}(F) \cong \mathcal{G}_{\Pi(F)}$. Si r est une représentation de degré n de ce groupe, π_L la représentation associée par Langlands, π_T celle donnée par la correspondance rationnelle, on a

$$\pi_L = \pi_T|\ |^{\frac{1-n}{2}}. \tag{4.5}$$

Réciproquement, si π est une représentation de $GL(n, \mathbb{A}_F)$, r_L et r_T les représentations de $\mathcal{G}_{Isob}(F)$ qui lui sont associées, on a

$$(4.6) \qquad r_T = r_L |\ |^{\frac{1-n}{2}}.$$

Dans [38 f], Langlands conjecture l'existence d'un autre groupe de Galois automorphe $\mathcal{G}_{\Pi^0(F)}$ associé aux représentations tempérées. On a conjecturalement un homomorphisme $\mathcal{G}_{\Pi(F)} \to \mathcal{G}_{\Pi^0(F)}$; mais, puisque la catégorie des représentations tempérées n'est pas stable par $\overset{T}{\boxplus}$, il n'y a pas de groupe analogue dans $\mathcal{G}_{Alg}$.

Notons quelques conséquences des traductions (4.5), (4.6): tout d'abord, si $\pi = \otimes \pi_v \in Alg(n)$, on peut considérer la représentation $r_v = r_{L,v} : W_{\mathbb{C}} \to GL(n, \mathbb{C})$ de 1.2.3. La Définition 1.8 s'exprime donc simplement par le fait que $r_{T,v} = r_{L,v} |\ |^{\frac{1-n}{2}}$ est algébrique.

En ce qui concerne les corps p-adiques, les considérations du paragraphe 3.4 sur la rationalité des représentations amènent à la conséquence suivante (cf. [53, §3.6]). Soit $W'_{F_v} = SL(2) \times W_{F_v}$. D'après Langlands, on conjecture que les représentations continues de degré n de W'_{F_v}, algébriques sur le facteur $SL(2)$, paramètrent les représentations admissibles irréductibles de $GL(n, F_v)$. Notons $Isob(F_v)$ la catégorie (tannakienne ?) de ces représentations.

On a une application $Alg(n, F) \to Isob(n, F_v)$ si F est une complétion de k. On peut supposer que cela sera un morphisme de catégories tannakiennes. On en déduirait alors une application

$W'_{F_v} \underset{j_v}{\longrightarrow} \mathcal{G}_{Alg,E}(F)$ pour tout E, passant par l'enveloppe algébrique de W'_{F_v}, [21 II.2.33]. Si la représentation globale π est définie sur E, et donc ses facteurs locaux, la représentation r, et donc $r \circ j_v$, l'est aussi. On est ainsi amené à conjecturer:

CONJECTURE 4.4. *Si $\pi \in Isob(F_v)$, soit r_L la représentation de W'_{F_v} associée à π par la conjecture de réciprocité locale, $r_T = r_L |\ |^{\frac{1-n}{2}} = r_T(\pi)$. Alors, pour tout $\sigma \in Aut\ (\mathbb{C}/\mathbb{Q})$,*

$$r_T({}^\sigma\pi) \cong {}^\sigma r_T(\pi).$$

(Cette hypothèse est bien connue des spécialistes de la conjecture locale).

4.3.- Motifs et représentations algébriques.

4.3.0.- la catégorie $Alg_{\mathbb{Q}}(F)$ est l'analogue naturel de la catégorie des motifs sur F; $Alg_E(F)$, celui de la catégorie des motifs sur F, à coefficients dans E. Néanmoins, les catégories de représentations automorphes et de motifs ont des propriétés très différentes du point de vue de la descente et de la rationalité. Cela implique que la relation entre motifs et représentations algébriques ne peut être complètement transparente.

Par exemple, soit M un motif sur F, à coefficients dans $E \subset \bar{\mathbb{Q}}$, et supposons que E est une extension galoisienne de $\mathbb{Q}$. Supposons que $M \cong {}^{\sigma}M$ pour tout $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$. Il n'est alors pas vrai, en général, que M est obtenu par extension des scalaires à E d'un motif sur $\mathbb{Q}$. Par exemple, M pourrait être un motif d'Artin [19 b]; la catégorie des motifs d'Artin s'identifie à celle des représentations à image finie de $\mathrm{Gal}(\bar{F}/F)$, et une représentation d'un groupe fini n'est pas nécessairement définie sur son corps de rationalité. Pourtant, c'est vrai des représentations automorphes (§3).

De même, les espaces $\mathrm{Hom}_{\mathcal{M}(F)}(M, M)$ et $\mathrm{Hom}_{Alg,\mathbb{Q}}(\pi, \pi)$ se comportent de façon différente. Avec la définition qu'on a proposée de $Alg_{\mathbb{Q}}(F)$ (§4.2), $\mathrm{Hom}_{Alg,\mathbb{Q}}(\pi, \pi)$ est toujours une algèbre de matrices, et un corps si π est $\mathbb{Q}$ -irréductible. Mais si M est un motif irréductible sur F (à coefficients rationnels), $\mathrm{Hom}_{\mathcal{M}(F)}(M, M)$ est en général une algèbre à division sur $\mathbb{Q}$.

On en déduit qu'il existe des représentations cuspidales irréductibles algébriques de $GL(n, \mathbb{A}_F)$, définies sur $\mathbb{Q}$, qui ne peuvent être associées à des motifs à coefficients rationnels.

Notons qu'il n'y a pas de problème - on obtient bien une correspondance entre motifs et représentations, au moins conjecturalement - si l'on étend les scalaires à $\bar{\mathbb{Q}}$: c'est la descente (des coefficients) qui fait obstacle.

4.3.1.- Considérons donc les catégories (tannakiennes, on l'espère) $\mathcal{M}_{\bar{\mathbb{Q}}}(F)$ et $Alg_{\bar{\mathbb{Q}}}(F)$. Elles sont munies d'actions naturelles de $\mathfrak{g} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$: c'est bien connu pour $\mathcal{M}_{\bar{\mathbb{Q}}}(F)$, et on le déduit de la construction 4.2 pour $Alg_{\bar{\mathbb{Q}}}(F)$. L'action de $\sigma \in \mathfrak{g}$ est σ-linéaire pour la structure $\bar{\mathbb{Q}}$-linéaire des catégories. On est alors amené à poser la question suivante:

4.3.2. Existe-t-il une équivalence ω de catégories tannakiennes sur $\bar{\mathbb{Q}}$ entre $\mathcal{M}_{\bar{\mathbb{Q}}}(F)$ et $Alg_{\bar{\mathbb{Q}}}(F)$ telle que $\omega({}^{\sigma}M) = \sigma(\omega(M))$ si $M \in \mathcal{M}_{\bar{\mathbb{Q}}}(F)$?.

Si c'était le cas, on pourrait aussi espérer une équivalence des foncteurs fibres (conjectural pour Alg !), d'où un isomorphisme entre le groupe de Galois $\mathcal{G}_{Alg}(F)$ et le groupe de Galois motivique sur $\bar{\mathbb{Q}}$.

On va reformuler la conjecture de façon un peu plus explicite. On considère seulement les représentations et motifs à coefficients dans $\bar{\mathbb{Q}}$.

Si $\pi \in Alg(F) = Alg_{\bar{\mathbb{Q}}}(F)$, on dira que π est pure de poids $w \in \mathbb{Z}$ si, pour toute place infinie v de F, la représentation $r_{T,v} = r_{L,v} | \ |_{\mathbb{C}}^{\frac{1-n}{2}}$ de $W_{\mathbb{C}}$ est somme de caractères $z^p(\bar{z})^q$ avec $p+q=w$ (cf. Def. 1.8 et §4.2.2). Rappelons qu'un motif est pur de poids w s'il intervient dans la cohomologie motivique de degré w. Un motif irréductible est pur.

D'après le Lemme 4.9 (vide infra), une représentation cuspidale algébrique est pure. Mentionnons plus généralement le fait suivant, qui ne sera pas utilisé ici puisque le corps des coefficients est $\bar{\mathbb{Q}}$. Soit $\pi = \overset{T}{\boxplus}_{\sigma} {}^{\sigma}\tau$ une représentation $\mathbb{Q}$-irréductible; τ est cuspidale algébrique, et σ parcourt $\mathfrak{g}/\mathfrak{g}_{\tau}$, où $\mathfrak{g}_{\tau}$ est le stabilisateur de τ. Si $\iota \in I$, et p_{ι} est le type de π en ι, on vérifie facilement que, pour τ pure de poids w:

$$\sum p_{\iota} + \sum p_{\bar{\iota}} = nw \qquad (\iota \text{ complexe})$$

$$2\sum p_{\iota} = nw \qquad (\iota \text{ réelle}),$$

où l'on a posé $\sum p_{\iota} = p_1 + \cdots + p_n$ si $p_{\iota} = (p_1, \ldots, p_n)$. Par conséquent, $\sum_{\iota \in I}(\sum p_{\iota}) = |I| \cdot \frac{n}{2} w$.

Si l'on admet la conjecture 3.7, on voit donc que le poids w est invariant par l'action de $\mathfrak{g}$ sur τ, puisque celle-ci permute les p_{ι}. Par conséquent, toutes les composantes irréductibles ${}^{\sigma}\tau$ d'une représentation $\mathbb{Q}$-irréductible π doivent avoir le même poids w. On dira alors que π est pure de poids w (pour les représentations dont les composantes cuspidales sont régulières, ceci est vrai inconditionnellement d'après le Thm. 3.13).

Soit $M \in \mathcal{M}_{\bar{\mathbb{Q}}}(F)$. Supposons qu'en fait $M \in M_E(F)$. Alors, si λ est une place de E, la cohomologie λ-adique $H_{\lambda}(M)$ reçoit une représentation de $\mathfrak{g}_F = \mathrm{Gal}(\bar{\mathbb{Q}}/F)$. Soit $L_v(M,s)$ la fonction L locale en une place finie v:

$$L_v(M,s) = \det(1 - \mathfrak{F}_v q_v^{-s} | H_{\lambda}(M)^{I_v})$$

où $\mathfrak{F}_v$ est un élément de Frobenius en v, et $I_v \subset \mathrm{Gal}(\bar{F}_v/F_v)$ le groupe d'inertie. Elle est à coefficients dans E, que l'on suppose toujours plongé dans $\bar{\mathbf{Q}}$. Elle ne dépend pas du choix de E, ni de λ si on fait une hypothèse de compatibilité stricte [19 b, (1.2.1)].

De même, si $\pi \in Alg(F)$, on sait définir les fonctions L locales $L_v(\pi, s)$ [29].

CONJECTURE 4.5. *Soit $\pi \in Alg(n, F)$ et supposons π cuspidale. Soit w le poids de π, $E \subset \bar{\mathbf{Q}}$ son corps de définition.*

Alors, il existe un motif irréductible de degré n, de poids w, $M \in \mathcal{M}_{E'}(F)$, E' étant une extension finie de E, tel que, pour toute place finie v de F:

$$L_v(\pi, s + \frac{1-n}{2}) = L_v(M, s) \tag{4.7}$$

(la fonction $L_v(\pi, s)$ est normalisée de la façon habituelle dans la théorie automorphe: équation fonctionnelle $s \mapsto 1 - s$, reliant π et sa contragrédiente $\tilde{\pi}$).

4.3.3.- La conjecture peut être précisée de la façon suivante, en considérant les divers avatars (= réalisations dans différentes théories cohomologiques) du motif:

• **Places archimédiennes.** Soit v une place infinie de F. D'après Serre [57 b], $L_v(M, s)$ est associée à une représentation $r_v(M)$ de W_{F_v}; soit $r_v^T(\pi)$ la représentation de W_{F_v} associée à π_v (§4.2.2). On doit avoir

$$r_v(M) \cong r_v^T(\pi). \tag{4.8}$$

• **Structure de Hodge.** En particulier, si $\iota : F \to \mathbf{C}$ est un plongement, $H_\iota(M) \otimes_{\mathbf{Q}} \mathbf{C}$ est muni d'une structure de Hodge, qui définit une représentation de $W_{\mathbf{C}} = \mathbf{C}^\times$. Cette représentation doit être isomorphe à $r_{\iota|\mathbf{C}^\times}$, où r_ι est la représentation r_v de W_{F_v} que l'on identifie à $W_{\mathbf{R}}$ ou $W_{\mathbf{C}}$ à l'aide de ι (§3.3).

• **Places finies ramifiées.** Pour toute place finie v de F, la représentation λ'-adique de $\mathfrak{g}_F$ (où λ' est une place de E') associée à M définit par restriction une représentation r_v de $D_v = \mathrm{Gal}(\bar{F}_v/F_v)$. Soit r_v^{ss} sa semi-simplifiée, une représentation de W_{F_v} [53, 4.2.2]. Par ailleurs, soit $r_T(\pi_v)$ la représentation de W'_{F_v} associée à π_v, normalisée à la

Tate (cf. conjecture 4.4): son corps de rationalité doit être, d'après cette conjecture, égal à E. Après une extension finie E'' du corps des coefficients, on devrait avoir

$$r_T(\pi_v) \otimes_{E''} E''_{\lambda''} \cong r_v^{ss} \otimes_{E'_{\lambda'}} E''_{\lambda''}, \tag{4.9}$$

λ'' étant une place de E'' au-dessus de λ', l'isomorphisme étant **entre représentations de W_{F_v}.**

4.3.4. Comportement fonctoriel, et vérifications.

Donnons tout d'abord un résultat portant sur la rationalité de $L(\pi, s)$. Revenons à la conjecture 4.3.2. Soit $\pi \in Alg_E(F)$, et soit $M \in \mathcal{M}_{E'}(F)$ (E' extension de E) associé à π. Si l'équivalence de catégories conjecturée est bien équivariante sous $\mathfrak{g} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, et si $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$, on doit avoir ${}^\sigma M \cong M$: en particulier, la fonction L de M devrait être à coefficients dans E. La propriété analogue est vraie dans le cas automorphe. Soit F un corps p-adique; rappelons que les fonctions $L(\pi, s)$, π étant une représentation admissible de $GL(n, F)$, sont définies comme p.g.c.d. d'intégrales

$$Z(\Phi, s + \frac{n-1}{2}, f) = \int_{G_n(F)} \Phi(g)\, |\det g|^{s+\frac{n-1}{2}}\, f(g)\, d^\times g, \tag{4.10}$$

[31, §1]; Φ est une fonction de Schwartz sur $M_v(F)$, f un coefficient de π.

Cela étant, on a le résultat suivant, qui implique en particulier que $L(\pi, s + \frac{1-n}{2})$ est une série de Dirichlet à coefficients dans E(*). Si $P \in \mathbb{C}[X]$ est un polynôme, notons ${}^\sigma P$ le polynôme obtenu en faisant agir $\sigma \in \mathrm{Aut}\,(\mathbb{C})$ sur les coefficients de E.

LEMME 4.6. *Soit π une représentation admissible irréductible de $GL(n, F)$. Ecrivons [31]:*

$$L(\pi, s + \frac{1-n}{2}) = P(q^{-s})^{-1} \text{ avec } P \in \mathbb{C}[X], \quad P(0) = 1.$$

Alors $L({}^\sigma\pi, s + \frac{1-n}{2}) = {}^\sigma P(q^{-s})^{-1}$.

Remarques. (1) La fonction L est celle de Langlands: équation fonctionnelle $s \mapsto 1 - s$, reliant π et $\tilde{\pi}$.

(2) Il devrait y avoir un résultat analogue pour les fonctions L de paires de [32 a]. Nous ne l'avons pas vérifié.

(*)Le résultat précis est plus fort: $L(\pi, s + \frac{1-n}{2})$ est un produit eulérien dont tous les facteurs sont définis sur E.

Démonstration. par étapes.

(a) Si $n = 1$, le lemme résulte de la théorie de Tate.

(b) Supposons π **supercuspidale**. Les coefficients $f(g)$ sont alors à support compact modulo le centre. L'intégrale (4.10) se réduit alors à une commission linéaire finie, dont les coefficients sont les intégrales des coefficients de π sur des translatés de $SL(n, F)$, d'intégrales de Tate associées au caractère central de π évaluées en $s + \frac{1-n}{2} + \frac{n-1}{2} = s$. Si l'on change π en ${}^\sigma\pi$, ces intégrales, ainsi que les coefficients, sont changés en leur tranformés par σ. Par ailleurs, $P(q^{-s})^{-1}$ est l'unique générateur $P \in \mathbb{C}[X]$ tel que $P(0) = 1$ de l'idéal fractionnaire I de $\mathbb{C}[q^{-s}, q^{-s}]$ engendré par les intégrales (4.10) avec s remplacé par $s + \frac{1-n}{2}$ ([31]). Si l'on remplace I par ${}^\sigma I$, P est remplacé par ${}^\sigma P$, q.e.d.

(c) Soit maintenant π de carré intégrable modulo le centre. On peut écrire π comme quotient de Langlands pour l'induction rationnelle (démonstration de la Prop. 3.2):

$$\pi = \mathcal{L}(b, \tau) \tag{4.11}$$

avec τ supercuspidale. On a alors ${}^\sigma\pi = \mathcal{L}(b, {}^\sigma\tau)$. Par ailleurs, τ étant une représentation de $GL(m)$, (4.11) s'écrit, pour l'induction **unitaire** :

$$\pi|\,|^{\frac{1-n}{2}} = L(b, \tau|\,|^{\frac{1-m}{2}}). \tag{4.12}$$

D'après Jacquet [31, Prop. 3.13], $L(\pi, s + \frac{1-n}{2})$ est alors un translaté, par un décalage entier de s dépendant du segment b, de $L(\tau, s + \frac{1-m}{2})$, d'où le résultat.

(d) Supposons maintenant π **générique** . On peut alors écrire (démonstration de la Prop. 3.2) π comme somme, pour l'induction rationnelle, de segments $\mathcal{L}(b, \tau_i)$; ${}^\sigma\pi$ est alors somme des $\mathcal{L}(b, {}^\sigma\tau_i)$; la fonction L de π est le produit de celle des $\mathcal{L}(b, \tau_i)$ [31], d'où le résultat.

(e) Enfin, si π est quelconque, on l'écrit (ibid) comme unique quotient d'une représentation générique ρ. On a $L(s, \pi) = L(s, \rho)$ ([31]). Par unicité, ${}^\sigma\pi$ s'obtient comme unique quotient de ${}^\sigma\rho$. Ceci ramène la démonstration au cas générique.

Passons maintenant au comportement fonctoriel de la Conjecture 4.5 par rapport au corps de base F. On dispose de foncteurs d'extension des scalaires $\mathcal{M}_{\bar{\mathbb{Q}}}(F) \to \mathcal{M}_{\bar{\mathbb{Q}}}(F_1)$ et de restriction des scalaires $\mathcal{M}_{\bar{\mathbb{Q}}}(F_1) \to \mathcal{M}_{\bar{\mathbb{Q}}}(F)$ pour une extension F_1/F de degré d de corps de nombres [19 b: 0.1.1].

Extension des scalaires: si $\pi \in Alg_{\bar{\mathbf{Q}}}(n,F)$ est associée à $M \in \mathcal{M}_{\bar{\mathbf{Q}}}(F)$, la représentation $\rho_{F/F_1}\pi = \pi_1$ (§1.1) est associée au motif $M_1 \in \mathcal{M}_{\bar{\mathbf{Q}}}(F_1)$ obtenu par extension des scalaires: l'extension des scalaires (motivique) correspond à la restriction du côté galoisien.

Restriction des scalaires : soit $\pi_1 \in Alg_{\bar{\mathbf{Q}}}(n,F_1)$ associée à M_1, $\pi = \iota_{F_1}^F \pi_1 \in Alg_{\bar{\mathbf{Q}}}(dn,F)$ la représentation automorphiquement induite (§1.1). Du point de vue motivique, cela correspond à l'opération de restriction des scalaires à la Weil [19: 0.1.1, 2.1]: π devrait être associée au motif $M \in \mathcal{M}_{\bar{\mathbf{Q}}}(F)$ obtenu par restriction des scalaires.

4.3.5.- Exemples.

Des approximations de la Conjecture 4.5 ont été démontrées dans les cas suivants:
(a) Formes modulaires paraboliques de poids $k \geq 2$ sur des sous-groupes de congruences de $SL(2,\mathbf{Z})$: il s'agit donc de représentation cuspidales π de $GL(2,\mathbf{A}_{\mathbf{Q}})$ dont le type à l'infini est de carré intégrable. Eichler, Shimura, Ihara, Deligne, Morita, Langlands et Carayol (cf. [19 a, 39 b, 17]) ont construit les systèmes de représentations ℓ-adiques associés à π; Deligne [19 b] décrit les différents avatars du "motif" M associé à π; enfin, tout récemment, Scholl [55] montre l'existence d'un motif sur $\mathbf{Q}$ (à coefficients dans $\mathbf{Q}$) au sens de Grothendieck (pour les correspondances définies par les cycles à équivalence rationnelle près) associé à toute la cohomologie parabolique de degré 1 des courbes modulaires.

(b) Formes modulaires paraboliques de poids 1: elles correspondent à des représentations de $GL(2,\mathbf{A}_{\mathbf{Q}})$ dont le type à l'infini correspond (pour la normalisation de Tate) à un couple de caractères $(\mathrm{sgn}^{\epsilon_1}, \mathrm{sgn}^{\epsilon_2})$ $W_{\mathbf{R}} \to \mathbf{R}^\times$ ($\epsilon_i = \pm 1$, $\epsilon_1\epsilon_2 = -1$). Deligne et Serre [22] leur associent des motifs d'Artin.

(c) Formes de Maass de valeur propre du laplacien $\lambda = \frac{1}{4}$: dans les articles [6 a,b] de Blasius, Ramakrishnan et l'auteur on leur associe des formes automorphes holomorphes sur $Gsp(4)$; on peut alors montrer l'algébricité des valeurs propres de Hecke [6 b, 7] et, modulo la solution des problèmes de L-indiscernabilité pour $GSp(4)$, leur associer des motifs d'Artin (= représentations galoisiennes) [8]. Voir les articles de Blasius et Harris dans ce volume.

(d) Caractères de Hecke algébriques χ **d'un corps** F **de type** CM. L'existence du motif $M = M(\chi)$ résulte alors des travaux de

Deligne, et en particulier de l'identification du groupe de Taniyama [38 f] et du groupe de Galois motivique associé à la catégorie tannakienne $(CM)_F$ des variétés abéliennes potentiellement de type CM [21]. Les motifs obtenus sont des objets de $\mathcal{M}_{HA}(F)$ (§4.1.1), non des motifs au sens de Grothendieck.

(e) Formes automorphes sur les variétés de Hilbert-Blumenthal. Dans ce cas, F est totalement réel, π une représentation cuspidales de $GL(2, \mathbb{A}_F)$ dont la composante à l'infini est de carré intégrable. Plutôt que le motif $M = M(\pi)$ associé à π, on avait d'abord construit une induite tensorielle convenable du système de représentations λ-adiques associé à π (Langlands [38 c], Brylinski-Labesse [15]); dans certains cas, le système de représentations λ-adiques lui-même avait été construit par Rogawski-Tunnell [44]. Tout récemment a été démontrée l'existence du système de représentations λ-adiques associé au motif $M(\pi)$ - et satisfaisant toutes les conditions de 4.3.3, y compris aux mauvaises places: cela résulte de la conjonction du travail de Wiles et Taylor [59, 54] et de celui de Blasius-Rogawski [9]. Voir l'exposé de Taylor dans ce volume.

4.3.6.- Problème.

Dans les cas (a, e), peut-on isoler les motifs sur F associés aux représentations $\mathbb{Q}$-irréductibles, précisant le résultat de Scholl? Peut-on même construire les motifs à coefficients associés aux représentations irréductibles (les obstructions décrites en 4.3.0 à associer des motifs dans $\mathcal{M}_E(F)$ aux représentations de $Alg_E(F)$ ne semblent pas apparaître dans ce cas)?

4.4. Conséquences du formalisme motivique.

Voici quelques conséquences, pour les représentations algébriques, du formalisme conjectural de 4.3.

4.4.1. Pureté.

Soit $\pi \in Alg_E^0(n, F)$. La Conjecture 4.5 a alors la conséquence suivante:

CONJECTURE 4.8. *Soit* $\pi \in Alg_E^0(n, F)$.

Alors il existe $w \in \mathbb{Z}$ *tel que, à toutes les places non-ramifiées de* π, $t_{\pi,v}^T$ *soit une matrice diagonale dont les valeurs propres* t_i *vérifient*

$$|t_i| = q_v^{w/2}.$$

Si $w \geq 0$, les t_i devraient être des entiers algébriques

($|z|$ désigne la valeur absolue usuelle sur $\mathbb{C}$: $|z| = |z|_{\mathbb{C}}^{\frac{1}{2}}$).

Comme on va le voir, le poids w est en fait défini par le type à l'infini de π. Il est probable que l'on peut alors démontrer, pour les représentations **régulières** telles que $w > 0$, l'**intégralité** des t_i à l'aide des méthodes du paragraphe 3.5. Il faudrait utiliser la cohomologie à coefficients entiers, cf. Harder [30 c].

La Conjecture 4.8 n'est, bien sûr, qu'une reformulation de la conjecture de Ramanujan 1.6.

Etudions maintenant la pureté aux places infinies . Dans ce cas, on peut démontrer le résultat suggéré par la Conjecture 4.5, et déjà utilisé dans le paragraphe 3.5:

LEMME 4.9 (LEMME DE PURETÉ). *Soit $\pi \in Alg^0(n)$. Alors il existe $w \in \mathbb{Z}$ tel que, pour toute place infinie v de f, la représentation $r_{T,v} = r_{L,v} \,|\;|_{\mathbb{C}}^{\frac{1-n}{2}}$ de $W_{\mathbb{C}}$ associée à π_v soit somme de caractères $z^p(\bar{z})^q$ avec $p + q = w$.*

En d'autres termes, $\pi \,|\;|^{\frac{1-n-w}{2}}$ est tempérée aux places infinies.

Démonstration : Supposons d'abord F totalement imaginaire. Soit v un place infinie. Puisque π_v est cuspidale, sa restriction à $SL(n, \mathbb{C})$ est unitaire; d'après Shalika [49], elle est générique; enfin, comme on l'a vu (**Remarque** après la Def. 1.8), son caractère infinitésimal est entier. Rappelons [35, 56 a] qu'une représentation irréductible τ de $Sl(n, \mathbb{C})$ est générique si et seulement si elle est **large** , c'est- à-dire si son annulateur est un idéal primitif minimal (pour son caractère infinitésimal) ou, ce qui est équivalent, si sa dimension de Gelfand-Kirillov est maximale [56 a, Thm. 6.2].

Si χ est un caractère unitaire de $\mathbb{C}^\times$, soit $J(\chi, m)$ la représentation $\chi(\det g)$ de $GL(m, \mathbb{C})$; soit, pour $\alpha \in \mathbb{R}$, $J(\chi, m, \alpha)$ la représentation $J(\chi, m) \,|\;|^{\alpha} \boxplus J(\chi, m) \,|\;|^{-\alpha}$ (induction unitaire). On sait alors que toute représentation unitaire de $GL(n, \mathbb{C})$ s'écrit alors comme somme (pour l'induction unitaire $\boxplus$) de représentations $J(\chi, m)$ ou $J(\chi, m, \alpha)$ avec $0 < \alpha < \frac{1}{2}$ ([56 b]; cf. [18 e, §7]).

Il est clair que les $J(\chi, m)$ pour $m > 1$ ne sont pas génériques. On en déduit alors [56 a, Cor. 6.6] qu'une représentation ainsi écrite comme induite par blocs ne peut être générique que si tous les blocs sont de dimension 1 ou 2 et les représentations induisantes des caractères abéliens χ ou des $J(\chi, 1, \alpha)$.

Identifions à $\mathbb{C}^2 \times \mathbb{C}^2$, de la façon usuelle, l'algèbre de Lie complexifiée d'un tore maximal de $GL(2, \mathbb{C})$. Si $\chi = z^p(\bar{z})^q$ $(p + q = 0)$ est un caractère unitaire, le caractère infinitésimal de $J(\chi, 1, \alpha)$ s'écrit alors $(p + \alpha, p - \alpha; q + \alpha, q - \alpha)$. La différence des deux premières composantes, 2α, ne peut être entière si $0 < \alpha < \frac{1}{2}$. Si le caractère infinitésimal de π_0 est entier, on voit enfin que π_v- tordue par une puissance $| \ |^s$ convenable du déterminant - doit être une somme de caractères abéliens unitaires, c'est-à-dire une représentation tempérée. Puisque π est cuspidale, $\pi | \ |^s$ est unitaire pour quelque $s \in \mathbb{R}$: celui-ci doit donc être le même pour toutes les places infinies, et ceci termine la démonstration.

Dans le cas général, on pourrait imiter cette démonstration en utilisant la classification du dual unitaire de $GL(n, \mathbb{R})$ ([56 b], cf. [18 c]). On peut aussi recourir au subterfuge suivant: soit F_1 une extension quadratique totalement imaginaire de F, π_1 le relèvement de π à $GL(n, \mathbb{A}_{F_1})$ par changement de base [2]. A torsion près par un caractère réel, π_1 est alors unitairement induite de cuspidale (§1.2.2). La démonstration précédente s'applique à π_1, et le Lemme de pureté pour π_1 implique le résultat pour π puisque les représentations tempérées se correspondent par changement de base [18 a].

Le résultat ci-dessus montre que toute représentation $\pi \in Alg^0(n)$ est pure, d'un certain poids w. En ce qui concerne les places finies, on a alors (notation de la Conjecture 4.8):

LEMME 4.10. *Soit $\pi \in Alg^0(n)$, pure de poids w.*

Alors, à toutes les places non ramifiées de π, les valeurs propres de $t^T_{\pi,v}$ satisfont

$$q_v^{\frac{w-1}{2}} < |t_i| < q_v^{\frac{w+1}{2}}.$$

Cela résulte de Jacquet-Shalika [33 b] ou, comme pour la démonstration du Lemme 4.9, de la classification du dual unitaire (Tadič [52]). Celle-ci donne même des estimées analogues aux places ramifiées de π .

Mentionnons une conséquence du Lemme de pureté, déjà remarquée en 3.4. Soit $\pi \in Alg^0(n)$, v une place complexe de F. Si $(p_v, q_v) = (p_\iota, p_{\bar{\iota}})$ sont les paramètres de π_v pour la classification rationnelle des représentations de $GL(n, F_v)$ (§3.3: ι, $\bar{\iota}$ sont les plongements définissant v), la représentation π_v est associée à des caractères $z^{p_i}\bar{z}^{q_i}$,

$p_\iota = (p_i)$, $p_{\bar\iota} = (q_i)$. A priori, le type à l'infini (algébrique) de π ne définit p_ι et $p_{\bar\iota}$ que modulo $\mathfrak{S}_n$. D'après le Lemme 4.9, on doit avoir $p_i + q_i = w$ pour tout i. On voit donc que le couple $(p_\iota, p_{\bar\iota})$ satisfaisant cette condition est bien défini modulo l'action **diagonale** de $\mathfrak{S}_n$ sur $\mathbf{Z}^n \times \mathbf{Z}^n$. En d'autres termes, les caractères $z^{p_i}\bar{z}^{q_i}$ sont bien définis à permutation près, donc aussi la représentation π_v. Utilisant un changement de base comme dans la démonstration du Lemme 4.9, on en déduit que si v est une place **réelle**, le relèvement de π_v à $GL(n,\mathbf{C})$ est bien défini. D'après les résultats de [18 a], ceci implique:

COROLLAIRE 4.11. *Soit $\pi \in Alg^0(n)$. Alors, pour toute place v de F, le caractère de π_v sur l'ensemble des carrés de $GL(n, F_v)$ est bien défini par le type à l'infini (algébrique) de π.*

4.4.3. Types à l'infini.

Les conjectures de 4.3 impliquent de fortes restrictions sur les types à l'infini des représentations algébriques.

Par exemple, soit $\pi \in Alg_{\mathbf{Q}}(n, F)$ et **supposons** que π est associée à un motif $M \in \mathcal{M}_{\mathbf{Q}}(F)$. Pour tout plongement $\iota : F \to \mathbf{C}$, considérons la décomposition de Hodge $H_\iota(M) \otimes \mathbf{C} = \oplus H_\iota(M)^{pq}$. Puisque celle-ci peut être obtenue à l'aide de la filtration de Hodge sur la cohomologie de De Rham, les nombres de Hodge $h_\iota^{pq}(M) = \dim H_\iota(M)^{pq}$ sont indépendants de ι.

L'analogue est vrai pour π:

PROPOSITION 4.12. *Supposons la Conjecture 3.7. Soit $h_\iota^{pq}(\pi)$ la multiplicité de $z^p(\bar{z})^q$ dans la représentation normalisée r_ι^T de $\mathbf{C}^\times$ associée à π. Si π est définie sur $\mathbf{Q}$, $h_\iota^{pq}(\pi)$ est indépendant de ι.*

COROLLAIRE 4.13. *Si $\pi \in Alg_{\mathbf{Q}}(n, F)$ est somme, dans $Alg(F)$, de représentations cuspidales régulières, $h_\iota^{pq}(\pi)$ est indépendant de ι.*

En effet, la Conjecture 3.7 est alors démontrée.

Demonstration de la Proposition 4.12: Supposons π $\mathbf{Q}$-irréductible. A cause du Lemme 4.9, on a $p + q = w$ pour tout caractère $z^p\bar{z}^q$ apparaissant dans r_ι^T, avec w indépendant de ι (voir aussi le §4.3.1). Il est alors facile de vérifier que $h_\iota^{pq} = \mathrm{mult}(p, p_\iota)$ (= le nombre d'occurences de p dans $p_\iota = (p_1, \ldots, p_n)$), que ι soit réelle ou complexe. Si π est $\mathbb{Q}$-rationnelle, la Conjecture 3.7 implique que $p_{\sigma\iota} = p_\iota$ pour tout $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Donc $h_\iota^{pq} = h_{\sigma\iota}^{pq}$.

4.5. Conjecture de Hasse-Weil et théorie réciproque pour les représentations algébriques.

4.5.1. Commençons par décrire la théorie réciproque pour les représentations automorphes de $GL(n, \mathbb{A}_F)$.

Soit π, τ des représentations admissibles de $GL(n, F)$ (**resp.** $GL(m, F)$) où F est un corps local. On leur associe des fonctions L de paires $L(\pi \times \tau, s)$. Si F est archimédien, $L(\pi \times \tau, s)$ est définie à l'aide de la classification de Langlands: π (resp. τ) est associée à un homomorphisme $r_\pi : W_F \to Gl(n, \mathbb{C})$ (resp.$r_\tau : W_F \to GL(m, \mathbb{C})$; $L(\pi \times \tau, s)$ est la fonction L associée à la représentation, de degré nm, $r_\pi \otimes r_\tau$ ([53]) (la correspondance est normalisée à la Langlands). Si F est p-adique, $L(\pi \times \tau, s)$ est définie dans [32 a].

A l'aide de l'équation fonctionnelle locale, on définit de plus des facteurs ϵ, $\epsilon(\pi \times \tau, \psi, s)$: ils dépendent d'un caractère additif ψ de F [32 a].

Soit alors F **global**, $\pi \in \mathrm{Aut}\,^0(n, F)$, $\tau \in \mathrm{Aut}\,^0(m, F)$ des représentations cuspidales. On pose

$$L(\pi \times \tau, s) = \prod_v L(\pi_v \times \tau_v, s) \tag{4.14}$$

(produit sur toutes les places). Ce produit est convergent pour Re $s \gg 0$ (si π et τ sont **unitaires**, il converge en fait pour Re $s > 1$).

Il admet un prolongement méromorphe à $\mathbb{C}$ tout entier (ceci est annoncé dans [32 a]); on peut aussi le déduire des travaux de Shahidi, cf. [2, démonstration de la Prop. 6.9]). D'après Jacquet (à paraître: voir aussi les notes de Jacquet dans ce volume), $L(\pi \times \tau, s)$ est holomorphe, à l'exception peut-être d'un pôle unique; on a un pôle simple en $s = 1$ si et seulement si $\pi \cong \tilde{\tau}$.

Soit ψ un caractère non-trivial de $F \backslash \mathbb{A}_F$, $\psi = \otimes_v \psi_v$. On pose

$$\epsilon(\pi \times \tau, s) = \prod_v \epsilon(\pi_v \times \tau_v, \psi_v, s). \tag{4.14}$$

Il ne dépend pas de ψ. On a alors l'équation fonctionnelle

$$L(\pi \times \tau, s) = \epsilon(\pi \times \tau, s) L(\tilde{\pi} \times \tilde{\tau}, 1 - s) \tag{4.15}$$

(voir [32 a], où ceci n'est pas complètement démontré; pour un argument reposant sur les travaux de Shahidi, voir encore [32 a, démonstration de la Prop. 6.9]).

Notons que le produit (4.13) est convergent pour Re $s \gg 0$ dès que π et τ sont des représentations admissibles et unitaires, à torsion près par un caractère du déterminant [10 a]. La **théorie réciproque pour** $GL(n)$ est alors l'assertion suivante (on n'a donné que l'assertion la plus faible: des variantes beaucoup plus fortes sont connues pour $GL(2)$ et $GL(3)$):

CONJECTURE 4.14. *Soit π une représentation admissible de $GL(n, \mathbb{A}_F)$. Supposons que, pour tout $m \leq n-1$ et $\tau \in Aut^0(m, F)$, les produits $L(\pi \times \tau, s)$ et $L(\tilde{\pi} \times \tau, s)$ sont convergents pour Re $s \gg 0$, qu'ils admettent un prolongement holomorphe au plan complexe, et vérifient l'équation fonctionnelle (4.15). Alors π est isomorphe à une représentation automorphe cuspidale de $GL(n, \mathbb{A}_F)$.*

Cette conjecture, d'après Jacquet, peut être démontré à l'aide des résultats de [32a,33a,b]. Nous l'admettrons (pour le cas des corps de fonctions, cf. Piatetski-Shapiro[40b]).

Il est naturel de se demander s'il existe une théorie réciproque dans le cadre de la catǵorie $Alg(F)$. Cela amène au problème suivant.

PROBLÈME 4.15. *Soit π une représentation admissible de $GL(n, \mathbb{A}_F)$, et supposons que $\pi_\infty = \otimes_{v \text{ infinie}} \pi_v$ est algébrique. Supposons que, pour tout $m \leq n-1$ et $\tau \in Alg^0(m, F)$, les produits $L(\pi \times \tau, s)$ et $L(\tilde{\pi} \times \tau, s)$ convergent pour* Res $\gg 0$, *admettent un prolongement holomorphe et l'équation fonctionelle (4.15); π est-elle alors isomorphe à une représentation de $Alg^0(n, F)$?*

Il serait intéressant d'étudier le problème dans le cas le plus simple: $n = 2$, F quadratique réel. Nous ne l'avons pas fait.

4.5.2. Relation avec la Conjecture de Hasse-Weil. Soit M un motif irréductible à coefficients dan $\bar{\mathbb{Q}}$. On a donc $M \in \mathcal{M}_E(F)$ pour quelque $E \subset \bar{\mathbb{Q}}$; supposant $\bar{\mathbb{Q}}$, et donc E, plongé dans $\mathbb{C}$, on en déduit la fonction $L(M, s)$. D'aprs̀ la Conjecture de Hasse-Weil, $L(M, s)$ devrait avoir un prolongement méromorphe au plan complexe; si M est irréductible et n'est pas une puissance du motif de Tate, on s'attend à ce que $L(M, s)$ soit holomorphe; enfin, elle devrait satisfaire une équation fonctionelle

$$L(M, s) = \epsilon(M, s) L(\check{M}, 1-s) \tag{4.16}$$

(noter que dans la correspondance de la Conjecture 4.5, $\check{M}$ n'est pas associé à $\tilde{\pi}$ mais à $\check{\pi}$ (Def. 1.11); ceci est compatible avec la translation introduite dans l'identité des fonctions L).

La Conjecture 4.5 ne précise la question 4.3.2 que dans un sens. L'hypothèse de Langlands, selon laquelle toutes les fonctions L de la géométrie algébrique devraient être automorphes, devrait s'exprimer par une correspondance en sens inverse:

QUESTION 4.16. *Soit $M \in \mathcal{M}_E(F)$ un motif absolument irréductible de rang n. Existe-t-il une représentation $\pi \in Alg_E(n, F)$, cuspidale, telle que M soit associé à π au sens de la Conjecture 4.5?*

Notons qu'à la différence de la Conjecture 4.5 qu'on peut déjà aborder, au moin sous une forme affaiblie, ceci paraît absolument hors d'atteinte: un cas particulier de ce problème est la conjecture de Taniyama-Weil! En fait, si l'on suppose résolu le Problème 4.15 et démontrées la conjecture de réciprocité locale de Langlands et la Conjecture 4.5, la question 4.16 est essentiellement équivalente à la conjecture (apparemment plus faible) de Hasse-Weil (donné le motif M, on lui associe la représentation admissible $\pi = \otimes_v \pi_v$ de $GL(n, \mathbb{A}_F)$ définie par ses avatars locaux; si l'on admet la conjecture de Hasse-Weil et la Conjecture 4.5, les fonctions $L(s, \pi \times \tau)$ auraient, pour toute représentation τ algébrique, cuspidale, de $GL(m, \mathbb{A})$ $(m < n)$, un prolongement holomorphe et une équation fonctionelle; d'après le Problème 4.15, s'il admet une solution positive, on en déduirait que π est alors cuspidale). Bien sûr, comme le fait remarquer Langlands, on ne sait démontrer **aucun** cas de la conjecture de Hasse-Weil sans démontrer d'abord l'existence de π.

5. LE CAS DES REPRÉSENTATIONS AUTODUALES

5.1. Le but de ce paragraphe est de montrer que dans le cas des représentations algébriques **autoduales** de $GL(n)$ sur un corps totalement réel, on peut démontrer une approximation de la Conjecture 4.5. Plus précisément, on peut associer des systèmes compatibles de représentations λ-adiques à de telles représentations, supposées de plus régulières et satisfaisant à certaines conditions locales. Ceci implique, par exemple, la conjecture de Ramanujan à presque toutes les places.

La démonstration n'est ici qu'esquissée: les détails seront donnés dans un article ultérieur. Pour simplifier, on s'est même limité au corps des rationnels. La démonstration repose entièrement sur plusieurs résultats fondamentaux de Kottwitz, en particulier: sa démonstration de la conjecture de Langlands pour certaines variétés de Shimura qui sont des variétés de modules (non publié, mais voir son article dans ce volume; une autre démonstration a été donnée par Reimann et Zink [41]); et son étude des variétés de Shimura associées aux algèbres à division munies d'involutions de seconde espèce [36].

Rappelons (§1, 2) que si $F/\mathbb{Q}$ est une extension quadratique, on sait associer à π une représentation π_F de $GL(n, \mathbb{A}_F)$ par changement de base ("restriction automorphe"). Si v est une place finie de F, on note $\mathfrak{F}_v$ un élément de Frobenius dans $\mathrm{Gal}(\bar{F}/F)$.

THEOREME 5.1. *Soit π une représentation cuspidale, algébrique, régulière et autoduale ($\pi \cong \tilde{\pi}$) de $GL(n, \mathbb{A}_{\mathbb{Q}})$. Supposons:*

(i) Si $4 \nmid n$, il existe p_0 tel que π_{p_0} est de carré intégrable.

(ii) Si $4|n$, il existe $p_0 \neq p_1$ tel que π_{p_0}, est de carré intégrable.

Soit F un corps quadratique imaginaire, déployé en p_0 et en p_1 si $4|n$. Il existe alors:

(iii) Un corps de nombres $E/\mathbb{Q}$.

(iv) Un ensemble fini S de nombres premiers.

(v) Un entier $a(\pi) > 0$.

(vi) Un système compatible $W = (W_\lambda, r_\lambda)$ de représentations λ-adiques de $\mathrm{Gal}(\bar{F}/F)$ non-ramifiées en-dehors des places divisant S, λ décrivant les places de E, pur de poids $n-1$, tel que, si $p \notin S$ et $\lambda \nmid p$, on ait pour toute place v de F divisant p:

$$\text{trace } r_\lambda(\mathfrak{F}_v^m) = a(\pi)\text{trace } (t^T_{\pi_{F,v}})^m q_v^{m(n-1)}$$

pour tout $m \geq 0$.

COROLLAIRE 5.2. *Soit π une représentation vérifiant les hypothèses du Théorème 5.1. Alors*

(i) Pour $p \notin S$, π_p est tempérée, ou, ce qui revient au-même, $t^T_{\pi,p}$ a des valeurs propres de valeur absolue complexe $p^{\frac{n-1}{2}}$.

(ii) Pour $p \notin S$, $p^{n-1} t^T_{\pi,p} = p^{\frac{n-1}{2}} t_{\pi,p}$ a pour valeurs propres des entiers algébriques.

Cela résulte en effet, aux normalisations indiquées près, des propriétés analogues, connues, des systèmes compatibles de représentations ℓ-adiques (... pour π_E: les formules décrivant le changement de base l'impliquent alors pour π).

5.2. Remarques.

5.2.1. La condition d'autodualité ($\pi \cong \tilde{\pi}$) est quelque peu étrange puisque la bonne dualité dans Alg est plutôt $\pi \mapsto \check{\pi}$. La démonstration devrait s'étendre au cas où $\pi \cong \tilde{\pi} \otimes \chi$ pour quelque caractère algébrique χ. Je n'ai pas vérifié les détails: cela sera étudié dans l'article final.

5.2.2. On peut utiliser la méthode de démonstration du Théorème 5.1 (voir §5.3) pour associer à certaines représentations algébriques singulières de $GL(2, \mathbb{A}_{\mathbb{Q}})$ des représentations automorphes d'un groupe unitaire dont la composante à l'infini est une limite de série discrète. Cela permet de retrouver par exemple, pour les formes de Maass de valeur propre $\lambda = \frac{1}{4}$ satisfaisant certaines conditions de ramification, les résultats d'algébricité démontrés dans [6a,b,7] à l'aide de $GSp(4)$.

5.2.3. La constante $a(\pi)$ devrait bien sûr être égale à 1. Il est possible que cela résulte de l'étude détaillée de la L-indiscernabilité pour les groupes unitaires introduits en 5.3: plus précisément, celle-ci devrait montrer que la représentation τ du Lemme 5.3 intervient avec multiplicité 1, cela impliquant que $a(\pi) = 1$.

5.2.4. Terminons en remarquant que l'on devrait, bien sûr, pouvoir se débarasser des conditions ancillaires (i - ii). Cela nécessiterait cependant l'utilisation - à la place des groupes unitaires décrits dans le paragraphe 4.3 - du groupe unitaire quasi-déployé, et la démonstration des résultats analogues nécessiterait la stabilisation de la formule des traces (cf. Rogawski [43 b] pour $U(2,1)$) ainsi que l'étude des variétés de Shimura de ces groupes-problèmes qui restent difficiles.

En revanche, on ne voit pas, même en principe, comment associer des objets “motiviques” à des représentations ne satisfaisant aucune condition d'autodualité. En fait, il semble que toutes les représentations de $GL(n, F)$ - pour F totalement réel - que l'on puisse associer, par les applications accessibles du principe de fonctorialité, à des “motifs” apparaissant dans la cohomologie d'intersection des variétés de Shimura, vérifient de fortes conditions d'autodualité. J'espère revenir sur cette question énigmatique dans un article ultérieur.

5.3. Esquisse de démonstration.

5.3.1. La première étape, on l'a vu, consiste à associer à π son changement de base π_F, F étant un corps quadratique imaginaire déployé en p_0 et p_1.

5.3.2. Soit alors D un algèbre à division de degré n^2 sur F, déployée à toutes les places de F différentes des deux places V_0' et v_0'' divisant p_0. L'algèbre D admet alors une involution de seconde espèce, i.e. un antiautomorphisme induisant sur F l'élé ment non trivial σ de $\mathrm{Gal}(F, \mathbb{Q})$. On la note $*$.

Si $*$ est fixée, soit U le groupe algébrique sur $\mathbb{Q}$ défini par

$$U(R) = \{x \in D \otimes_{\mathbb{Q}} R : xx^* = 1\}, \tag{5.1}$$

R étant une $\mathbb{Q}$-algèbre. Si l'on remplace $*$ par une autre involution de seconde espèce, on remplace le groupe V par une de ses formes intérieures. On peut choisir $*$ de façon que:

(5.2) $U(\mathbb{R}) \cong U(n, 1)$, le groupe unitaire d'une forme hermitienne de signature $(n, 1)$.

(5.3) Si $4 \nmid n$, $U \times_{\mathbb{Q}} \mathbb{Q}_p$ est quasi-déployé en toute place finie $p \neq p_0, p_1$.

Si $4|n$, $U \times_{\mathbb{Q}} \mathbb{Q}_p$ est quasi-déployé en toute place finie $p \neq p_0, p_1$.

On supposera dorénavant $*$ ainsi choisie.

L'idée de la démonstration est d'associer à π une représentation automorphe de $U(\mathbb{A})$ ($\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$) qui "interviendra" dans la cohomologie d'une variété de Shimura pour un groupe G déduit de U, suivant le diagramme suivant:

$$\begin{array}{cccccc} D^\times & \text{——} & & GL(n)/F & & \\ | & & \tau_F & \text{——} & | \text{——} & \pi_F \\ U & & \tau & GL(n)/\mathbb{Q} & & \pi \end{array}$$

où les flèches verticales sont des changements de base, et τ_F est associée à π_F par la correspondance de Jacquet-Langlands.

On commence par construire la représentation automorphe τ_F de $D^\times(\mathbb{A}_F)$ associée à π_F. Son existence résulte de la comparaison de formules des traces démontrée dans [2, Ch. II] et du travail de M.-F. Vignéras [55]. La représentation $\tau_F = \otimes_v \tau_{F,v}$ jouit des propriétés suivantes:

(5.4) Si $v|p_0$ (de sorte que $D_v \cong M_n(F_v)$), $\tau_{F_v} \cong \pi_{F_v}$. En particulier, si $4|n$, $\tau_{F,v}$ appartient à la série discrète quand v est l'une des places divisant p_1.

(5.5) En $v = v_0'$ ou v_0'', $\tau_{F,v}$ est la représentation de la série discrète de $D^\times(F_v)$ associée par la correspondance de Jacquet Langlands [43 a, 20] à $\pi_{F,v}$.

(5.6) La représentation τ_F est *fortement de multiplicité 1* dans $L^2(D^\times(F)\backslash D^\times(\mathbb{A}_F),\omega)$, ω désignant le caractère central de π_F ou τ_F: cela signifie que toute sous-représentation irréductible de L^2, coïncidant avec τ_F à presque toutes les places, est égale à une unique sous-représentation de L^2 (cf. Thm 1.1 pour $GL(n)$).

(5.7) Notons σ l'action galoisienne sur $D^\times(F) = U(F)$ définie par la $\mathbb{Q}$ -forme U: on a donc $\sigma g = (g^*)^{-1}$, $g \in D^\times(F)$. Si ${}^\sigma\tau_F$ désigne la transformée de τ_F par σ, on a alors $\tau_F \cong {}^\sigma\tau_F$.

5.3.3. L'étape suivante consiste à descendre, par changement de base, de $\tau_F \cong {}^\sigma\tau_F$ à une représentation τ de $U(\mathbb{A})$. On utilise la comparaison de la formule des traces pour $L^2(D^\times(F)\backslash D^\times(\mathbb{A}_F),\omega)$, tordue par l'action galoisienne de σ, et de la formule des traces pour $L^2(U(\mathbb{Q})\backslash U(\mathbb{A}),\eta)$; η est l'unique caractère du centre $Z_{\mathbb{A}}$ de $U(\mathbb{A})$, isomorphe au noyau $\mathbb{A}^\times(F)_1$ de l'homomorphisme norme: $\mathbb{A}^\times(F) \to \mathbb{A}^\times(\mathbb{Q})$, et tel que $\omega(z) = \eta(z/z^\sigma)$ pour z appartenant au centre $\mathbb{A}^\times(F)$ de $D^\times(\mathbb{A}_F)$ (il résulte de la définition de ω qu'il se factorise ainsi).

En général, bien sûr, le changement de base pour un groupe arbitraire est un problème difficile et non résolu, à cause de la L-indiscernabilité. Dans le cas qui nous intéresse, cependant, Kottwitz a remarqué, à la suite de Rapoport et Zink, que la formule des traces pour $U(\mathbb{Q})$ s'exprimait directement en termes d'intégrales orbitales stables [36 a, §4: Pseudo-stabilization of the trace formula for G]. Kottwitz montre qu'une formule analogue est vraie pour la formule des traces **tordue** correspondante [36 a]. On en déduit que, pour pouvoir comparer les formules des traces, il suffit de pouvoir comparer les intégrales orbitales **stables** (au sens de [36 c], ainsi que de l'analogue tordu [36 b]) de fonctions C^∞ à support compact sur $U(\mathbb{A})$ et $D^\times(\mathbb{A}_F)$.

Le résultat obtenu par descente est alors le suivant:

LEMME 5.3. *Il existe une représentation irréductible $\tau = \tau_\infty \otimes \otimes_p \tau_p$ de $U(\mathbb{A})$, intervenant dans $L^2(U(\mathbb{Q})\backslash U(\mathbb{A}),\eta)$ et ayant les propriétés suivantes:*

(i) τ_∞ est l'un des éléments d'un L-paquet de séries discrètes de $U(\mathbb{R}) \cong U(n-1,1)$ associé par changement de base [18 a] *à $\tau_{F,\infty}$.*

(ii) En presque tout p, τ_p est la représentation non ramifiée de $U(\mathbb{Q}_p)$ associée à la représentation non-ramifiée $\tau_{F,p}$ de $D^\times(F_p) =$

$\prod_{v|p} D^\times(F_v)$.

Précisons seulement le sens des assertions du Lemme. Pour (i), rappelons que les L-paquets de représentations irréductibles d'un groupe réel H sont paramétrés par les homomorphismes $W_{\mathbb{R}} \to {}^LH = {}^LH^0 \times W_{\mathbb{R}}$ commutant avec la projection sur $W_{\mathbb{R}}$; il en est de même pour un groupe complexe, si l'on remplace $W_{\mathbb{R}}$ par $W_{\mathbb{C}}$. Le groupe réel $U(n-1,1) \cong U \times_{\mathbb{Q}} \mathbb{R}$ donne, par extension des scalaires, le groupe complexe $GL(n,\mathbb{C}) \cong D^\times \times_F \mathbb{C}$. On vérifie alors que la représentation $\tau_{F,\infty}$ de $GL(n,\mathbb{C})$ est associée à un homomorphisme $W_{\mathbb{C}} \to {}^L(D^\times \times_F \mathbb{C}) \cong GL(n,\mathbb{C}) \times W_{\mathbb{C}}$ qui provient, par restriction, d'un homomorphisme $W_{\mathbb{R}} \to {}^L(U \times_{\mathbb{R}} \mathbb{R}) \cong GL(n,\mathbb{C}) \times W_{\mathbb{R}}$. La représentation τ_∞ est l'un des éléments du L-paquet associé à cet homomorphisme.

Pour (ii), notons qu'en presque tout p, le groupe $U_p = U \times_{\mathbb{Q}} \mathbb{Q}_p$ est non-ramifié; on a, suivant que p se déploie ou non dans F, $U_p \times_{\mathbb{Q}_p} F_p \cong D^\times_{v'} \times D^\times_{v''}$ (p se déploie en v', v'') ou $U_p \times_{\mathbb{Q}_p} F_p \cong D^\times_v$ (p reste inerte, d'où la place v). Dans les deux cas, il y a un homomorphisme des algèbres de Hecke (relatives à un compact hyperspécial) $b : \mathcal{H}(U_p \times_{\mathbb{Q}_p} F_p) \to \mathcal{H}(U_p)$. La correspondance mentionnée dans (ii) est la correspondance duale entre représentations non ramifiées.

A cause du phénomène de multiplicité 1 fort (5.6), le Lemme 5.3 peut se démontrer en ne comparant les termes géométriques des deux formules des traces que pour les éléments **réguliers**. On a alors besoin de deux résultats:

(5.8) Le *lemme fondamental* comparant les intégrales orbitales stables (resp. les intégrales orbitales tordues stables) des fonctions des algèbres de Hecke associées aux éléments réguliers de $U(\mathbb{Q}_p)$ (resp. $D^\times(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$). Nous en donnerons la démonstration dans un article ultérieur [18f](*)

(5.9) Soit $\tau_{F,\infty}$ le facteur de τ_F en $F_\infty = \mathbb{C}$. Cette représentation est "σ -discrète", i.e., c'est un point isolé du dual tempéré σ-stable de $GL(n,\mathbb{C})$ (cf. [2, §I.2]). Le théorème de Paley-Wiener tordu (cf. [2, §I.7] quand σ est la conjugaison de $GL(n,\mathbb{C})$ par rapport à $GL(n,\mathbb{R})$) a été démontré dans ce cas par Delorme: il permet de construire un pseudo-coefficient tordu ϕ (cf. [2, Cor. I.2.10]) de $\tau_{F,\infty}$. On peut alors montrer, à l'aide de résultats de Bouaziz et Shelstad, que les

(*)Une démonstration erronnée a été publiée par Flicker.

intégrales orbitales tordues stables de ϕ sont égales aux intégrales orbitales stables d'un pseudo-coefficient du L-paquet de τ_∞.

5.3.4. La fin de l'argument repose sur les travaux de Kottwitz sur les représentations λ -adiques associées aux groupes de similitudes unitaires issus des involutions de seconde espèce. Pour ceci, on remplace le groupe U par le groupe G défini, avec les notations de (5.1), par

$$G(R) = \{x \in D \underset{\mathbf{Q}}{\otimes} R : xx^* \in R^\times\}. \tag{5.10}$$

C'est une extension de U par $GL(1)/\mathbf{Q}$. Des arguments simples (cf. Labesse-Schwermer[37]) montrent que τ est l'une des composantes d'une représentation $\tilde{\tau}$ de $G(\mathbf{A})$ restreinte à $U(\mathbf{A})$. Kottwitz [36 a] démontre alors l'existence d'un système compatible de représentations λ -adiques vérifiant les conditions du Théorème 5.1 (vi).

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Shimura Varieties and λ-adic Representations

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The Hasse-Weil zeta functions of Shimura varieties have been studied intensively, especially for GL_2 and its inner forms, beginning with the work of Eichler, Shimura, Kuga, Sato and Ihara. At that time, however, it must have been difficult to imagine the precise form the zeta function might take for more general Shimura varieties. After introducing L-groups and the principle of functoriality into the theory of automorphic forms, Langlands **[L3]**, **[L5]**, **[L6]** was able to predict the contribution of tempered automorphic representations to the zeta function. However, non-tempered automorphic representations and their contribution remained mysterious. Arthur **[A1]**, **[A3]**, **[A4]** realized that with the introduction of the group SL_2 (the same SL_2, so to speak, as the one giving the Lefschetz decomposition of the cohomology of smooth projective varieties), the theory of non-tempered automorphic representations should look almost like the theory in the tempered case. This suggests that Langlands's predictions ought to remain valid in the non-tempered case, once the group SL_2 is brought into play. There are two ways to check that this idea is reasonable. One is to verify that it is consistent with what one knows about continuous cohomology, as has been done by Arthur **[A3]**. The other is to verify that it is consistent with what one knows about ℓ-adic cohomology, as we will do in Part II of this article.

In fact this article has several goals. The first is to give a conjectural formula for the number of points modulo p on a Shimura variety (more generally we allow Hecke operators away from p as well as local systems). After a preliminary construction in §2, the conjecture is stated in §3; it is a direct descendant of a conjecture of Langlands **[L2]**. Some useful lemmas are given in §§5, 6.

The second goal is to give a conjectural stabilization of the conjectural expression (3.1) for the number of points modulo p. This is begun in §4 and is continued in §7, which gives a detailed discussion

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of the relevant functions on the endoscopic groups and culminates in the desired stabilization (Theorem 7.2).

This stabilization, together with the stable trace formulas for the endoscopic groups, expresses the number of points modulo p in terms of automorphic representations on the endoscopic groups (see formula (10.1)). In order to destabilize (10.1) it is necessary to discuss the relevant functions on the endoscopic groups again, this time from the point of view of their stable character values rather than their stable orbital integrals. This is the content of §9 (it is interesting to note the symmetry between the formulas in §7 and the "dual" formulas in §9). It is also necessary to review some of Arthur's ideas; we do this in §8, which will undoubtedly overlap with **[A4]**. In §10 we use §§8,9 to carry out the destabilization of (10.1), obtaining (conjecturally) the formula (10.5). This formula suggests a precise generalization of Langlands's prediction, which is the last topic discussed in §10.

The last goal is to motivate the conjecture in §3 by discussing a simple example: the moduli space of principally polarized abelian varieties of dimension g with level N structure. This is the purpose of §§11, 12. The approach taken here differs slightly from the one suggested by Langlands **[L2]**, which has reappeared in updated form in work of Langlands–Rapoport **[L-R]** and has been proved in certain PEL cases by Zink **[Z]** and Reimann-Zink **[R-Z]**.

In this article we will follow Deligne's sign conventions **[D]**.

Part I. Points modulo p on Shimura varieties

§1. Statement of problem

Let G be a connected reductive group over $\mathbb{Q}$ and let X_∞ be a $G(\mathbb{R})$-conjugacy class of homomorphisms

$$h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$$

satisfying conditions (2.1.1.1)-(2.1.1.3) of **[D]**. We assume further that the derived group G_{der} of G is simply connected and that the maximal $\mathbb{Q}$-split torus in the center of G coincides with the maximal $\mathbb{R}$-split torus in the center of G. For any sufficiently small compact open subgroup $K \subset G(\mathbb{A}_f)$ we get a smooth Shimura variety S_K over $\mathbb{C}$, whose complex points are given by

$$S_K(\mathbb{C}) = G(\mathbb{Q})\backslash(X_\infty \times (G(\mathbb{A}_f)/K)).$$

For $h \in X_\infty$ we write μ_h for the restriction of $h_{\mathbb{C}} : (R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} \to G_{\mathbb{C}}$ to the first factor of $(R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$ (the two factors correspond to the two $\mathbb{R}$-algebra homomorphisms $\mathbb{C} \to \mathbb{C}$, and we order the two factors so that the first one corresponds to the identity map from $\mathbb{C}$ to $\mathbb{C}$). The $G(\mathbb{C})$-conjugacy class of μ_h is well-defined, and its field of definition, a number field E contained in $\mathbb{C}$, is called the reflex field. It is known that there is a canonical model of S_K over E.

Let L be a number field and let ξ be a finite dimensional representation of G on an L-vector space. Then ξ gives rise to a local system $\mathcal{F}$ of L-vector spaces on $S_K(\mathbb{C})$. For any finite place λ of L the local system $\mathcal{F} \otimes L_\lambda$ comes from a smooth L_λ-sheaf $\mathcal{F}_\lambda$ on S_K over E.

Let $\overline{S}_K$ be the Baily-Borel-Satake compactification of S_K; it too has a canonical model over E. Consider the intersection cohomology groups

$$W^i = IH^i(\overline{S}_K(\mathbb{C}), \mathcal{F}).$$

The Hecke algebra $\mathcal{H}_L$ of locally constant K-bi-invariant L-valued functions on $G(\mathbb{A}_f)$ acts on W^i. For any finite place λ of L the λ-adic vector space $W^i \otimes_L L_\lambda$ has algebraic meaning and carries an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ commuting with the action of $\mathcal{H}_L$. One would like to understand these commuting actions.

Let p be a rational prime. Assume that G and K are unramified at p. This means simply that G is quasi-split over $\mathbb{Q}_p$ and split over an unramified extension of $\mathbb{Q}_p$, and that K is of the form $K^p \cdot K_p$, where K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$ and $K_p = G(\mathbb{Z}_p)$ for some extension of G to a flat group scheme over $\mathbb{Z}_p$ with connected reductive geometric fibers. Note that E is necessarily unramified at p. Let $\mathfrak{p}$ be a place of E lying over p. One expects that for every finite place λ of L not lying over p, the representations $W^i \otimes L_\lambda$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_{\mathfrak{p}})$ are unramified. Granting this, a first step in understanding the commuting representations of $\mathcal{H}_L$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on $W^i \otimes L_\lambda$ would be to find a suitable expression for the trace of $f \times \Phi_{\mathfrak{p}}^j$ on the virtual representation

$$W_\lambda := \oplus_{i=0}^{2 \dim S_K} (-1)^i W^i \otimes L_\lambda,$$

where $f \in \mathcal{H}_L$, $\Phi_{\mathfrak{p}}$ denotes a geometric Frobenius element of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_{\mathfrak{p}})$, and j is a positive integer. Complications arise unless f is of the form $f^p f_{K_p}$, where f^p is a function on $G(\mathbb{A}_f^p)$ and f_{K_p}

denotes the characteristic function of K_p in $G(\mathbb{Q}_p)$, divided by the measure of K_p; from now on we assume that f has this special form.

One expects to have an expression for $\operatorname{tr}(f \times \Phi_{\mathfrak{p}}^j; W_\lambda)$ resembling Arthur's formula (Thm. 6.1 of [**A2**]) for $\operatorname{tr}(f; W_\lambda)$. In particular the expression should start with a sum over conjugacy classes of parabolic subgroups P of G. Our purpose here is simply to state a conjectural expression for the summand indexed by $P = G$. In other words we are leaving aside the contribution of the boundary components of $\overline{S}_K$. If G_{der} is anisotropic over $\mathbb{Q}$, then there are no other summands, and the conjecture is complete; of course $\overline{S}_K = S_K$ in this case, and intersection cohomology is just ordinary cohomology. The conjecture is stated in §3, but first we need some preparation.

§2. Construction of an invariant $\alpha(\gamma_0; \gamma, \delta)$

We retain the notation of §1. In this section we construct an invariant $\alpha(\gamma_0; \gamma, \delta)$ that is needed in order to formulate the conjectural expression for the contribution of $P = G$ to $\operatorname{tr}(f \times \Phi_{\mathfrak{p}}^j; W_\lambda)$.

Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ containing $E_{\mathfrak{p}}$. As an algebraic closure $\overline{\mathbb{Q}}_\infty$ of $\mathbb{Q}_\infty = \mathbb{R}$ we take $\mathbb{C}$. Then for $v = p, \infty$ we have embeddings $E \to \overline{\mathbb{Q}}_v$. For every place v of $\mathbb{Q}$ other than p, ∞ we choose any algebraic closure $\overline{\mathbb{Q}}_v$ of $\mathbb{Q}_v$. We also choose an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing E and for each place v of $\mathbb{Q}$ an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$. For $v = p, \infty$ we assume that $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ extends $E \to \overline{\mathbb{Q}}_v$. We denote by Γ (resp. $\Gamma(v)$) the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$ (resp. $\overline{\mathbb{Q}}_v/\mathbb{Q}_v$). Our choice of embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ allows us to regard each $\Gamma(v)$ as a subgroup of Γ.

Recall that we are considering the j-th power of $\Phi_{\mathfrak{p}}$. Let F denote the unramified extension of degree j of $E_{\mathfrak{p}}$ in $\overline{\mathbb{Q}}_p$. Since the hypotheses on p made in §1 guarantee that $E_{\mathfrak{p}}/\mathbb{Q}_p$ is unramified, the extension $F/\mathbb{Q}_p$ is also unramified. We write $\mathbb{Q}_p^{\mathrm{un}}$ for the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, and L for the completion of $\mathbb{Q}_p^{\mathrm{un}}$. We write σ for the Frobenius element of $\operatorname{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$ and also for the corresponding automorphism of L over $\mathbb{Q}_p$. Reusing the symbol L should not cause too much confusion, since the field denoted by L in §1 does not appear in §2.

What is the nature of the elements γ_0, γ, δ entering into the invariant $\alpha(\gamma_0; \gamma, \delta)$? The element γ_0 is any semisimple element of $G(\mathbb{Q})$ that is elliptic in $G(\mathbb{R})$ (this means that γ_0 is contained in $T(\mathbb{R})$ for some elliptic maximal $\mathbb{R}$-torus T of G). We write I_0 for the centralizer of γ_0 in G. Since γ_0 is semisimple and G_{der} is simply connected, the $\mathbf{Q}$-group I_0 is connected and reductive.

We choose a complex group $\widehat{G}$ of dual reductive type to G (so that the L-group of G is formed using $\widehat{G}$) and write $Z(\widehat{G})$ for its center. The Galois group Γ and its subgroups $\Gamma(v)$ act on $Z(\widehat{G})$. The same remarks apply to $I_0, \widehat{I}_0$ and $Z(\widehat{I}_0)$.

The element γ is an element of $G(\mathbf{A}_f^p)$. For a place v of $\mathbf{Q}$ other than p, ∞ we write γ_v for the v-component of γ. We assume that for all such v the elements γ_0 and γ_v are conjugate under $G(\overline{\mathbf{Q}}_v)$.

The element δ is an element of $G(F)$. We make two assumptions on δ. The first is that the norm $N\delta$ of δ is conjugate to γ_0 under $G(\overline{\mathbf{Q}}_p)$, where $N\delta$ is defined by

$$N\delta = \delta \cdot \sigma(\delta) \dots \sigma^{r-1}(\delta) \in G(F)$$

with $r = [F : \mathbf{Q}_p]$. Before stating the second assumption we need to recall a definition from **[K5]**. The set $B(G_{\mathbf{Q}_p})$ is defined to be the set of σ-conjugacy classes in $G(L)$. In §6 below we show that there is a canonical map

$$B(G_{\mathbf{Q}_p}) \to X^*(Z(\widehat{G})^{\Gamma(p)}).$$

The second assumption on δ is that the image in $X^*(Z(\widehat{G})^{\Gamma(p)})$ of the σ-conjugacy class of δ is equal to the restriction of $-\mu_1 \in X^*(Z(\widehat{G}))$ to $Z(\widehat{G})^{\Gamma(p)}$, where μ_1 is obtained from X_∞ as follows.

Choose $h \in X_\infty$. As before we get $\mu_h : \mathbf{G}_m \to G_{\mathbf{C}}$. Using $\overline{\mathbf{Q}} \to \mathbf{C}$, we get a well-defined $G(\overline{\mathbf{Q}})$-conjugacy class of homomorphisms $\mu_0 : \mathbf{G}_m \to G_{\overline{\mathbf{Q}}}$. Choose a maximal torus $\widehat{T}$ of $\widehat{G}$. Then dual to μ_0 is a Weyl group orbit of elements $\mu^* \in X^*(\widehat{T})$. We define μ_1 to be the restriction of μ^* to $Z(\widehat{G})$. Clearly μ_1 is independent of the choices of $h, \widehat{T}$ and μ^*.

This condition we are imposing on δ seems to depend on the choices of embeddings $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_v$ for $v = p, \infty$. In fact it does not, since these embeddings are only allowed to change by an element of $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$, and E is the field of definition of the $G(\mathbf{C})$-conjugacy class of μ_h. In particular μ_1 does not depend on the choice of $\overline{\mathbf{Q}} \to \mathbf{C}$.

For any finite abelian group A we denote by A^D the dual finite abelian group $\mathrm{Hom}(A, \mathbf{C}^\times)$. For any $(\gamma_0; \gamma, \delta)$ as above we are going to define an element $\alpha(\gamma_0; \gamma, \delta)$ of $\mathfrak{K}(I_0/\mathbf{Q})^D$. The finite abelian group $\mathfrak{K}(I_0/\mathbf{Q})$ was defined in **[K6]** 4.6, generalizing a definition in **[L7]** for regular semisimple γ_0. Let us recall the definition. There is a natural embedding $Z(\widehat{G}) \to Z(\widehat{I}_0)$, and the exact sequence

$$1 \to Z(\widehat{G}) \to Z(\widehat{I}_0) \to Z(\widehat{I}_0)/Z(\widehat{G}) \to 1$$

induces a homorphism

$$\pi_0((Z(\widehat{I}_0)/Z(\widehat{G}))^\Gamma) \to H^1(\mathbf{Q}, Z(\widehat{G})).$$

Let $\ker^1(\mathbf{Q}, Z(\widehat{G}))$ denote the kernel of

$$H^1(\mathbf{Q}, Z(\widehat{G})) \to \prod_v H^1(\mathbf{Q}_v, Z(\widehat{G})),$$

where the product is taken over all places v of $\mathbf{Q}$. Then $\mathfrak{K}(I_0/\mathbf{Q})$ is defined to be the subgroup of $\pi_0((Z(\widehat{I}_0)/Z(\widehat{G}))^\Gamma)$ consisting of elements whose image in $H^1(\mathbf{Q}, Z(\widehat{G}))$ belongs to $\ker^1(\mathbf{Q}, Z(\widehat{G}))$. Since γ_0 is elliptic, the identity component of $(Z(\widehat{I}_0)/Z(\widehat{G}))^\Gamma$ is trivial. Therefore $\mathfrak{K}(I_0/\mathbf{Q})$ is equal to the quotient

$$\Big(\bigcap_v Z(\widehat{I}_0)^{\Gamma(v)} Z(\widehat{G})\Big)/Z(\widehat{G})$$

(use the Chebotarev density theorem to conclude that if $s \in \bigcap_v Z(\widehat{I}_0)^{\Gamma(v)} Z(\widehat{G})$, then the image of s in $Z(\widehat{I}_0)/Z(\widehat{G})$ is fixed by Γ).

For each place v of $\mathbf{Q}$ we will define an element $\alpha_v(\gamma_0; \gamma, \delta)$ of $X^*(Z(\widehat{I}_0)^{\Gamma(v)})$. We will usually abbreviate $\alpha_v(\gamma_0; \gamma, \delta)$ to α_v. First consider places v other then p, ∞. Choose $g \in G(\overline{\mathbf{Q}}_v)$ such that $g\gamma_0 g^{-1} = \gamma_v$. Then $\tau \mapsto g^{-1}\tau(g)$ ($\tau \in \Gamma(v)$) is a 1-cocycle of $\Gamma(v)$ in $I_0(\overline{\mathbf{Q}}_v)$; its cohomology class is an element of

$$\ker[H^1(\mathbf{Q}_v, I_0) \to H^1(\mathbf{Q}_v, G)].$$

From §§1,4 of **[K6]** we have a commutative diagram

$$\begin{array}{ccc} H^1(\mathbf{Q}_v, I_0) & \to & H^1(\mathbf{Q}_v, G) \\ \downarrow & & \downarrow \\ \pi_0(Z(\widehat{I}_0)^{\Gamma(v)})^D & \to & \pi_0(Z(\widehat{G})^{\Gamma(v)})^D \end{array}$$

in which the vertical arrows are bijections. Thus we get an element $\alpha_v \in X^*(Z(\widehat{I}_0)^{\Gamma(v)})$ whose restrictions to $Z(\widehat{G})^{\Gamma(v)}$ and the identity component of $Z(I_0)^{\Gamma(v)}$ are trivial.

Next we consider the place $v = p$. We are assuming that $N\delta$ is conjugate to γ_0 under $G(\overline{\mathbf{Q}}_p)$. By a theorem of Steinberg $H^1(L, I_0)$ is trivial, and therefore $N\delta$ is conjugate under $G(L)$ to γ_0. Choose $c \in G(L)$ such that $c\gamma_0c^{-1} = N\delta$. Define $b \in G(L)$ by putting $b = c^{-1}\delta\sigma(c)$. Applying σ to the equation $c\gamma_0c^{-1} = N\delta$, we find that $b \in I_0(L)$. Since c is well-defined up to right multiplication by an element of $I_0(L)$, the element $b \in I_0(L)$ is well-defined up to σ-conjugacy in $I_0(L)$ and hence determines a well-defined element of $B((I_0)_{\mathbf{Q}_p})$. Using the map

$$B((I_0)_{\mathbf{Q}_p}) \to X^*(Z(\widehat{I}_0)^{\Gamma(p)})$$

of Lemma 6.1 below, we get the desired element

$$\alpha_p \in X^*(Z(\widehat{I}_0)^{\Gamma(p)}).$$

Because of our second assumption on δ, the restriction of α_p to $Z(\widehat{G})^{\Gamma(p)}$ is equal to the restriction of $-\mu_1 \in X^*(Z(\widehat{G}))$ (use Lemma 6.2 below).

Finally we consider the place $v = \infty$. Choose an elliptic maximal $\mathbf{R}$-torus T of G containing γ_0. Then T is a maximal torus in I_0 as well. Choose $h \in X_\infty$ such that h factors through T. Then μ_h belongs to $X_*(T) = X^*(\widehat{T})$, where $\widehat{T}$ is the complex torus dual to T, and μ_h is well-defined up to the Weyl group of $T(\mathbf{R})$ in $G(\mathbf{R})$. By Lemma 5.1 below the image of μ_h in $X^*(\widehat{T}^{\Gamma(\infty)})$ is independent of the choice of h. Using the canonical embedding $Z(\widehat{I}_0) \to \widehat{T}$, we get from μ_h an element

$$\alpha_\infty \in X^*(Z(\widehat{I}_0)^{\Gamma(\infty)}).$$

The element α_∞ is independent of the choice of T since any two elliptic maximal tori in I_0 are conjugate under $I_0(\mathbf{R})$. Clearly the restriction of α_∞ to $Z(\widehat{G})^{\Gamma(\infty)}$ is equal to the restriction of $\mu_1 \in X^*(Z(\widehat{G}))$ to $Z(\widehat{G})^{\Gamma(\infty)}$.

We are now finished with the construction of α_v for all places v of $\mathbf{Q}$. Proposition 7.1 of **[K6]** implies that α_v is trivial for all but a finite number of v. For each place v of $\mathbf{Q}$ we denote by β_v the unique extension of α_v to a character on $Z(\widehat{I}_0)^{\Gamma(v)}Z(\widehat{G})$ such that the restriction of β_v to $Z(\widehat{G})$ is

$$\begin{array}{lll} \mu_1 & \text{if} & v = \infty, \\ -\mu_1 & \text{if} & v = p, \\ \text{trivial} & \text{if} & v \neq p, \infty. \end{array}$$

Clearly β_v is trivial for all but a finite number of v. Restrict each β_v to $\bigcap_v Z(\widehat{I}_0)^{\Gamma(v)}Z(\widehat{G})$ and form the product over all places v of $\mathbb{Q}$ of these restrictions. In this way we get a character on $\bigcap_v Z(\widehat{I}_0)^{\Gamma(v)}Z(\widehat{G})$ whose restriction to $Z(\widehat{G})$ is trivial; in other words, we get a character α on the finite abelian group $\mathfrak{K}(I_0/\mathbb{Q})$. This is the desired element $\alpha(\gamma_0;\gamma,\delta)$ of $\mathfrak{K}(I_0/\mathbb{Q})^D$.

Suppose that $\gamma_0' \in G(\mathbb{Q})$ is conjugate to γ_0 in $G(\overline{\mathbb{Q}})$. Let I_0' denote the centralizer of γ_0' in G. By choosing $g \in G(\overline{\mathbb{Q}})$ such that $g\gamma_0 g^{-1} = \gamma_0'$, we get an inner twisting $\psi := \mathrm{Int}\,(g)$ from I_0 to I_0'. Using ψ, we identify $Z(\widehat{I}_0')$ with $Z(\widehat{I}_0)$ and $\mathfrak{K}(I_0'/\mathbb{Q})$ with $\mathfrak{K}(I_0/\mathbb{Q})$. Then it is not hard to check that

$$\alpha(\gamma_0;\gamma,\delta) = \alpha(\gamma_0',\gamma,\delta).$$

In fact, using Lemma 1.4 of **[K6]** and Lemmas 5.1 and 6.3 below, one shows that for all v

$$\alpha_v(\gamma_0;\gamma,\delta) = \alpha_v(\gamma_0';\gamma,\delta) + \lambda_v,$$

where λ_v denotes the image under

$$H^1(\mathbb{Q}, I_0) \to H^1(\mathbb{Q}_v, I_0) \to \pi_0(Z(\widehat{I}_0)^{\Gamma(v)})^D$$

of the class of the 1-cocyle $\tau \mapsto g^{-1}\tau(g)$ of Γ in $I_0(\overline{\mathbb{Q}})$. Let I_1 denote the centralizer of γ_0 in G_{sc}. Then

$$Z(\widehat{I}_1) = Z(\widehat{I}_0)/Z(\widehat{G}).$$

Choose $g_1 \in G_{\mathrm{sc}}(\overline{\mathbb{Q}})$ such that $g_1 g^{-1}$ is central in G, let λ_v' denote the image under

$$H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}_v, I_1) \to \pi_0(Z(\widehat{I}_1)^{\Gamma(v)})^D$$

of the 1-cocyle $\tau \mapsto g_1^{-1}\tau(g_1)$ of Γ in $I_1(\overline{\mathbb{Q}})$, and let λ_v'' denote the restriction of λ_v' to

$$Z(\widehat{I}_0)^{\Gamma(v)}/Z(\widehat{G})^{\Gamma(v)} = (Z(\widehat{I}_0)^{\Gamma(v)}Z(\widehat{G}))/Z(\widehat{G}).$$

Then

$$\beta_v(\gamma_0;\gamma,\delta) = \beta_v(\gamma_0';\gamma,\delta) + \lambda_v'',$$

and it follows from Proposition 2.6 of [**K6**] that

$$\alpha(\gamma_0; \gamma, \delta) = \alpha(\gamma_0'; \gamma, \delta).$$

Suppose that $\gamma' = (\gamma_v')_{v \neq p, \infty}$ belongs to $G(\mathbb{A}_f^p)$ and that γ_v' is conjugate to γ_v under $G(\overline{\mathbb{Q}}_v)$ for all $v \neq p, \infty$. Let $I(v)$ denote the centralizer of γ_v in $G_{\mathbb{Q}_v}$ and choose $g_v \in G(\overline{\mathbb{Q}}_v)$ such that $g_v \gamma_0 g_v^{-1} = \gamma_v$. Then $\operatorname{Int}(g_v)$, restricted to I_0, is an inner twisting

$$\psi_v : I_0 \to I(v),$$

well-defined up to inner automorphisms of I_0, allowing us to identify $Z(\widehat{I}(v))$ with $Z(\widehat{I}_0)$. Choose $x_v \in G(\overline{\mathbb{Q}}_v)$ such that $x_v \gamma_v x_v^{-1} = \gamma_v'$ and consider the 1-cocycle $\tau \mapsto x_v^{-1} \tau(x_v)$ of $\Gamma(v)$ in $I(v)(\overline{\mathbb{Q}}_v)$. We denote by $\operatorname{inv}(\gamma_v, \gamma_v')$ the image of this 1-cocycle under

$$H^1(\mathbb{Q}_v, I(v)) \to \mathfrak{K}(I_0/\mathbb{Q}_v)^D,$$

where $\mathfrak{K}(I_0/\mathbb{Q}_v)$ is the finite abelian group

$$\operatorname{coker}[\pi_0(Z(\widehat{G})^{\Gamma(v)}) \to \pi_0(Z(\widehat{I}_0)^{\Gamma(v)})],$$

so that $\mathfrak{K}(I_0/\mathbb{Q}_v)^D$ equals

$$\ker[\pi_0(Z(\widehat{I}_0)^{\Gamma(v)})^D \to \pi_0(Z(\widehat{G})^{\Gamma(v)})^D].$$

It is easy to see that

$$\alpha(\gamma_0; \gamma', \delta) = \alpha(\gamma_0; \gamma, \delta) + \sum_v \operatorname{inv}(\gamma_v, \gamma_v');$$

here the sum is taken over all $v \neq p, \infty$, and we use the canonical maps

$$\mathfrak{K}(I_0/\mathbb{Q}) \to \mathfrak{K}(I_0/\mathbb{Q}_v)$$

to regard $\operatorname{inv}(\gamma_v, \gamma_v')$ as an element of $\mathfrak{K}(I_0/\mathbb{Q})^D$.

Suppose that $\delta' \in G(F)$ and that δ' satisfies the same assumptions as δ. Let $R = R_{F/\mathbb{Q}_p}(G_F)$. There is a $\mathbb{Q}_p$-automorphism θ of R corresponding to $\sigma \in \operatorname{Gal}(F/\mathbb{Q}_p)$ (it induces σ on $R(\mathbb{Q}_p) = G(F)$). We use the equality $R(\mathbb{Q}_p) = G(F)$ to regard δ, δ' as elements of $R(\mathbb{Q}_p)$. Let $I(p) = \{x \in R | x^{-1}\delta\theta(x) = \delta\}$. Lemma 5.8 of [**K1**] gives us an inner twisting $\psi_p : I_0 \to I(p)$, canonical up to inner automorphisms of I_0; therefore $Z(\widehat{I}(p)) = Z(\widehat{I}_0)$. Choose $x \in R(L)$ such that $x\delta\theta(x)^{-1} = \delta'$. Then $x^{-1}\sigma(x)$ belongs to $I(p)(L)$; the σ-conjugacy class of $x^{-1}\sigma(x)$ is an element of $B(I(p))$ that in fact lies in the image of the canonical injection (see **[K5]**)

$$H^1(\mathbf{Q}_p, I(p)) \to B(I(p)).$$

Using the map

$$B(I(p)) \to X^*(Z(\widehat{I}_0)^{\Gamma(p)}),$$

we get an element

$$\mathrm{inv}(\delta,\delta') \in X^*(Z(\widehat{I}_0)^{\Gamma(p)})$$

that is in fact trivial on the identity component of $Z(\widehat{I}_0)^{\Gamma(p)}$. Since δ' and δ have the same image in $B(G_{\mathbf{Q}_p})$, the restriction of $\mathrm{inv}(\delta,\delta')$ to $Z(\widehat{G})^{\Gamma(p)}$ is trivial, and therefore $\mathrm{inv}(\delta,\delta')$ may be regarded as an element of $\mathfrak{K}(I_0/\mathbf{Q}_p)^D$. It is easy to see that

$$\alpha(\gamma_0;\gamma,\delta') = \alpha(\gamma_0;\gamma,\delta) + \mathrm{inv}(\delta,\delta'),$$

where we use the canonical map

$$\mathfrak{K}(I_0/\mathbf{Q}) \to \mathfrak{K}(I_0/\mathbf{Q}_p)$$

to regard $\mathrm{inv}(\delta,\delta')$ as an element of $\mathfrak{K}(I_0/\mathbf{Q})^D$.

The last task in this section is to check that $\alpha(\gamma_0;\gamma,\delta)$ is independent of the choice of embeddings $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_v$. This is clearly the case for all but the finite number of places v where α_v is non-trivial, and therefore it suffices to change one embedding at a time. Fix v and suppose that $i : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_v$ is replaced by $i' = i \circ \rho$ for $\rho \in \Gamma$; if $v = p, \infty$ then ρ must belong to $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$. Using i (resp. i'), we get objects $\Gamma(v), \alpha_v, \beta_v, \alpha$ (resp. $\Gamma(v)', \alpha'_v, \beta'_v, \alpha'$). It is obvious that $\Gamma(v)' = \rho^{-1}\Gamma(v)\rho$, $Z(\widehat{I}_0)^{\Gamma(v)'}Z(\widehat{G}) = \rho^{-1}(Z(\widehat{I}_0)^{\Gamma(v)}Z(\widehat{G}))$, and $\beta'_v = \beta_v \circ \rho$ (use that $\rho \in \mathrm{Gal}(\overline{\mathbf{Q}}/E)$ if $v = p, \infty$). Moreover $\bigcap_w Z(\widehat{I}_0)^{\Gamma(w)}Z(\widehat{G})$ is the same for i' as for i and is left stable by ρ. Therefore $\alpha' = \alpha \circ \rho$ (as characters on $\bigcap_w Z(\widehat{I}_0)^{\Gamma(w)}Z(\widehat{G})$). But α' and α are trivial on $Z(\widehat{G})$, and ρ acts trivially on the quotient

$$\Big(\bigcap_w Z(\hat{I}_0)^{\Gamma(w)}Z(G)\Big)/Z(G).$$

It follows that $\alpha' = \alpha$.

§3. STATEMENT OF CONJECTURE

Recall that we want to give a conjecture for the contribution of $P = G$ to $\mathrm{tr}\,(f \times \Phi_{\mathfrak{p}}^{j}; W_\lambda)$. The conjectural expression has the form

$$\sum_{\gamma_0} \sum_{(\gamma,\delta)} c(\gamma_0;\gamma,\delta) \cdot O_\gamma(f^p) \cdot TO_\delta(\phi_j) \cdot \mathrm{tr}\,\xi(\gamma_0). \tag{3.1}$$

It turns out that $c(\gamma_0;\gamma,\delta), TO_\delta(\phi_j) \in \mathbf{Q}$, $O_\gamma(f^p), \mathrm{tr}\,\xi(\gamma_0) \in L$, and that the sum has only finitely many non-zero terms. Therefore the expression makes sense and yields a number in L. Quite a bit of explanation is needed. The first sum runs over a set of representatives for the $G(\overline{\mathbf{Q}})$-conjugacy classes of semisimple $\gamma_0 \in G(\mathbf{Q})$ that are elliptic in $G(\mathbf{R})$. Fix such an element γ_0 and let I_0 denote its centralizer in G.

The index set for the second sum involves certain pairs $(\gamma, \delta) \in G(\mathbf{A}_f^p) \times G(F)$, with F as in §2. We consider only those pairs satisfying all the conditions of §2, so that the invariant $\alpha(\gamma_0;\gamma,\delta)$ is defined, and we retain only those pairs such that $\alpha(\gamma_0;\gamma,\delta)$ is trivial. We say that two such pairs $(\gamma_1,\delta_1), (\gamma_2,\delta_2)$ are equivalent if γ_1, γ_2 are conjugate under $G(\mathbf{A}_f^p)$ and δ_1, δ_2 are σ-conjugate under $G(F)$. The second sum is taken over a set of representatives for these equivalence classes. Fix one of these representatives (γ, δ).

As in §2 the pair (γ,δ) determines, for each finite place v of $\mathbf{Q}$, a group $I(v)$ over $\mathbf{Q}_v$ and an inner twisting $\psi_v : I_0 \to I(v)$, well-defined up to inner automorphisms of I_0 over $\overline{\mathbf{Q}}_v$. There are analogous objects at the infinite place. Choose an elliptic maximal torus T of $G_{\mathbf{R}}$ containing γ_0, so that T is also a maximal torus of I_0. Choose $h \in X_\infty$ such that h factors through T. Let Z denote the center of G. Then $\mathrm{Int}\,(h(i))$ induces a Cartan involution on G/Z and preserves I_0; therefore its restriction to I_0 induces a Cartan involution on I_0/Z, which we use to twist I_0 over $\mathbf{R}$, obtaining an inner twisting

$$\psi_\infty : I_0 \to I(\infty),$$

with $I(\infty)/Z$ anisotropic over $\mathbf{R}$.

Fix a triple $(I, \psi, (j_v))$ consisting of a $\mathbf{Q}$-group I, an inner twisting $\psi : I_0 \to I$ and for each place v of $\mathbf{Q}$ an isomorphism $j_v : I \to I(v)$ over

$\mathbf{Q}_v$, unramified almost everywhere, such that $j_v \circ \psi$ and ψ_v differ by an inner automorphism of I_0 over $\overline{\mathbf{Q}}_v$. Let I_2 denote the adjoint group of I_0. The existence of such a triple is equivalent to the existence of an element $x \in H^1(\mathbf{Q}, I_2)$ such that for every place v of $\mathbf{Q}$ the image of x in $H^1(\mathbf{Q}_v, I_2)$ is equal to the element $x_v \in H^1(\mathbf{Q}_v, I_2)$ determined by the inner twisting $\psi_v : I_0 \to I(v)$. It is not hard to see that the image of x_v in $\pi_0(Z(\widehat{I}_2)^{\Gamma(v)})^D$ is equal to image of $\alpha_v(\gamma_0; \gamma, \delta)$ under

$$X^*(Z(\widehat{I}_0)^{\Gamma(v)}) \to X^*(Z(\widehat{I}_2)^{\Gamma(v)}).$$

The existence of x then follows from Proposition 2.6 of [**K6**] and our assumption that $\alpha(\gamma_0; \gamma, \delta)$ is trivial.

The element x is unique by the Hasse principle for adjoint groups (see Cor. 5.4 of [**Sa**], noting that G has no factors of type E_8, and hence that the same is true for I_2). Therefore any triple as above is isomorphic to a triple of the form $(I, \psi, (j_v \circ \operatorname{Int}(h_v)))$ for some $(h_v) \in I_{\mathrm{ad}}(\mathbf{A})$, where I_{ad} denotes the adjoint group of I.

Let dx (resp. dy) denote the Haar measure on $G(\mathbf{A}_f^p)$ (resp. $G(F)$) giving measure 1 to K^p (resp. K_F), where K_F denotes the subgroup $G(\mathfrak{o}_F)$ of $G(F)$ ($\mathfrak{o}_F$ is the valuation ring of F). Choose a Haar measure di^p (resp. di_p) on $I(\mathbf{A}_f^p)$ (resp. $I(\mathbf{Q}_p)$) that gives rational measure to compact open subgroups of $I(\mathbf{A}_f^p)$ (resp. $I(\mathbf{Q}_p)$), and use the isomorphisms j_v to transport these measures to $G(\mathbf{A}_f^p)_\gamma$ (the centralizer of γ in $G(\mathbf{A}_f^p)$) and $I(p)(\mathbf{Q}_p)$. The resulting measure does not change if (j_v) is modified by an element of $I_{\mathrm{ad}}(\mathbf{A})$.

Since I/Z is anisotropic over $\mathbf{R}$, and since the split components of Z over $\mathbf{R}$ and $\mathbf{Q}$ are the same, $I(\mathbf{Q})$ is a discrete subgroup of $I(\mathbf{A}_f)$. We define $c_1(\gamma_0; \gamma, \delta)$ to be the volume of $I(\mathbf{Q})\backslash I(\mathbf{A}_f)$, a rational number by our assumption on di^p and di_p. We define $c_2(\gamma_0)$ to be the cardinality of the finite set

$$\ker[\ker^1(\mathbf{Q}, I_0) \to \ker^1(\mathbf{Q}, G)]$$

and $c(\gamma_0; \gamma, \delta)$ to be the product of $c_1(\gamma_0; \gamma, \delta)$ and $c_2(\gamma_0)$.

We write $d\overline{x}$ for the quotient of dx by di^p and define $O_\gamma(f^p)$ to be the orbital integral

$$\int_{G(\mathbf{A}_f^p)_\gamma \backslash G(\mathbf{A}_f^p)} f^p(x^{-1}\gamma x)\, d\overline{x},$$

an element of L. We write $d\overline{y}$ for the quotient of dy by di_p and define $TO_\delta(\phi_j)$ to be the twisted orbital integral

$$\int_{I(p)(\mathbb{Q}_p)\backslash G(F)} \phi_j(y^{-1}\delta\sigma(y))d\overline{y},$$

where ϕ_j is the following element of the Hecke algebra of $G(F)$ with respect to K_F.

Since F is an extension field of $E_\mathfrak{p}$, the $G(\mathbb{C})$-conjugacy class of homomorphisms $\mu_h : \mathbb{G}_m \to G_\mathbb{C}$ gives us a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of homomorphisms $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$, fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Choose a maximal $\mathfrak{o}_F$-split torus S in G over $\mathfrak{o}_F$. Then S is also a maximal F-split torus in G over F, and by Lemma (1.1.3) of **[K3]** we can choose μ so that it factors through S. Let ϕ_j be the characteristic function of the double coset $K_F a K_F$, where

$$a = \mu(\pi_F^{-1})$$

for some uniformizing element π_F in F. Note that the double coset $K_F a K_F$ is independent of the choice of S, μ, and π_F.

Clearly $TO_\delta(\phi_j)$ is a rational number, and $\mathrm{tr}\,\xi(\gamma_0)$ is a number in L. As mentioned before there are only finitely many triples $(\gamma_0; \gamma, \delta)$ (up to equivalence) such that $O_\gamma(f^p)\ TO_\delta(\phi_j)$ is non-zero. This can be proved in almost the same way as Proposition 8.2 of **[K6]**. Our formulation of the conjecture is now complete.

§4. Stabilization

Our next goal is to stabilize the expression (3.1). We fix an embedding $L \to \mathbb{C}$ and write $\xi_\mathbb{C}$ for $\xi \otimes_L \mathbb{C}$ and $f^p_\mathbb{C}$ for the $\mathbb{C}$-valued function obtained from f^p via $L \to \mathbb{C}$. There is a unique Haar measure di_∞ on $I(\mathbb{R})$ so that the product of di^p, di_p, di_∞ is the canonical measure on $I(\mathbb{A})$ (the one used to define the Tamagawa number $\tau(I)$). Let A_G denote the maximal $\mathbb{Q}$-split torus in the center of G and let $A_G(\mathbb{R})^0$ denote the identity component of the topological group $A_G(\mathbb{R})$. We use the usual Haar measure on $A_G(\mathbb{R})^0$, coming from a basis of the $\mathbb{Z}$-module of rational characters of G, so that

$$\tau(I) = \mathrm{vol}\,(I(\mathbb{Q})A_G(\mathbb{R})^0\backslash I(\mathbb{A})).$$

Then $c_1(\gamma_0; \gamma, \delta)$ is equal to

$$\tau(I) \cdot \mathrm{vol}\,(A_G(\mathbb{R})^0\backslash I(\mathbb{R}))^{-1},$$

since $A_G(\mathbf{R})^0\backslash I(\mathbf{R})$ is a compact group. Moreover $\tau(I) = \tau(I_0)$ (see [**K7**]). Just as in §9 of [**K6**] we see that

$$c_2(\gamma_0) \cdot |\mathfrak{K}(I_0/\mathbf{Q})|^{-1} = \tau(G) \cdot \tau(I_0)^{-1};$$

therefore

$$c(\gamma_0; \gamma, \delta) = \tau(G) \cdot |\mathfrak{K}(I_0/\mathbf{Q})| \cdot \mathrm{vol}\,(A_G(\mathbf{R})^0\backslash I(\mathbf{R}))^{-1}.$$

Fix γ_0 and consider a pair (γ, δ) satisfying the conditions of §2, so that $\alpha(\gamma_0; \gamma, \delta)$ is defined. The number

$$|\mathfrak{K}(I_0/\mathbf{Q})|^{-1} \sum_{\kappa \in \mathfrak{K}(I_0/\mathbf{Q})} \langle \alpha(\gamma_0; \gamma, \delta), \kappa \rangle \tag{4.1}$$

is 1 if $\alpha(\gamma_0; \gamma, \delta)$ is trivial and is 0 otherwise. Therefore in (3.1) we can sum over all equivalence classes of pairs (γ, δ) satisfying the conditions of §2, so long as we multiply the summand indexed by (γ, δ) by the expression (4.1).

Recall the signs $e(I(v))$ of [**K2**]. Define a sign $e(\gamma, \delta)$ by

$$e(\gamma, \delta) = \prod_v e(I(v)).$$

If $\alpha(\gamma_0; \gamma, \delta) = 1$, so that there exists a group I over $\mathbf{Q}$ whose localizations are the groups $I(v)$, then $e(\gamma, \delta) = 1$ by the main result of [**K2**].

These considerations show that (3.1) can be rewritten as

$$\tau(G) \sum_{\gamma_0} \sum_{\kappa} \sum_{(\gamma,\delta)} \langle \alpha(\gamma_0; \gamma, \delta), \kappa \rangle \cdot e(\gamma, \delta) \cdot O_\gamma(f^p_{\mathbf{C}}) \cdot TO_\delta(\phi_j) \cdot \tag{4.2}$$

$$\mathrm{tr}\, \xi_{\mathbf{C}}(\gamma_0) \cdot \mathrm{vol}\,(A_G(\mathbf{R})^0\backslash I(\infty)(\mathbf{R}))^{-1}.$$

The first sum is taken over the same set of elements $\gamma_0 \in G(\mathbf{Q})$ as before. The second sum is taken over $\kappa \in \mathfrak{K}(I_0/\mathbf{Q})$. The third sum is taken over a set of representatives for the equivalence classes of elements (γ, δ) satisfying the conditions of §2. It was legitimate to interchange the order of the second and third sums since the triple sum has only finitely many non-zero terms.

§5. A LEMMA ON REAL GROUPS

The purpose of this section is to prove a slight refinement of a fact about real groups that is stated in the introduction to [**L5**]. Let G be a connected reductive group over $\mathbb{R}$. Assume that there exists an elliptic maximal torus T of G and a homomorphism

$$h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G$$

such that $\operatorname{Int}(h(i))$ is a Cartan involution of G_{ad} and Lie (G) is of Hodge type $\{(1,-1),(0,0),(-1,1)\}$. Define $\mu_h \in X_*(T)$ as before. Write G_{sc} for the simply connected cover of the derived group of G and write T_{sc} for the inverse image of T in G_{sc}. Let Ω be the Weyl group of $T_{\mathrm{sc}}(\mathbb{C})$ in $G_{\mathrm{sc}}(\mathbb{C})$, let Ω_0 denote the Weyl group of $T_{\mathrm{sc}}(\mathbb{R})$ in $G_{\mathrm{sc}}(\mathbb{R})$, and let Ω_1 denote the Weyl group of $T(\mathbb{R})$ in $G(\mathbb{R})$. Since T_{sc} is anisotropic, Ω acts on T_{sc} by $\mathbb{R}$-automorphisms. For $\omega \in \Omega$ choose a representative $g \in G_{\mathrm{sc}}(\mathbb{C})$ for ω in the normalizer of $T_{\mathrm{sc}}(\mathbb{C})$. Then $\tau \mapsto g\tau(g)^{-1}$ ($\tau \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})$) is a 1-cocycle of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ in $T_{\mathrm{sc}}(\mathbb{C})$. Its class in $H^1(\mathbb{R}, T_{\mathrm{sc}})$ is well-defined. Using the homomorphism

$$H^1(\mathbb{R}, T_{\mathrm{sc}}) \to \pi_0(\hat{T}_{\mathrm{sc}}^{\Gamma(\infty)})^D,$$

where $\Gamma(\infty) = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$, we get an element $a_\omega \in X^*(\hat{T}_{\mathrm{sc}}^{\Gamma(\infty)}) = \pi_0(\hat{T}_{\mathrm{sc}}^{\Gamma(\infty)})^D$. The cocharacter $\omega\mu_h - \mu_h \in X_*(T)$ belongs to the subgroup $X_*(T_{\mathrm{sc}})$ of $X_*(T)$ and hence determines an element of $X^*(\hat{T}_{\mathrm{sc}})$, which we can restrict to $\hat{T}_{\mathrm{sc}}^{\Gamma(\infty)}$ to get an element $b_\omega \in X^*(\hat{T}_{\mathrm{sc}}^{\Gamma(\infty)})$.

LEMMA 5.1. *For all $\omega \in \Omega$ the element a_ω is equal to b_ω. Moreover a_ω depends only on the image of ω in Ω/Ω_0, and the image of a_ω in $X^*(\hat{T}^{\Gamma(\infty)})$ depends only on the image of ω in Ω/Ω_1.*

The second statement is obvious from the definition of a_ω. Now we prove $a_\omega = b_\omega$. For $\omega_1, \omega_2 \in \Omega$ we have $a_{\omega_1\omega_2} = \omega_1(a_{\omega_2}) \cdot a_{\omega_1}$ and $b_{\omega_1\omega_2} = \omega_1(b_{\omega_2}) \cdot b_{\omega_2}$. Therefore it is enough to prove that $a_\omega = b_\omega$ for simple reflections ω. So let α be a root of T and let ω be the corresponding reflection. Consider the connected subgroup G_α of G whose Lie algebra consists of Lie (T) plus the root spaces for α and $-\alpha$. Note that G_α is defined over $\mathbb{R}$. Then ω comes from G_α and in proving the theorem we may replace G by G_α. Clearly we may replace G_α by its adjoint group. The problem has now been reduced to the case in which $G_{\mathbf{C}}$ is isomorphic to PGL_2 and ω is the non-trivial element of Ω, a group of order 2. Note that $X^*(\hat{T}_{\mathrm{sc}}^{\Gamma(\infty)})$ is also of order 2. There are two cases. If $G(\mathbf{R})$ is compact, then a_ω and b_ω are both trivial. If $G(\mathbf{R})$ is non-compact, then a_ω and b_ω are both non-trivial.

§6. Lemmas concerning $B(G)$

In this section F denotes a p-adic field, $\overline{F}$ an algebraic closure of F, Γ the Galois group of $\overline{F}/F$, and G a connected reductive group over F. We write L for the completion of the maximal unramified extension of F in $\overline{F}$, and σ for the Frobenius automorphism of L over F. The purpose of this section is to prove several lemmas about $B(G)$, the set of σ-conjugacy classes in $G(L)$.

For any torus T over F there is a functorial isomorphism (see §2 of [**K5**])

$$B(T) \to X^*(\hat{T}^\Gamma). \tag{6.1}$$

Consider the functors $G \mapsto B(G)$ and $G \mapsto X^*(Z(\hat{G})^\Gamma)$ from the category of connected reductive groups over F and normal homomorphisms (see §5 of [**K5**]) to the category of sets.

LEMMA 6.1. *There is a unique functorial map*

$$B(G) \to X^*(Z(\hat{G})^\Gamma)$$

extending the functorial map (6.1) for tori.

The proof is essentially the same as that of Proposition 5.6 in [**K5**]. Of course the map $B(G) \to X^*(Z(\hat{G})^\Gamma)$ extends the bijection

$$B(G)_b \to X^*(Z(\hat{G})^\Gamma)$$

of that proposition.

Suppose that H is a connected reductive subgroup of G having the property that any maximal torus of H is a maximal torus of G. There is a canonical Γ-equivariant embedding $Z(\hat{G}) \to Z(\hat{H})$, constructed in the same way as the embedding $Z(\hat{G}) \to Z(\hat{T})$ of [**K6**], 4.2.

LEMMA 6.2. *The following diagram commutes:*

$$\begin{array}{ccc} B(H) & \to & B(G) \\ \downarrow & & \downarrow \\ X^*(Z(\widehat{H})^\Gamma) & \to & X^*(Z(\widehat{G})^\Gamma). \end{array}$$

Since $H \to G$ is not a normal homomorphism, we cannot apply the functoriality of $B(G) \to X^*(Z(\widehat{G})^\Gamma)$ directly. However, if G_{der} is simply connected, then we can use the functoriality for $G \to G/G_{\text{der}}$ and $H \to G/G_{\text{der}}$ to prove the lemma, remembering, of course, that in this case $Z(\widehat{G})$ is the complex torus dual to the torus G/G_{der}. In the general case we choose a central extension G' of G such that

(a) the kernel C of $G' \to G$ is a torus,

(b) G'_{der} is simply connected.

Let H' denote the fiber product of H and G' over G. Then $H' \to H$ is a central extension with kernel C, and H' is a connected reductive subgroup of G' such that any maximal torus of H' is a maximal torus of G'. The surjectivity of $B(H') \to B(H)$ reduces us to the case of $H' \to G'$, which has been treated already.

The last lemma concerns twists of G. Let $\tau \mapsto a_\tau$ be a 1-cocycle of $\mathrm{Gal}(F^{\text{un}}/F)$ in $G(F^{\text{un}})$, where F^{un} denotes the maximal unramified extension of F in $\overline{F}$. Let G^* denote the twist of G by a_τ, so that

$$G^*(L) = G(L),$$

$$\sigma^* = \mathrm{Int}(a_\sigma) \circ \sigma.$$

We identify $Z(\widehat{G}^*)$ with $Z(\widehat{G})$. Let $b \in G^*(L)$. Then the σ-conjugacy class of $c := ba_\sigma$ depends only on the σ^*-conjugacy class of b in $G^*(L)$. Let a', b', c' denote the images of a_σ, b, c in $X^*(Z(\widehat{G})^\Gamma)$.

LEMMA 6.3. *The element c' is the product of a' and b'.*

The proof is similar to that of Lemma 1.4 of [**K6**].

§7. STABILIZATION CONTINUED

We would like to rewrite (4.2) in terms of elliptic endoscopic triples (H, s, η_0) for G (using the same conventions regarding endoscopic groups as in §7 of [K4]). Fix such a triple and extend η_0 to an L-homomorphism $\eta : {}^LH \to {}^LG$. We use the Weil group of $\overline{\mathbb{Q}}/\mathbb{Q}$ to form the L-groups; therefore η exists [L4] (since G_{der} is simply connected).

Granting three standard conjectures on transfer of functions on p-adic groups, we now construct a function h on $H(\mathbb{A})$, depending on s and η (as well as $f^p_{\mathbb{C}}, j$ and $\xi_{\mathbb{C}}$); although the function itself is not well-defined, its stable orbital integrals are. We define h to be 0 unless H is unramified at p and the elliptic maximal tori of $G_{\mathbb{R}}$ come from $H_{\mathbb{R}}$. For H satisfying these two conditions we define h as a product of functions h^p, h_p, h_∞ on $H(\mathbb{A}^p_f), H(\mathbb{Q}_p), H(\mathbb{R})$ respectively. The stable orbital integrals of h_p, h_∞ depend on s, not just the class of s modulo $Z(\widehat{G})$, although those of h (and h^p) do not. Moreover the stable orbital integrals of h^p, h_p, h_∞ depend on a choice of local transfer factors $\Delta_v(\gamma_H, \gamma_G)$ for (H, s, η), subject (as usual) to the "global hypothesis" [L7], [L-S] on the adelic transfer factor $\Delta := \prod_v \Delta_v$. As in [K6] we consider all (G, H)-regular semisimple elements γ_H in H, and so we need the global hypothesis in the form given in 6.10 (b) of [K6]. There is a choice of sign implicit in the transfer factors of [L-S]: one is free to replace s by s^{-1} in the formulas of [L-S]. In fact we will use this alternative normalization, which is the one compatible with the form of the global hypothesis given in 6.10 (b) of [K6].

We start with h^p. We write Δ^p for the product

$$\Delta^p = \prod_{v \neq p, \infty} \Delta_v.$$

Two standard conjectures, Conjecture 5.5 of [K6] (an extension to (G, H)-regular elements of a conjecture in [L7]) and the "fundamental lemma" for H, G and the unit element of the unramified Hecke algebra of G at all but a finite number of places of $\mathbb{Q}$, imply the existence of a function $h^p \in C^\infty_c(G(\mathbb{A}^p_f))$ such that for every (G, H)-regular semisimple $\gamma_H \in H(\mathbb{A}^p_f)$

$$SO_{\gamma_H}(h^p) = \sum_\gamma \Delta^p(\gamma_H, \gamma) \cdot e^p(\gamma) \cdot O_\gamma(f^p_{\mathbb{C}}). \tag{7.1}$$

The sum is taken over $G(\mathbb{A}^p_f)$-conjugacy classes of semisimple elements $\gamma \in G(\mathbb{A}^p_f)$ that come from γ_H, and $e^p(\gamma)$ denotes the product over all $v \neq p, \infty$ of the signs $e(G_{\gamma_v})$ (G_{γ_v} is the centralizer in G of the v-component γ_v of γ). What does (G, H)-regular semisimple mean for an element γ_H of $H(\mathbb{A}_f^p)$? It means that each component $\gamma_{H,v}$ of γ_H $(v \neq p, \infty)$ is (G,H)-regular semisimple and that for all but a finite number of v the element $\gamma_{H,v}$ determines a (G,H)- regular element over the residue field of $\mathbb{Q}_v$. Note that any (G,H)- regular semisimple element of $H(\mathbb{Q})$ is also (G,H)-regular semisimple as an element of $H(\mathbb{A}_f^p)$. The sum on the right side of (7.1) has only finitely many non-zero terms (use Proposition 7.1 of **[K6]**). Clearly neither side of (7.1) changes if s is multiplied by an element of $Z(\widehat{G})$.

Next we consider h_p. We first treat the case in which $s \in Z(\widehat{H})^{\Gamma(p)}$ (not just $Z(\widehat{H})^{\Gamma(p)}Z(\widehat{G})$). As before we write R for $R_{F/\mathbb{Q}_p}(G_F)$ and θ for the automorphism of R corresponding to $\sigma \in \mathrm{Gal}\,(F/\mathbb{Q}_p)$. We now use ${}^LG, {}^LR, {}^LH$ to denote the L-groups obtained from $G_{\mathbb{Q}_p}, R, H_{\mathbb{Q}_p}$ (${}^LG = \widehat{G} \rtimes W_{\mathbb{Q}_p}$ and so on). Our global η induces a local L-homomorphism

$$\eta : {}^LH \to {}^LG.$$

We are going to use η and s to define an "allowed" embedding (see **[Sh3]**)

$$\widetilde{\eta} : {}^LH \to {}^LR.$$

The dual group $\widehat{R}$ is equal to $\widehat{G} \times \ldots \times \widehat{G}$, where the product has r factors (recall that $r = [F : \mathbb{Q}_p]$) and the i-th factor corresponds to $\sigma^{r-i} \in \mathrm{Gal}\,(F/\mathbb{Q}_p)$. The group $W_{\mathbb{Q}_p}$ acts on $\widehat{G} \times \ldots \times \widehat{G}$ through its unramified quotient, and the Frobenius element σ of $\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p$ acts by

$$\sigma(x_1, \ldots, x_r) = \widehat{\theta}(\sigma(x_1), \ldots, \sigma(x_r))$$

for $(x_1, \ldots, x_r) \in \widehat{G} \times \ldots \times \widehat{G}$, where $\widehat{\theta}$ denotes the automorphism $(x_1, \ldots, x_r) \mapsto (x_2, \ldots, x_r, x_1)$ of $\widehat{G} \times \ldots \times \widehat{G}$.

We have assumed that the element s belongs to $Z(\widehat{H})^{\Gamma(p)}$. Choose elements $t_i \in Z(\widehat{H})^{\Gamma(p)}$ $(i = 1, \ldots, r)$ such that $t_1 \ldots t_r = s^{-1}$, and let t be the element $(t_1, \ldots, t_r)$ of $\widehat{G} \times \ldots \times \widehat{G}$. We are going to define $\widetilde{\eta}$ by using t to twist the composition $\widetilde{\eta}_0 : {}^LH \to {}^LR$ of $\eta : {}^LH \to {}^LG$ and the canonical embedding ${}^LG \to {}^LR$ (the one induced by the diagonal map $\widehat{G} \to \widehat{G} \times \ldots \times \widehat{G}$, which is $W_{\mathbb{Q}_p}$-equivariant). Since the centralizer of $\widetilde{\eta}_0(\widehat{H})$ in $\widehat{G} \times \ldots \times \widehat{G}$ is $Z(\widehat{H}) \times \ldots \times Z(\widehat{H})$, we can twist $\widetilde{\eta}_0$ by the unramified 1-cocycle of $W_{\mathbb{Q}_p}$ taking σ to t. This gives us the desired $\eta : {}^LH \to {}^LR$; note that $\eta(\sigma) = t \cdot \eta_0(\sigma) = t \rtimes \sigma$ and that $\eta(\sigma^r) = \eta_0(s^{-1}\sigma^r)$. Since any two choices for t are σ-conjugate by an element of $Z(\widehat{H})^{\Gamma(p)} \times \ldots \times Z(\widehat{H})^{\Gamma(p)}$ (on which σ acts by $\widehat{\theta}$), the pair $(t, \widetilde{\eta})$ is well-defined up to replacement by $(ut\widehat{\theta}(u)^{-1}, \mathrm{Int}\,(u) \circ \widetilde{\eta})$ for $u \in \widehat{R}$. Moreover one sees easily that $\widetilde{\eta}(\widehat{H})$ is the identity component of the $\widehat{\theta}$-centralizer of t in $\widehat{R}$, and thus $(H, t, \widetilde{\eta})$ is a twisted endoscopic datum for (R, θ).

We are already making one temporary assumption (that $s \in Z(\widehat{H})^{\Gamma(p)}$). Now we make a second temporary assumption, namely that $\eta : {}^L H \to {}^L G$ is unramified, by which we mean that η comes by inflation from an L-homomorphism

$$\widehat{H} \rtimes W(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p) \to \widehat{G} \rtimes W(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p).$$

Note that unramified extensions η of η_0 always exist locally at p. Under this assumption, $\widetilde{\eta}$ is also unramified and hence induces a homomorphism of Hecke algebras

$$\mathcal{H}(G(F), K_F) \to \mathcal{H}(H(\mathbf{Q}_p), K_H)$$

for any hyperspecial maximal compact subgroup K_H of $H(\mathbf{Q}_p)$. We define h_p to be the image of $\phi_j \in \mathcal{H}(G(F), K_F)$ under this homomorphism. Since any two choices for K_H are conjugate under $H_{\mathrm{ad}}(\mathbf{Q}_p)$, the stable orbital integrals of h_p are independent of the choice of K_H.

Assuming the validity of the conjectural "fundamental lemma" for this homomorphism of Hecke algebras, the stable orbital integrals of h_p for (G, H)-regular semisimple $\gamma_H \in H(\mathbf{Q}_p)$ are given by

$$SO_{\gamma_H}(h_p) = \sum_{\delta} \langle \alpha(\gamma_0; \delta), s\rangle \cdot \Delta_p(\gamma_H, \gamma_0) \cdot e(I) \cdot TO_\delta(\phi_j) \tag{7.2}$$

for a certain normalization of Δ_p determined by the $\mathbf{Z}_p$-structure on $G_{\mathbf{Q}_p}$; we assume temporarily that Δ_p is so normalized. The right side of (7.2) requires some explanation. We choose a semisimple element $\gamma_0 \in G(\mathbf{Q}_p)$ that comes from γ_H; this is possible since G is quasi-split. We write I_0 for the centralizer of γ_0. The sum is taken over a set of representatives for the σ-conjugacy classes of elements $\delta \in G(F)$ such that the norm $N\delta$ of δ is conjugate to γ_0 under $G(\overline{\mathbf{Q}}_p)$. We define an element

$$\alpha(\gamma_0; \delta) \in X^*(Z(\widehat{I}_0)^{\Gamma(p)})$$

in the same way we defined $\alpha_p(\gamma_0; \gamma, \delta)$. In order to pair s with $\alpha(\gamma_0; \delta)$ we use the canonical embedding $Z(\widehat{H}) \to Z(\widehat{I}_0)$ as in **[K6]** (5.6). Note that the product

$$\langle \alpha(\gamma_0; \delta), s\rangle \Delta_p(\gamma_H, \gamma_0)$$

is independent of the choice of γ_0 (with the normalization of transfer factors that is being used in this paper). The group I is the θ-centralizer of δ:

$$I = \{x \in R | x^{-1}\delta\theta(x) = \delta\},$$

and $e(I)$ is the sign attached to the $\mathbf{Q}_p$-group I.

It is clear from the definition of ϕ_j that $TO_\delta(\phi_j)$ vanishes unless δ satisfies the following condition: the image of the σ-conjugacy class of δ under $B(G_{\mathbf{Q}_p}) \to X^*(Z(\widehat{G})^{\Gamma(p)})$ is equal to the restriction of $-\mu_1 \in X^*(Z(\widehat{G}))$ to $Z(\widehat{G})^{\Gamma(p)}$ (μ_1 was defined in §2). Therefore in (7.2) we may as well sum over only those δ satisfying this additional restriction. For such δ we can extend $\alpha(\gamma_0;\delta)$ to a character $\beta(\gamma_0;\delta)$ on $Z(\widehat{I}_0)^{\Gamma(p)}Z(\widehat{G})$ by defining $\beta(\gamma_0;\delta)$ to be equal to $-\mu_1$ on $Z(\widehat{G})$, as in §2. We can rewrite (7.2) as

$$SO_{\gamma_H}(h_p) = \sum_\delta \langle \beta(\gamma_0;\delta), s\rangle \cdot \Delta_p(\gamma_H,\gamma_0) \cdot e(I) \cdot TO_\delta(\phi_j). \tag{7.3}$$

Now we remove our temporary assumptions on η, s and Δ_p. We still want h_p to satisfy (7.3), which is why we had to rewrite (7.2) as (7.3): $\langle \beta(\gamma_0;\delta), s\rangle$ makes sense for any $s \in Z(\widehat{H})^{\Gamma(p)}Z(\widehat{G})$, but $\langle\alpha(\gamma_0;\delta),s\rangle$ does not. If s is replaced by sz for $z \in Z(\widehat{G})$, the right side of (7.3) is multiplied by $\mu_1(z)^{-1}$. Therefore we can multiply the original h_p by $\mu_1(z)^{-1}$ to get a function that works for sz. Multiplying Δ_p by a constant multiplies h_p by the same constant. Any two choices for η differ by an element of $H^1(W_{\mathbf{Q}_p}, Z(\widehat{H}))$, and this element determines a quasi-character χ on $H(\mathbf{Q}_p)$. The transfer factors for these two choices of η are related by the factor $\chi(\gamma_H)$. In order to get a function on $H(\mathbf{Q}_p)$ that works for general η we start with one that works for an unramified η and multiply it by the quasi-character χ on $H(\mathbf{Q}_p)$ that relates the two choices of η.

Let us summarize what we have done. Granting the fundamental lemma for the endoscopic groups of (R,θ), we have constructed a function $h_p \in C_c^\infty(H(\mathbf{Q}_p))$ satisfying (7.3) for every (G,H)-regular semisimple element $\gamma_H \in H(\mathbf{Q}_p)$. If η is unramified, then h_p belongs to $\mathcal{H}(H(\mathbf{Q}_p),K_H)$; in general it is a quasi-character on $H(\mathbf{Q}_p)$ times a function in $\mathcal{H}(H(\mathbf{Q}_p),K_H)$.

Our next task is to construct the required function $h_\infty \in C^\infty(H(\mathbf{R}))$. Let T be an elliptic maximal torus in $G_{\mathbf{R}}$. Recall that we are assuming that T comes from a maximal torus T_H in $H_{\mathbf{R}}$. In particular the split component of the center of $H_{\mathbf{R}}$ is equal to A_G. We want the function h_∞ to be compactly supported modulo $A_G(\mathbf{R})^0$. In fact we will construct h_∞ as a suitable linear combination of pseudo-coefficients for certain discrete series representations of $H(\mathbf{R})$. We want h_∞ to have the following stable orbital integrals for (G,H)-regular semisimple $\gamma_H \in H(\mathbf{R})$: $SO_{\gamma_H}(h_\infty) = 0$ unless γ_H is elliptic in $H(\mathbf{R})$, in which case

$$(7.4)\quad SO_{\gamma_H}(h_\infty) = \langle \beta(\gamma_0), s\rangle \cdot \Delta_\infty(\gamma_H, \gamma_0) \cdot e(I) \cdot \operatorname{tr} \xi_{\mathbf{C}}(\gamma_0) \cdot \mathrm{vol}^{-1}.$$

The right side of (7.4) requires some explanation. We choose an element $\gamma_0 \in T(\mathbf{R})$ that comes from γ_H, and we write I_0 for the centralizer of γ_0. We define an element

$$\beta(\gamma_0) \in X^*(Z(\widehat{I}_0)^{\Gamma(\infty)})$$

in the same way we defined $\beta_\infty(\gamma_0; \gamma, \delta)$ (using μ_h for $h \in X_\infty$ factoring through T), and we pair s with $\beta(\gamma_0)$ by means of the canonical embedding $Z(\widehat{H}) \to Z(\widehat{I}_0)$ as in **[K6]** (5.6). The product

$$\langle \beta(\gamma_0), s\rangle \Delta_\infty(\gamma_H, \gamma_0)$$

is independent of the choice of γ_0. The group I is the inner form of I_0 such that I/A_G is anisotropic over $\mathbf{R}$, and $e(I)$ is the sign attached to the $\mathbf{R}$-group I. We write vol as an abbreviation for $\mathrm{vol}\,(A_G(\mathbf{R})^0 \backslash I(\mathbf{R}))$. Note that replacing s by sz for $z \in Z(\widehat{G})$ multiplies h_∞ by $\mu_1(z)$.

Constructing h_∞ as a linear combination of pseudo-coefficients is an easy exercise in using the work of Clozel-Delorme and Shelstad. First we need a review of endoscopy for discrete series representations of $G(\mathbf{R})$. We now write ${}^LG$ for $\widehat{G} \rtimes W_{\mathbf{R}}$.

Let φ be a Langlands parameter $W_{\mathbf{R}} \to {}^LG$. We write C_φ for the centralizer of $\varphi(W_{\mathbf{R}})$ in $\widehat{G}$ and S_φ for $C_\varphi Z(\widehat{G})$. Now assume that φ is elliptic, in the sense that $S_\varphi/Z(\widehat{G})$ is finite. Langlands **[L1]** associates to φ an L-packet $\prod(\varphi)$ consisting of discrete series representations of $G(\mathbf{R})$, in the following way. First of all φ determines a pair $(\widehat{S}, \widehat{B})$ consisting of a maximal torus $\widehat{S}$ of $\widehat{G}$ and a Borel subgroup $\widehat{B}$ of $\widehat{G}$ containing $\widehat{S}$. The maximal torus $\widehat{S}$ is simply the centralizer of $\varphi(W_{\mathbf{C}})$ in $\widehat{G}$. Since $W_{\mathbf{C}}$ is abelian, the composition

$$\mathbf{C}^\times = W_{\mathbf{C}} \xrightarrow{\varphi} G \times W_{\mathbf{C}} \to G$$

factors through $\widehat{S}$ and can be written as $z \mapsto z^{\Lambda}\bar{z}^{\Lambda'}$ for unique $\Lambda, \Lambda' \in X_*(\widehat{S}) \otimes \mathbb{C}$ such that $\Lambda - \Lambda' \in X_*(\widehat{S})$ ($z^{\Lambda}\bar{z}^{\Lambda'}$ is to be interpreted as the unique element of $\widehat{S}$ such that for every $\lambda \in X^*(\widehat{S})$ $\lambda(z^{\Lambda}\bar{z}^{\Lambda'}) = z^{\langle \Lambda - \Lambda', \lambda\rangle}(z\bar{z})^{\langle \Lambda', \lambda\rangle}$). The group $\widehat{B}$ is the unique Borel subgroup containing $\widehat{S}$ such that $\langle \Lambda, \alpha\rangle$ is positive for every root α of $\widehat{S}$ that is positive for $\widehat{B}$. Of course the $\widehat{G}$-conjugacy class of $(\widehat{S}, \widehat{B})$ depends only on the equivalence class of φ.

Recall that we have fixed an elliptic maximal torus T of G. Write $\mathcal{B}$ for the set of Borel subgoups of $G_{\mathbb{C}}$ containing T. An L-homomorphism ${}^LT \to {}^LG$ is said to be an admissible embedding if the restriction $\widehat{T} \to \widehat{G}$ belongs to the canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{T} \to \widehat{G}$. For $B \in \mathcal{B}$ there is an admissible embedding $\eta_B : {}^LT \to {}^LG$, unique up to $\widehat{G}$-conjugacy, such that

(a) For all $z \in \mathbb{C}^\times = W_{\mathbb{C}} \subset {}^LT$

$$\eta_B(z) = \eta_B(z^{\delta}\bar{z}^{-\delta}) \times z \in \widehat{G} \times W_{\mathbb{C}},$$

where δ is the half sum of the B-positive roots for T, viewed as an element of $X_*(\widehat{T}) \otimes \mathbb{C}$.

(b) The restriction of η_B to $W_{\mathbb{R}} \subset {}^LT$ factors through ${}^L(G_{\mathrm{ad}}) \to {}^LG$.

The restriction of η_B to $W_{\mathbb{R}}$ is the Langlands parameter for the L-packet consisting of discrete series representations having the same infinitesimal and central characters as the trivial representation. For arbitrary elliptic $\varphi : W_{\mathbb{R}} \to {}^LG$ and for $B \in \mathcal{B}$ we let $\pi(\varphi, B)$ denote the unique (up to isomorphism) irreducible discrete series representation π of $G(\mathbb{R})$ whose character Θ_π is given on $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$ by

$$(-1)^{q(G)} \sum_{\omega \in \Omega_{G(\mathbb{R})}} \chi_{\omega(B)}(\gamma) \cdot \Delta_{\omega(B)}(\gamma)^{-1}.$$

This expression requires some explanation. The integer $q(G)$ is half the real dimension of the symmetric space attached to $G(\mathbb{R})$. We write Ω for the Weyl group $\Omega(T(\mathbb{C}), G(\mathbb{C}))$ and $\Omega_{G(\mathbb{R})}$ for the real Weyl group $\Omega(T(\mathbb{R}), G(\mathbb{R}))$. For any $B \in \mathcal{B}$ we define a complex-valued function $\Delta_B(\gamma)$ on $T(\mathbb{R})$ by

$$\Delta_B(\gamma) = \prod_{\alpha}(1 - \alpha(\gamma)^{-1})$$

where α runs through the B-positive roots of T. For any $B \in \mathcal{B}$ we define a quasi-character $\chi_B = \chi(\varphi, B)$ on $T(\mathbb{R})$ as follows. The pairs $(\widehat{S}, \widehat{B})$, (T, B) determine an isomorphism $\widehat{T} \xrightarrow{\sim} \widehat{S}$. Conjugate η_B by $\widehat{G}$ so that it agrees with $\widehat{T} \xrightarrow{\sim} \widehat{S}$ on $\widehat{T} \subset {}^L T$, and write $\varphi = \eta_B \circ \varphi_B$ where φ_B is a Langlands parameter for T. Then χ_B is the quasi-character on $T(\mathbb{R})$ corresponding to φ_B.

It is clear that $\pi(\varphi, B) \approx \pi(\varphi, \omega(B))$ $(\omega \in \Omega)$ if and only if $\omega \in \Omega_{G(\mathbb{R})}$. The L-packet $\prod(\varphi)$ is equal to

$$\{\pi(\varphi, B) | B \in \Omega_{G(\mathbb{R})} \backslash \mathcal{B}\}.$$

Now we review Shelstad's results on endoscopy for discrete series representations. Fix an elliptic maximal torus T_H in our $\mathbb{R}$-elliptic endoscopic group H, and write $\mathcal{B}_H$ for the set of Borel subgroups of $H_{\mathbb{C}}$ containing T_H. There is a canonical set J of isomorphisms $j : T_H \to T$, permuted simply transitively by Ω. Each pair $(j, B) \in J \times \mathcal{B}$ determines an element $B_H \in \mathcal{B}_H$; in this way we get a map $(j, B) \mapsto B_H$. Given (j, B), each element $\omega \in \Omega$ can be decomposed uniquely as $\omega_H \omega_*$ for $\omega_H \in \Omega_H$, $\omega_* \in \Omega_*$, where $\Omega_H = \Omega(T_H(\mathbb{C}), H(\mathbb{C}))$, viewed as a subgroup of Ω via j, and Ω_* is the set of $\omega \in \Omega$ such that $(j, \omega(B)), (j, B)$ have the same image under $J \times \mathcal{B} \to \mathcal{B}_H$. Given (j, B) there is a unique normalization $\Delta_{j,B}$ of the transfer factors Δ_∞ such that for all (G, H)-regular $\gamma_H \in T_H(\mathbb{R})$

$$\Delta_{j,B}(\gamma_H, \gamma) = (-1)^{q(G)+q(H)} \cdot \chi_{G,H}(\gamma) \cdot \Delta_B(\gamma^{-1}) \cdot \Delta_{B_H}(\gamma_H^{-1})^{-1}$$

where $\gamma = j(\gamma_H)$ and $(j, B) \mapsto B_H$. Here $\chi_{G,H}$ is the quasi-character on $T(\mathbb{R})$ corresponding to the following 1-cocycle a of $W_{\mathbb{R}}$ in $\widehat{T}$. The group B determines $\eta_B : {}^L T \to {}^L G$. The group B_H determines $\eta_{B_H} : {}^L T_H \to {}^L H$. Conjugate η_B by $\widehat{G}$ to arrange that $\eta \circ \eta_{B_H} \circ \widehat{j}$ and η_B agree on $\widehat{T}$; then the 1-cocycle a is defined by $\eta \circ \eta_{B_H} \circ \widehat{j} = \eta_B \cdot a$.

Let $\omega \in \Omega$ and write $\omega = \omega_H \omega_*$ with $\omega_H \in \Omega_H$ and $\omega_* \in \Omega_*$ (with $\Omega_H \hookrightarrow \Omega$ and Ω_* defined by (j, B)). Then it is not hard to see that $\Delta_{j,\omega(B)} = \det(\omega_*)\Delta_{j,B}$, where $\det(\omega_*) = \det(\omega_*; X^*(T))$.

Let $\varphi_H : W_{\mathbb{R}} \to {}^L H$ be a Langlands parameter for H such that $\varphi := \eta \circ \varphi_H$ is elliptic. Then φ_H is itself elliptic, and as before we get pairs $(\widehat{S}, \widehat{B})$ in $\widehat{G}$ and $(\widehat{S}_H, \widehat{B}_H)$ in $\widehat{H}$. We say that $(j, B, B_H) \in J \times \mathcal{B} \times \mathcal{B}_H$ is *aligned* with φ_H if the following diagram of isomorphisms commutes

$$\begin{array}{ccc} \widehat{T} & \rightarrow & \widehat{S} \\ \widehat{j} \downarrow & & \uparrow \eta \\ \widehat{T}_H & \rightarrow & \widehat{S}_H, \end{array}$$

where $\widehat{T} \to \widehat{S}$ (resp., $\widehat{T}_H \to \widehat{S}_H$) is determined by $(B, \widehat{B})$ (resp., $(B_H, \widehat{B}_H)$); this notion depends only on the equivalence class of φ_H. Note that if (j, B, B_H) is aligned with φ_H, then $(j, B) \mapsto B_H$.

Shelstad **[Sh1]** (Thm. 4.1.1 and (4.4.3)) has shown that if the transfer factors $\Delta_\infty(\gamma_H, \gamma)$ are used to transfer the stable distribution

$$S\Theta_{\varphi_H} := \sum_{\pi_H \in \Pi(\varphi_H)} \Theta_{\pi_H}$$

on $H(\mathbb{R})$ to a distribution Tran $(S\Theta_{\varphi_H})$ on $G(\mathbb{R})$, then there exist unique complex numbers $\Delta_\infty(\varphi_H, \pi)$ $(\pi \in \Pi(\varphi))$ such that

$$\mathrm{Tran}\,(S\Theta_{\varphi_H}) = \sum_{\pi \in \Pi(\varphi)} \Delta_\infty(\varphi_H, \pi)\Theta_\pi.$$

More precisely, if (j, B, B_H) is aligned with φ_H and if Δ_∞ is taken to be $\Delta_{j,B}$, then

$$\Delta_\infty(\varphi_H, \pi(\varphi, \omega^{-1}(B))) = \langle a_\omega, s\rangle,$$

where s is viewed as an element of $\widehat{T}^{\Gamma(\infty)}Z(\widehat{G})$ via j and a_ω is the character on $(\widehat{T}/Z(\widehat{G}))^{\Gamma(\infty)}$ described in §5.

Now we are in a position to construct the desired function h_∞ on $H(\mathbb{R})$. Fix $(j, B) \in J \times \mathcal{B}$ and let B_H be the image of (j, B) under $J \times \mathcal{B} \to \mathcal{B}_H$. Without loss of generality we assume that $\Delta_\infty = \Delta_{j,B}$. Let $\varphi : W_{\mathbb{R}} \to {}^LG$ be an elliptic Langlands parameter whose L-packet $\prod(\varphi)$ consists of the discrete series representations of $G(\mathbb{R})$ having the same infinitesimal and central characters as the contragredient of $\xi_{\mathbb{C}}$. Thus

$$(-1)^{q(G)}S\Theta_\varphi(\gamma^{-1}) = \mathrm{tr}\,\xi_{\mathbb{C}}(\gamma)$$

for all $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$. Let $\Phi_H(\varphi)$ denote the set of equivalence classes of Langlands parameters $\varphi_H : W_{\mathbb{R}} \to {}^LH$ such that $\eta \circ \varphi_H$ is equivalent to φ. For $\varphi_H \in \Phi_H(\varphi)$ there exists a unique $\omega_*(\varphi_H) \in \Omega_*$ such that $(\omega_*^{-1} \circ j, B, B_H)$ is aligned with φ_H; this construction yields a bijection between $\Phi_H(\varphi)$ and Ω_*.

As usual we choose $h \in X_\infty$ factoring through T and obtain $\mu_h \in X_*(T) = X^*(\widehat{T})$. Using j to regard s as an element of $\widehat{T}^{\Gamma(\infty)}Z(\widehat{G})$, we can pair μ_h and s, obtaining $\langle \mu_h, s\rangle \in \mathbb{C}^\times$. Using Lemma 5.1, we see that $\langle \mu_h, s\rangle$ is independent of the choice of h.

For $\varphi_H \in \Phi_H(\varphi)$ we define a function $h(\varphi_H)$ on $H(\mathbf{R})$ by taking

$$h(\varphi_H) = |\Omega_H/\Omega_{H(\mathbf{R})}|^{-1} \sum_{\pi_H} h(\pi_H),$$

where π_H ranges over the L-packet $\Pi(\varphi_H)$ and $h(\pi_H)$ is a pseudo-coefficient [C-D] for π_H. The restriction to $A_G(\mathbf{R})^\circ$ of the central character of π_H is independent of π_H, φ_H; therefore each function $h(\varphi_H)$ transforms under $A_G(\mathbf{R})^0$ by the same quasi-character. The desired function h_∞ turns out to be

$$h_\infty = \langle \mu_h, s\rangle \cdot \sum_{\varphi_H \in \Phi_H(\varphi)} \det(\omega_*(\varphi_H)) \cdot (-1)^{q(G)} \cdot h(\varphi_H).$$

It is easy to see that h_∞ has the right stable orbital integrals for (G, H)-regular semisimple $\gamma_H \in H(\mathbf{R})$. A continuity argument using Lemma 2.9.3 of [**Sh2**] shows that it is enough to consider $\gamma_H \in T_H(\mathbf{R})$ such that $\gamma := j(\gamma_H)$ is regular in G. The stable orbital integral is zero unless γ_H is elliptic. For elliptic G-regular γ_H the stable orbital integral $SO_{\gamma_H}(h_\infty)$ equals

$$\langle \mu_h, s\rangle \cdot \mathrm{vol}^{-1} \cdot \sum_{\varphi_H} \det(\omega_*) \cdot (-1)^{q(G)} \cdot S\Theta_{\varphi_H}(\gamma_H^{-1}).$$

Moreover

$$S\Theta_{\varphi_H}(\gamma_H^{-1}) = (-1)^{q(H)} \sum_{\omega_H \in \Omega_H} \chi_{\omega_H(B_H)}(\gamma_H^{-1}) \Delta_{\omega_H(B_H)}(\gamma_H^{-1})^{-1}$$

$$= (-1)^{q(G)} \sum_{\omega_H \in \Omega_H} \Delta_{j,\omega(B)}(\gamma_H, \gamma) \chi_{\omega(B)}(\gamma^{-1}) \cdot \Delta_{\omega(B)}(\gamma^{-1})^{-1},$$

where $\omega := \omega_H \omega_*$. Since $\Delta_{j,\omega(B)} = \det(\omega_*)\Delta_{j,B}$, we see that $SO_{\gamma_H}(h_\infty)$ equals

$$\langle \mu_h, s\rangle \cdot \mathrm{vol}^{-1} \cdot \Delta_{j,B}(\gamma_H, \gamma) \cdot \sum_{\omega \in \Omega} \chi_{\omega(B)}(\gamma^{-1}) \cdot \Delta_{\omega(B)}(\gamma^{-1})^{-1}$$

$$= \langle \mu_h, s\rangle \cdot \mathrm{vol}^{-1} \cdot \Delta_{j,B}(\gamma_H, \gamma) \cdot \mathrm{tr}\, \xi_{\mathbf{C}}(\gamma),$$

in agreement with (7.4).

This is a convenient moment to evaluate $S\Theta_{\varphi_H}(h_\infty)$ for $\varphi_H \in \Phi_H(\varphi)$. Let $(\widehat{S}, \widehat{B})$ be the pair associated to φ as before. For π in the L-packet of φ we now define a character $\lambda_\pi \in X^*(S_\varphi)$, where $S_\varphi = C_\varphi \cdot Z(\widehat{G})$ as before. Choose $B \in \mathcal{B}$ such that $\pi = \pi(\varphi, B)$. Then $B, \widehat{B}$ determine an isomorphism $\widehat{T} \tilde{\to} \widehat{S}$, and this isomorphism restricts to an isomorphism

$$\widehat{T}^{\Gamma(\infty)} Z(\widehat{G}) \tilde{\to} S_\varphi.$$

As before $\mu_h \in X^*(\widehat{T})$ restricts to a well-defined character on $\widehat{T}^{\Gamma(\infty)} Z(\widehat{G})$; we transport this character to S_φ and denote it by λ_π.

Choose $g \in \widehat{G}$ such that $\varphi = \operatorname{Int}(g) \circ \eta \circ \varphi_H$. Then $\epsilon := g\eta(s)g^{-1}$ belongs to S_φ and is independent of the choice of g since S_φ is abelian; we can pair λ_π and ϵ, obtaining $\langle \lambda_\pi, \epsilon \rangle \in \mathbb{C}^\times$.

LEMMA 7.1. *The product* $\langle \lambda_\pi, \epsilon \rangle \cdot \Delta_\infty(\varphi_H, \pi)$ *is the same for all* $\pi \in \Pi(\varphi)$. *Moreover*

$$S\Theta_{\varphi_H}(h_\infty) = (-1)^{q(G)} \langle \lambda_\pi, \epsilon \rangle \cdot \Delta_\infty(\varphi_H, \pi).$$

Let $\pi \in \Pi(\varphi)$ and choose $B \in \mathcal{B}$ such that $\pi = \pi(\varphi, B)$. Choose j, B_H so that (j, B, B_H) is aligned with φ_H, and use the transfer factors $\Delta_\infty = \Delta_{j,B}$ to make the calculation. Our expression for h_∞ in terms of pseudo-coefficients implies that

$$S\Theta_{\varphi_H}(h_\infty) = \langle \mu_h, s \rangle \cdot (-1)^{q(G)}$$

(since $\omega_* = 1$). Moreover $\Delta_{j,B}(\varphi_H, \pi) = 1$ (a special case of the result of Shelstad mentioned before). Therefore we have only to show that $\langle \lambda_\pi, \epsilon \rangle = \langle \mu_h, s \rangle$ with the choices that we have made. We may as well assume that $\varphi = \eta \circ \varphi_H$. Since (j, B, B_H) is aligned with φ_H, the following diagram commutes:

$$\begin{array}{ccc} \widehat{T} & \to & \widehat{S} \\ \downarrow & & \uparrow \\ \widehat{T}_H & \to & \widehat{S}_H. \end{array}$$

Therefore s maps to ϵ under $\widehat{T}^{\Gamma(\infty)} Z(\widehat{G}) \tilde{\to} S_\varphi$, and since μ_h corresponds to λ_π under $\widehat{T}^{\Gamma(\infty)} Z(\widehat{G}) \tilde{\to} S_\varphi$ by the definition of λ_π, it follows that $\langle \lambda_\pi, \epsilon \rangle$ is indeed equal to $\langle \mu_h, s \rangle$.

We can finally define a function h on $H(\mathbb{A})$ by taking $h = h^p h_p h_\infty$. Replacing s by sz for $z \in Z(\widehat{G})$ has no effect on h. Moreover the adelic transfer factor Δ is canonical (although it depends on η, not just η_0). Thus the function h (or, more precisely, its stable orbital integrals) depends only on H, s, η. Let γ_H be a (G, H)-regular semisimple element of $H(\mathbb{Q})$. Then $SO_{\gamma_H}(h) = 0$ unless γ_H is elliptic in $H(\mathbb{R})$ and appears in $G(\mathbb{Q}_v)$ for every place v of $\mathbb{Q}$ (this last condition is automatic at p since G is quasi-split over $\mathbb{Q}_p$ and G_{der} is simply connected [**K1**] Thm. 4.1, and is automatic at ∞ if γ_H is elliptic at ∞, since we are assuming that the elliptic maximal tori of $G_{\mathbb{R}}$ come from $H_{\mathbb{R}}$). Suppose that γ_H satisfies these two conditions. Then γ_H appears in $G(\mathbb{Q})$, as we will now check.

Choose $\gamma \in G(\mathbb{A})$ coming from γ_H. Choose an inner twisting $\psi : G^* \to G$ with G^* quasi-split and let γ^* be an element of $G^*(\mathbb{Q})$ coming from γ_H (use [**K1**] Thm. 4.1 again). Then γ and γ^* determine an element $\mathrm{obs}(\gamma) \in \mathfrak{K}(I^*/\mathbb{Q})^D$ [**K6**] (6.5), where I^* denotes the centralizer of γ^* in G^*. Note that the identity component of $Z(\widehat{I}^*)^{\Gamma(\infty)}$ is contained in $Z(\widehat{G})$, since γ^* is elliptic in $G^*(\mathbb{R})$. By the Chebotarev density theorem the same is true for $Z(\widehat{I}^*)^{\Gamma(v)}$ for some finite place v of $\mathbb{Q}$. Therefore $\mathfrak{K}(I^*/\mathbb{Q}) \to \mathfrak{K}(I^*/\mathbb{Q}_v)$ is injective and $\mathfrak{K}(I^*/\mathbb{Q}_v)^D \to \mathfrak{K}(I^*/\mathbb{Q})^D$ is surjective. Therefore, replacing the v-component of γ by some stable conjugate, we may assume that $\mathrm{obs}(\gamma)$ is trivial, and then from Theorem 6.6 of [**K6**] we conclude that γ is $G(\mathbb{A})$-conjugate to an element of $G(\mathbb{Q})$.

Choose $\gamma_0 \in G(\mathbb{Q})$ coming from γ_H. Then γ_0 is elliptic in $G(\mathbb{R})$, so that the considerations of §2 apply. In particular, when (γ, δ) satisfies the conditions of §2 we have the element $\alpha(\gamma_0; \gamma, \delta) \in \mathfrak{K}(I_0/\mathbb{Q})^D$. Using the "global hypothesis" (that $\Delta(\gamma_H, \gamma_0) = 1$), we see that $SO_{\gamma_H}(h)$ equals

$$(7.5) \qquad \sum_{(\gamma,\delta)} \langle \alpha(\gamma_0; \gamma, \delta), s \rangle \cdot e(\gamma, \delta) \cdot O_\gamma(f^p_{\mathbb{C}}) \cdot TO_\delta(\phi_j) \cdot$$

$$\operatorname{tr} \xi_{\mathbb{C}}(\gamma_0) \cdot \operatorname{vol}(A_G(\mathbb{R})^0 \backslash I(\infty)(\mathbb{R}))^{-1}$$

with $\sum_{(\gamma,\delta)}$ and $e(\gamma, \delta)$ as in (4.2).

Now we can write (4.2) in terms of elliptic endoscopic triples. Choose a set $\mathcal{E}$ of representatives for the isomorphism classes of elliptic endoscopic triples (H, s, η_0) for G, and for each one choose an L-homomorphism $\eta : {}^L H \to {}^L G$ extending η_0 and write h for the function on $H(\mathbb{A})$ that we have just discussed. Write $ST_e^*(h)$ for the (G,H)-regular $\mathbb{Q}$-elliptic part of the stable trace formula for (H,h):

$$ST_e^*(h) = \sum_{\gamma_H} |(H_{\gamma_H}/H_{\gamma_H}^0)(\mathbb{Q})|^{-1} \cdot \tau(H) \cdot SO_{\gamma_H}(h),$$

where γ_H runs over a set of representatives for the (G,H)-regular semisimple $\mathbb{Q}$-elliptic stable conjugacy classes in $H(\mathbb{Q})$. As usual [**L7**], [**K4**] we write $\iota(G,H)$ for the positive rational number

$$\iota(G,H) = \tau(G) \cdot \tau(H)^{-1} \cdot |\mathrm{Aut}\,(H,s,\eta)/H_{\mathrm{ad}}\,(\mathbb{Q})|^{-1}.$$

THEOREM 7.2. *Assume the existence of functions h having the stable orbital integrals prescribed above. Then the expression (3.1) (conjecturally the contribution of $P = G$ to* $\mathrm{tr}\,(f \times \Phi_{\mathfrak{p}}^j; W_\lambda)$*) is equal to*

$$\sum_{\mathcal{E}} \iota(G,H) ST_e^*(h).$$

Comparing (4.2) and (7.5), we see that the theorem follows from Lemma 9.7 of [**K6**] and the following two remarks. Suppose that $(H,s,\eta,\gamma_H) \to (\gamma_0,\kappa)$ in the sense of §9 of [**K6**]. The first remark is that γ_0 is elliptic in $G(\mathbb{R})$ if and only if γ_H is elliptic in $H(\mathbb{R})$ and the elliptic maximal tori of $G_{\mathbb{R}}$ come from $H_{\mathbb{R}}$. The second remark is that for fixed γ the sum

$$\sum_{\delta} \langle \alpha(\gamma_0;\gamma,\delta),\kappa\rangle \cdot e(I(p)) \cdot TO_\delta(\phi_j)$$

vanishes unless H is unramified at p (use the proof of Proposition 7.5 of [**K6**]).

Part II. λ-adic representations

§8. REVIEW OF ARTHUR'S CONJECTURES

Let χ be a quasi-character on $A_G(\mathbb{R})^0$. One expects [**L-L**], [**K4**] that the cuspidal tempered part of $L^2_\chi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is isomorphic to

$$\bigoplus_{[\varphi]} \bigoplus_{\pi\in\Pi(\varphi)} m(\varphi,\pi)\cdot\pi, \qquad (8.1)$$

where $m(\varphi,\pi) = |\mathfrak{S}_\varphi|^{-1} \sum_{x\in\mathfrak{S}_\varphi} \langle x,\pi\rangle$.

Here $[\varphi]$ stands for the equivalence class of an admissible homomorphism $\varphi : \mathcal{L} \to {}^LG$ ($\mathcal{L}$ denotes the conjectural Langlands group of $\overline{\mathbb{Q}}/\mathbb{Q}$), and the first direct sum is taken over all equivalence classes of elliptic φ such that the associated quasi-character on $A_G(\mathbb{R})^0$ equals χ. We write $\Pi(\varphi)$ for the (conjectural) L-packet of φ. As in §10 of **[K4]** we write S_φ for the group of self-equivalences of φ and $\mathfrak{S}_\varphi$ for the quotient of S_φ by $S_\varphi^0 \cdot Z(\widehat{G})$.

Let $\psi : \mathcal{L} \times SL_2(\mathbb{C}) \to {}^LG$ be a homomorphism. We say that ψ is an Arthur parameter if the restriction of ψ to $\mathcal{L}$ is an essentially tempered Langlands parameter, and we say that ψ is elliptic if S_ψ^0 is contained in $Z(\widehat{G})$. Arthur **[A2]**, **[A3]** conjectures that the discrete part of $L^2_\chi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is isomorphic to

$$\bigoplus_{[\psi]} \bigoplus_{\pi\in\Pi(\psi)} m(\psi,\pi)\cdot\pi, \tag{8.2}$$

where $m(\psi,\pi) = |\mathfrak{S}_\psi|^{-1} \sum_{x\in\mathfrak{S}_\psi} \epsilon_\psi(x)\langle x,\pi\rangle$.

Here the first sum is taken over all equivalence classes of elliptic Arthur parameters ψ such that the associated quasi-character on $A_G(\mathbb{R})^0$ equals χ. We write $\Pi(\psi)$ for the (conjectural) Arthur packet of ψ. The notion of equivalence and the groups S_ψ, $\mathfrak{S}_\psi$ are defined in the same way as for φ. The new ingredient is the character $\epsilon_\psi : \mathfrak{S}_\psi \to \{\pm 1\}$ introduced by Arthur (see §8 of **[A3]**).

Let f be a smooth function on $G(\mathbb{A})$, compactly supported modulo $A_G(\mathbb{R})^0$, transforming by χ^{-1} under $A_G(\mathbb{R})^0$. According to (8.2), the trace $T_d(f)$ of f on the discrete part of $L^2_\chi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ should be

$$\sum_{[\psi]} |\mathfrak{S}_\psi|^{-1} \sum_{\pi\in\Pi(\psi)} \sum_{x\in\mathfrak{S}_\psi} \epsilon_\psi(xs_\psi)\cdot\langle xs_\psi,\pi\rangle \cdot \operatorname{tr}\pi(f). \tag{8.3}$$

Here s_ψ denotes the element of S_ψ obtained as the image under ψ of the non-trivial central element of $SL_2(\mathbb{C})$. The reason for introducing s_ψ is the following. Let (H, s, η_0) be an elliptic endoscopic triple for G, and let $\eta : {}^LH \to {}^LG$ be an L-homomorphism extending η_0. Let ψ_H be an Arthur parameter for H, and let $\psi = \eta\circ\psi_H$. The element $s \in Z(\widehat{H})$ determines an element $x = \eta(s) \in S_\psi$, and Arthur conjectures that

$$\sum_{\pi_H\in\Pi(\psi_H)} \langle s_{\psi_H}, \pi_H\rangle \operatorname{tr}\pi_H(f^H) = \sum_{\pi\in\Pi(\psi)} \langle xs_\psi,\pi\rangle \operatorname{tr}\pi(f), \tag{8.4}$$

where f^H is a function on $H(\mathbb{A})$ obtained from f in the usual way (matching orbital integrals). Of course it is part of his conjecture that the distribution taking a function h on $H(\mathbb{A})$ to

$$\sum_{\pi_H \in \Pi(\psi_H)} \langle s_{\psi_H}, \pi_H \rangle \operatorname{tr} \pi_H(h)$$

is stable. Thus it is natural to expect that the contribution $ST_d(f)$ of the discrete part of $L^2_\chi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ to the stable trace formula for (G, f) is equal to

$$\sum_{[\psi]} |\mathfrak{S}_\psi|^{-1} \sum_{\pi \in \Pi(\psi)} \epsilon_\psi(s_\psi) \cdot \langle s_\psi, \pi \rangle \cdot \operatorname{tr} \pi(f). \tag{8.5}$$

Arthur [**A4**] has stabilized all of the discrete terms on the spectral side of the trace formula. However, it is worth noting here that the stabilization of $T_d(f)$ is relatively simple. Let $\mathcal{E}$ be a set of representatives for the elliptic endoscopic triples for G, and choose an extension $\eta : {}^L H \to {}^L G$ of η_0 for each $(H, s, \eta_0) \in \mathcal{E}$. Then the method of §§11-12 of [**K4**] shows that

$$T_d(f) = \sum_{\mathcal{E}} \iota(G, H) ST_d^*(f^H), \tag{8.6}$$

where $ST_d^*(f^H)$ is the part of the sum (8.5) (relative to (H, f^H)) indexed by $[\psi_H]$ such that $\eta \circ \psi_H$ is elliptic for G. The only new ingredient in the proof is Arthur's identity [**A4**]:

$$\epsilon_{\psi_H}(s_{\psi_H}) = \epsilon_\psi(x s_\psi), \tag{8.7}$$

where $x = \eta(s)$ as before. This compatibility between ϵ_{ψ_H} and ϵ_ψ makes it possible to prove (8.6) in the same way as the identity (12.2) of [**K4**] is proved: use 11.2.1 and 11.3.2 of [**K4**], which apply to Arthur parameters since S_ψ is unchanged if $\mathcal{L} \times SL_2(\mathbb{C})$ is replaced by $\mathcal{L} \times SU_2(\mathbb{R})$, and the latter group enjoys all the properties of $W_{\mathbb{Q}}$ that were used in §11 of [**K4**].

§9. Stable character values on h

In order to apply Arthur's conjectures to our situation we need to return to the function h on $H(\mathbb{A})$ discussed in §7. Recall that

$h = h^p \cdot h_p \cdot h_\infty$ and that we prescribed the stable orbital integrals of h on (G,H)-regular semisimple elements of $H(\mathbb{A})$. Now we face the dual problem. Let ψ_H be an elliptic Arthur parameter for H (giving rise to the right quasi-character on $A_G(\mathbb{R})^0$), and let $\psi = \eta \circ \psi_H$. We need to evaluate

$$\sum_{\pi_H \in \Pi(\psi_H)} \langle s_{\psi_H}, \pi_H \rangle \operatorname{tr} \pi_H(h) \tag{9.1}$$

in terms of $\psi, f^p_{\mathbb{C}}, j, \xi_{\mathbb{C}}$. The number $\langle s_{\psi_H}, \pi_H \rangle$ should factor canonically as

$$\langle s_{\psi_H}, \pi^p_H \rangle \langle s_{\psi_H}, \pi_{H,p} \rangle \; \langle s_{\psi_H}, \pi_{H,\infty} \rangle,$$

where $\pi_H = \pi^p_H \otimes \pi_{H,p} \otimes \pi_{H,\infty}$ is the decomposition corresponding to $H(\mathbb{A}) = H(\mathbb{A}^p_f) \times H(\mathbb{Q}_p) \times H(\mathbb{R})$. Therefore our problem is to evaluate

$$\sum_{\pi^p_H} \langle s_{\psi_H}, \pi^p_H \rangle \operatorname{tr} \pi^p_H(h^p), \tag{9.2}$$

$$\sum_{\pi_{H,p}} \langle s_{\psi_H}, \pi_{H,p} \rangle \operatorname{tr} \pi_{H,p}(h_p), \tag{9.3}$$

$$\sum_{\pi_{H,\infty}} \langle s_{\psi_H}, \pi_{H,\infty} \rangle \operatorname{tr} \pi_{H,\infty}(h_\infty). \tag{9.4}$$

In the case of (9.2) there is almost nothing to do. By our choice of h^p one expects (9.2) to equal

$$\sum_{\pi^p} \Delta^p(\psi_H, \pi^p) \operatorname{tr} \pi^p(f^p_{\mathbb{C}}), \tag{9.5}$$

where π^p runs through $\Pi^p(\psi)$, the Arthur packet on $G(\mathbb{A}^p_f)$ corresponding to ψ. In fact this identity (for all $f^p_{\mathbb{C}}$) should define the numbers $\Delta^p(\psi_H, \pi^p)$. Of course there should be analogous numbers $\Delta_p(\psi_H, \pi_p), \Delta_\infty(\psi_H, \pi_\infty)$ and for $\pi = \pi^p \otimes \pi_p \otimes \pi_\infty \in \Pi(\psi)$ one expects to define $\langle \cdot, \pi \rangle$ so that

$$\langle s_\psi x, \pi \rangle = \Delta^p(\psi_H, \pi^p) \Delta_p(\psi_H, \pi_p) \Delta_\infty(\psi_H, \pi_\infty), \tag{9.6}$$

where $x = \eta(s) \in S_\psi$.

Next we consider (9.3). Write ψ_p for the restriction of ψ to the local group $\mathcal{L}_{\mathbf{Q}_p} \times SL_2(\mathbf{C})$. Assume that ψ_p is unramified, which means simply that ψ_p comes from an admissible L-homomorphism

$$W(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p) \times SL_2(\mathbf{C}) \to \widehat{G} \rtimes W(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p).$$

Let $\phi_p : W(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p) \to \widehat{G} \rtimes W(\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p)$ be the unramified Langlands parameter associated to ψ_p (as in §6 of [**A3**], where it is denoted by ϕ_{ψ_p}), and let π_p denote the corresponding K_p-spherical representation of $G(\mathbf{Q}_p)$. Then π_p belongs to $\Pi(\psi_p)$. It follows from the definition of h_p and Thm. 2.1.3 of [**K3**] that if ψ_p is unramified, then (9.3) equals

$$\Delta_p(\psi_H, \pi_p) \cdot p^{r \dim S_K/2} \cdot \mathrm{tr}\,(x\phi_p(\Phi_{\mathfrak{p}}^j); V). \tag{9.7}$$

This expression requires some explanation.

As before $x = \eta(s) \in S_\psi$. As for V, it is the usual irreducible representation of ${}^L(G_E)$ determined by X_∞ (see [**L5**]). It is characterized up to isomorphism by the following two properties. The restriction of V to $\widehat{G}$ is irreducible with extreme weight $-\mu$; here $\mu = \mu_h$ ($h \in X_\infty$), viewed as a character on some maximal torus of $\widehat{G}$, well-defined up to the Weyl group. Let y be a splitting (see [**K4**], §1) of $\widehat{G}$ and assume that y is fixed by W_E. Then the subgroup W_E of ${}^L(G_E)$ acts trivially on the highest weight space of V corresponding to y.

Note that Theorem 2.1.3 of [**K3**] was stated using the arithmetic Frobenius. In this paper we are using Deligne's normalization of the reciprocity law for local fields, and therefore the geometric Frobenius $\Phi_{\mathfrak{p}}$ appears instead.

There is an assumption implicit in our formula (9.7). We are assuming that unramified Arthur packets on $H(\mathbf{Q}_p)$ contain exactly one K_H-spherical representation. It also seems reasonable to assume that ramified Arthur packets on $H(\mathbf{Q}_p)$ contain no K_H-spherical representations. If so, then (9.3) is equal to (9.7) if ψ_p is unramified and is equal to 0 if ψ_p is ramified.

We are left with (9.4). We write $\psi_{H,\infty}$ for the restriction of ψ_H to $W_{\mathbf{R}} \times SL_2(\mathbf{C})$. Of course any $\pi_{H,\infty}$ in the Arthur packet of $\psi_{H,\infty}$ should be essentially unitary. Moreover our explicit expression for h_∞ in terms of pseudo-coefficients shows that $\mathrm{tr}\,\pi_{H,\infty}(h_\infty)$ vanishes unless the elliptic maximal tori of $G_{\mathbf{R}}$ come from $H_{\mathbf{R}}$ and

$$\pi_{H,\infty} \otimes \xi_H \otimes \nu_H$$

has non-vanishing continuous cohomology for some finite dimensional representation ξ_H of $H(\mathbf{C})$ and some quasi-character ν_H on $H(\mathbf{R})$ (use [**C-D**]). Therefore it follows from [**V-Z**] that the only relevant Arthur parameters $\psi_{H,\infty}$ are those studied by Adams-Johnson [**A-J**] (see also [**A3**]), and calculating (9.4) is an exercise in using their results.

Adams and Johnson study what we will refer to as (essentially) *cohomological* Arthur parameters ψ for $G_{\mathbf{R}}$. They can be characterized by the following three conditions. Let $\widehat{M}$ denote the centralizer of $\psi(W_{\mathbf{C}})$ in $\widehat{G}$. Then $\widehat{M}$ is a Levi subgroup of $\widehat{G}$ and $\Gamma(\infty)$ acts on $\widehat{M}$ (use the restriction of ψ to $W_{\mathbf{R}}$). The first condition is that the identity component of $Z(\widehat{M})^{\Gamma(\infty)}$ be contained in $Z(\widehat{G})$. The second is that the image under ψ of any regular unipotent element of $SL_2(\mathbf{C})$ be a regular unipotent element of $\widehat{M}$ (note that ψ automatically maps $SL_2(\mathbf{C})$ into $\widehat{M}$, and even into $\widehat{M}^{\Gamma(\infty)}$). The restriction of ψ to $W_{\mathbf{C}} = \mathbf{C}^\times$ has the following form. There exist $\Lambda, \Lambda' \in X_*(Z(\widehat{M})) \otimes \mathbf{C}$ with $\Lambda - \Lambda' \in X_*(Z(\widehat{M}))$ such that for $z \in \mathbf{C}^\times$

$$\psi(z) = z^{\Lambda}\bar{z}^{\Lambda'} \times z \in Z(\widehat{M}) \times W_{\mathbf{C}}.$$

Let $\widehat{T}$ be a maximal torus in $\widehat{M}$ and let $\delta_M \in X_*(\widehat{T}) \otimes \mathbf{Q}$ be the half sum of the positive coroots for $\widehat{T}$ in $\widehat{M}$ (for some order on the root system of $\widehat{T}$ in $\widehat{M}$). The third condition is that $\langle \Lambda + \delta_M, \alpha \rangle$ be non-zero for every root α of $\widehat{T}$ in $\widehat{G}$. This condition is independent of the choice of order on the root system of $\widehat{T}$ in $\widehat{M}$.

The theory of these cohomological parameters exactly parallels the theory for Langlands parameters for the discrete series of $G_{\mathbf{R}}$. One sees easily that there exists a unique parabolic subgroup $\widehat{P}$ of $\widehat{G}$ with Levi subgroup $\widehat{M}$ such that $\langle \Lambda, \alpha \rangle$ is positive for every root α of $Z(\widehat{M})$ in the unipotent radical of $\widehat{P}$. The pair $(\widehat{M}, \widehat{P})$ determines a $G(\mathbf{C})$-conjugacy class of pairs (L, Q), where Q is a parabolic subgroup of $G_{\mathbf{C}}$ and L is a Levi subgroup of Q. There is a canonical $\widehat{M}$-conjugacy class of isomorphisms $\rho : \widehat{L} \to \widehat{M}$. We say that (L, Q) is *relevant* if L is defined over $\mathbf{R}$ and $\tau(\rho), \rho$ are conjugate under $\widehat{M}$ for any $\tau \in \Gamma(\infty)$.

Using that $G_{\mathbf{R}}$ has elliptic maximal tori, one sees without much trouble that relevant pairs exist. It is elementary to check that for a fixed relevant pair (L, Q) the $G(\mathbf{R})$-conjugacy classes of relevant pairs (L', Q') are parametrized by

$$\ker[H^1(\mathbf{R}, L) \to H^1(\mathbf{R}, G)].$$

The Arthur packet $\Pi(\psi)$ consists of representations $\pi(\psi, L, Q)$ ($\pi(L,Q)$ for short) obtained by cohomological induction from a suitable 1-dimensional representation of $L(\mathbf{R})$. Here (L,Q) ranges through all relevant pairs. Two relevant pairs give rise to isomorphic representations if and only if the two pairs are conjugate under $G(\mathbf{R})$.

It is clear that the group S_ψ is equal to $Z(\widehat{M})^{\Gamma(\infty)}Z(\widehat{G})$. As for discrete series parameters we can use X_∞ to define a map $\Pi(\psi) \to X^*(S_\psi)$. Let $\pi \in \Pi(\psi)$ and choose a relevant pair (L,Q) such that $\pi = \pi(L,Q)$. Choose an elliptic maximal torus T of G such that $T \subset L$ (it is not hard to see such T exist). Choose $h \in X_\infty$ such that h factors through T and define $\mu_h \in X_*(T)$ in the usual way. Then, regarding μ_h as a character on $\widehat{T}$, we can restrict μ_h to the subgroup $Z(\widehat{M}) = Z(\widehat{L})$ of $\widehat{T}$, and then restrict it further to the subgroup S_ψ of $Z(\widehat{M})$, thus obtaining $\lambda_\pi \in X^*(S_\psi)$. Then λ_π is independent of the choices made during its construction, and $\pi \mapsto \lambda_\pi$ is the desired map.

LEMMA 9.1. *Let $\pi = \pi(L,Q)$. Then*

$$\langle \lambda_\pi, s_\psi \rangle = (-1)^{q(L)}.$$

As before $s_\psi \in S_\psi$ is the image under ψ of the non-trivial central element of $SL_2(\mathbf{C})$. The lemma is a slight refinement of Lemma 5.1 of [**A3**]. The lemma follows from the following two facts. On the one hand, for T, h as above, h satisfies the usual axioms relative to L, and it is well-known that $q(L) = \langle \mu_h, \nu \rangle$, where ν is the sum of the positive roots of T in L relative to an order on the root system for which μ_h is dominant. On the other hand it is clear that $s_\psi = \nu(-1)$, where ν is now regarded as a coweight of $\widehat{T}$.

We continue to let T denote an elliptic maximal torus T of G such that $T \subset L$. Then $\rho : \widehat{L} \to \widehat{M}$ and $\psi|_{W_\mathbf{R}}$ together define an L-homomorphism

$$\eta_L : {}^L L \to {}^L G.$$

Note that this L-homomorphism depends on ψ. If $\Pi(\psi)$ consists of representations having the same infinitesimal and central characters as the trivial representation, then η_L agrees with the L-homomorphism ${}^L L \to {}^L G$ considered by Arthur [**A3**]; in general the two embeddings differ by a 1-cocycle of $W_\mathbf{R}$ in $Z(\widehat{L})$.

Let φ_L be the elliptic Langlands parameter for L whose L-packet consists of the discrete series representations of $L(\mathbf{R})$ having the same infinitesimal and central characters as the trivial representation. Then we define a Langlands parameter φ for G as the composition of η_L and φ_L. It is easy to see that φ is elliptic for G. Prop. 6.4 of **[V-Z]** states that for $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$

$$(-1)^{q(L)}\Theta_{\pi(L,Q)}(\gamma) = \sum_B \Theta_{\pi(T,B)}(\gamma).$$

Here, as usual, Θ_π denotes the character of π. We write $\pi(T,B)$ as an abbreviation for $\pi(\varphi,T,B)$. The sum on the right side is taken over $G(\mathbb{R})$-conjugacy classes of Borel subgroups B in $G_{\mathbb{C}}$ such that $T \subset B \subset Q$.

Define $S\Theta_\psi$ by $\sum_{(L,Q)} e(L)\cdot\Theta_{\pi(L,Q)}$. Then Adams-Johnson **[A-J]** have shown that $S\Theta_\psi$ is a stable distribution on $G(\mathbb{R})$. Let L_0 denote a quasi-split inner form of L. The result of Vogan-Zuckerman that we just reviewed implies that

$$(-1)^{q(L_0)}S\Theta_\psi(\gamma) = S\Theta_\varphi(\gamma)$$

for all $\gamma \in T_{\mathrm{reg}}(\mathbb{R})$.

The groups S_ψ, S_φ are related as follows. As in §7 the elliptic Langlands parameter φ determines a maximal torus $\widehat{S}$ of $\widehat{G}$ and a Borel subgroup $\widehat{B}$ of $\widehat{G}$ containing $\widehat{S}$. It is easy to see that $\widehat{S} \subset \widehat{M}$ and $\widehat{P} \supset \widehat{B}$. The group S_ψ is the subgroup $Z(\widehat{M})^{\Gamma(\infty)}Z(\widehat{G})$ of $S_\varphi = \widehat{S}^{\Gamma(\infty)}Z(\widehat{G})$. It is also easy to see that if B is a Borel subgroup of $G_{\mathbb{C}}$ such that $T \subset B \subset Q$, then $\lambda_{\pi(L,Q)}$ is the restriction of $\lambda_{\pi(T,B)} \in X^*(S_\varphi)$ to S_ψ.

Now let us return to the endscopic triple (H,s,η_0) and L-homomorphism $\eta : {}^LH \to {}^LG$. For the moment we work locally at $\mathbb{R}$, so that the L-groups are formed using $W_{\mathbb{R}}$. Consider an Arthur parameter ψ_H for H such that $\psi := \eta \circ \psi_H$ is cohomological. Since ψ takes regular unipotent elements of $SL_2(\mathbb{C})$ to regular unipotent elements of the centralizer of $\psi(W_{\mathbb{C}})$ in $\widehat{G}$, the homomorphism η_0 induces an isomorphism between the centralizer of $\psi(W_{\mathbb{C}})$ in $\widehat{G}$ and its centralizer in $\widehat{H}$; we use $\widehat{M}$ to denote both of these groups. In particular T must come from H. Let L be a real group provided with a $\Gamma(\infty)$-equivariant isomorphism $\widehat{L} \xrightarrow{\sim} \widehat{M}$. As before we use ψ, ψ_H to get L-homomorphisms $\eta_{L,G} : {}^LL \to {}^LG$ and $\eta_{L,H} : {}^LL \to {}^LH$. Clearly $\eta_{L,G} = \eta \circ \eta_{L,H}$, and therefore we may take $\varphi = \eta \circ \varphi_H$ (φ and φ_H are the elliptic Langlands parameters associated to ψ and ψ_H respectively).

Adams and Johnson have shown that there exist complex numbers $\Delta_\infty(\psi_H, \pi)$, for each $\pi \in \Pi(\psi)$, such that when the transfer factors $\Delta_\infty(\cdot, \cdot)$ are used to transfer $S\Theta_{\psi_H}$ to a distribution Tran $(S\Theta_{\psi_H})$ on $G(\mathbb{R})$ the following equality holds:

$$\operatorname{Tran}(S\Theta_{\psi_H}) = \sum_{\pi \in \Pi(\psi)} \Delta_\infty(\psi_H, \pi) \cdot \Theta_\pi .$$

In fact they have evaluated the numbers $\Delta_\infty(\psi_H, \pi)$. For our purposes it is more convenient to evaluate $\Delta_\infty(\psi_H, \cdot)$ in terms of $\Delta_\infty(\varphi_H, \cdot)$. Since we already know by **[A-J]** that the numbers $\Delta_\infty(\psi_H, \cdot)$ exist, it is just a question of calculating them, and for this is suffices to look on $T_{\text{reg}}(\mathbb{R})$. Using the character formula of Vogan-Zuckerman reviewed earlier, we see that if B is a Borel subgroup of $G_{\mathbb{C}}$ such that $T \subset B \subset Q$, then

$$\text{(9.8)} \qquad \Delta_\infty(\psi_H, \pi(Q, L)) = (-1)^{e(L)} \cdot \Delta_\infty(\varphi_H, \pi(T, B)) .$$

Now we are in a position to evaluate $S\Theta_{\psi_H}(h_\infty)$ for the function h_∞ on $H(\mathbb{R})$ defined in §7. We continue to assume that $\psi := \eta \circ \psi_H$ is cohomological. Assume further that the associated discrete series L-packet $\Pi(\varphi)$ consists of representations having the same infinitesimal and central characters as the contragredient of $\xi_{\mathbb{C}}$. Write x for the image of $s \in Z(\widehat{H})$ under η, and note that $x \in S_\psi \subset S_\varphi$.

LEMMA 9.2. *The product $\langle \lambda_\pi, x s_\psi \rangle \cdot \Delta_\infty(\psi_H, \pi)$ is the same for all $\pi \in \Pi(\psi)$. Moreover*

$$S\Theta_{\psi_H}(h_\infty) = (-1)^{q(G)} \langle \lambda_\pi, x s_\psi \rangle \cdot \Delta_\infty(\psi_H, \pi).$$

Since the stable orbital integrals of h_∞ vanish off the elliptic set in $H(\mathbb{R})$, we have

$$S\Theta_{\psi_H}(h_\infty) = (-1)^{q(L_0)} S\Theta_{\varphi_H}(h_\infty).$$

Now use Lemmas 7.1 and 9.1, as well as the equality (9.8).

We are finally ready to evaluate (9.4). Of course (9.4) is by definition $S\Theta_{\psi_H}(h_\infty)$, where we now abbreviate $\psi_{H,\infty}$ to ψ_H. In our earlier discussion we already reduced to the case in which T comes from $H_{\mathbb{R}}$ and ψ_H is cohomological. The explicit expression for h_∞ in terms of pseudo-coefficients shows that (9.4) vanishes unless $\eta \circ \varphi_H$

parametrizes the discrete series L-packet of $G(\mathbb{R})$ consisting of representations with the same infinitesimal and central characters as the contragredient of $\xi_{\mathbf{C}}$, in which case $\psi := \eta \circ \psi_H$ is cohomological and Lemma 9.2 applies, yielding the desired expression for (9.4). To see that ψ is necessarily cohomological requires a little thought. One first shows that the centralizer $\widehat{M}$ of $\psi_H(W_{\mathbf{C}})$ in $\hat{H}$ is a Levi subgroup of $\widehat{G}$ (use η to identify $\hat{H}$ with a subgroup of $\widehat{G}$) and then checks that $\widehat{M}$ is also the centralizer of $\psi(W_{\mathbf{C}})$ in $\widehat{G}$; the rest is easy.

At this point we have dealt with (9.2), (9.3), (9.4) separately, and all that is left is to take their product, which is the desired expression for (9.1). Of course (9.1) is 0 unless H satisfies the following two conditions:

(a) H is unramified at p,

(b) The elliptic maximal tori of $G_{\mathbf{R}}$ come from $H_{\mathbf{R}}$.

Moreover we have seen that (9.1) should be 0 unless ψ_H satisfies the following two conditions (with $\psi := \eta \circ \psi_H$):

(c) ψ_p is unramified,

(d) ψ_∞ is cohomological for $G_{\mathbf{R}}$, and the associated discrete series L-packet is the one consisting of all discrete series representations having the same infinitesimal and central characters as the contragredient of $\xi_{\mathbf{C}}$.

Note that (c) implies (a) and that (d) implies (b). To have a convenient way to refer to (d) we say that a parameter ψ_∞ is cohomological for $\xi_{\mathbf{C}}$ if it satisfies condition (d). Suppose that ψ satisfies (c) and (d). Then, multiplying (9.5), (9.7) and the expression in Lemma 9.2, we find that (9.1) is equal to

$$\sum_{\pi_f} \operatorname{tr} \pi_f(f_{\mathbf{C}}) \cdot \langle x s_\psi, \pi \rangle \cdot \langle \lambda_{\pi_\infty}, x s_\psi \rangle \cdot A(x, \psi_p), \tag{9.9}$$

where

$$A(x, \psi_p) = (-1)^{q(G)} p^{r(\dim S_K)/2} \operatorname{tr} (x \phi_p(\Phi_{\mathfrak{p}}^j); V).$$

Here π_f runs through the Arthur packet for $G(\mathbb{A}_f)$ parametrized by ψ. As before, $x = \eta(s) \in S_\psi$, and π_∞ is any representation of $G(\mathbb{R})$ in the Arthur packet parametrized by ψ_∞. We are writing π for $\pi_f \otimes \pi_\infty$. To get (9.9) we also used (9.6) and our supposition that there is only one K_p-spherical representation in the Arthur packet of ψ_p.

§10. Conjectural description of $IH^{\cdot}(\overline{S}_K, \mathcal{F})$

Theorem 7.2 suggests that $\operatorname{tr}(f \times \Phi_{\mathfrak{p}}^{j}; W_\lambda)$ is equal to

$$\sum_{\mathcal{E}} \iota(G,H) ST_d(h), \tag{10.1}$$

where, as before, $ST_d(h)$ denotes the contribution of the (essentially) discrete part of $L^2(H(\mathbb{Q})\backslash H(\mathbb{A}))$ to the stable trace formula for (H,h). In order to arrive at a conjectural description of W_λ we need to use Arthur's conjectures (see §8) to rewrite (10.1) in terms of Arthur parameters for G.

According to (8.5), the expression (10.1) should be equal to

$$\sum_{\mathcal{E}} \iota(G,H) \sum_{[\psi_H]} |\mathfrak{S}_{\psi_H}|^{-1} \epsilon_{\psi_H}(s_{\psi_H}) \sum_{\pi_H} \langle s_{\psi_H}, \pi_H\rangle \operatorname{tr} \pi_H(h). \tag{10.2}$$

Write ψ for $\eta \circ \psi_H$. Then the results of §9 imply that the contribution of H, s, η, ψ_H to (10.2) should be 0 unless ψ_p is unramified and ψ_∞ is cohomological for $\xi_{\mathbb{C}}$, in which case the contribution should be $\iota(G,H)|\mathfrak{S}_{\psi_H}|^{-1}\epsilon_{\psi_H}(s_{\psi_H})$ times (9.9). Using Arthur's identity (8.7) together with **[K4]**, 11.2.1 and 11.3.2, and arguing as in §12 of **[K4]**, we see that (10.2) should be equal to

$$\sum_{[\psi]} \sum_{\pi_f} \operatorname{tr} \pi_f(f_{\mathbb{C}}) |\mathfrak{S}_\psi|^{-1} \sum_{\overline{x} \in \mathfrak{S}_\psi} B(\overline{x}, \psi, \pi_f), \tag{10.3}$$

where $[\psi]$ ranges over the equivalence classes of Arthur parameters ψ such that ψ_p is unramified and ψ_∞ is cohomological for $\xi_{\mathbb{C}}$, and π_f ranges over the Arthur packet of ψ. Note that such a parameter ψ is elliptic and that S_ψ is abelian, as is the quotient $\mathfrak{S}_\psi = S_\psi/Z(\widehat{G})$. Here $B(\overline{x}, \psi, \pi_f)$ is defined to be

$$\epsilon_\psi(\overline{x}) \cdot \langle \overline{x}, \pi\rangle \cdot \langle \lambda_{\pi_\infty}, x\rangle \cdot A(x s_\psi, \psi_p),$$

where x denotes a representative of $\overline{x}$ in S_ψ, and (as before) $\pi = \pi_f \otimes \pi_\infty$ for some $\pi_\infty \in \Pi(\psi_\infty)$.

Let $\mathcal{L}_E$ denote the Langlands group of $\overline{\mathbb{Q}}/E$ and ψ_E denote the restriction of ψ to $\mathcal{L}_E \times SL_2(\mathbb{C})$. Recall the representation V of ${}^L(G_E)$ defined in §9. Via ψ_E the group $\mathcal{L}_E \times SL_2(\mathbb{C})$ acts on V. This action should commute with the action of S_ψ on V, as we will now check. Let

$s \in S_\psi$. Then s commutes with ψ_E up to a 1-cocycle of $\mathcal{L}_E$ in $Z(\widehat{G})$ that is locally a coboundary. Since V is irreducible as a representation of $\widehat{G}$, the image of $Z(\widehat{G})$ in $GL_{\mathbb{C}}(V)$ is contained in the center $\mathbb{C}^\times$ of $GL_{\mathbb{C}}(V)$, and therefore s commutes with the action of $\mathcal{L}_E \times SL_2(\mathbb{C})$ on V up to a locally trivial quasi-character of $\mathcal{L}_E$, and such a quasi-character should come from a locally trivial quasi-character on W_E and hence should be trivial.

The subgroup $Z(\widehat{G})$ of S_ψ acts on V by the inverse of the character μ_1 defined in §2. Let ν be a character on S_ψ whose restriction to $Z(\widehat{G})$ is μ_1. We write V_ν for the largest subspace of V on which S_ψ acts by the inverse of ν, so that $V = \bigoplus_\nu V_\nu$ as representations of $\mathcal{L}_E \times SL_2(\mathbb{C})$. Let ϕ be the Langlands parameter associated to ψ (as in §6 of [**A3**], where it is denoted by ϕ_ψ), and let ϕ_E be its restriction to $\mathcal{L}_E$. Then ϕ_E gives us another action of $\mathcal{L}_E$ on each V_ν.

Let π_f belong to the Arthur packet of $G(\mathbb{A}_f)$ parametrized by ψ. For any character ν as above we define a non-negative integer $m(\pi_f, \nu)$ by choosing $\pi_\infty \in \Pi(\psi_\infty)$ and taking $m(\pi_f, \nu)$ to be the multiplicity with which the 1-dimensional character $\nu - \lambda_{\pi_\infty}$ occurs in the character $\overline{x} \mapsto \epsilon_\psi(\overline{x}) \cdot \langle \overline{x}, \pi \rangle$ of $\mathfrak{S}_\psi$ (one expects that the function $\overline{x} \mapsto \epsilon_\psi(\overline{x}) \cdot \langle \overline{x}, \pi \rangle$ is the character of some finite dimensional representation of $\mathfrak{S}_\psi$). Note that $m(\pi_f, \nu)$ should be independent of the choice of π_∞.

It is clear that

$$|\mathfrak{S}_\psi|^{-1} \sum_{\overline{x}} B(\overline{x}, \psi, \pi_f) = \sum_\nu m(\pi_f, \nu) \cdot A(\psi_p, \nu), \tag{10.4}$$

where

$$A(\psi_p, \nu) = (-1)^{q(G)} \cdot \nu(s_\psi) \cdot p^{r(\dim S_K)/2} \cdot \operatorname{tr}(\phi_p(\Phi_{\mathfrak{p}}^j); V_\nu).$$

Of course $\nu(s_\psi)$ is ± 1.

For ψ and ν as above we write $V(\psi, \nu)$ for the twist of the representation V_ν of $\mathcal{L}_E$ (acting via ϕ_ψ) by the unramified quasi-character $|\ |^{-(\dim S_K)/2}$ of $\mathcal{L}_E$. We have shown that $\operatorname{tr}(f \times \Phi_{\mathfrak{p}}^j; W_\lambda)$ should be equal to

$$\sum_{[\psi]} \sum_{\pi_f} \operatorname{tr} \pi_f(f_{\mathbb{C}}) \sum_\nu m(\pi_f, \nu) \cdot (-1)^{q(G)} \cdot \nu(s_\psi) \cdot \operatorname{tr}(\Phi_{\mathfrak{p}}^j; V(\psi, \nu)). \tag{10.5}$$

Let $\mathcal{M}_E$ denote the motivic Galois group of $\overline{\mathbb{Q}}/E$ relative to the Betti fiber functor and the inclusion of E in $\mathbb{C}$. Then $V(\psi, \nu)$ should come from a complex representation of $\mathcal{M}_E$, and, after L is enlarged suitably, this complex representation should come from a representation of $\mathcal{M}_E$ on an L-vector space $V(\psi,\nu)_L$ (here we use our choice of embedding $L \to \mathbb{C}$). Then, letting $V(\psi,\nu)_\lambda$ denote the λ-adic completion of $V(\psi,\nu)_L$, we should get a λ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on $V(\psi,\nu)_\lambda$, and (10.5) suggests that the virtual representation $\bigoplus_{i=0}^{2\dim S_K} (-1)^i IH^i(\overline{S}_K, \mathcal{F}) \otimes L_\lambda$ of $\mathcal{H}_L$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ is equal to

$$\bigoplus_{[\psi]} \bigoplus_{\pi_f} \bigoplus_{\nu} n(\pi_f,\nu)(\pi_f^K \otimes V(\psi,\nu)_\lambda),$$

where $[\psi]$ now ranges over the equivalence classes of Arthur parameters ψ such that ψ_∞ is cohomological for $\xi_{\mathbb{C}}$. Here π_f^K denotes the space of K-fixed vectors in π_f, and $n(\pi_f,\nu) = (-1)^{q(G)} \cdot \nu(s_\psi) \cdot m(\pi_f,\nu)$. The action of SL_2 on $V(\psi,\nu)$ should give the Lefschetz decomposition. The individual space $IH^i(\overline{S}_K, \mathcal{F}) \otimes L_\lambda$ can be reconstructed from the Euler characteristic by taking $(-1)^i$ times the part of weight i; in terms of ψ this amounts to taking weight spaces for SL_2 (up to a shift by $\dim S_K$).

Part III. Counting principally polarized abelian varieties over finite fields

§11. Points of $M_{g,N}$ in $\mathbb{C}$

Let g be a positive integer. Choose a pair $(V, \langle\ ,\ \rangle)$ consisting of a free $\mathbb{Z}$-module V of rank $2g$ and a non-degenerate $\mathbb{Z}$-valued alternating form $\langle\ ,\ \rangle$ on V; such a pair is unique up to isomorphism. Let G denote the $\mathbb{Z}$-group of symplectic similitudes of $(V, \langle\ ,\ \rangle)$, so that for any commutative ring R with 1 the group $G(R)$ consists of all $g \in \mathrm{Aut}_R(V \otimes R)$ for which there exists $c \in R^\times$ such that $\langle gv, gw\rangle = c\langle v, w\rangle$ for all $v, w \in V \otimes R$. The element $c \in R^\times$ such that $\langle gv, gw\rangle = c\langle v, w\rangle$ is uniquely determined by g, and sending g to c yields a homomorphism $c : G \to \mathbb{G}_m$.

Let N be an integer such that $N \geq 3$ and let $M_{g,N}$ be the moduli space of principally polarized abelian varieties of dimension g with level N structure. For any field k in which N is invertible $M_{g,N}(k)$ is the set of isomorphism classes of triples (A, λ, φ), where A is an abelian variety of dimension g over k, λ is a principal polarization of

A, and φ is a level N structure on A (compatible with λ). Recall that a principal polarization of A is an isomorphism $\lambda : A \to \widehat{A}$ ($\widehat{A}$ denotes the dual abelian variety $\mathrm{Pic}(A)^0$) such that λ comes from some ample invertible sheaf on A over an algebraic closure $\overline{k}$ of k. Also recall that a level N structure φ on A (compatible with λ) is a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$\varphi : H^1(A_{\overline{k}}, \mathbf{Z}/N\mathbf{Z}) \xrightarrow{\sim} V \otimes \mathbf{Z}/N\mathbf{Z}$$

for which there exists a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$c_\varphi : \mathbf{Z}/N\mathbf{Z}(-1) \xrightarrow{\sim} \mathbf{Z}/N\mathbf{Z}$$

such that φ and c_φ carry the $\mathbf{Z}/N\mathbf{Z}$-valued alternating form $\langle\ ,\ \rangle$ on $V \otimes \mathbf{Z}/N\mathbf{Z}$ over to the $\mathbf{Z}/N\mathbf{Z}(-1)$-valued alternating form $\langle\ ,\ \rangle_\lambda$ on $H^1(A_{\overline{k}}, \mathbf{Z}/N\mathbf{Z})$ obtained from λ. Of course $\mathrm{Gal}(\overline{k}/k)$ acts trivially on $V \otimes \mathbf{Z}/N\mathbf{Z}$ and $\mathbf{Z}/N\mathbf{Z}$. The set of level N structures on (A, λ) is either empty or a principal homogeneous space under $G(\mathbf{Z}/N\mathbf{Z})$.

There is a classical description of $M_{g,N}(\mathbf{C})$. Let X^∞ be the set of pairs (Λ, α) where Λ is a lattice in $V \otimes \mathbf{A}_f$ such that $\Lambda^\perp = c\Lambda$ for some $c \in \mathbf{A}_f^\times$ and α is an isomorphism

$$\Lambda/N\Lambda \xrightarrow{\sim} V \otimes \mathbf{Z}/N\mathbf{Z}$$

carrying the form $\langle\ ,\ \rangle$ on $V \otimes \mathbf{Z}/N\mathbf{Z}$ into the form on $\Lambda/N\Lambda$ induced by $c\langle\ ,\ \rangle$ for some $c \in \mathbf{A}_f^\times$ such that $\Lambda^\perp = c\Lambda$. Note that $G(\mathbf{A}_f)$ acts transitively on X^∞ and that the stabilizer of the obvious base point in X^∞ is $K_N := \ker[G(\widehat{\mathbf{Z}}) \to G(\mathbf{Z}/N\mathbf{Z})]$.

Let X_∞ be the set of Hodge structures of type $\{(0,1),(1,0)\}$ on $V \otimes \mathbf{R}$ for which $\langle\ ,\ \rangle$ or $\langle\ ,\ \rangle$ is a polarization of the Hodge structure. Note that $G(\mathbf{R})$ acts transitively on X_∞ and that X_∞ is a union of two Hermitian symmetric domains. Now we can recall the classical description of $M_{g,N}(\mathbf{C})$. Let (A, λ, φ) belong to $M_{g,N}(\mathbf{C})$ and choose an isomorphism

$$H^1(A, \mathbf{Q}) \xrightarrow{\sim} V \otimes \mathbf{Q}$$

which carries $\langle\ ,\ \rangle$ into a scalar multiple of the alternating form $\langle\ ,\ \rangle_\lambda$ on $H^1(A, \mathbf{Q})$ obtained from λ. The lattice $H^1(A, \widehat{\mathbf{Z}})$ in $H^1(A, \mathbf{A}_f)$ together with the level N structure on (A, λ) gives us an element of X^∞ while the Hodge structure on $H^1(A, \mathbf{R})$ gives us an element of

X_∞. Taking into account the choice of isomorphism from $H^1(A,\mathbb{Q})$ to $V \otimes \mathbb{Q}$ we get from (A,λ,φ) a well-defined element of the quotient $G(\mathbb{Q})\backslash(X^\infty \times X_\infty)$, and the classical theory says simply that

$$M_{g,N}(\mathbb{C}) \xrightarrow{\sim} G(\mathbb{Q})\backslash(X^\infty \times X_\infty)$$

is an isomorphism of complex manifolds. Therefore $M_{g,N}(\mathbb{C})$ is the set of complex points of the Shimura variety associated to (G,X_∞) and the compact open subgroup K_N of $G(\mathbb{A}_f)$.

§12. Points of $M_{g,N}$ in finite fields

Let k be a finite field whose characteristic p does not divide N, and let $q = p^r$ be the cardinality of k. Our goal is to write the cardinality of $M_{g,N}(k)$ in the form (3.1) (with trivial ξ and f^p, of course). To this end it is convenient to enlarge the notion of isogeny. Let R be a commutative ring with 1. The category of abelian varieties up to R-isogeny is the R-linear category whose objects are abelian varieties over k and whose R-module of morphisms from A to B is

$$\operatorname{Hom}_k(A,B)\otimes R,$$

where $\operatorname{Hom}_k(A,B)$ is the $\mathbb{Z}$-module of k-homomorphisms from A to B. An *R-isogeny from A to B* is an isomorphism in the category of abelian varieties up to R-isogeny. The homorphism $f \mapsto \widehat{f}$ from $\operatorname{Hom}_k(A,B)$ to $\operatorname{Hom}_k(\widehat{B},\widehat{A})$ extends to an R-module map

$$\operatorname{Hom}_k(A,B)\otimes R \to \operatorname{Hom}_k(\widehat{B},\widehat{A})\otimes R.$$

For any $f \in \operatorname{Hom}_k(A,B)\otimes R$ and any $\lambda \in \operatorname{Hom}_k(B,\widehat{B})\otimes R$ we write $f^*\lambda$ for

$$\widehat{f}\circ\lambda\circ f \in \operatorname{Hom}_k(A,\widehat{A})\otimes R.$$

An *R-isogeny from one polarized abelian variety (A,λ) to another (A',λ')* is an R-isogeny $f: A \to A'$ for which there exists $c \in R^\times$ such that $f^*\lambda' = c\lambda$.

Now fix a polarized abelian variety up to $\mathbb{Q}$-isogeny (A_0,λ_0) of dimension g. For any commutative $\mathbb{Q}$-algebra R with 1 let $I(R)$ be the group of R-isogenies from (A_0,λ_0) to itself. It is easy to see that the functor I is representable by an affine algebraic group over $\mathbb{Q}$. Let $W^p = H^1(\overline{A}_0,\mathbb{A}_f^p)$ and $W_p = H^1_{\text{cris}}(A_0/W(k))\otimes K(k)$, where $\overline{A}_0$ is the base change of A_0 from k to $\bar{k}$, and $K(k)$ is the field of fractions of the Witt ring $W(k)$. The category of abelian varieties A plus a $\mathbb{Q}$-isogeny from A to A_0 is equivalent to the category of pairs (Λ^p, Λ_p), where Λ^p is a lattice in W^p fixed by $\mathrm{Gal}(\bar{k}/k)$ and Λ_p is a lattice in W_p such that $F(\Lambda_p) \subset \Lambda_p$ and $V(\Lambda_p) \subset \Lambda_p$. Here F is the usual σ-linear (σ is the Frobenius element of $\mathrm{Gal}(K(k)/\mathbb{Q}_p)$) automorphism of W_p and $V = pF^{-1}$. Therefore giving a principally polarized abelian variety with level N structure (A, λ, φ) plus a $\mathbb{Q}$-isogeny from (A_0, λ_0) to (A, λ) is the same as giving an element of $Y^p \times Y_p$, where Y^p is the set of fixed points of $\mathrm{Gal}(\bar{k}/k)$ in the set X^p of pairs $(\Lambda, \varphi_\Lambda)$ consisting of a lattice Λ in W^p such that $\Lambda^\perp = c\Lambda$ for some $c \in (\mathbb{A}_f^p)^\times$ and a symplectic similitude $\varphi_\Lambda : \Lambda/N\Lambda \xrightarrow{\sim} V \otimes \mathbb{Z}/N\mathbb{Z}$, and where Y_p is the set of lattices Λ in W_p such that $\Lambda^\perp = c\Lambda$ for some $c \in K(k)^\times$ and $F\Lambda \subset \Lambda$ and $V\Lambda \subset \Lambda$. Let X_p be the set of lattices Λ in W_p such that $\Lambda^\perp = c\Lambda$ for some $c \in K(k)^\times$. It is easy to see that Y_p is equal to

$$\{\Lambda \in X_p \mid F\Lambda \subset \Lambda \ \text{ and } \ \Lambda/F\Lambda \approx k^g \text{ as } W(k)\text{-modules}\}.$$

Therefore there is a bijection from the subset of $M_{g,N}(k)$ consisting of (A, λ, φ) such that (A, λ) is $\mathbb{Q}$-isogenous to (A_0, λ_0) and the set

$$I(\mathbb{Q})\backslash(Y^p \times Y_p).$$

Choose symplectic similitudes from W^p to $V \otimes \mathbb{A}_f^p$ and from W_p to $V \otimes K(k)$. Then the geometric Frobenius element of $\mathrm{Gal}(\bar{k}/k)$ can be carried over to an element $\gamma \in G(\mathbb{A}_f^p)$ and the σ-linear automorphism F of W_p can be carried over to $V \otimes K(k)$ and written as $\delta\sigma$ for some $\delta \in G(K(k))$. We then have

$$Y^p = \{g \in G(\mathbb{A}_f^p)/K_N^p \mid g^{-1}\gamma g \in K_N^p\}$$

where $K_N^p = K_N \cap G(\mathbb{A}_f^p)$ and

$$Y_p = \{g \in G(K(k))/G(W(k)) \mid g^{-1}\delta\sigma(g) \in G(W(k))aG(W(k))\},$$

where $a \in G(K(k))$ has the property that

$$a(V \otimes W(k)) \subset V \otimes W(k)$$

and

$$V \otimes W(k)/a(V \otimes W(k)) \approx k^g$$

as $W(k)$-modules.

It follows immediately that the cardinality of this subset of $M_{g,N}(k)$ is equal to

$$\int_{I(\mathbb{Q})\backslash(G(\mathbb{A}_f^p)\times G(K(k)))} f^p(g^{-1}\gamma g)\cdot\phi_r(g^{-1}\delta\sigma(g))$$

where f^p is the characteristic function of K_N^p in $G(\mathbb{A}_f^p)$ and ϕ_r is the characteristic function of $G(W(k))aG(W(k))$ in $G(K(k))$. It follows from a theorem of Tate [**T1**] that $I(\mathbb{A}_f^p)$ is equal to the centralizer of γ in $G(\mathbb{A}_f^p)$ and $I(\mathbb{Q}_p)$ is equal to the twisted centralizer of δ in $G(K(k))$. Therefore our integral can be rewritten as

$$\operatorname{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))\cdot O_\gamma(f^p)\cdot TO_\delta(\phi_r).$$

We have now counted the triples (A,λ,φ) that are $\mathbb{Q}$-isogenous to a fixed (A_0,λ_0). Next we need to classify the pairs (A_0,λ_0). As a first step we classify the $\mathbb{Q}$-isogeny classes of pairs (A,λ) that are $\overline{\mathbb{Q}}$-isogenous to (A_0,λ_0). Given such a pair (A,λ) we choose a $\overline{\mathbb{Q}}$-isogeny

$$f:(A_0,\lambda_0)\to(A,\lambda)$$

and get a 1-cocycle $\tau\mapsto f^{-1}\circ\tau(f)\quad(\tau\in\Gamma)$ with values in $I(\overline{\mathbb{Q}})$. As before I denotes the $\mathbb{Q}$-group of isogenies from (A_0,λ_0) to itself. In this way we get a bijection from the set of $\mathbb{Q}$-isogeny classes of (A,λ) that are $\overline{\mathbb{Q}}$-isogenous to (A_0,λ_0) to the set

$$\ker[H^1(\mathbb{Q},I)\to H^1(\mathbb{R},I)].$$

To prove this one considers the semisimple $\mathbb{Q}$-algebra $E=\operatorname{End}(A_0)\otimes\mathbb{Q}$ and the obvious embedding $I\to E^\times$ and then uses:

(1) $H^1(\mathbb{Q},E^\times)=\{1\}$.

(2) Let H denote the $\mathbb{Z}$-module of symmetric homomorphisms from A_0 to $\widehat{A}_0$. Then $\lambda\in H$ is a polarization of A_0 if and only if λ belongs to the orbit of λ_0 under $E^\times(\mathbb{R})$ ($E^\times(\mathbb{R})$ acts on $H\otimes\mathbb{R}$ by $\lambda\mapsto\widehat{e}\circ\lambda\circ e$ for $\lambda\in H\otimes\mathbb{R}$ and $e\in E^\times(\mathbb{R})$) (see [M], §21).

As before we can associate to (A_0,λ_0) an element $\gamma\in G(\mathbb{A}_f^p)$, well-defined up to conjugacy in $G(\mathbb{A}_f^p)$ and an element $\delta\in G(K(k))$, well-defined up to σ-conjugacy in $G(K(k))$. In the same way we can associate elements γ',δ' to any (A,λ) that is $\overline{\mathbb{Q}}$-isogenous to (A_0,λ_0), and it turns out that γ' is $G(\mathbb{A}_f^p)$-conjugate to γ and δ' is σ-conjugate to δ if and only if the element of $H^1(\mathbf{Q}, I)$ obtained from (A, λ) is trivial in $H^1(\mathbf{Q}_v, I)$ for every place v of $\mathbf{Q}$.

As our next step let us classify the $\overline{\mathbf{Q}}$-isogeny classes of pairs (A_0, λ_0). It turns out that (A, λ) is $\overline{\mathbf{Q}}$-isogenous to (A_0, λ_0) if and only if A is $\mathbf{Q}$-isogenous to A_0. To prove this use that $H^1(\mathbf{Q}, E^\times) = \{1\}$ (again $E = \mathrm{End}\,(A_0) \otimes \mathbf{Q}$) and that the map $x \mapsto xx^*$ maps $(E \otimes \overline{\mathbf{Q}})^\times$ onto $\{y \in (E \otimes \overline{\mathbf{Q}})^\times | y^* = y\}$, where $x \mapsto x^*$ is the Rosati involution (a general fact about a semisimple algebra with involution over an algebraically closed field of characteristic different from 2).

Therefore our problem is to classify $\mathbf{Q}$-isogeny classes of g-dimensional abelian varieties. For this we use the classification of simple abelian varieties up to isogeny, due to Honda and Tate **[T1]**, **[T2]**. Given a g-dimensional abelian variety A we associate to it a semisimple element $\gamma_0 \in G(\mathbf{Q})$, well-defined up to conjugacy in $G(\overline{\mathbf{Q}})$, characterized by

(a) $c(\gamma_0) = q$.

(b) The characteristic polynomial of γ_0 is equal to the characteristic polynomial of Frobenius for A.

Then one checks that $A \mapsto \gamma_0$ induces a bijection from the set of g-dimensional abelian varieties up to $\mathbf{Q}$-isogeny to the set of $G(\overline{\mathbf{Q}})$-conjugacy classes of semisimple $\gamma_0 \in G(\mathbf{Q})$ such that

(1) $c(\gamma_0) = q$.

(2) Every complex eigenvalue of γ_0 has absolute value $\sqrt{q}$.

(3) Every eigenvalue of γ_0 in $\overline{\mathbf{Q}}_\ell$ ($\ell \neq p$) is an ℓ-adic unit.

(4) The image of γ_0 in $GL(V \otimes K(k))$ is the norm of an element $\delta \in GL(V \otimes K(k))$ for which there exists a lattice Λ in $V \otimes K(k)$ such that $\delta\sigma\Lambda \subset \Lambda$ (recall that the norm of δ is equal to $\delta\sigma(\delta)\ldots\sigma^{r-1}(\delta) \in GL(V \otimes K(k))$).

We are not quite finished. Given (A, λ) up to $\mathbf{Q}$-isogeny, we have an associated triple $(\gamma_0; \gamma, \delta)$ with $\gamma_0 \in G(\mathbf{Q})$ up to $G(\overline{\mathbf{Q}})$-conjugacy, $\gamma \in G(\mathbb{A}_f^p)$ up to $G(\mathbb{A}_f^p)$-conjugacy, and $\delta \in G(K(k))$ up to σ-conjugacy in $G(K(k))$. The three elements are related by:

(5) For every place v of $\mathbf{Q}$ other than p, ∞ the v-component of γ is conjugate to γ_0 under $G(\overline{\mathbf{Q}}_v)$.

(6) The norm $N\delta$ of δ is conjugate to γ_0 under $G(\overline{K(k)})$.

Since any A is polarizable, any γ_0 satisfying (1)–(4) comes from some (A, λ). However it is not true that every triple $(\gamma_0; \gamma, \delta)$ satisfying (1)–(6) comes from some (A, λ). In fact any such triple $(\gamma_0; \gamma, \delta)$ satisfies all the conditions in §2 (the additional condition on δ imposed in §2 is automatically satisfied), and therefore the invariant $\alpha(\gamma_0; \gamma, \delta)$ is defined.

THEOREM 12.1. *The invariant $\alpha(\gamma_0; \gamma, \delta)$ is trivial if and only if $(\gamma_0; \gamma, \delta)$ comes from some (A, λ).*

We leave the proof of this theorem for another occasion; Reimann and Zink [**R-Z**] have proved a related result. Note that $\ker^1(\mathbf{Q}, I)$ classifies the $\mathbf{Q}$-isogeny classes of (A', λ') such that $(A', \lambda'), (A, \lambda)$ give rise to the same triple $(\gamma_0; \gamma, \delta)$ (up to equivalence). This accounts for the factor $c_2(\gamma_0)$ of §3 (use that $|\ker^1(\mathbf{Q}, I)| = |\ker^1(\mathbf{Q}, I_0)|$ and that $\ker^1(\mathbf{Q}, G)$ is trivial). In order to conclude that $|M_{g,N}(k)|$ has the form (3.1), it remains only to observe that if $(\gamma_0; \gamma, \delta)$ indexes a non-zero term in (3.1), then γ_0 satisfies (1)–(4).

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The Present State of the Trace Formula

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1. INTRODUCTION

The Selberg trace formula now available in full generality after the work of J. Arthur, amounts to the equality of two different expressions, the geometric and the spectral expansion, for the "renormalized trace" of the convolution operator by a compactly supported smooth function on $G(\mathbb{A}_F)^1$ in $L^2(G(F)\backslash G(\mathbb{A}_F)^1)$ where G is a connected reductive group defined over a global field F (for the definition of $G(\mathbb{A}_F)^1$ see §3 below).

In general the trace has to be renormalized since the convolution operator is not of trace class, this being due to the fact that the space $G(F)\backslash G(\mathbb{A}_F)^1$ is of finite volume but not compact. The first step of the renormalization is obtained via a cutoff of the divergent integral over the diagonal of the kernel associated to the convolution operator. As can be expected the cutoff procedure destroys the invariance of the trace. In a second step the invariance can eventually be restored and it is our duty to try to explain how Arthur achieves this. One can then apply the trace formula to functions that are known only through their scalar Fourier transform, for example pseudo-coefficients of discrete series. We shall not discuss further applications like base change.

In what follows we shall restrict ourselves to fields of zero characteristic since, although it would be of interest to overcome this restriction, all our references [A1,..,9] are written in this setting. We shall also consider only connected groups G ; the extension to the non connected case also known as the twisted case, necessary for applications to base change for instance, has been the subject of **[CLL]** and after 1984 Arthur has worked in this more general context, but here this would only add technicalities without giving more insight.

Nothing will be said either on the stabilization which is the subject of **[A11]** or on Jacquet's relative trace formula. We shall assume the reader familiar with the content of survey articles **[A3]**, **[A4]** or **[L]**.

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2. The classical trace formula

Let us explain, once again, the trace formula when is G anisotropic, in order to introduce in this simple setting the notations and the results one wants to generalize. Consider $f \in \mathcal{C}_c^\infty(G(\mathbb{A}_F))$ a smooth compactly supported function on the group of adelic points, and let $\rho(f)$ be the operator defined by f acting via the right regular representation in $L^2(G(F)\backslash G(\mathbb{A}_F))$. When G is anisotropic $G(F)\backslash G(\mathbb{A}_F)$ is compact and $\rho(f)$ is of trace class; one has two expressions for its trace $I(f)$ which can be computed by integrating over the diagonal the kernel $K(x,y)$ associated to $\rho(f)$:

$$I(f) = \int_{G(F)\backslash G(\mathbb{A}_F)} K(x,x)\, dx = \int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in G(F)} f(x^{-1}\gamma x)\, dx\ .$$

The geometric term is obtained by expanding the kernel according to conjugacy classes:

$$I(f) = \sum_{\gamma\in\{G(F)\}} a^G(\gamma)\ I_G(\gamma, f)$$

where the sum is over $\{G(F)\}$ a set of representatives of conjugacy classes in $G(F)$; if as usual G_γ denotes the centralizer of γ in G , then

$$a^G(\gamma) = \mathrm{vol}\ (G_\gamma(F)\backslash G_\gamma(\mathbb{A}_F))$$

is a Tamagawa number if the quotient measure is suitably chosen; the distribution

$$I_G(\gamma, f) = \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)\, d\dot{x}$$

is the γ-orbital integral; when f is decomposable, as a product of functions on the local groups, $I_G(\gamma, f)$ is a product of local distributions.

The spectral term is given by the spectral decomposition of ρ :

$$I(f) = \mathrm{tr}\ \rho(f) = \sum_{\pi\in\Pi(G)} a^G(\pi)\ I_G(\pi, f)$$

where $\Pi(G)$ is the unitary dual of $G(\mathbb{A}_F)$, the integer $a^G(\pi)$ is the multiplicity of π in $L^2(G(F)\backslash G(\mathbb{A}_F))$ and the distribution $I_G(\pi, f) = \mathrm{tr}\ \pi(f)$ is the character of π .

3. The cutoff trace

We return now to the general case. We fix a minimal parabolic subgroup P_0 with Levi subgroup M_0 ; in the sequel we shall consider only parabolic subgroups P with Levi subgroup M_P such that $M_0 \subset M_P$.

Let P be a parabolic subgroup of G and let $X(P)$ be the group of its rational characters; we denote by $\mathfrak{a}_P$ the vector space $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{X(P)}, \mathbb{R})$. There is a natural map:

$$H_P \; : \; P(\mathbb{A}_F) \longrightarrow \mathfrak{a}_P$$

whose kernel is $P(\mathbb{A}_F)^1$. If $P \subset Q$ we denote by $\mathfrak{a}_P^Q$ the kernel of the natural map $\mathfrak{a}_P \to \mathfrak{a}_Q$. The map H_P has a section whose image lies in the center of a Levi subgroup $M_P(F \otimes \mathbb{R})$ of $P(F \otimes \mathbb{R})$, and will be denoted A_P . This allows to define sections for the maps $\mathfrak{a}_P \to \mathfrak{a}_Q$.

Then one defines $\hat{\tau}_P$ as the caracteristic function of the the open cone in $\mathfrak{a}_P^G$ generated by $\check{\Delta}_P$ the simple co-roots of the pair (P, S_P) where S_P is the split component of the center of M_P. We shall also need the functions $\hat{\tau}_P^Q$ on $\mathfrak{a}_P^Q$ defined using co-roots of $(P \cap M_Q, S_P)$, and the functions of two variables $\Gamma_Q(H, X)$ on $\mathfrak{a}_Q^G \times \mathfrak{a}_Q^G$ such that

$$\hat{\tau}_P(H - X) = \sum_{Q \supset P} (-1)^{a_Q^G} \; \hat{\tau}_P^Q(H) \; \Gamma_Q(H, X)$$

where a_Q^G is the dimension of $\mathfrak{a}_Q^G$.

Choose a "good" maximal compact subgroup $K \subset G(\mathbb{A}_F)$ then one has Iwasawa decompositions: $x \in G(\mathbb{A}_F)$ can be written $x = p a_P(x) k$ with $p \in P(\mathbb{A}_F)^1$, $k \in K$ and where $a_P(x) \in A_P$ is uniquely defined. Then H_P can be extended to a map on $G(\mathbb{A}_F)$ such that $H_P(x) = H_P(a_P(x))$; notice that this depends strongly on the choice of K .

Given $T \in \mathfrak{a}_0$ sufficiently deep in the positive Weyl chamber one can define **[A1]** the cutoff trace denoted $J^T(f)$ – or $J^{T,G}(f)$ if one wants to emphasize the dependance on G :

$$J^T(f) = \int_{G(F)\backslash G(\mathbb{A}_F)^1} k^T(x) \; dx$$

where

$$k^T(x) = \sum_{P \supset P_0} (-1)^{a_P^G} \sum_{\delta \in P(F)\backslash G(F)} \hat{\tau}_P(H_P(\delta x) - T) \; K_P(\delta x, \delta x)$$

with $a_P^G = \dim \mathfrak{a}_P^G$ and

$$K_P(x,y) = \sum_{\gamma \in M_P(F)} \int_{N_P(\mathbb{A}_F)} f(x^{-1}\gamma n y)\, dn \quad .$$

As a function of T the cutoff trace defined on an open set of $\mathfrak{a}_0$ is a polynomial. In fact if one changes the cutoff parameter T to $T+X$ then one sees easily that

$$J^{T+X}(f) = \sum_{Q \supset P_0} J^{T,Q}(f_Q)\gamma_Q(X)$$

where $J^{T,Q}(f_Q) = J^{T,M_Q}(f_Q)$ is the cutoff trace for M_Q applied to

$$f_Q(m) = \delta_Q(m)^{1/2} \int_K \int_{N_Q(\mathbb{A}_F)} f(k^{-1}mnk)\, dk\, dn$$

here δ_Q is the module of $Q(\mathbb{A}_F)$ and

$$\gamma_Q(X) = \int_{\mathfrak{a}_Q^G} \Gamma_Q(H,X)\, dH$$

is a polynomial in X ; when X is "positive" this is the volume of a compact convex subset in $\mathfrak{a}_Q/\mathfrak{a}_G$.

The cutoff trace J^T can then be continued to all $T \in \mathfrak{a}_0$ as a polynomial. One defines

$$T_0 = \frac{1}{w^G} \sum_{s \in W^G} H_{P_0}({w_s}^{-1})$$

where w_s is a representative in $G(F)$ of s in W^G the Weyl group of G , and w^G the cardinal of W^G . The element T_0 is such that

$$T_0 - s^{-1}T_0 = H_{P_0}({w_s}^{-1}) \; .$$

The value at $T = T_0$ of $J^{T,Q}$ will be denoted J^Q or even simply J when $Q = G$. Even at $T = T_0$ the cutoff trace is not invariant, the distribution J still depends on the choice of M_0 and K . More precisely let $f^y(x) = f(yxy^{-1})$ and for the sake of simplicity assume that f is K-central that is $f^k = f$ for all $k \in K$ then a simple computation shows that:

$$J(f^y) = \sum_{Q \supset P_0} J^Q(f_Q)u_Q(y)$$

where

$$u_Q(y) = \int_K \int_{\mathfrak{a}_Q^G} \Gamma_Q(X, -H_Q(ky))\, dk\, dX \; .$$

4. The geometric term

The geometric term is obtained by expanding $J(f)$ as a sum over $\mathfrak{O}$ the set of equivalence classes of elements in $G(F)$ the equivalence being the conjugacy of their semisimple part:

$$J(f) = \sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}(f)$$

where $J_{\mathfrak{o}}(f)$ is defined as $J(f)$ above but with K_P replaced by $K_{P,\mathfrak{o}}$ itself defined by

$$K_{P,\mathfrak{o}}(x,y) = \sum_{\gamma\in M_P(F)\cap\mathfrak{o}} \int_{N_P(\mathbb{A}_F)} f(x^{-1}\gamma ny)\, dn \quad .$$

If $\mathfrak{o}$ is the class of a semisimple element γ whose centralizer is connected and anisotropic modulo its center, then $J_{\mathfrak{o}}$ has a simple expression given by a weighted orbital integral: let M be a Levi subgroup that intersects $\mathfrak{o}$ and is minimal for this property; we may assume that M contains γ , then $G_\gamma = M_\gamma$ and

$$J_{\mathfrak{o}}(f) = a^M(\gamma)\; J_M(\gamma, f)$$

where

$$a^M(\gamma) = \mathrm{vol}\; (M_\gamma(F)\backslash M_\gamma^1(\mathbb{A}_F))$$

and

$$J_M(\gamma, f) = \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)\; v_M(x)\; d\dot{x}$$

is a weighted orbital integral. The weight v_M is defined as follows:

$$v_M(x) = \sum_{Q\in\mathcal{P}(M)} \gamma_Q(T_0 - H_Q(x))$$

where the sum is over $\mathcal{P}(M)$ the set of parabolic subgroups with Levi subgroup M .

But in general there does not seem to be any simple explicit expression for $J_{\mathfrak{o}}(f)$ when the $\gamma \in \mathfrak{o}$ may have non trivial unipotent parts: already for $SL(2)$ one must first stabilize the trace formula to get reasonably simple expressions for the contribution of unipotent elements, like those for $GL(2)$; moreover for groups of higher rank the geometry of nilpotent orbits comes into play to mess up the situation. Nevertheless for a restricted set of functions f the distribution $J_{\mathfrak{o}}(f)$ can be shown to be a linear combination of generalized weighted orbital integrals that will be defined below.

5. The weighted orbital integrals

Let S be a finite set of places of F and let f be a function in $\mathcal{C}_c^\infty(G(F_S)^1)$ where F_S is the product of completions of F over places in S . One can define **[A6]** a generalized notion of weighted orbital integral for all $\gamma \in M(F_S)$. Assume first that $G_\gamma = M_\gamma$, and let G_γ^0 be the neutral component of G_γ , then introduce

$$J_M(S,\gamma,f) = |D(S,\gamma)|^{1/2} \int_{G_\gamma^0(F_S)\backslash G(F_S)} f(x^{-1}\gamma x)\ v_M(S,x)\ d\dot{x}$$

with

$$D(S,\gamma) = \prod_{v\in S} \det\{(1-ad\gamma_s)|\mathfrak{g}/\mathfrak{g}_{\gamma_s}\}_v$$

here $\mathfrak{g}$ is the lie algebra of G and γ_s is the semi simple part of the Jordan decomposition $\gamma = \gamma_s\gamma_u$; the weight is

$$v_M(S,x) = \sum_{Q\in\mathcal{P}(M)} \gamma_Q(T_{0,S} - H_Q(x_S)) \ .$$

In general one defines $J_M(\gamma, f)$ as a limit over regular $a \in A_M$ of a sum over $\mathcal{L}(M)$ the set of Levi subgroups containing M :

$$J_M(S,\gamma,f) = \lim_{a\to 1} \sum_{L\in\mathcal{L}(M)} r_M^L(S,\gamma,a)\ J_L(S,a\gamma,f)$$

where the $r_M^L(S,\gamma,a)$ are themselves defined as limits over regular $\lambda \in \mathfrak{a}_M^*$ of sums over $\mathcal{P}^L(M)$ the set of parabolic subgroups with Levi subgroup M :

$$r_M^L(S,\gamma,a) = \lim_{\lambda\to 0} \sum_{R\in\mathcal{P}^L(M)} r_R^L(S,\lambda,\gamma,a)\ \theta_R(\lambda)^{-1}$$

with

$$\theta_R(\lambda) = \mathrm{vol}(\mathfrak{a}_R/\mathbb{Z}(\check{\Delta}_R))^{-1} \prod_{\alpha\in\Delta_R} \lambda(\check{\alpha})$$

and

$$r_R^L(S,\lambda,\gamma,a) = \prod_{v\in S}\prod_{\beta\in\Delta_R^L} |a^\beta - a^{-\beta}|_v^{\lambda(\check{\beta})\rho(\beta,\gamma_u)}$$

the $\rho(\beta, \gamma_u)$ being so chosen that the limits exist and do not vanish identically.

Now assume that f is the product of a function in $C_c^\infty(G(F_{S_0})^1)$ by the characteristic function of the maximal compact subgroup outside S_0 a finite set of places containing archimedean ones; it can be shown **[A7]** that the distributions $J_\mathfrak{o}$ are linear combinations of S-weighted orbital integrals provided that $S \supset S_0$ is big enough.

This yields the so called "fine $\mathfrak{O}$ expansion":

PROPOSITION. – *Assume that $S \supset S_0$ is big enough, then there exist uniquely defined numbers $a^L(S, \gamma)$ such that*

$$J(f) = \sum_{L \in \mathcal{L}} \frac{w^L}{w^G} \sum_{\gamma \in \{L(F)\}_{L,S}} a^L(S,\gamma)\ J_L(S,\gamma,f)$$

where $\{L(F)\}_{L,S}$ is a set of representatives for $\gamma \in L(F)$ up to conjugacy under $L(F)$ and for fixed semi simple part γ_s up to conjugacy under $L_{\gamma_s}(F_S)$ of the unipotent part of γ .

The existence proof for the coefficients $a^L(S, \gamma)$ proceeds by induction on the Levi subgroups $L \in \mathcal{L}$. But it does not give in general a procedure to compute them explicitly. Happily the following particular case seems to be enough for many applications. We need first a definition: consider γ semisimple and contained in a Levi subgroup M, we shall say that γ is M-elliptic if its M-centralizer M_γ contains a torus maximal in M and anisotropic modulo the center. Let $i^M(\gamma)$ be the number of connected components of M_γ with rational points.

PROPOSITION. – *Assume γ is semisimple and S big enough then*

$$a^L(S,\gamma) = i^L(\gamma)^{-1} \mathrm{vol}(L^0_\gamma(F) \backslash L^0_\gamma(\mathbb{A}_F)^1)$$

if γ is L-elliptic and equals 0 otherwise.

6. Intertwining operators

Before studying the spectral term we need to review some facts on intertwining operators. Given two parabolic subgroups P and Q and $s \in W$ such that $s(\mathfrak{a}_P) = \mathfrak{a}_Q$ one defines for ϕ on $N_P(\mathbb{A}_F)M_P(F)\backslash G(\mathbb{A}_F)$

$$(M_{Q|P}(s,\lambda)\phi)(x) = \int_{N(Q|P,s;\mathbb{A}_F)} \phi(w_s^{-1}nx)\ \beta(w_s^{-1}nx, x)\ dn$$

where $N(Q|P,s) = N_Q \cap w_s N_P w_s^{-1} \backslash N_Q$ and

$$\beta(y,x) = exp(<\lambda + \rho_P, H_P(y)> - <s\lambda + \rho_Q, H_Q(x)>)$$

with ρ_P the half sum of roots of P in N_P . This converges for good ϕ and $Re(\lambda)$ in a certain chamber and can be meromorphically continued.

We shall now define multidimensional logarithmic derivatives of intertwining operators [**A5**]. Consider

$$\mathcal{M}_Q(P,\lambda,\Lambda) = M_{Q|P}(1,\lambda)^{-1} M_{Q|P}(1,\lambda+\Lambda)e^{\Lambda(T_0)}$$

now assume $Q \subset R$ and restrict λ and Λ to be in $\mathfrak{a}_R$ then $\mathcal{M}_Q(P,\lambda,\Lambda)$ is independent of Q and will be denoted $\mathcal{M}_R(P,\lambda,\Lambda)$. Let L be a Levi subgroup, then the logarithmic derivative is defined to be

$$\mathcal{M}_L(P,\lambda) = \lim_{\Lambda\to 0} \sum_{R\in\mathcal{P}(L)} \mathcal{M}_R(P,\lambda,\Lambda)\theta_R(\Lambda)^{-1} .$$

Given $\pi \in \Pi(M_P)$ and $\lambda \in \mathfrak{a}^*_{M_P}$ we denote by π_λ the representation of $P(\mathbb{A}_F)$ defined by $\pi_\lambda(nam) = a^\lambda \pi(m)$ and by $\mathcal{I}_P(\pi_\lambda)$ the unitarily induced representation from $P(\mathbb{A}_F)$ up to $G(\mathbb{A}_F)$ by π_λ . The restriction of operators $M_{Q|P}(1,\lambda)$ to the subspace of vectors in $\mathcal{I}_P(\pi_\lambda)$ will be denoted $M_{Q|P}(\pi_\lambda)$

Intertwining operators $M_{Q|P}(\pi_\lambda)$ are the meromorphic continuation of the product over all places v of F of local operators. At a place v the local operator can be written as a product of a scalar function, the normalizing factor $n_{Q|P}(\pi_\lambda)$ meromorphic in λ, and a "normalized" intertwining operator $R_{Q|P}(\pi_\lambda)_v$. The normalized operators satisfy stronger functional equations than the ordinary ones (the cocycle relation will hold with no condition on the length for Weyl group elements). Moreover the following properties should hold:

- $R_{Q|P}(\pi_\lambda)_v$ is a rational function of λ if the place is archimedean or of $\{q_v^{<\lambda,\check{\alpha}>} \mid \alpha \in \Delta_P\}$ if the place is finite,
- the normalizing factor $n_{Q|P}(\pi_\lambda)$ has neither zero nor pole if π is tempered and Re(λ) is in the positive Weyl chamber defined by P ;
- at finite places v where G and π are unramified and K_v hyperspecial, the normalized operator induces the identity on K_v-fixed vectors.

For archimedean places, using results of [**KS**], or when $G = GL(n)$ thanks to [**S1**] the normalizing factors can be chosen as quotients of

L-functions. In general, such a natural normalization is not available but would follow from Conjecture 7.1 and Theorem 9.5 in **[S2]**. Nevertheless the existence of normalized operators can be proved. This is due to Langlands (see **[CLL]** Lecture 15 and **[A8]**).

Using normalized operators instead of ordinary ones logarithmic derivatives $\mathcal{R}_L(P,\pi_\lambda)$ are defined by the same procedure as $\mathcal{M}_L(P,\lambda)$ above and are used to define weighted characters **[A8]** :

$$J_L(\pi,f) = \mathrm{tr}\ (\mathcal{R}_L(P,\pi)\ \mathcal{I}_P(\pi,f))\ .$$

The logarithmic derivatives of normalizing factors $\mathbf{n}_L(P,\pi_\lambda)$ will also be needed in a relative situation and we shall use a splitting formula:

$$\mathcal{M}_M(P,\pi_\lambda) = \sum_{L\in\mathcal{L}(M)} \mathbf{n}_M^L(P,\pi_\lambda)\mathcal{R}_L(P,\pi_\lambda)\ .$$

7. The spectral term

For sufficiently large T the cutoff trace can be shown **[A2]** to be equal (at least for connected groups G) to the trace of the operator induced by the operator $\rho(f)$ in the image of a projector Λ^T called the truncation operator:

$$J^T(f) = \mathrm{tr}\ \Lambda^T\rho(f)\Lambda^T = \int_{G(F)\backslash G(\mathbb{A}_F)} \Lambda^T K(x,x)\ dx$$

where

$$\Lambda^T K(x,y) = \sum_{P\supset P_0} (-1)^{a_P^G} \sum_{\delta\in P(F)\backslash G(F)} \hat{\tau}_P(H_P(\delta x)-T)\ C_P K(\delta x,y)$$

and

$$C_P K(x,y) = \int_{N_P(F)\backslash N_P(\mathbb{A}_F)} K(nx,y)\ dn\ .$$

We shall now always assume f left and right K-finite. Using the spectral decomposition and the scalar product formula for truncated Eisenstein series one gets **[A5]** an explicit formula for $J(f)$. The spectral expansion of $J(f)$ is the sum

- over real positive numbers t ,
- over chains of Levi subgroups $M\subset L\in\mathcal{L}$,

- over classes modulo W^M of Weyl group elements $s \in W^L$ such that $s(\mathfrak{a}_M) = \mathfrak{a}_M$ and without lines of fixed vectors in $\mathfrak{a}_M^L$,
- over π in a maximal family of orthogonal irreducible subspace in $L^2(M(F)\backslash M(\mathbb{A}_F)^1)$, whose infinitesimal character has an imaginary part of modulus t and such that $s(\pi) \simeq \pi$,

of the following terms:

$$\frac{w^M}{w^G}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1} \int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} \operatorname{tr}\,(\mathcal{M}_L(P,\lambda)M_{P|P}(s,0)\mathcal{I}_P(\pi_\lambda,f))\,d\lambda$$

where P is any parabolic subgroup with Levi subgroup M . One can probably omit the partial summation indexed by the real parameter t since the trace class property for the discrete spectrum has now been established by W. Müller [M] but we didn't check the necessary uniform estimates to prove the convergence.

The "discrete" part $J_{disc,t}(f)$ of the spectral expansion of $J(f)$ is the sum over M, s, π as above of terms with $L = G$:

$$\frac{w^M}{w^G}|\det(s-1)_{\mathfrak{a}_M^G}|^{-1}\operatorname{tr}\,(M_{P|P}(s,0)\mathcal{I}_P(\pi_\lambda,f)) \;.$$

This contains in particular the contribution of representations that occur discretely in $L^2(G(F)\backslash G(\mathbb{A}_F)^1)$: they correspond to terms such that $M = G$. One can write $J_{disc,t}(f)$ as a sum of characters of irreducible representations of $G(\mathbb{A}_F)^1$ and we shall denote by $\Pi_{disc}(G,t)$ the subset of $\Pi(G)$ that contributes non trivially:

$$J_{disc,t}(f) = \sum_{\Pi_{disc}(G,t)} a^G_{disc}(\pi)\operatorname{tr}\pi(\mathrm{f}) \;.$$

When π occurs in the discrete automorphic spectrum for G then $a^G_{disc}(\pi)$ is simply its multiplicity. Given $\pi \in \Pi_{disc}(M,t)$ and $\lambda \in i\mathfrak{a}_M^*/\mathfrak{a}_L^*$ we define

$$a^L(\pi_\lambda) = a^M_{disc}(\pi)\mathbf{n}^L_M(\pi_\lambda) \;.$$

The above expression for $J(f)$ can now be rewritten **[A9]**,[**II**] with the help of weighted characters and of the splitting formula of §6 :

$$J(f) = \sum_t \sum_{L\in\mathcal{L}} \frac{w^L}{w^G}\int_{\Pi(L,t)} a^L(\pi)\;J_L(\pi,f)\;d\pi$$

the measure $(\Pi(L,t), d\pi)$ is defined by

$$\int_{\Pi(L,t)} \phi(\pi)\, d\pi = \sum_{M \subset L} \frac{w^M}{w^L} \sum_{\sigma \in \Pi_{disc}(M,t)} \int_{i\mathfrak{a}_M^*/\mathfrak{a}_L^*} \phi(\sigma_\lambda)\, d\lambda$$

The non invariant trace formula is the equality of the geometric and the spectral expansions of $J(f)$ described above:

THEOREM. – *For S large enough*

$$\sum_{L \in \mathcal{L}} \frac{w^L}{w^G} \sum_{\gamma \in \{L(F)\}_{L,S}} a^L(S,\gamma)\ J_L(S,\gamma,f)$$

$$= \sum_t \sum_{L \in \mathcal{L}} \frac{w^L}{w^G} \int_{\Pi(L,t)} a^L(\pi)\ J_L(\pi,f)\, d\pi\ .$$

8. THE INVARIANT TRACE FORMULA

We shall consider below global or S-objects using the same notations. When $L \neq G$ the distributions $J_L(*,f)$ are non invariant; more precisely if f is K-central one has

$$J_L(*,f^y) = \sum_{Q \in \mathcal{F}(L)} \frac{w^Q}{w^G}\ J_L^Q(*,f_Q)\ u_Q(y)$$

where $\mathcal{F}(L)$ is the set of parabolic subgroups Q of G with Levi subgroup M_Q containing L .

The key to the invariant form for the trace formula is to assume that f is left and right K-finite and to show, thanks to scalar Paley-Wiener theorems ([**CD**], [**BDK**] and [**R**]), that there exist functions $\phi_L(f)$ in $\mathcal{C}^\infty(L(F_S))$ with compact support modulo the center such that

$$\mathrm{tr}\ \pi(\phi_L(f)) = J_L(\pi,f)$$

for $\pi \in \Pi(L)$ tempered up to twist by quasi-characters; the functions $\phi_L(f)$ are defined up to addition of a function with vanishing scalar Fourier transform; of course we may and shall take $\phi_G(f) = f$.

Invariant distributions I_* "supported on characters" – i.e. such that $I_*(f) = 0$ whenever $\mathrm{tr}\ \pi(f) = 0$ for all $\pi \in \Pi_{temp}(G)$ – corresponding to the various non invariant distributions J_*, are then uniquely defined, inductively on L, by the following requirements:

$$J_M(*,f) = \sum_{L \in \mathcal{L}(M)} I_M^L(*,\phi_L(f))$$

the induction begins by taking $I_M^M = J_M^M$; of course this is consistent with the anisotropic case. The existence proof for ϕ_L and the I_* uses an intricate local-global inductive argument [**A9**].

Since up to now the extension of the scalar Paley-Wiener theorem [**CD**] to non connected real algebraic groups is not known the proof has been completed only when G is connected or is an inner twist of the semi-direct product of $GL(n)^r$ by the cyclic permutation of the r factors.

The following lemma is easy but important for applications:

LEMMA. – *If everything is unramified at v and f_v is in the spherical Hecke algebra then $I_M(\gamma, f_v) = J_M(\gamma, f_v)$.*

To see this one may use that $\mathcal{R}(\pi)$ which is a logarithmic derivative of normalized intertwining operators vanishes on the K_v-fixed vectors; since we assumed everything unramified and f spherical then $\phi_L(f_v)$ can be taken to be zero if $L \neq G$.

In the same vein one has the following

LEMMA. – *If π is tempered $I_L(\pi, f) = 0$ if $L \neq G$.*

The existence of an invariant form of the trace formula is a formal consequence of the existence of distributions I_* . The geometric and spectral terms have the same expression as in the non invariant form except that the I_* replace the J_* :

THEOREM. – *For S large enough*

$$I(f) = \sum_{L \in \mathcal{L}} \frac{w^L}{w^G} \sum_{\gamma \in \{L(F)\}_{L,S}} a^L(S,\gamma)\, I_L(S,\gamma,f)$$

$$= \sum_t \sum_{L \in \mathcal{L}} \frac{w^L}{w^G} \int_{\Pi(L,t)} a^L(\pi)\, I_L(\pi, f)\, d\pi.$$

9. The trace of Hecke operators

The invariant form of the trace formula is well suited to a classical problem: to compute the trace of Hecke operators in a space of automorphic forms satisfying given local conditions. This is a fundamental step in the study of the L-functions of Shimura varieties (cf. [**Ko**]).

For the sake of simplicity assume G semisimple connected and defined over $\mathbf{Q}$. Given a finite dimensional irreducible complex representation V of $G_\infty = G(\mathbf{R})$ one defines a local system on $X_K = G(\mathbf{Q})\backslash G(\mathbf{A})/K$ by

$$\mathcal{F}_V = G(\mathbf{Q})\backslash(V \times G(\mathbf{A})/K) .$$

Assume that G_∞ is such that $rk\ G_\infty = rk\ K_\infty$ then the L^2-cohomology $H^*_{(2)}(X_K, \mathcal{F}_V)$ is finite dimensional [**BC**] and can be shown to be equal to the relative Lie algebra cohomology of the space of smooth vectors in the discrete spectrum of the right regular representation of $G(\mathbf{A})$ in $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$:

$$H^*_{(2)}(X_K, \mathcal{F}_V) = H^*(\mathfrak{g}, K_\infty, L^2_{disc}(G(\mathbf{Q})\backslash G(\mathbf{A}))^\infty \otimes V) .$$

Consider $h \in C_c^\infty(G(\mathbf{A}_{fin}))$ a smooth compactly supported function on the group of finite adeles and let $\Lambda(h, V)$ be the alternating sum of the trace of the operator h^* induced in the L^2-cohomology:

$$\Lambda(h, V) = \sum(-1)^i \operatorname{tr}(h^* | H^i_{(2)}(X_K, \mathcal{F}_V)) .$$

The above equality between cohomology groups shows that

$$\Lambda(h, V) = \sum(-1)^i \operatorname{tr}(h^* | H^i(\mathfrak{g}, K_\infty, L^2_{disc}(G(\mathbf{Q})\backslash G(\mathbf{A})^1)^\infty \otimes V)) .$$

It can be shown to have a nice "geometric" expression using the trace formula. First it is not difficult (for example using [**CD**]) to prove that there exist Euler-Poincaré functions i.e. functions $f_V \in C_c^\infty(G(\mathbf{R}))$ such that

$$\operatorname{tr} \pi(f_V) = \sum(-1)^i \dim H^i(\mathfrak{g}, K_\infty, \pi \otimes V)$$

and hence $\Lambda(h, V)$ equals tr $\rho_{disc}(f_V \otimes h)$ the trace of the convolution operator defined by $f_V \otimes h$ in the discrete spectrum. Since f_V has zero trace in generalized principal series, the spectral term in the invariant trace formula is very simple, in fact:

$$I(f_V \otimes h) \ = \ \operatorname{tr} \rho_{disc}(f_V \otimes h)$$

The invariant trace formula tells us that for $S = \{\infty\} \cup S_{fin}$ large enough

$$\Lambda(h,V) = \sum_{M\in\mathcal{L}} \frac{w^M}{w^G} \sum_{\gamma\in\{M(F)\}_{M,S}} a^M(S,\gamma)\, I_M(S,\gamma,f_V\otimes h)$$

Then, using splitting formulas and the vanishing of real orbital integrals for non elliptic elements, we may seperate the contributions of the finite adeles and of the reals:

$$I_M(S,\gamma,f_V\otimes h) = I_M(\gamma,f_V)\; I_M^M(S_{fin},\gamma,h_M)$$

Over the finite adeles the contribution of γ is an orbital integral integral on $M(\mathbb{A}_{fin})$: when γ is semisimple

$$I_M^M(S_{fin},\gamma,h_M) = |D^M(S_{fin},\gamma)|^{1/2}\; h(M,\gamma)$$

where

$$h(M,\gamma) = \int_{M_\gamma^0(\mathbb{A}_{fin})\backslash M(\mathbb{A}_{fin})} h_Q(m^{-1}\gamma m)\; d\dot{m}$$

with h_Q deduced from h as explained in §3 but with integration over the finite adeles only, and Q is any parabolic subgroup with Levi subgroup M .

The contribution of the real place $I_M(\gamma,f_V)$ after some hard work can be shown to be particularly simple since f_V is "discrete" and "stable", in fact up to a sign f_V is the sum of pseudo-coefficients f_{π_∞}, where π_∞ runs over the L-packet of discrete series attached to V . It can be shown ([**A8**],[**II**], [**A10**]) that $I_M(\gamma,f_V)$ is zero unless γ is an $M(\mathbb{R})$-elliptic element; for such a γ it can be computed explicitly using Θ_V the sum of the characters for V-discrete series. For regular γ define

$$\Phi_M(\gamma,V) = (-1)^{q(G)}|D_M^G(\{\infty\},\gamma)|^{1/2}\Theta_V(\gamma)$$

where as usual $q(G) = \dim G_\infty^1/K_\infty$. These functions can be continuously continued to all $M(\mathbb{R})$-elliptic elements.

Now introduce the real form compact modulo the center $\overline{M_\gamma^0}$ of the connected centralizer, and define a variant of the Tamagawa numbers:

$$\chi(M_\gamma) = (-1)^{q(M_\gamma)}d(M_\gamma)\frac{\mathrm{vol}(A_M M_\gamma^0(\mathbb{Q})\backslash M_\gamma^0(\mathbb{A}))}{\mathrm{vol}(A_M\backslash\overline{M_\gamma^0}(\mathbb{R}))}$$

where $d(M_\gamma)$ is the cardinal of L-packets of discrete series on M^0_γ .
We can now state the final formula [**A10**]:

$$\Lambda(h,V) = \sum_{M\in\mathcal{L}} (-1)^{a^G_M} \frac{w^M}{w^G} \sum_{\gamma\in\{M(\mathbb{Q})_{\mathbf{R}-ell}\}} i^M(\gamma)^{-1}\chi(M_\gamma)\Phi_M(\gamma,V)h_{M,\gamma} .$$

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Faisceaux Automorphes Liés aux Séries d'Eisenstein

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0. INTRODUCTION

L'objet principal de cet exposé est de donner une interprétation géométrique des séries d'Eisenstein (principales et partout non ramifiées) pour GL_n sur un corps de fonctions. Cette interprétation géométrique nous permet alors de définir de nouveaux faisceaux automorphes qui viennent s'ajouter aux faisceaux automorphes cuspidaux construits par Drinfeld pour GL_2.

Plus précisément, le contenu de cet exposé est le suivant. Au numéro 1, nous rappelons certains traits marquants de la théorie des faisceaux caractères de Lusztig: cette théorie nous sert de modèle.

Au numéro 2, nous rappelons la théorie du corps de classes abélien géométrique de Lang et Rosenlicht (dans le cas partout non ramifié): il s'agit de la théorie des faisceaux automorphes pour GL_1 sur un corps de fonctions.

Puis nous donnons, au numéro 3, l'interprétation géométrique des séries d'Eisenstein (principales et partout non ramifiées) pour GL_n sur un corps de fonctions ainsi que du produit de ces séries d'Eisenstein par leur "dénominateur" usuel (un produit de fonctions L).

Cette interprétation géométrique conduit tout naturellement à définir un processus d'induction parabolique pour les faisceaux automorphes, ce que nous faisons au numéro 4.

Au numéro 5, nous examinons en détail la cas où le corps des fonctions est celui d'une droite projective. Dans ce cas, on peut dresser la liste complète de tous les faisceaux automorphes (partout non ramifiés) du fait de l'absence totale de faisceaux automorphes cuspidaux (partout non ramifiés).

Enfin, au numéro 6, nous donnons quelques exemples de faisceaux automorphes (partout non ramifiés) sur un corps de fonctions arbitraire et nous énonçons une conjecture sur le rang générique des faisceaux automorphes cuspidaux (partout non ramifiés), conjecture qui précise celles de [Lau 2].

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1. Quelques Aspects de la Theorie des Faisceaux Caractères de Lusztig

(1.0) Soient k un corps algébriquement clos, ℓ un nombre premier inversible dans k et $\bar{\mathbb{Q}}_\ell$ une clôture algébrique de $\mathbb{Q}_\ell$.

(1.1) Pour tout groupe réductif connexe G sur k, Lusztig a défini un ensemble particulier de classes d'isomorphie de $\bar{\mathbb{Q}}_\ell$-faisceaux pervers irréductibles G-équivariants sur G (G agit par conjugaison sur lui-même): les faisceaux caractères (cf. [Lu 1] pour une présentation de cette théorie et une bibliographie).

Si G est un tore, les faisceaux caractères sont les $\bar{\mathbb{Q}}_\ell$-faisceaux pervers de la forme

$$\mathcal{F}[\dim G]$$

où $\mathcal{F}$ est un $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1 sur G d'ordre fini premier à la caractéristique p de k si $p > 0$ (i.e. il existe un entier $N \geq 1$, premier à p si $p > 0$, tel que $\mathcal{F}^{\otimes N}$ soit isomorphe au faisceau constant $\bar{\mathbb{Q}}_\ell$).

Si $G = GL_{n,k}$ pour un entier $n \geq 1$ les faisceaux caractères sur G sont tous obtenus par la construction suivante (cette construction qui vaut pour G arbitraire ne donne en général qu'une partie des faisceaux caractères). Soient $T \subset G$ un tore maximal de G et $T \subset B \subset G$ un sous-groupe de Borel de G admettant T comme facteur de Levi. On note W le groupe de Weyl de $(G\ ,\ T\ )$. On a alors un diagramme d'induction parabolique

$$\begin{tikzcd} \tilde{G} \arrow[d] \arrow[dr, "\rho"] & \\ G & T \end{tikzcd} \qquad (1.1.1)$$

où

$$\tilde{G} = \{(g, hB) \in G \times (G/B) | h^{-1}gh \in B\}$$
$$\pi(g, hB) = g$$
$$p(g, hB) = \overline{h^{-1}gh}$$

(on a noté

$$B \twoheadrightarrow T, \quad b \mapsto \bar{b}$$

la projection canonique). Il est important de remarquer que, si l'on note $[H/ad(H)]$ le champ algébrique sur k des classes de conjugaison dans un groupe algébrique H sur k (le quotient de H par l'action de H sur lui-même par conjugaison, [De-Mu] (4.8) et [Lau 3] (4.14.1.1)), le diagramme d'induction s'insère dans le diagramme commutatif suivant

$$\begin{tikzcd} G \arrow[d] & \tilde{G} \arrow[l, "\pi"'] \arrow[r, "\rho"] \arrow[d] & T \arrow[d] \\ {[G/ad(G)]} & {[B/ad(B)]} \arrow[l] \arrow[r] & {[T/ad(T)]} \end{tikzcd}$$

où les flèches verticales sont les projections canoniques pour les deux extrêmes et le morphisme induit par

$$\tilde{G} \to B, \quad (g, hB) \mapsto h^{-1}gh$$

pour la médiane, où les flèches horizontales de la ligne du bas sont induites par l'inclusion $B \hookrightarrow G$ et la projection $B \twoheadrightarrow T$ et où le premier carré est cartésien.

LEMME 1.1.2.

(1) *$\tilde{G}$ est une variété quasi-projective, lisse et connexe sur k, de dimension égale à celle de G.*
(2) *ρ est lisse.*
(3) *π est projective et "small" au sens de Goresky-MacPherson ([Go-Ma](6.2)); en particulier, au-dessus de l'ouvert dense G_{rss} de G formé des éléments réguliers semi-simples, π est un revêtement fini étale galoisien de groupe de Galois W.*

Rappelons l'argument de Lusztig pour prouver que π est "small". Il suffit de démontrer que

$$Z = \tilde{G} \times_G \tilde{G}$$

est de dimension au plus égale à celle de G et que chaque composante irréductible de Z de dimension égale à celle de G domine G . Mais

$$Z = \{(g, h_1B, h_2B) | g \in h_1Bh_1^{-1} \cap h_2Bh_2^{-1}\}$$

admet une stratification $(Z_w)_{w \in W}$ où, pour chaque $w \in W$, Z_w est la partie localement fermée de Z définie par

$$Z_w = \{(g, h_1B, h_2B) | h_2^{-1}h_1 \in B\dot{w}B, g \in h_1Bh_1^{-1} \cap h_2Bh_2^{-1}\}$$

($\dot{w} \in N_G(T)$ est un représentant arbitraire de w). On vérifie alors facilement que chaque Z_w est lisse, connexe sur k, de dimension égale à celle de G et que la projection naturelle

$$Z_w \to G$$

est un isomorphisme au-dessus de G_{rss}.

Alors, si $A = \mathcal{F}[\dim T]$ est un faisceau caractère sur T, on peut former le "complexe de $\bar{\mathbb{Q}}_\ell$-faisceaux"

$$K_A = R\pi_* \rho^* A[\dim G - \dim T] \tag{1.1.3}$$

et il résulte aussitôt du lemme que K_A est en fait un $\bar{\mathbb{Q}}_\ell$-faisceau pervers et que, plus précisément,

$$K_A = j_{!*}(\pi_{rss})_* \rho_{rss}^* \mathcal{F}[\dim G]$$

où $\pi_{rss} : \tilde{G}_{rss} \to G_{rss}$ et $\rho_{rss} : \tilde{G}_{rss} \to T$ sont les restrictions de π et ρ respectivement à G_{rss} et où $j : G_{rss} \hookrightarrow G$ est l'inclusion, $(\pi_{rss})_* \rho_{rss}^* \mathcal{F}$ étant un $\bar{\mathbb{Q}}_\ell$-faisceau lisse semi-simple (en fait à monodromie finie) sur G_{rss}.

Par définition, les constituants irréductibles de K_A pour chaque choix de A sont des faisceaux caractères (que ces constituants soient G-équivariants résulte des considérations précédant le lemme).

(1.2) Supposons maintenant que k soit une clôture algébrique du corps fini $\mathbb{F}_q$ et que G et T soient définis sur $\mathbb{F}_q$ (on ne suppose pas que B est défini sur $\mathbb{F}_q$ ni que T est déployé sur $\mathbb{F}_q$). Si A est un faisceau caractère sur T qui est défini sur $\mathbb{F}_q$ au sens des faisceaux de Weil ([De 2] (1.1.10)), i.e. muni d'un isomorphisme

$$\varphi : \mathrm{Frob}_q^* A \tilde{\to} A$$

alors K_A est lui aussi défini sur $\mathbb{F}_q$, i.e. muni d'un isomorphisme

$$\psi : \mathrm{Frob}_q^* A \tilde{\to} A$$

induit par φ. On rigidifie φ de telle sorte que

$$tr(\varphi_1) = (-1)^{\dim T}$$

(i.e. on rigidifie $\mathcal{F}$ par $\mathcal{F}_1 \simeq \bar{\mathbb{Q}}_\ell$ où $1 \in T$ est l'identité); alors la fonction "trace de Frobenius" ([SGA 5] XV)

$$\chi_{(A,\varphi)} : T(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$$

définie par

$$\chi_{(A,\varphi)}(t) = tr(\varphi_t)$$

est de la forme

$$\chi_{(A,\varphi)} = (-1)^{\dim T}\theta_{(A,\varphi)}$$

où

$$\theta_{(A,\varphi)} : T(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell^\times$$

est un caractère du tore fini $T(\mathbb{F}_q)$.

D'après Deligne-Lusztig ([De-Lu]), le caractère $\theta_{(A,\varphi)}$ induit cohomologiquement une représentation virtuelle

$$R_T^{\theta_{(A,\varphi)}}$$

de $G(\mathbb{F}_q)$ et Lusztig a montré que (modulo certaines restrictions sur q, cf. [Lu] (9.2) pour un énoncé précis).

THÉORÈME 1.2.1. *La fonction "trace de Frobenius" de* (K_A, ψ),

$$\chi_{(K_A,\psi)} : G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell,$$

définie par

$$\chi_{(K_A,\psi)}(g) = tr(\psi_g)$$

est liée au caractère virtuel de $R_T^{\theta_{(A,\varphi)}}$ *par la relation suivante*

$$\chi_{(K_A,\psi)}(g) = (-1)^{\dim T} tr(R_T^{\theta_{(A,\varphi)}}(g))$$

pour tout $g \in G(\mathbb{F}_q)$.

(1.3) Supposons k de caractéristique nulle. Soient $\mathfrak{G}$ l'algèbre de Lie de G et $\mathfrak{G}^*$ le k-espace vectoriel dual de G; G agit par l'action adjointe $Ad : G \to GL(\mathfrak{G})$ sur $\mathfrak{G}$ et l'action coadjointe $Ad^* : G \to GL(\mathfrak{G}^*)$ sur $\mathfrak{G}^*$.

Identifions T^*G et $G \times \mathfrak{G}^*$ par la translation à gauche.

Le cône nilpotent

$$\Lambda_G \subset T^*G = G \times \mathfrak{G}^*$$

est le fermé lagrangien défini par

$$\Lambda_G = \{(g, \xi^*) \in G \times \mathfrak{G}^* | Ad^*(g)(\xi^*) = \xi^* \text{et} \xi^* \in \mathcal{N}^*\}$$

où

$$\mathcal{N}^* = \bigcup_{h \in G} Ad^*(h)(b^\perp)$$

avec $b^\perp \subset \mathfrak{G}^*$ l'orthogonal de l'algèbre de Lie $b \subset \mathfrak{G}$ de $B \subset G$.

Le résultat suivant est dû à Mirkovic et Vilonen ([Mi-Vi]) et indépendamment à Ginsburg ([Gi](1.4.2) et (1.6.1)).

THÉORÈME 1.3.1. *Un $\bar{\mathbb{Q}}_\ell$ -faisceau pervers irréductible G-équivariant A sur G est un faisceau caractère si et seulement si sa variété caractéristique est contenue dans Λ_G.*

2. Corps de Classes Abelien Geometrique

(2.0) On rappelle brièvement dans ce numéro la théorie du corps de classes abélien géométrique due à Lang ([Lan]) et Rosenlicht ([Ro]) en suivant la présentation qu'en ont donnée Serre ([Se]) puis Deligne ([De 1]). On se limitera au cas partout non ramifié.

(2.1) Soit X une courbe projective, lisse et géométriquement connexe sur le corps fini $\mathbb{F}_q$ à q éléments; on note g le genre de X. Pour simplifier, on munit X d'un point base ∞ rationnel sur $\mathbb{F}_q$, $\infty \in X(\mathbb{F}_q)$.

Soit $\mathrm{Pic}_{X/\mathbb{F}_q}$ le schéma de Picard qui paramètre les classes d'isomorphie de $\mathcal{O}_X$-Modules inversibles $\mathcal{L}$. C'est un $\mathbb{F}_q$-schéma en groupes abéliens (pour le produit tensoriel des $\mathcal{O}_X$-Modules inversibles), son groupe des composantes connexes est canoniquement isomorphe à $\mathbb{Z}$ par l'application degré

$$\deg : \mathrm{Pic}_{X/\mathbb{F}_q} \to \mathbb{Z}$$

(pour tout $\mathcal{O}_X$-Module inversible $\mathcal{L}$, on a

$$\chi(X, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g$$

par Riemann-Roch) et sa composante neutre $Pic^o_{X/\mathbb{F}_q}$ est une variété abélienne sur $\mathbb{F}_q$ de dimension g (la "jacobienne" de X). Le point $\infty \in X(\mathbb{F}_q)$ fournit un scindage de la suite exacte

$$1 \to \mathrm{Pic}^o_{X/\mathbb{F}_q} \to \mathrm{Pic}_{X/\mathbb{F}_q} \xrightarrow{\deg} \mathbb{Z} \to 0 :$$

à $\ell \in \mathbb{Z}$, on associe la classe d'isomorphie du $\mathcal{O}_X$-Module inversible $\mathcal{O}_X(\ell.\infty)$.

On note F le corps des fonctions de X; son corps des constantes est exactement $\mathbb{F}_q$. On identifie l'ensemble des places de F à l'ensemble $|X|$ des points fermés de X. Pour chaque $x \in |X|$, on note F_x le complété de F en la place x, $v_x : F_x - \{0\} \to \mathbb{Z}$ la valuation discrète correspondante, $\mathcal{O}_x \subset F_x$ l'anneau de cette valuation, m_x l'idéal maximal de $\mathcal{O}_x$ et $\deg(x)$ le degré du corps résiduel $\mathcal{O}_x/m_x$ sur $\mathbb{F}_q$. On peut former alors l'anneau des adèles de F,

$$\mathbb{A} = \prod_{x\in|X|}(F_x, \mathcal{O}_x)$$

et plonger F diagonalement dans $\mathbb{A}$. On note

$$\mathcal{O} = \prod_{x \in |X|} \mathcal{O}_x$$

le sous-anneau compact maximal de $\mathbb{A}$ et

$$\deg : \mathbb{A}^\times \to \mathbb{Z}$$

l'homomorphisme défini par

$$\deg((a_x)_{x\in|X|}) = \sum_{x\in|X|} \deg(x) v_x(a_x);$$

$F^\times$ et $\mathcal{O}^\times$ sont contenus dans le noyau $(\mathbb{A}^\times)^0$ de cet homomorphisme.

On a un isomorphisme canonique de suites exactes

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Pic}^0_{X/\mathbb{F}_q}(\mathbb{F}_q) \arrow[r] \arrow[d] & \mathrm{Pic}_{X/\mathbb{F}_q}(\mathbb{F}_q) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & F^\times \backslash (\mathbb{A}^\times)^0 / \mathcal{O}^\times \arrow[r] & F^\times \backslash (\mathbb{A}^\times) / \mathcal{O}^\times \arrow[r, "\deg"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

et le scindage donné par

$$\ell \mapsto (\text{classe d'isomorphie de } \mathcal{O}_X(\ell.\infty))$$

correspond au scindage donné par

$$\ell \mapsto F^\times . a^\ell . \mathcal{O}^\times$$

où $a = (a_x)_{x\in|X|}$ avec $a_x = 1$ pour tout $x \neq \infty$ et $v_\infty(a_\infty) = 1$.

Enfin, on a un morphisme naturel

$$\imath : X \to Pic^0_{X/\mathbb{F}_q}$$

donné par

$$\imath(x) = \text{classe d'isomorphie de } \mathcal{O}_X(x - \infty)$$

et ι est un plongement dès que $g \geq 1$.

(2.2) On fixe maintenant un nombre premier ℓ distinct de la caractéristique p de $\mathbb{F}_q$ et une clôture algébrique $\bar{\mathbb{Q}}_\ell$ de $\mathbb{Q}_\ell$.

On considère les trois ensembles suivants:

(1) l'ensemble des $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de rang $1, \mathcal{F}$, sur X munis d'une rigidification $\infty^*\mathcal{F} \simeq \bar{\mathbb{Q}}_\ell$ en $\infty : \mathrm{Spec}(\mathbb{F}_q) \hookrightarrow X$;

(2) l'ensemble des $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de rang 1, $\mathcal{G}$, sur $\mathrm{Pic}^0_{X/\mathbb{F}_q}$ munis d'une rigidification $1^*\mathcal{G} \simeq \bar{\mathbb{Q}}_\ell$ en $1 : \mathrm{Spec}(\mathbb{F}_q) \hookrightarrow Pic^0_X/\mathbb{F}_q$ (l'origine);

(3) l'ensemble des caractères

$$\chi : F^\times\backslash(\mathbb{A}^\times)^0/\mathcal{O}^\times \to \bar{\mathbb{Q}}_\ell^\times .$$

On a des applications naturelles entre ces trois ensembles,

$$(3) \to (2) \to (1).$$

A χ, on associe le $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1, $\mathcal{G}_\chi$, sur $\mathrm{Pic}^0_{X/\mathbb{F}_q}$ rigidifié en 1 obtenu en poussant le $\mathrm{Pic}^0_{X/\mathbb{F}_q}(\mathbb{F}_q)$-torseur de Lang à l'aide de χ^{-1},

$$\begin{array}{c}\mathrm{Pic}^0_{X/\mathbb{F}_q}\\ \mathcal{L} \mapsto \mathrm{Frob}_q^*\mathcal{L} \otimes \mathcal{L}^{\otimes -1} \ \big\downarrow\ \big) \ \mathrm{Pic}^0_{X/\mathbb{F}_q}(\mathbb{F}_q) \xrightarrow{\sim} F^\times\backslash(\mathbb{A}^\times)^0/\mathcal{O}^\times \xrightarrow{\chi^{-1}} \bar{\mathbb{Q}}_\ell^\times\\ \mathrm{Pic}^0_{X/\mathbb{F}_q}\end{array}$$

où $\mathrm{Frob}_q : \mathrm{Pic}^0_{X/\mathbb{F}_q} \to \mathrm{Pic}^0_{X/\mathbb{F}_q}$ est l'endomorphisme de Frobenius relatif à $\mathbb{F}_q$; à $\mathcal{G}$, on associe le $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1, $\iota^*\mathcal{G}$, sur X avec sa rigidification naturelle en ∞.

THÉORÈME (2.2.1). *Les applications ci-dessus sont des bijections entre les ensembles (1), (2) et (3) ci-dessus.*

PREUVE: On va construire des applications inverses

$$(1) \to (2) \to (3).$$

Pour tout entier $d \geq 1$, on note $X^{(d)}$ la puissance symétrique de X et

$$\iota^{(d)} : X^{(d)} \to \mathrm{Pic}^0_{X/\mathbb{F}_q}$$

le morphisme défini par

$$\iota^{(d)}(D) = \text{classe d'isomorphie de } \mathcal{O}_X(D - d.\infty)$$

(on identifie les points $X^{(d)}$ aux diviseurs effectifs de degré d sur X); pour $d > 2g-2, \iota^{(d)}$ est un fibré projectif. A $\mathcal{F}$ (dans l'ensemble (1)), on associe dans un premier temps le $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1, $\mathcal{F}^{(d)}$, sur $X^{(d)}$ de fibre en

$$D = \Sigma_{i \in I} d_i . x_i \in X^{(d)}$$

($x_i \neq x_j$ si $i \neq j \in I$, $\Sigma_{i \in I} d_i . deg(x_i) = d$) le $\bar{\mathbb{Q}}_\ell$-espace vectoriel

$$\mathcal{F}_D^{(d)} = \bigotimes_{i \in I} \mathrm{Sym}^{d^i}(\mathcal{F}_{x_i})$$

(voir [SGA 4](XVII, 5.5) pour une construction en forme); $\mathcal{F}^{(d)}$ est naturellement rigidifié en $d.\infty \in X^{(d)}(\mathbb{F}_q)$. Comme les espaces projectifs sont simplement connexes, dès que $d > 2g-2, \mathcal{F}^{(d)}$ est nécessairement de la forme

$$\iota^{(d)^*} \mathcal{G}$$

pour un $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1, $\mathcal{G}$, sur $\mathrm{Pic}^0_{X/\mathbb{F}_q}$ rigidifié en 1; $\mathcal{G}$ est uniquement déterminé par $\mathcal{F}^{(d)}$ et il n'est pas difficile de vérifier que $\mathcal{G}$ est indépendant de $d > 2g-2$ et que

$$\iota^{(d)^*} \mathcal{G} = \mathcal{F}^{(d)}$$

pour tout $d \geq 1$ et en particulier pour $d = 1$. Ceci achève la construction de l'application (1) $\rightarrow$ (2).

Soit maintenant $\mathcal{G}$ dans l'ensemble (2) et soit

$$\chi : \mathrm{Pic}^0_{X/\mathbb{F}_q}(\mathbb{F}_q) \rightarrow \bar{\mathbb{Q}}_\ell^\times$$

sa fonction "trace de Frobenius" ([SGA 5] XV). Compte-tenu de l'isomorphisme

$$\mathrm{Pic}^0_{X/\mathbb{F}_q}(\mathbb{F}_q) \tilde{\rightarrow} F^\times \backslash (\mathbb{A}^\times)^0 / \mathcal{O}^\times,$$

pour achever la construction de l'application (2) $\rightarrow$ (3), il ne reste plus qu'à vérifier que χ est un caractère. Or, si

$$pr_1, pr_2 \text{ et } m : \mathrm{Pic}^0_{X/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{Pic}^0_{X/\mathbb{F}_q} \rightarrow \mathrm{Pic}^0_{X/\mathbb{F}_q}$$

sont respectivement les deux projections canoniques et la loi de composition, on a un isomorphisme canonique de $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de rang 1 rigidifiés

$$m^* \mathcal{G} \simeq pr_1^* \mathcal{G} \otimes pr_2^* \mathcal{G}$$

($\mathrm{Pic}^0_{X/\mathbb{F}_q}$ est une variété abélienne, d'où l'assertion).

La vérification que l'on a bien des applications inverses l'une de l'autre est laissée au lecteur (cf. [SGA $4\frac{1}{2}$] [Sommes trig.]1).

(2.3) On a la variante suivante des résultats du numéro (2.2). On considère les trois ensembles:

(1′) l'ensemble des classes d'isomorphie de $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de Weil de rang 1, $\mathcal{F}$, sur X;

(2′) l'ensemble des classes d'isomorphie de $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de Weil de rang 1, $\mathcal{G}$, sur $\mathrm{Pic}^0_{X/\mathbb{F}_q}$ munis d'un isomorphisme de $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de Weil

$$m^*\mathcal{G} \simeq pr_1^*\mathcal{G} \otimes pr_2^*\mathcal{G}$$

compatible à l'associativité et la commutativité où

$$pr_1, pr_2 \text{ et } m : \mathrm{Pic}^0_{X/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathrm{Pic}^0_{X/\mathbb{F}_q} \to \mathrm{Pic}^0_{X/\mathbb{F}_q}$$

sont respectivement les deux projections canoniques et la loi de composition;

(3′) l'ensemble des caractères

$$\chi : F^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times \to \bar{\mathbb{Q}}_\ell^\times.$$

(Pour la notion de $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de Weil, on renvoie à [De 2] (1.1.10)).

THÉORÈME 2.3.1. *Les applications*

$$(1') \leftarrow (2') \rightarrow (3'),$$

définie par

$$\mathcal{G} \mapsto \iota'^*\mathcal{G}$$

où

$$\iota' : X \to \mathrm{Pic}^1_{X/\mathbb{F}_q} \hookrightarrow Pic_{X/\mathbb{F}_q}$$

envoie x sur la classe d'isomorphie du $\mathcal{O}_X$-Module inversible $\mathcal{O}_X(x)$, et

$$\mathcal{G} \mapsto \chi$$

où χ est la "trace de Frobenius" de $\mathcal{G}$ (compte-tenu de l'identification

$$Pic_{X/\mathbb{F}_q}(\mathbb{F}_q) \to F^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times),$$

sont bijectives.

PREUVE: A $\mathcal{F}$ (dans l'ensemble (1')), on associe $\mathcal{G}$ (dans l'ensemble (2')) de la façon suivante. Le $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1

$$\mathcal{F} \otimes (\infty^* \mathcal{F})^{\otimes -1}$$

sur X ($\infty^*\mathcal{F}$ est considéré comme faisceau de Weil géométriquement constant sur X) est canoniquement rigidifié en ∞ et il lui correspond par (2.2.1) un unique $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1, $\mathcal{G}^0$, sur $\mathrm{Pic}^0_{X/\mathbb{F}_q}$ rigidifié en 1. Pour tout entier $\ell \in \mathbb{Z}$, on a un isomorphisme

$$\epsilon_\ell : \mathrm{Pic}^\ell_{X/\mathbb{F}_q} \to \mathrm{Pic}^0_{X/\mathbb{F}_q}, \mathcal{L} \mapsto \mathcal{L}(-\ell.\infty)$$

et on définit $\mathcal{G}$ par

$$\mathcal{G}|\mathrm{Pic}^\ell_{X/\mathbb{F}_q} = (\epsilon_\ell^* \mathcal{G}^0) \otimes (\infty^* \mathcal{F})^{\otimes \ell}.$$

Le reste de la preuve du théorème est laissé au lecteur.

Remarques (2.3.2.1) Si l'on fixe une clôture algébrique $\bar{F}$ de F (le corps des fonctions de X), l'ensemble (1') s'identifie canoniquement à l'ensemble des classes d'isomorphie de caractères ℓ-adiques

$$W(\bar{F}/F) \to \bar{\mathbb{Q}}_\ell^\times$$

qui sont partout non ramifiés ($W(\bar{F}/F) \subset \mathrm{Gal}(\bar{F}/F)$ est le groupe de Weil, cf. [De 2] (1.1.10)).

(2.3.2.2) Dans l'énoncé (2.3.1), le point ∞ ne joue plus aucun rôle.

(2.3.2.3) Soit $\mathbb{F}_{q^n} \supset \mathbb{F}_q$ une extension finie du corps fini $\mathbb{F}_q$; on note $F_n = F.\mathbb{F}_{q^n}$ le corps des fonctions de $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. On a un homomorphisme "norme"

$$N_{\mathbb{F}_{q^n}/\mathbb{F}_q} : \mathbb{A}_n^\times \to \mathbb{A}^\times$$

où $\mathbb{A}_n^\times$ est l'anneau des adèles de F_n. Si $\mathcal{F}$ (dans l'ensemble (1')), $\mathcal{G}$ (dans l'ensemble (2')) et χ (dans l'ensemble (3')) se correspondent par les bijections ci-dessus pour X, alors $\mathcal{F}|X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, $\mathcal{G}|Pic_{X/\mathbb{F}_q} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ et $\chi \circ N_n$ se correspondent par les bijections ci-dessus pour $X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ (cf. [SGA $4\frac{1}{2}$] [Sommes trig.] (1.7.7)).

(2.4) Soient $\mathcal{F}$ (dans l'ensemble (1')), $\mathcal{G}$ (dans l'ensemble (2')) et χ (dans l'ensemble (3')) se correspondant par les bijections ci-dessus. On a alors une fonction L attachée à $\mathcal{F}$

$$L(\mathcal{F}, T) = \frac{P_1(\mathcal{F}, T)}{P_0(\mathcal{F}, T) P_2(\mathcal{F}, T)}$$

où

$$P_i(\mathcal{F},T) = \det(1 - T\,\mathrm{Frob}_q^*, H^i(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}))$$

($\bar{\mathbb{F}}_q$ est une clôture algébrique de $\mathbb{F}_q$ et Frob_q est le Frobenius géométrique de $W(\bar{\mathbb{F}}_q/\mathbb{F}_q)$).

D'après la formule des traces de Grothendieck (cf. [SGA 5] (XV, §3 n° 2) et [SGA $4\frac{1}{2}$] [Rapport] (4.10)) on a:

Proposition 2.4.1. *La fraction rationnelle $L(\mathcal{F},T) \in \bar{\mathbb{Q}}_\ell(T)$ admet le développement en série formelle*

$$L(\mathcal{F},T) = \Sigma_{d\geq 0}(\Sigma_{D\in X^{(d)}(\mathbb{F}_q)}\chi(\mathcal{O}_X(D)))T^d$$

dans $\bar{\mathbb{Q}}_\ell[[T]]$.

3. Series d'Eisenstein Principales partout non Ramifies pour GL_n sur un Corps de Fonctions

(3.0) On va donner une interprétation géométrique des séries d'Eisenstein principales partout non ramifiées pour GL_n sur un corps de fonction ainsi que de leurs "numérateurs".

(3.1) On conserve les notations de (2.1). On fixe un entier $n \geq 1$ et on considère le champ sur $\mathbb{F}_q$

$$\mathrm{Fib}_{X/\mathbb{F}_q,n}$$

des fibrés vectoriels de rang n, $\mathcal{L}$, sur X (ou $\mathcal{O}_X$-Modules localement libres de rang n). C'est un champ algébrique sur $\mathbb{F}_q$ dont les composantes connexes sont paramétrées par $\mathbb{Z}$: pour $\ell \in \mathbb{Z}$,

$$\mathrm{Fib}^{\ell}_{X/\mathbb{F}_q,n}$$

classifie les fibrés vectoriels sur X de rang n et de degré ℓ. Chacune de ces composantes connexes est localement de type fini, lisse et de dimension $n^2(g-1)$ sur $\mathbb{F}_q$ (cf. [Lau 3]5 pour la notion de dimension d'un champ).

On considère aussi le champ sur $\mathbb{F}_q$

$$Fib_{X/\mathbb{F}_q,(1^n)}$$

des drapeaux

$$\mathcal{L}_\cdot = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_n = \mathcal{L})$$

où chaque $\mathcal{L}_i$ est un fibré vectoriel de rang i sur X, localement facteur direct de $\mathcal{L}_{i+1}$ en tant que $\mathcal{O}_X$-Module localement libre ($i = 0, \ldots, n-1$). C'est un champ algébrique sur $\mathbf{F}_q$; ses composantes connexes sont paramétrées par les suites d'entiers

$$\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n :$$

la composante connexe

$$\mathrm{Fib}^{\underline{\lambda}}_{X/\mathbf{F}_q,(1^n)}$$

classifie les drapeaux $\mathcal{L}.$ comme ci-dessus tels que

$$\mathcal{A}_i = \mathcal{L}_i/\mathcal{L}_{i-1}$$

soit un $\mathcal{O}_X$-Module inversible de degréλ_i, pour $i = 1, \ldots, n$; elle est localement de type fini, lisse sur $\mathbf{F}_q$ et

$$\dim(\mathrm{Fib}^{\underline{\lambda}}_{X/\mathbf{F}_q,(1^n)}) = \frac{n(n+1)}{2}(g-1) + \sum_{1\leq i<j\leq n}(\lambda_j - \lambda_i)$$

(cf. [Lau 1]2).

On notera

$$\begin{array}{ccc} \mathrm{Fib}_{X/\mathbf{F}_q,(1^n)} & & \\ \downarrow \pi & \searrow^{\rho} & \\ \mathrm{Fib}_{X/\mathbf{F}_q,n} & & (\mathrm{Pic}_{X/F_q})^n \end{array}$$

les projections naturelles

$$\begin{aligned} \pi(\mathcal{L}.) &= \mathcal{L} \\ \rho(\mathcal{L}.) &= (\mathcal{L}_1/\mathcal{L}_0, \ldots, \mathcal{L}_n/\mathcal{L}_{n-1}) = (\mathcal{A}_1, \ldots, \mathcal{A}_n); \end{aligned}$$

on notera

$$\pi^{\underline{\lambda}} : \mathrm{Fib}^{\underline{\lambda}}_{X/\mathbf{F}_q,(1^n)} \to Fib^{\ell}_{X/\mathbf{F}_q,n}$$

et

$$\rho^{\underline{\lambda}} : \mathrm{Fib}^{\underline{\lambda}}_{X/\mathbf{F}_q,(1^n)} \to \prod_{i=1}^{n} \mathrm{Pic}^{\lambda_i}_{X/\mathbf{F}_q}$$

les restrictions de π et ρ respectivement à $\mathrm{Fib}^{\underline{\lambda}}_{X/\mathbf{F}_q,(1^n)}$ où $\ell = \lambda_1 + \cdots + \lambda_n$; π est représentable et localement de type fini et en fait $\pi^{\underline{\lambda}}$ est représentable et quasi-projective pour chaque $\underline{\lambda} \in \mathbb{Z}^n$.

Suivant une remarque de Weil, on a les équivalences de catégories

$$Fib_{X/\mathbb{F}_q,n}(\mathbb{F}_q) \tilde{\to} [GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})]$$

et

$$Fib_{X/\mathbb{F}_q,(1^n)}(\mathbb{F}_q) \tilde{\to} [B_n(F)/GL_n(\mathbb{A})/GL_n(\mathcal{O})]$$

où $B_n \subset GL_n$ est le sous-groupe de Borel des matrices triangulaires supérieures et où, pour tout groupe G agissant sur l'ensemble X la catégorie $[G\backslash X]$ est la catégorie des G-torseurs P munis d'un G-morphisme $P \to X$. De plus, π s'identifie à la projection canonique

$$[B_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})] \to [GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})].$$

Enfin, l'inclusion $B_n \hookrightarrow GL_n$ induit une équivalence de catégories

$$[B_n(F)\backslash B_n(\mathbb{A})/B_n(\mathcal{O})] \tilde{\to} [B_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})]$$

et ρ s'identifie à la projection

$$\begin{aligned}[][B_n(F)\backslash B_n(\mathbb{A})/B_n(\mathcal{O})] &\to (F^\times\backslash \mathbb{A}^\times/\mathcal{O}^\times)^n\\ B_n(F)bB_n(\mathcal{O}) &\mapsto (F^\times b_{11}\mathcal{O}^\times, \ldots, F^\times b_{nn}\mathcal{O}^\times).\end{aligned}$$

(3.2) On choisit $\bar{\mathbb{Q}}_\ell$ comme en (2.2). Soient $\mathcal{F}_1, \ldots, \mathcal{F}_n$ des $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de Weil de rang 1 sur X. On note $\mathcal{G}_1, \ldots, \mathcal{G}_n$ les $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de Weil de rang 1 sur $\mathrm{Pic}_{X/\mathbb{F}_q}$ correspondants et $\chi_1, \ldots, \chi_n$ les fonctions "trace de Frobenius" de $\mathcal{G}_1, \ldots, \mathcal{G}_n$ respectivement (cf. (2.3.1)).

Soient $T_1, \ldots, T_n$ des indéterminées indépendantes. Pour tout $\mathcal{L} \in \mathrm{ob}\,\mathrm{Fib}_{X/\mathbb{F}_q,n}(\mathbb{F}_q)$, on forme la série d'Eisenstein

$$E(\mathcal{L}, \underline{\mathcal{F}}, \underline{T}) = \sum_{\underline{\lambda}\in\mathbb{Z}^n} k^{\underline{\lambda}}(\mathcal{L}, \underline{\mathcal{F}}) \prod_{i=1}^{n} T_i^{\lambda_i} \tag{3.2.1}$$

où on a posé, pour tout $\underline{\lambda} \in \mathbb{Z}^n$,

$$k^{\underline{\lambda}}(\mathcal{L}, \underline{\mathcal{F}}) = \Sigma_{\mathcal{L}.\in(\pi^{\underline{\lambda}})^{-1}(\mathcal{L})(\mathbb{F}_q)} \prod_{i=1}^{n} (\chi_i(\mathcal{A}_i) q^{-i\lambda_i}).$$

Il résulte immédiatement de la formule des traces de Grothendieck [SGA 5] (XV, §3 n° 2) que:

LEMME 3.2.2. *Pour tout $\underline{\lambda} \in \mathbb{Z}^n$, posons*

$$K^{\underline{\lambda}}(\underline{\mathcal{F}}) = R(\pi^{\underline{\lambda}})_!(\rho^{\underline{\lambda}})^* \boxtimes_{i=1}^{n} (\mathcal{G}_i | \operatorname{Pic}^i_{X/\mathbb{F}_q})(i\lambda_i)$$

(c'est un objet de $D^b_c(\operatorname{Fib}^{\lambda_1+\cdots+\lambda_n}_{X/\mathbb{F}_q}, \bar{\mathbb{Q}}_\ell)$ ou plutôt de sa variante à la Weil). Alors, $k^{\underline{\lambda}}(\mathcal{L}, \underline{\mathcal{F}})$ est la fonction "trace de Frobenius" de $K^{\underline{\lambda}}(\underline{\mathcal{F}})$.

(3.3) Considérons maintenant le produit de fonctions L

$$D(\underline{\mathcal{F}}, \underline{T}) = \prod_{1 \leq i < j \leq n} L(\mathcal{F}_j \otimes \mathcal{F}_i^{-1}, q^{-1}T_jT_i^{-1}).$$

D'après (2.4.1), ce produit de fractions rationnelles admet le développement en série formelle

$$D(\underline{\mathcal{F}}, \underline{T}) = \Sigma_{(D_{ji})_{(1\leq i<j\leq n)}} \prod_{1 \leq i < j \leq n} (\chi_j\chi_i^{-1})(\mathcal{O}_X(D_{ji}))(q^{-1}T_jT_i^{-1})^{d_{ji}})$$

où les D_{ji} parcourent les diviseurs effectifs sur X rationnels sur $\mathbb{F}_q$ et où d_{ji} désigne le degré de D_{ji}. On forme le produit

(3.3.1) $$\bar{E}(\mathcal{L}, \underline{\mathcal{F}}, \underline{T}) = D(\underline{\mathcal{F}}, \underline{T}).E(\mathcal{L}, \underline{\mathcal{F}}, \underline{T});$$

c'est une série formelle

$$\bar{E}(\mathcal{L}, \underline{\mathcal{F}}, \underline{T}) = \sum_{\underline{\lambda}' \in \mathbb{Z}^n} \bar{k}^{\underline{\lambda}'}(\mathcal{L}, \underline{\mathcal{F}}) \prod_{j=1}^{n} T_j^{\lambda'_j}$$

et on se propose de donner une interprétation cohomologique de ses coefficients $\bar{k}^{\underline{\lambda}'}(\mathcal{L}, \underline{\mathcal{F}})$ analogue à (3.3.2).

Pour cela, considérons le champ sur $\mathbb{F}_q$

$$\overline{\operatorname{Fib}}_{X/\mathbb{F}_q,(1^n)}$$

des drapeaux généralisés

$$\mathcal{L}.' = (0 = \mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_n = \mathcal{L})$$

où maintenant chaque $\mathcal{L}'_j$ est un $\mathcal{O}_X$-Module localement libre de rang j mais pas nécessairement localement facteur direct de $\mathcal{L}'_{j+1}(j =$

$0, \ldots, n-1)$. C'est un champ algébrique localement de type fini et lisse sur $\mathbf{F}_q$, contenant $\mathrm{Fib}_{X/\mathbf{F}_q,(1^n)}$ comme ouvert dense. On note

$$\begin{array}{ccc} \overline{\mathrm{Fib}}_{X/\mathbf{F}_q,(1^n)} & & \\ \bar{\pi}\downarrow & \searrow\bar{\rho} & \\ \mathrm{Fib}_{X/\mathbf{F}_q,n} & & (\mathrm{Pic}_{X/\mathbf{F}_q})^n \end{array}$$

les deux morphismes de champs sur $\mathbf{F}_q$ définis comme suit

$$\begin{aligned} \bar{\pi}(\mathcal{L}_\cdot') &= \mathcal{L}_n' = \mathcal{L} \\ \bar{\rho}(\mathcal{L}_\cdot') &= (\det(\mathcal{A}_1'), \ldots, \det(\mathcal{A}_n')) \end{aligned}$$

où, pour $i = 1, \ldots, n$,

$$\mathcal{A}_j' = \mathcal{L}_j'/\mathcal{L}_{j-1}'$$

est un $\mathcal{O}_X$-Module cohérent de rang générique 1 et où

$$\det(\mathcal{A}_j') = (\overset{j}{\Lambda}\, \mathcal{L}_j') \otimes (\overset{j-1}{\Lambda}\, \mathcal{L}_{j-1}')^{\otimes -1}$$

est le $\mathcal{O}_X$-Module inversible "déterminant" de $\mathcal{A}_j'$ (cf. [Kn-Mu]).

Bien entendu, les composantes connexes de $\overline{\mathrm{Fib}}_{X/\mathbf{F}_q,(1^n)}$ sont encore paramétrées par $\mathbb{Z}^n$ et on note

$$\bar{\pi}^{\underline{\lambda}'} : \overline{\mathrm{Fib}}^{\underline{\lambda}}_{X'/\mathbf{F}_q,(1^n)} \to \mathrm{Fib}^{\lambda_1'+\cdots+\lambda_n'}_{X/\mathbf{F}_q}$$

et

$$\bar{\rho}^{\underline{\lambda}'} : \overline{\mathrm{Fib}}^{\underline{\lambda}}_{X'/\mathbf{F}_q,(1^n)} \to \prod_{j=1}^{n} \mathrm{Pic}^{\lambda_j'}_{X/\mathbf{F}_q}$$

les restrictions de $\bar{\pi}$ et $\bar{\rho}$ respectivement à la composante connexe d'indice $\underline{\lambda}' \in \mathbb{Z}^n$. Le morphisme de champ $\bar{\pi}$ est représentable et localement de type fini et en fait $\bar{\pi}^{\underline{\lambda}'}$ est représentable et projectif pour chaque $\underline{\lambda}' \in \mathbb{Z}^n$.

THÉORÈME 3.3.2. *Pour tout $\underline{\lambda}' \in \mathbb{Z}^n$, posons*

$$\bar{K}^{\underline{\lambda}'}(\underline{\mathcal{F}}) = R(\bar{\pi}^{\underline{\lambda}'})_*(\bar{\rho}^{\underline{\lambda}'})^* \boxtimes_{j=1}^n (\mathcal{G}_j |\, \mathrm{Pic}^{\lambda_j'}_{X/\mathbf{F}_q})(j\lambda_j')$$

(c'est un objet de $D_c^b(\mathrm{Fib}_{X/\mathbb{F}_q,n}^{\lambda'_1+\cdots+\lambda'_n}, \bar{\mathbb{Q}}_\ell)$ ou plutôt de sa variante de Weil). Alors, $\bar{k}^{\underline{\lambda}'}(\mathcal{L}, \underline{\mathcal{F}})$ est la fonction "trace de Frobenius" de $\bar{K}^{\underline{\lambda}'}(\underline{\mathcal{F}})$.

PREUVE: Compte-tenu de la formule des traces de Grothendieck ([SGA 5] (XV, §3 n° 2)), on doit comparer, pour tout $\underline{\lambda}' \in \mathbb{Z}^n$, les deux expressions suivantes:

$$\sum_{\mathcal{L}.'} \prod_{j=1}^{n} \chi_j(\det \mathcal{A}'_j),$$

où $\mathcal{L}.'$ parcourt $(\bar{\pi}^{\underline{\lambda}'})^{-1}(\mathcal{L})(\mathbb{F}_q)$, et

$$\sum_{\mathcal{L}.,(D_{ji})_{1\leq i<j\leq n}} q^{\sum_{1\leq i<j\leq n}(j-i-1)d_{ji}} \prod_{j=1}^{n} \chi_j(\mathcal{A}_j(-\sum_{\ell=j+1}^{n} D_{\ell j} + \sum_{k-1}^{j-1} D_{jk})),$$

où $\mathcal{L}.$ parcourt $(\pi^{\underline{\lambda}})^{-1}(\mathcal{L})(\mathbb{F}_q)$ et les D_{ji} parcourent $X^{(d_{ji})}(\mathbb{F}_q)$ pour tous les $\underline{\lambda} \in \mathbb{Z}^n$ et $(d_{ji})_{1\leq i<j\leq n}, d_{ji} \in \mathbb{N}$, tels que

$$\lambda'_j = \lambda_j - \sum_{l=j+1}^{n} d_{lj} + \sum_{k=1}^{j-1} d_{jk} \quad (j = 1, \ldots, n)$$

(sous ces conditions, on a

$$\sum_{j=1}^{n} j\lambda'_j = \sum_{j=1}^{n} j\lambda_j + \sum_{1\leq i<j\leq n} (j-i)d_{ji}).$$

Pour cela, on va construire une application

$$\mathcal{L}.' \mapsto (\mathcal{L}., (D_{ji})_{1\leq i<j\leq n})$$

dont le cardinal de la fibre en $(\mathcal{L}., (D_{ji})_{1\leq i<j\leq n})$ est

$$q^{\sum_{1\leq i<j\leq n}(j-i-1)d_{ji}}$$

et telle que

$$\det \mathcal{A}'_j = \mathcal{A}_j(-\sum_{l=j+1}^{n} D_{lj} + \sum_{k=1}^{j-1} D_{jk})$$

si $\mathcal{L}.'$ a pour image $(\mathcal{L}., (D_{ji})_{1\leq i<j\leq n})$.

Soit donc $\mathcal{L}.' \in (\bar{\pi}^{\underline{\lambda}'})^{-1}(\mathcal{L})(\mathbb{F}_q)$ un drapeau généralisé. Pour chaque $i = 1, \ldots, n$, le sous-$\mathcal{O}_X$-Module $\mathcal{L}'_i \subset \mathcal{L}$ engendre un unique sous-fibré vectoriel de rang i, $\mathcal{L}'_i \subset \mathcal{L}_i \subset \mathcal{L}$, et, comme $\mathcal{L}'_{i-1} \subset \mathcal{L}'_i$, on a aussi $\mathcal{L}_{i-1} \subset \mathcal{L}_i$. Par suite, on a attaché à $\mathcal{L}.'$ un drapeau

$$\mathcal{L}. = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_n = \mathcal{L}).$$

On regardera $\mathcal{L}$ comme un objet bi-filtré: on a les filtrations

$$F. = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_n = \mathcal{L})$$

et

$$F.' = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_n = \mathcal{L})$$

avec

$$gr_i^F \mathcal{L} = \mathcal{A}_i$$

un $\mathcal{O}_X$-Module inversible de degré λ_i et

$$gr_j^{F'} \mathcal{L} = \mathcal{A}'_j$$

un $\mathcal{O}_X$-Module cohérent de rang générique 1 et de degré λ'_j. La filtration $F.'$ induit une filtration

$$0 = F'_0\mathcal{A}_i = \cdots = F'_{i-1}\mathcal{A}_i \subset F'_i\mathcal{A}_i \subset \cdots \subset F'_n\mathcal{A}_i = \mathcal{A}_i$$

sur $\mathcal{A}_i$ et la filtration $F.$ induit une filtration

$$0 = F_0\mathcal{A}'_j \subset F_1\mathcal{A}'_j \subset \cdots \subset F_j\mathcal{A}'_j = F_{j+1}\mathcal{A}'_j = \cdots = F_n\mathcal{A}'_j = \mathcal{A}'_j$$

sur $\mathcal{A}'_j$ (on a $\mathcal{L}'_i \subset \mathcal{L}_i$ pour tout i). De plus, on a

$$gr_j^{F'} \mathcal{A}_i = gr_i^F \mathcal{A}'_j = gr_j^{F'}\, gr_i^F \mathcal{L}$$

pour tous i, j.

On définit alors des diviseurs effectifs $D_{ji} (1 \leq i < j \leq n)$ rationnels sur $\mathbb{F}_q$ par

$$F'_j\mathcal{A}_j = \mathcal{A}_i(-D_{j+1i} - \cdots - D_{ni}) \quad (1 \leq i < j \leq n)$$

ou encore

$$gr_j^{F'}\mathcal{A}_i \simeq \mathcal{O}_X/\mathcal{O}_X(-D_{ji}) \quad (1 \leq i < j \leq n).$$

Comme

$$gr_j^F\mathcal{A}_i = \begin{cases} \mathcal{O}_X/\mathcal{O}_X(-D_{jk}), & \text{si } k < j \\ \mathcal{A}_j(-\sum_{l=j+1}^n D_{lj}), & \text{si } k = j \\ 0, & \text{si } k > j \end{cases}$$

on a bien

$$\det \mathcal{A}'_j = \mathcal{A}_j(-\sum_{l=j+1}^{n} D_{lj} + \sum_{k=1}^{j-1} D_{jk})$$

et

$$\lambda'_j = \lambda_j - \sum_{l=j+1}^{n} d_{lj} + \sum_{k=1}^{j-1} d_{jk}$$

où d_{ji} est le degré de D_{ji}.

Fixons enfin $\mathcal{L}.$ et $(D_{ji})_{1\leq i<j\leq n}$ et calculons le nombre $\mathcal{L}.'$ donnant $(\mathcal{L}., (D_{ji})_{1\leq i<j\leq n})$ par le procédé ci-dessus. On doit montrer que le nombre de ces $\mathcal{L}.'$ est q^N où

$$N = \sum_{1\leq i<j\leq n} (j-i-1)d_{ji}$$

i.e.

$$N = \sum_{i=1}^{n} N_i$$

où

$$N_i = \sum_{\substack{1\leq k\leq i-1 \\ i+1\leq l\leq n}} d_{lk}$$

Pour cela, on va montrer par récurrence sur i que le nombre de filtrations

$$0 = F'_0\mathcal{L}_i \subset F'_1\mathcal{L}_i \subset \cdots \subset F'_n\mathcal{L}_i = \mathcal{L}_i$$

de la forme

$$F'_j\mathcal{L}_i = \mathcal{L}'_j \cap \mathcal{L}_i \quad (j = 0, \ldots, n)$$

pour un $\mathcal{L}.'$ comme ci-dessus est égal à

$$q^{N_1+\ldots+N_i}$$

ce qui achèvera bien entendu la démonstration du théorème.

Tout revient à voir que pour chaque diagramme

$$
\begin{array}{ccccc}
\mathcal{L}_{i-1} & \hookrightarrow & \mathcal{L}_i & \twoheadrightarrow & \mathcal{A}_i \\
\| & & & & \| \\
F'_n\mathcal{L}_{i-1} & & & & \mathcal{A}_i \\
\uparrow & & & & \uparrow \\
F'_{n-1}\mathcal{L}_{i-1} & & & & \mathcal{A}_i(-D_{ni}) \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
F'_i\mathcal{L}_{i-1} & & & & \mathcal{A}_i(-D_{i+1,i} - \cdots - D_{ni}) \\
\uparrow & & & & \uparrow \\
F'_{i-1}\mathcal{L}_{i-1} & & & & 0 \\
\uparrow & & & & \| \\
\vdots & & & & \vdots \\
\uparrow & & & & \| \\
F'_1\mathcal{L}_{i-1} & & & & 0
\end{array}
\tag{3.3.2.1}
$$

il y a q^{N_i} façons de le compléter en un diagramme

$$
\begin{array}{ccccc}
\mathcal{L}_{i-1} & \hookrightarrow & \mathcal{L}_i & \twoheadrightarrow & \mathcal{A}_i \\
\| & & \| & & \| \\
F'_n\mathcal{L}_{i-1} & \hookrightarrow & F'_n\mathcal{L}_i & \twoheadrightarrow & \mathcal{A}_i \\
\uparrow & & \uparrow & & \uparrow \\
F'_{n-1}\mathcal{L}_{i-1} & \hookrightarrow & F'_{n-1}\mathcal{L}_i & \twoheadrightarrow & \mathcal{A}_i(-D_{ni}) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
F'_i\mathcal{L}_{i-1} & \hookrightarrow & F'_i\mathcal{L}_i & \twoheadrightarrow & \mathcal{A}_i(-D_{i+1i} - \cdots - D_{ni}) \\
\uparrow & & \uparrow & & \uparrow \\
F'_{i-1}\mathcal{L}_{i-1} & \hookrightarrow & F'_{i-1}\mathcal{L}_i & \twoheadrightarrow & 0 \\
\uparrow & & \uparrow & & \| \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \| \\
F'_1\mathcal{L}_{i-1} & = & F'_1\mathcal{L}_i & \twoheadrightarrow & 0
\end{array}
\tag{3.3.2.2}
$$

à lignes des suites exactes courtes. Or, le "RHom filtré"

$$R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))$$

donne lieu à un triangle distingué

$$\begin{tikzcd}
 & \frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.')) \arrow[dl] & \\
F'_0\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.')) \arrow[rr] & & R\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}_i, \mathcal{L}_{i-1}) \arrow[ul]
\end{tikzcd}$$

(cf. [Il] (V, 2.2) et [B-B-D] (3.1)) et pour que l'on puisse compléter le diagramme (3.3.2.1) en un diagramme (3.3.2.2), il faut et il suffit que l'image de la classe d'extension

$$(\mathcal{L}_{i-1} \hookrightarrow \mathcal{L}_i \twoheadrightarrow \mathcal{A}_i) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{A}_i, \mathcal{L}_{i-1})$$

dans

$$H^1\Big(\frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))\Big)$$

soit nulle, le nombre des diagrammes (3.3.2.2) complétant le diagramme (3.3.2.1) étant alors égal au cardinal de l'espace vectoriel de dimension finie sur $\mathbb{F}_q$

$$H^0\Big(\frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))\Big).$$

Le théorème est donc conséquence du lemme:

LEMME 3.3.2.3. *Pour tout entier $p \neq 0$,*

$$H^p\Big(\frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))\Big) = 0$$

et

$$\chi\Big(\frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))\Big) = N_i.$$

PREUVE DU LEMME: On a (cf. [Il] (V, 2.2))

$$\frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.')))$$

$$= R\Gamma\Big(X, \frac{F'_n}{F'_0}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))\Big).$$

Or

$$\frac{F'_n}{F'_0} R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.'))$$

n'a de cohomologie qu'en degré 0 car

$$R\underline{Hom}_{\mathcal{O}_X}(\mathcal{A}_i, \mathcal{L}_{i-1})$$

n'a de cohomologie qu'en degré 0 et car

$$F'_0 R\underline{Hom}_{\mathcal{O}_X}(\mathcal{A}_i, \mathcal{L}_{i-1})$$

n'a de cohomologie qu'en degrés 0 et 1($\mathcal{A}_i$ et $\mathcal{L}_{i-1}$ sont localement libres et $\dim(X) = 1$). En outre

$$\underline{H}^0(\frac{F'_n}{F'_0} R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'), (\mathcal{L}_{i-1}, F.')))$$

est à support fini sur X car il en est ainsi de

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(gr_k^{F'} \mathcal{A}_i, gr_\ell^{F'} \mathcal{L}_{i-1})$$

pour tous $k < l$ (seul $gr_k^{F'} \mathcal{A}_i$ n'est pas à support fini mais $gr_\ell^{F'} \mathcal{L}_{i-1}$ est à support fini pour tout $l > i$ comme on le voit aisément par récurrence sur i). D'où la première assertion.

Pour ce qui est de la seconde assertion, on a

$$\begin{aligned}
\chi(\frac{F'_n}{F'_0} R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{A}_i, F.'),(\mathcal{L}_{i-1}, F.')))& \\
&= \sum_{0 \leq k \leq l \leq n} \chi(R\mathrm{Hom}_{\mathcal{O}_X}(gr_k^{F'} \mathcal{A}_i, gr_\ell^{F'} \mathcal{L}_{i-1})) \\
&= \sum_{i+1 \leq l \leq n} \deg(gr_\ell^{F'} \mathcal{L}_{i-1}) \\
&= \sum_{\substack{1 \leq k \leq i-1 \\ i+1 \leq l \leq n}} d_{lk} \\
&= N_i
\end{aligned}$$

d'où le lemme.

4. Induction Parabolique

(4.0) Sur le modèle des constructions géométriques du numéro précédant, nous allons maintenant définir et étudier un processus général d'induction parabolique géométrique.

(4.1) Nous nous placerons dans le cadre suivant. Soit k un corps algébriquement clos de caractéristique $p \geq 0$ et soit X une courbe projective, lisse et connexe sur k de genre g. On fixe une clôture algébrique $\bar{\mathbb{Q}}_\ell$ de $\mathbb{Q}_\ell$ pour un nombre premier $\ell \neq p$.

Soient n un entier ≥ 1 et

$$\underline{\nu} = (\nu_1, \ldots, \nu_s)$$

une partition de n $(\nu_1, \ldots, \nu_s \in \mathbb{Z}, \nu_1, \ldots, \nu_s > 0$ et $\nu_1 + \cdots + \nu_s = n)$. On a alors un diagramme d'induction

$$\text{(4.1.1)} \qquad \begin{array}{ccc} & \mathrm{Fib}_{X/k,\underline{\nu}} & \\ & \pi_{\underline{\nu}} \downarrow & \searrow^{\rho_{\underline{\nu}}} \\ & \mathrm{Fib}_{X/k,n} & \prod_{i=1}^{s} \mathrm{Fib}_{X/k,\nu_s} \end{array}$$

où $\mathrm{Fib}_{X/k,n}$ (resp. $\mathrm{Fib}_{X/k,\nu_i} (i = 1, \ldots, s)$) est le champ sur k des fibrés vectoriels sur X de rang n (resp. ν_i) et où $Fib_{X/k,\nu.}$ est le champ sur k des drapeaux

$$\mathcal{L}. = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_s = \mathcal{L})$$

avec $\mathcal{L}_i$ un fibré vectoriel de rang ν_i sur X, localement facteur direct de $\mathcal{L}_{i+1}$ en tant que $\mathcal{O}_X$-Module localement libre $(i = 0, \ldots, s-1)$; $\pi_{\underline{\nu}}$ et $\rho_{\underline{\nu}}$ sont les projections définies par

$$\pi_{\underline{\nu}}(\mathcal{L}.) = \mathcal{L}$$

et

$$\rho_{\underline{\nu}}(\mathcal{L}.) = (\mathcal{L}_1/\mathcal{L}_0, \ldots, \mathcal{L}_s/\mathcal{L}_{s-1}).$$

Si l'on fixe les degrés ℓ de $\mathcal{L}$ et λ_i de $\mathcal{L}_i/\mathcal{L}_{i-1}$ $(i = 1, \ldots, s)$, on obtient un diagramme

$$\text{(4.1.2)} \qquad \begin{array}{ccc} & \mathrm{Fib}^{\underline{\lambda}}_{X/k,\underline{\nu}} & \\ & \pi^{\underline{\lambda}}_{\underline{\nu}} \downarrow & \searrow^{\rho^{\underline{\lambda}}_{\underline{\nu}}} \\ & \mathrm{Fib}^{\ell}_{X/k,n} & \prod_{i=1}^{s} \mathrm{Fib}^{\lambda_i}_{X/k,\nu_i} \end{array}$$

$(\underline{\lambda} = (\lambda_1, \ldots, \lambda_s),\ \ell = \lambda_1 + \cdots + \lambda_s,\ \lambda_i \in \mathbb{Z}\ (i = 1, \ldots, s))$.

Rappelons que:

LEMME 4.1.3.

(1) *Le champ* $\mathrm{Fib}^{\ell}_{X/k,n}$ *est algébrique, lisse et connexe de dimension* $n^2(g-1)$.

(2) *Le champ* $\mathrm{Fib}^{\underline{\lambda}}_{X/k,\underline{\nu}}$ *est algébrique, lisse et connexe de dimension*

$$\sum_{1\leq i\leq j\leq s}(\nu_i\nu_j(g-1)+\nu_i\lambda_j-\nu_j\lambda_i).$$

(3) *Le morphisme* $\pi^{\underline{\lambda}}_{\underline{\nu}}$ *est représentable et quasi-projectif.*

PREUVE: Voir [Lau1]2.

Remarque 4.1.4. Le morphisme $\rho^{\underline{\lambda}}_{\underline{\nu}}$ n'est pas représentable mais il est facile de vérifier qu'il est lisse, de dimension relative

$$\sum_{1\leq i<j\leq s}(\nu_i\nu_j(g-1)+\nu_i\lambda_j-\nu_j\lambda_i).$$

Pour toute suite $\underline{\lambda}=(\lambda_1,\dots,\lambda_s)\in\mathbb{Z}^s$ et toute suite

$$\underline{A}=(A_1,\dots,A_s)$$

où

$$A_i\in\ \mathrm{ob}\ D^b_c(\mathrm{Fib}^{\lambda_i}_{X/k,\nu_i},\bar{\mathbb{Q}}_\ell)\quad(i=1,\dots,s),$$

on peut alors former

$$K^{\underline{\lambda}}_{\underline{\nu}}(\underline{A})=R(\pi^{\underline{\lambda}}_{\underline{\nu}})_!(\rho^{\underline{\lambda}}_{\underline{\nu}})^*\boxtimes_{i=1}^{s}A_i\in\ \mathrm{ob}\ D^b_c(\mathrm{Fib}^{\ell}_{X/k,n},\bar{\mathbb{Q}}_\ell)\tag{4.1.5}$$

où $\ell=\lambda_1+\cdots+\lambda_s$.

(4.2) Le morphisme $\pi^{\underline{\lambda}}_{\underline{\nu}}$ de (4.1.2) est représentable et quasi-projectif mais non projectif en général. Cependant, il admet une compactification naturelle

$$\begin{array}{ccc} & \overline{\mathrm{Fib}}^{\underline{\lambda}}_{X/k,\underline{\nu}} & \\ & \bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}\downarrow\quad\ \searrow\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}} & \\ & \mathrm{Fib}^{\ell}_{X/k,n}\qquad \prod_{i=1}^{s}\mathrm{Coh}^{\lambda_i}_{X/k,\nu_i} & \end{array}\tag{4.2.1}$$

où $\overline{\mathrm{Fib}}^{\underline{\lambda}}_{X/k,\underline{\nu}}$ est le champ sur k des drapeaux généralisés

$$\mathcal{L}_\cdot=(0=\mathcal{L}_0\subset\mathcal{L}_1\subset\cdots\subset\mathcal{L}_n=\mathcal{L})$$

avec $\mathcal{L}_i$ un $\mathcal{O}_X$-Module localement libre de rang $\nu_1+\cdots+\nu_i$ et de degré $\lambda_1+\cdots+\lambda_i$ pour $i=1,\ldots,s$ (on n'exige plus que $\mathcal{L}_i$ soit localement facteur direct de $\mathcal{L}_{i+1}$) et où $\mathrm{Coh}^{\lambda_i}_{X/k,\nu_i}$ est le champ sur k des $\mathcal{O}_X$-Modules cohérents de rang générique ν_i et de degré λ_i ($i=1,\ldots,s$); les morphismes $\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}$ et $\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}}$ sont définis par

$$\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}(\mathcal{L}.)=\mathcal{L}$$

et

$$\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}}(\mathcal{L}.)=(\mathcal{L}_1/\mathcal{L}_0,\ldots,\mathcal{L}_s/\mathcal{L}_{s-1}).$$

De plus, le diagramme (4.2.1) s'étend naturellement en

$$\begin{array}{lcc} & \mathrm{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}} & \\ (4.2.2) & \bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}\downarrow & \searrow^{\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}}} \\ & \mathrm{Coh}^{\ell}_{X/k,n} & \prod_{i=1}^{s}\mathrm{Coh}^{\lambda_i}_{X/k,\nu_i} \end{array}$$

où $\mathrm{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}}$ est le champ sur k des drapeaux

$$\mathcal{L}.=(0=\mathcal{L}_0\subset\mathcal{L}_1\subset\cdots\subset\mathcal{L}_n=\mathcal{L})$$

avec $\mathcal{L}_i$ un $\mathcal{O}_X$-Module cohérent de rang générique $\nu_1+\cdots+\nu_i$ et de degré $\lambda_1+\cdots+\lambda_i$ pour $i=1,\ldots,s$ et où $\mathrm{Coh}^{\ell}_{X/k,n}$ est le champ sur k des $\mathcal{O}_X$-Modules cohérents de rang générique n et de degré ℓ; les morphismes $\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}$ et $\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}}$ sont définis comme ci-dessus.

LEMME 4.2.3.

(1) *Le champ $\mathrm{Coh}^{\ell}_{X/k,n}$ est algébrique, lisse et connexe, de dimension $n^2(g-1)$ et contient le champ $\mathrm{Fib}^{\ell}_{X/k,n}$ comme ouvert dense.*

(2) *Le champ $\mathrm{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}}$ est algébrique, lisse et connexe, de dimension*

$$\sum_{1\leq i\leq j\leq s}(\nu_i\nu_j(g-1)+\nu_i\lambda_j-\nu_j\lambda_i)$$

et

$$\mathrm{Fib}^{\underline{\lambda}}_{X/k,\underline{\nu}}\subset\overline{\mathrm{Fib}}^{\underline{\lambda}}_{X/k,\underline{\nu}}\subset\mathrm{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}}$$

sont des ouverts denses.

(3) *Le morphisme $\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}$ est représentable et projectif.*

PREUVE: Voir [Lau 1]2 (le fait d'avoir remplacé $\mathrm{Fib}^{\ell}_{X/k,n}$ par $\mathrm{Coh}^{\ell}_{X/k,n}$ ne change en rien les arguments de loc. cit.).

Remarque (4.2.4). Le morphisme $\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}}$ n'est pas représentable mais est lisse, de dimension relative

$$\sum_{1\leq i<j\leq s} (\nu_i\nu_j(g-1)+\nu_i\lambda_j-\nu_j\lambda_i).$$

Pour toute suite $\underline{\lambda}=(\lambda_1,\ldots,\lambda_s)\in\mathbb{Z}^s$ et toute suite

$$\underline{B}=(B_1,\ldots,B_s),$$

où

$$B_i\in \text{ ob } D^b_c(\mathrm{Coh}^{\lambda_i}_{X/k,\nu_i},\bar{\mathbb{Q}}_\ell) \quad (i=1,\ldots,s),$$

on peut alors former

$$\bar{K}^{\underline{\lambda}}_{\underline{\nu}}(\underline{B})=R(\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}})_*(\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}})^*\boxtimes_{i=1}^s B_i\in \text{ ob } D^b_c(\mathrm{Coh}^{\ell}_{X/k,n},\bar{\mathbb{Q}}_\ell) \tag{4.2.5}$$

où $\ell=\lambda_1+\cdots+\lambda_s$.

DEFINITION 4.2.6. *On dira qu'un $\bar{\mathbb{Q}}_\ell$-faisceau pervers irréductible B sur $\mathrm{Coh}^{\ell}_{X/k,n}$ est obtenu à partir des $\bar{\mathbb{Q}}_\ell$-faisceaux pervers irréductibles $B_1,\ldots,B_s$ sur $\mathrm{Coh}^{\lambda_1}_{X/k,\nu_1},\ldots,\mathrm{Coh}^{\lambda_s}_{X/k,\nu_s}$ respectivement par induction parabolique si B est isomorphe à un sous-quotient de*

$${}^p\mathcal{H}^i(\bar{K}^{\underline{\lambda}}_{\underline{\nu}}(\underline{B}))$$

pour au moins un $j\in\mathbb{Z}$ (on a bien entendu $\underline{\nu}$ qui est une partition de n et $\ell=\lambda_1+\cdots+\lambda_n$).

(4.3) Nous allons maintenant étudier la géométrie des morphismes $\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}}$; celle-ci est en partie contrôlée par la dimension des composantes irréductibles des produits fibrés

$$\mathrm{Coh}^{\underline{\lambda}'}_{X'/k,\underline{\nu}'}\times_{\mathrm{Coh}^{\ell}_{X/k,n}}\mathrm{Coh}^{\underline{\lambda}''}_{X/k,\underline{\nu}''}$$

où $\underline{\nu}'$ et $\underline{\nu}''$ sont des partitions de n, $\underline{\nu}'=(\nu'_1,\ldots,\nu'_{s'})$ et $\underline{\nu}''=(\nu''_1,\ldots,\nu''_{s''})$, et où $\underline{\lambda}'=(\lambda'_1,\ldots,\lambda'_{s'})$ et $\underline{\lambda}''=(\lambda''_1,\ldots,\lambda''_{s''})$ avec $\lambda'_1+\cdots+\lambda'_{s'}=\ell=\lambda''_1+\cdots+\lambda''_{s''}$.

Fixons $\underline{\nu}'$, $\underline{\nu}''$,$\underline{\lambda}'$ et $\underline{\lambda}''$ et notons $\mathcal{Z}$ le produit fibré ci-dessus; ce produit fibré s'interprète comme le champ sur k des triples

$$(\mathcal{L}, \mathcal{L}'_., \mathcal{L}''_.)$$

où $\mathcal{L}$ est un $\mathcal{O}_X$-Module cohérent de rang générique n et de degré ℓ et où

$$\mathcal{L}'_. = (0 = \mathcal{L}'_0 \in \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_{s'} = \mathcal{L})$$

et

$$\mathcal{L}''_. = (0 = \mathcal{L}''_0 \in \mathcal{L}''_1 \subset \cdots \subset \mathcal{L}''_{s''} = \mathcal{L})$$

sont des drapeaux de sous-$\mathcal{O}_X$-Modules cohérents de $\mathcal{L}$ avec

$$\mathrm{rang}(\mathcal{L}'_i/\mathcal{L}'_{i-1}) = \nu'_i, \mathrm{deg}(\mathcal{L}'_i/\mathcal{L}'_{i-1}) = \lambda'_i$$

pour $i = 1, \ldots, s'$ et

$$\mathrm{rang}(\mathcal{L}''_j/\mathcal{L}''_{j-1}) = \nu''_j, \mathrm{deg}(\mathcal{L}''_j/\mathcal{L}''_{j-1}) = \lambda''_j$$

pour $j = 1, \ldots, s''$. Ce champ est algébrique mais n'est ni lisse ni même irréductible. Cependant on va voir que l'on peut stratifier naturellement $\mathcal{Z}$ en sous-champs localement fermés qui eux sont lisses sur k (et probablement connexes ou vides mais je n'ai pas su vérifier ce dernier point).

Pour cela, considérons l'ensemble W des couples

$$w = ((\nu_{ij})_{\substack{1\leq i\leq s' \\ 1\leq j\leq s''}}, (\lambda_{ij})_{\substack{1\leq i\leq s' \\ 1\leq j\leq s''}})$$

où $\nu_{ij} \in \mathbb{N}, \lambda_{ij} \in \mathbb{Z}$ et où

$$\begin{cases} \nu_{i1} + \cdots + \nu_{is''} = \nu'_i, & (i = 1, \ldots, s') \\ \nu_{ij} + \cdots + \nu_{s'j} = \nu''_j, & (j = 1, \ldots, s'') \end{cases}$$

et

$$\begin{cases} \lambda_{i1} + \cdots + \lambda_{is''} = \lambda'_i, & (i = 1, \ldots, s') \\ \lambda_{ij} + \cdots + \lambda_{s'j} = \lambda''_j, & (j = 1, \ldots, s'') \end{cases}$$

(on peut aussi imposer $\lambda_{ij} \geq 0$ si $\nu_{ij} = 0$). Chaque triple $(\mathcal{L}, \mathcal{L}'_., \mathcal{L}''_.)$ définit un $\mathcal{O}_X$-Module cohérent bifiltré

$$(\mathcal{L}, F'_., F''_.)$$

de manière évidente, de bigradué

$$gr_i^{F'} gr_j^{F''} \mathcal{L} = (\mathcal{L}'_i \cap \mathcal{L}''_j)/((\mathcal{L}'_{i-1} \cap \mathcal{L}''_j) + (\mathcal{L}'_i \cap \mathcal{L}''_{j-1})).$$

Alors, pour chaque $w \in W$, on introduit le sous-champ

$$\mathcal{Z}_w \subset \mathcal{Z}$$

des triples $(\mathcal{L}, \mathcal{L}'_., \mathcal{L}''_.)$ tels que

$$\begin{cases} \operatorname{rang}(gr_i^{F'} gr_j^{F''} \mathcal{L}) = \nu_{ij} \\ \deg(gr_i^{F'} gr_j^{F''} \mathcal{L}) = \lambda_{ij} \end{cases}$$

pour $1 \leq i \leq s'$, $1 \leq j \leq s''$.

PROPOSITION 4.3.1. *Pour chaque $w \in W$, $\mathcal{Z}_w$ est un sous-champ algébrique localement fermé de $\mathcal{Z}$, lisse sur k, purement de dimension*

$$\sum_{\substack{1 \leq i \leq k \leq s' \\ 1 \leq j \leq l \leq s''}} (\nu_{ij}\nu_{k\ell}(g-1) + \nu_{ij}\lambda_{k\ell} - \nu_{k\ell}\lambda_{ij}).$$

De plus $\mathcal{Z}$ est réunion disjointe des $\mathcal{Z}_w$.

PREUVE: Le fait que $\mathcal{Z}_w$ soit localement fermé et que $\mathcal{Z}$ soit réunion disjointe des $\mathcal{Z}_w$ résulte aussitôt du théorème de platitude générique et du théorème de semi-continuité du polynôme de Hilbert ([EGA] (IV, 6.9.1) et (III, 7.9); [Mu] Lecture 8).

La théorie du complexe cotangent ([Il] II) entraîne alors la proposition. En effet, la fibre en $(\mathcal{L}, \mathcal{L}'_., \mathcal{L}''_.) \in \mathcal{Z}_w$ du complexe cotangent $L_{\mathcal{Z}_w/k}$ s'identifie canoniquement au complexe

$$(F'_{s'}F''_{s''}/(F'_{-1} + F''_{-1}))\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L}, F'_., F''_.), \Omega^1_X \otimes_{\mathcal{O}_X} (\mathcal{L}, F'_., F''_.))$$

(il s'agit d'un "RHom bifiltré", cf. [Il] (V, 2.2) et [De 3] (7.1)). Or ce dernier complexe est d'amplitude parfaite [0,1] et donc $\mathcal{Z}_w$ est lisse en $(\mathcal{L}, \mathcal{L}'_., \mathcal{L}''_.)$ de dimension la caractéristique d'Euler-Poincaré de ce dernier complexe, soit encore

$$\sum_{\substack{1 \leq i \leq k \leq s' \\ 1 \leq j \leq l \leq s''}} \chi(R\mathrm{Hom}_{\mathcal{O}_X}(gr_i^{F'} gr_j^{F''} \mathcal{L}, \Omega^1_X \otimes_{\mathcal{O}_X} gr_k^{F'} gr_\ell^{F''} \mathcal{L}))$$

et il ne reste plus qu'à appliquer le théorème de Riemann-Roch pour les courbes.

Avant de discuter plus en détails deux cas particuliers de la proposition ci-dessus, rappelons le critère suivant:

LEMME 4.3.2. *Soit $f : \mathcal{X} \to \mathcal{Y}$ un morphisme représentable et projectif entre deux champs algébriques connexes et lisses sur k. Notons $\mathcal{Z}$ le produit fibré $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ et $(\mathcal{Z}_\alpha)_{\alpha \in A}$ les composantes irréductibles de $\mathcal{Z}$. Alors:*

(1) *pour que le morphisme*

$$f : \mathcal{X} \to f(\mathcal{X})$$

soit "semi-small" au sens de Goresky-MacPherson (cf. [Go-Ma] (6.2) (et en particulier génériquement fini), il faut et il suffit que

$$\dim \mathcal{Z}_\alpha \leq \dim \mathcal{X}, \forall \alpha \in A;$$

(2) *pour que le morphisme*

$$f : \mathcal{X} \to f(\mathcal{X})$$

soit "small" au sens de Goresky-MacPherson (cf. [Go-Ma] (6.2)), il faut et il suffit que

$$\dim \mathcal{Z}_\alpha \leq \dim \mathcal{X}, \forall \alpha \in A,$$

et que, pour chaque $k \in A$ tel que

$$\dim \mathcal{Z}_\alpha = \dim \mathcal{X}$$

le morphisme de projection sur $\mathcal{Y}$

$$z_\alpha \to f(\mathcal{X})$$

soit dominant.

Si $\underline{\nu}' = \underline{\nu}'' = (1^n)$, la donnée de $(\nu_{ij})_{1 \leq i,j \leq n}$ avec $\nu_{ij} \in \mathbb{N}$ et

$$\begin{aligned} \nu_{i1} + \cdots + \nu_{in} &= 1 \qquad (i = 1, \ldots, n) \\ \nu_{ij} + \cdots + \nu_{nj} &= 1 \qquad (i = 1, \ldots, n) \end{aligned}$$

équivaut à celle de

$$\sigma \in \mathfrak{S}_n$$

(on a

$$\nu_{ij} = \begin{cases} 0 \ si \ j \neq \sigma(i) \\ 1 \ si \ j = \sigma(i) \end{cases}$$

pour tous $i, j = 1, \ldots, n$).

COROLLAIRE 4.3.3. *Si* $\underline{\nu}' = \underline{\nu}'' = (1^n)$ *et* $\underline{\lambda}' = \underline{\lambda}'' = \underline{\lambda}$, *pour chaque*

$$w = (\sigma, (\lambda_{ij})_{1\leq i,j\leq n}) \in W,$$

$\mathcal{Z}_w$ *est lisse sur* k, *purement de dimension*

$$d_w = \frac{n(n+1)}{2}(g-1) + \sum_{1\leq i\leq j\leq n} (\lambda_j - \lambda_i) + \sum_{\substack{1\leq i\leq j\leq n \\ \sigma(i)>\sigma(j)}} (\lambda_{j\sigma(i)} - \lambda_{i\sigma(j)} + 1 - g).$$

Un autre cas particulier intéressant de la proposition (4.3.1) est celui où $\underline{\nu}' = \underline{\nu}'' = (1, n-1)$; alors, pour

$$w = ((\nu_{ij})_{1\leq i,j\leq 2}, (\lambda_{ij})_{1\leq i,j\leq 2}) \in W,$$

on a soit

$$\nu_{11} = 1, \nu_{12} = \nu_{21} = 0, \nu_{22} = n-1$$

soit

$$\nu_{11} = 0, \nu_{12} = \nu_{21} = 1, \nu_{22} = n-2$$

COROLLAIRE 4.3.4. *Si* $\underline{\nu}' = \underline{\nu}'' = (1, n-1)$ *et* $\underline{\lambda}' = \underline{\lambda}'' = (\lambda_1, \lambda_2)$, $\mathcal{Z}_w$ *est lisse sur* k, *purement de dimension*

$$d_w = \begin{cases} (n^2-n+1)(g-1) \\ \qquad + \lambda_2 - (n-1)\lambda_1 - (n-2)\lambda_{12}, & \text{si } \nu_{11} = 1 \\ (n^2-2n+2)(g-1) \\ \qquad + 2\lambda_2 - 2(n-1)\lambda_1 - (n-2)\lambda_{11}, & \text{si } \nu_{11} = 0 \end{cases}$$

pour tout $w \in W$.

Remarque 4.3.5. Si $\underline{\nu}' = \underline{\nu}'' = (1, n-1), \underline{\lambda}' = \underline{\lambda}'' = (\lambda_1, \lambda_2)$ et

$$\lambda_2 - (n-1)\lambda_1 \leq (n-1)(g-1)$$

(resp.

$$\lambda_2 - (n-1)\lambda_1 < (n-1)(g-1) \text{ et } n > 2),$$

alors

$$\pi_{\underline{\nu}}^{\underline{\lambda}} : \overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1\lambda_2)} \to \pi_{\underline{\nu}}^{\underline{\lambda}} : \overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1\lambda_2)}$$

est "semi-small" (resp. "small") au sens de Goresky-MacPherson. En effet, on a

$$\begin{aligned}\dim \overline{\mathrm{Fib}}^{(\lambda_1\lambda_2)}_{X/k,(1,n-1)}\\ &= (n^2-n+1)(g-1)+\lambda_2-(n-1)\lambda_1,\\ (n^2-2n+2)(g-1)+2\lambda_2-2(n-1)\lambda_1 &\leq (n^2-n+1)(g-1)\\ &\qquad +\lambda_2-(n-1)\lambda_1\end{aligned}$$

(resp.

$$\begin{aligned}(n^2-2n+2)(g-1)+2\lambda_2-2(n-1)\lambda_1 &< (n^2-n+1)(g-1)\\ &\qquad +\lambda_2-(n-1)\lambda_1),\\ &\qquad \lambda_{12}\geq 0 \text{ si } \nu_{11}=1\end{aligned}$$

et

$$\lambda_{11}\geq 0 \text{ si } \nu_{11}=1$$

(en effet, si $\nu_{11}=1, \lambda_{12}$ est le degré du $\mathcal{O}_X$-Module de torsion $\mathcal{L}'_1/(\mathcal{L}'_1\cap\mathcal{L}''_1)$ et, si $\nu_{11}=0, \mathcal{L}'_1\cap\mathcal{L}''_1$ est de torsion et donc nul puisque $\mathcal{L}$ est sans torsion).
(4.4) Si k est de caractéristique nulle, on peut aussi étudier la géométrie des morphismes $\bar{\pi}^{\underline{\lambda}}$ d'un point de vue microlocal.

Pour chaque morphisme de champs algébriques

$$f:\mathcal{X}\to\mathcal{Y}$$

on a un triangle distingué correspondant

$$f^*L_{\mathcal{Y}/k}\to L_{\mathcal{X}/k}\to L_{\mathcal{X}/\mathcal{Y}}\to$$

dans $D^{[-\infty,1]}_{q\,\mathrm{Coh}}(\mathcal{O}_X)$ par la théorie du complexe cotangent ([Il] (II, 2.1)); on notera simplement $L(f)$ ce triangle distingué dans la suite de cet article.

LEMME 4.4.1. *Soit* $\mathcal{L}.\in$ ob $\mathrm{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}}(k)$. *Alors la fibre en* $\mathcal{L}.$ *de* $L(\bar{\pi}^{\underline{\lambda}}_{\underline{\nu}})$ *s'identifie canoniquement au triangle distingué*

$$\begin{tikzcd}[column sep=tiny] & F_{-1}\,R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L},F.),\Omega^1_X\otimes_{\mathcal{O}_X}(\mathcal{L},F.))[1] \arrow[dl, "\cdot"'] \arrow[dr] & \\ R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L},\Omega^1_X\otimes_{\mathcal{O}_X}\mathcal{L}) \arrow[rr] & & (\frac{F_s}{F_{-1}})\,R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L},F.),\Omega^1_X\otimes_{\mathcal{O}_X}(\mathcal{L},F.))\end{tikzcd}$$

et celle de $L(\bar{\rho}^{\lambda}_{\underline{\nu}})$ au triangle distingué

$$\begin{tikzcd}[column sep=small] & (\frac{F_s}{F_0})\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L},F.),\Omega^1_X\otimes_{\mathcal{O}_X}(\mathcal{L},F.)) \arrow[dl, "\cdot"'] \arrow[dr] & \\ \bigoplus_{i=1}^{s} R\mathrm{Hom}_{\mathcal{O}_X}(\frac{\mathcal{L}_i}{\mathcal{L}_{i-1}},\Omega^1_X\otimes_{\mathcal{O}_X}\frac{\mathcal{L}_i}{\mathcal{L}_{i-1}}) \arrow[rr] & & (\frac{F_s}{F_{-1}})\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L},F.),\Omega^1_X \otimes_{\mathcal{O}_X}(\mathcal{L},F.)) \end{tikzcd}$$

(on a muni $\mathcal{L}$ de la filtration

$$F. = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_s = \mathcal{L})$$

et les RHom sont filtrés).

PREUVE: Cela résulte aussitôt de la théorie de la déformation ([Il] IV) (voir aussi [Lau 1]2).

Remarque 4.4.2. Comme

$$\frac{F_s}{(F_0)}\, R\mathrm{Hom}_{\mathcal{O}_X}((\mathcal{L},F.),\Omega^1_X\otimes_{\mathcal{O}_X}(\mathcal{L},F.))$$

est d'amplitude parfaite $[0,1]$, on retrouve bien le fait que $\bar{\rho}^{\lambda}_{\underline{\nu}}$ est lisse (cf. (4.1.4) et (4.2.4)).

Le champ $T^*\,\mathrm{Coh}^{\ell}_{X/k,n}$ s'identifie naturellement au champ sur k des couples

$$(\mathcal{L},u)$$

avec $\mathcal{L}$ un $\mathcal{O}_X$-Module cohérent de rang générique n et de degré ℓ et

$$u : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$$

un homomorphisme de $\mathcal{O}_X$-Modules; c'est un champ algébrique sur k (cf. [Lau 1] (1.1)).

On a défini dans [Lau 1] (1.14) un fermé conique lagrangien

$$\Lambda^{\ell}_{X/k,n} \subset T^*\,\mathrm{Fib}^{\ell}_{X/k,n},$$

le **cône nilpotent**. On étend $\Lambda^{\ell}_{X/k,n}$ à $\mathrm{Coh}^{\ell}_{X/k,n}$ de la façon suivante. On considère le sous-champ

$$\bar{\Lambda}^{\ell}_{X/k,n} \subset T^*\,\mathrm{Coh}^{\ell}_{X/k,n} \tag{4.4.3}$$

des couples $(\mathcal{L}, u)$ tels que

$$(u \otimes id_{(\Omega^1_X)^{\otimes(n-1)}}) \circ \cdots \circ (u \otimes id_{\Omega^1_X}) \circ u = 0$$

(u est nilpotent d'ordre n).

Les arguments de loc. cit. se généralisent sans difficultés et donnent:

PROPOSITION 4.4.4. *Le sous-champ* $\bar{\Lambda}^{\ell}_{X/k,n}$ *de* $T^*\operatorname{Coh}^{\ell}_{X/k,n}$ *est un fermé conique lagrangien.*

Soient

$$\begin{tikzcd}[column sep=small] & \mathcal{Z} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{X} & & \mathcal{Y} \end{tikzcd}$$

une correspondance entre champs algébriques sur k; on suppose que $\mathcal{X}$, $\mathcal{Y}$ et $\mathcal{Z}$ sont des champs algébriques lisses sur k. Alors, on a un diagramme correspondant au niveau des fibrés cotangents

$$\begin{tikzcd}[column sep=small] & T^*\mathcal{X} \times_{\mathcal{X}} \mathcal{Z} \arrow[dl, "\bar{f}"'] \arrow[dr, "F"] & & T^*\mathcal{Y} \times_{\mathcal{Y}} \mathcal{Z} \arrow[dl, "G"'] \arrow[dr, "\bar{g}"] & \\ T^*\mathcal{X} & & T^*\mathcal{Z} & & T^*\mathcal{Y} \end{tikzcd}$$

Supposons de plus f représentable et propre et g lisse et soit $\Lambda \subset T^*\mathcal{Y}$ un fermé conique lagrangien. On appelle **image de** Λ par la correspondance ci-dessus le fermé conique isotrope

$$\bar{f}(F^{-1}(G(\bar{g}^{-1}(\Lambda))))$$

(G est propre car g est lisse et f est propre car f l'est).

Le résultat suivant est crucial pour le calcul des variétés caractéristiques des faisceaux automorphes.

PROPOSITION 4.4.5. *Soient* $\underline{\nu} = (\nu_1, \ldots, \nu_s)$ *une partition de* n *et* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathbb{Z}^s$ *avec* $\lambda_1 + \cdots + \lambda_s = \ell$. *Alors, l'image du fermé conique lagrangien*

$$\prod_{i=1}^{s} \bar{\Lambda}^{\lambda_i}_{X/k,\nu_i} \subset \prod_{i=1}^{s} T^*\operatorname{Coh}^{\lambda_i}_{X/k,\nu_i}$$

par la correspondance

$$\begin{tikzcd}[column sep=small] & \operatorname{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}} \arrow[dl, "\pi^{\underline{\lambda}}_{\underline{\nu}}"'] \arrow[dr, "\bar{\rho}^{\underline{\lambda}}_{\underline{\nu}}"] & \\ \operatorname{Coh}^{\ell}_{X/k,n} & & \prod_{i=1}^{s}\operatorname{Coh}^{\lambda_i}_{X/k,\nu_i} \end{tikzcd}$$

est contenue dans $\bar{\Lambda}^{\lambda}_{X/k,n}$.

PREUVE: Compte-tenu de (4.4.1), on est ramené à prouver l'assertion suivante: soit $\mathcal{L}.$ un drapeau dans $\mathrm{Coh}^{\underline{\lambda}}_{X/k,\underline{\nu}}$ et soit $u : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$ un homomorphisme de $\mathcal{O}_X$-Modules ($\mathcal{L} = \mathcal{L}_s$) tels que

$$u(\mathcal{L}_i) \subset \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}_i (i = 1, \ldots, s)$$

et que l'homomorphisme induit par u

$$\frac{\mathcal{L}_i}{\mathcal{L}_{i-1}} \to \Omega^1_X \otimes_{\mathcal{O}_X} \frac{\mathcal{L}_i}{\mathcal{L}_{i-1}}$$

est nilpotent d'ordre $\nu_i (i = 1, \ldots, s)$, alors u est nilpotent d'ordre n. Mais cette assertion est claire.

5. Faisceaux Automorphes sur $\mathbb{P}^1_k$.

(5.0) Dans ce numéro, on va dresser la liste complète des "faisceaux automorphes" sur la droite projective sur k et calculer leur fonction "trace de Frobenius" quand $k = \mathbb{F}_q$.

(5.1) Soient $X = \mathbb{P}^1_k$ la droite projective sur k obtenue en adjoignant le point $\infty \in \mathbb{P}^1_k(k)$ à la droite affine $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$.

D'après Grothendieck ([Gr]), tout fibré vectoriel de rang n sur X est isomorphe à

$$\mathcal{O}(\underline{\lambda}) = \mathcal{O}_X(\lambda_1) \oplus \cdots \oplus \mathcal{O}_X(\lambda_n)$$

pour une unique suite

$$\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$$

avec $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$; le degré de $\mathcal{O}(\underline{\lambda})$ est

$$\ell = \lambda_1 + \cdots + \lambda_n.$$

Le groupe des automorphismes $G(\underline{\lambda})$ de $\mathcal{O}(\underline{\lambda})$ est un groupe algébrique sur k admettant le dévissage

$$1 \to G_u(\underline{\lambda}) \to G(\underline{\lambda}) \to \bar{G}(\underline{\lambda}) \to 1$$

où $G_u(\underline{\lambda})$ est le radical unipotent de $G(\underline{\lambda})$ et $\bar{G}(\underline{\lambda})$ est réductif. Plus précisément, soient $B_n \subset GL_n$ le sous-groupe de Borel des matrices triangulaires supérieures et $T_n \subset B_n \subset GL_n$ le tore maximal des matrices diagonales. Chaque $\underline{\lambda} \in \mathbb{Z}^n$ définit un sous-groupe à 1 paramètre noté encore

$$\underline{\lambda} : \mathbb{G}_m \to T_n, z \mapsto \mathrm{diag}(z^{\lambda_1}, \dots, z^{\lambda_n})$$

et la condition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ équivaut à la condition

$$< \alpha_i, \underline{\lambda} > \geq 0 \qquad (i = 1, \dots, n-1),$$

où $\alpha_1, \dots, \alpha_{n-1}$ sont les racines simples de (GL_n, T_n) relativement à B_n. Par suite, chaque $\underline{\lambda} \in \mathbb{Z}^n$ tel que $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ définit un sous-groupe parabolique $P(\underline{\lambda})$ de GL_n contenant B_n, à savoir le sous-groupe engendré par le centralisateur $L(\underline{\lambda})$ de $\underline{\lambda}$ dans GL_n et B_n; $L(\underline{\lambda})$ est le facteur de Levi de $P(\underline{\lambda})$ contenant T_n; on note $U(\underline{\lambda})$ le radical unipotent de $P(\underline{\lambda})$. On a alors

$$\bar{G}(\underline{\lambda}) = L(\underline{\lambda})_k$$

et

$$G_u(\underline{\lambda}) = H^0(X, \mathcal{U}(\underline{\lambda}))$$

où $\mathcal{U}(\underline{\lambda})$ est le $U(\underline{\lambda})_k$-torseur sur X associé au $\mathbb{G}_{m,k}$-torseur tautologique

$$\begin{array}{c} \mathbb{A}_k^2 - \{(0,0)\} \\ \downarrow \\ \mathbb{P}_k^1 \end{array}$$

par l'action de $\mathbb{G}_{m,k}$ sur $U(\underline{\lambda})_k$ suivante:

$$(z, u) \mapsto \underline{\lambda}(z)^{-1} u \underline{\lambda}(z).$$

En particulier, si $R \subset X^*(T_n)$ est l'ensemble des racines de (GL_n, T_n), on a

$$\dim \bar{G}(\underline{\lambda}) = n + \#\{\alpha \in R | < \alpha, \underline{\lambda} >= 0\}$$

et

$$\dim G_u(\underline{\lambda}) = \sum_{\substack{\alpha \in R \\ <\alpha, \underline{\lambda}> > 0}} (1+ < \alpha, \underline{\lambda} >).$$

Sur $X_*(T_n)$ et donc sur l'ensemble des $\underline{\lambda} \in \mathbb{Z}^n$ tels que $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ on a un ordre partiel $\underline{\lambda} \geq \underline{\mu}$ défini par $\underline{\lambda} - \underline{\mu}$ combinaison linéaire à coefficients ≥ 0 de coracines simples pour (GL_n, T_n) relativement à B_n, i.e.

$$\begin{cases} \lambda_1 \geq \mu_1 \\ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2 \\ \quad \cdots\cdots \\ \lambda_1 + \cdots + \lambda_{n-1} \geq \mu_1 + \cdots + \mu_{n-1} \\ \lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n \end{cases}$$

Pour chaque $\ell \in \mathbb{Z}$, l'ensemble des $\underline{\lambda} \in \mathbb{Z}^n$ tels que $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ et $\lambda_1 + \cdots + \lambda_n = \ell$ admet alors un plus petit élément pour cet ordre partiel à savoir

$$\underline{\lambda}(\ell) = (\lambda + 1, \ldots, \lambda + 1, \lambda, \ldots, \lambda)$$

(r fois $\lambda + 1$, $n - r$ fois λ où

$$\ell = n\lambda + r \quad (\lambda \in \mathbb{Z}, 0 \leq r \leq n - 1)).$$

Le champ algébrique connexe et lisse sur k, de dimension $-n^2$, des fibrés vectoriels de rang n et degré ℓ sur X, $\mathrm{Fib}^{\ell}_{X/k,n}$, admet donc la stratification de Harder-Narasimhan-Shatz suivante (cf. [Sh] et [Ra]). On a

$$\mathrm{Fib}^{\ell}_{X/k,n} = \bigcup_{\substack{\underline{\lambda} \in \mathbb{Z}^n \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = \ell}} \mathcal{S}^{\underline{\lambda}}$$

(réunion disjointe), où chaque strate $\mathcal{S}^{\underline{\lambda}}$ est localement fermée dans $\mathrm{Fib}^{\ell}_{X/k,n}$ et isomorphe au champ algébrique sur k

$$BG(\underline{\lambda})$$

classifiant les $G(\underline{\lambda})$-torseurs; en particulier, chaque strate est connexe et lisse, de dimension

$$-\dim G(\underline{\lambda}) = -n - \sum_{\substack{\alpha \in R \\ <\alpha, \underline{\lambda}> \geq 0}} (1 + <\alpha, \underline{\lambda}>)$$

sur k. La strate ouverte est $\mathcal{S}^{\underline{\lambda}(\ell)}$ et, pour tout $\underline{\lambda}$, l'adhérence de la strate $\mathcal{S}^{\underline{\lambda}}$ est la réunion disjointe de strates

$$\bar{\mathcal{S}}^{\underline{\lambda}} = \bigcup_{\underline{\lambda} \leq \underline{\mu}} \mathcal{S}^{\underline{\mu}}.$$

On note

$$\mathcal{S}^{\underline{\lambda}} \overset{j^{\underline{\lambda}}}{\hookrightarrow} \mathrm{Fib}^{\ell}_{X/k,n}$$

l'inclusion.

(5.2) Pour chaque $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$ avec$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ et $\lambda_1 + \cdots + \lambda_n = \ell$, on a alors le complexe d'intersection

$$A^{\underline{\lambda}} = \underline{IC}(\bar{\mathcal{S}}^{\underline{\lambda}}, \bar{\mathbb{Q}}_\ell)$$

qui, à un décalage près de $-\dim G(\underline{\lambda})$, est le $\bar{\mathbb{Q}}_\ell$-faisceau pervers irréductible $j^{\underline{\lambda}}_{!*}\bar{\mathbb{Q}}_\ell[\dim G(\underline{\lambda})]$ sur $\mathrm{Fib}^{\ell}_{X/k,n}$ supporté par $\bar{\mathcal{S}}^{\underline{\lambda}}$.

On note $\bar{A}^{\underline{\lambda}}$ le prolongement intermédiaire de $A^{\underline{\lambda}}$ par l'immersion ouverte

$$\mathrm{Fib}^{\ell}_{X/k,n} \hookrightarrow \mathrm{Coh}^{\ell}_{X/k,n};$$

c'est aussi, à un décalage près de $-\dim G(\underline{\lambda})$, un $\bar{\mathbb{Q}}_\ell$-faisceau pervers irréductible sur $\mathrm{Coh}^{\ell}_{X/k,n}$ supporté par l'adhérence de $\bar{\mathcal{S}}^{\underline{\lambda}}$ dans $\mathrm{Coh}^{\ell}_{X/k,n}$.

PROPOSITION 5.2.1. *Pour chaque* $\underline{\lambda}, A^{\underline{\lambda}}$ *est facteur direct dans* D^b_c $(\mathrm{Coh}^{\ell}_{X/k,n}, \bar{\mathbb{Q}}_\ell)$ *du complexe*

$$\bar{K}^{\underline{\lambda}}_{(1n)}(\bar{\mathbb{Q}}_\ell) = R(\bar{\pi}^{\underline{\lambda}}_{(1n)})_* \bar{\mathbb{Q}}_\ell$$

défini en (4.2.5).

PREUVE: C'est une conséquence du théorème de décomposition ([B-B-D] (6.2.5)). En effet, pour $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ avec $\lambda_1 \geq \cdots \geq \lambda_n$ et $\lambda_1 + \cdots + \lambda_n = \ell$, l'image de $\overline{\mathrm{Fib}}^{\underline{\lambda}}_{X/k,(1^n)}$ dans $\mathrm{Fib}^{\ell}_{X/k,n}$ par $\bar{\pi}^{\underline{\lambda}}_{(1^n)}$ n'est autre que l'adhérence $\bar{\mathcal{S}}^{\underline{\lambda}}$ de la strate $\mathcal{S}^{\underline{\lambda}}$ et la restriction de $\bar{\pi}^{\underline{\lambda}}_{(1^n)}$ à la strate $\mathcal{S}^{\underline{\lambda}}$ (qui n'est autre que la restriction de $\bar{\pi}^{\underline{\lambda}}_{(1^n)}$ à $\mathcal{S}^{\underline{\lambda}}$) est un morphisme représentable, projectif et lisse, à fibres des produits de variétés de drapeaux usuelles sur k (si l'on a

$$\lambda_1 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} > \cdots$$
$$\cdots > \lambda_{n_1+\dots n_{r-1}+1} = \cdots = \lambda_{n_1+\dots n_r} = \lambda_n$$

la fibre type de $\bar{\pi}^{\underline{\lambda}}_{(1^n)}$ en un point de $\mathcal{S}^{\underline{\lambda}}$ n'est autre que

$$\prod_{j=1}^{r} GL_{n_j}/B_{n_j}$$

où $B_{n_j} \subset GL_{n_j}$ est le sous-groupe de Borel standard des matrices triangulaires supérieures). Par suite $\bar{\mathbb{Q}}_\ell$ est facteur direct dans $D_c^b(\mathcal{S}^{\underline{\lambda}}, \bar{\mathbb{Q}}_\ell)$ de $R(\bar{\pi}^{\underline{\lambda}}_{(1^n)})_*\bar{\mathbb{Q}}_\ell|\mathcal{S}^{\underline{\lambda}}$.

COROLLAIRE 5.2.2. *Si k est de caractéristique nulle la variété caractéristique de $\bar{A}^{\underline{\lambda}}$ est contenue dans le "cône nilpotent" $\bar{\Lambda}^\ell_{X/k,n} \subset T^*\operatorname{Coh}^\ell_{X/k,n}$ pour chaque $\underline{\lambda}$.*

PREUVE: C'est une conséquence directe de (5.2.1) et (4.4.5).

La proposition (5.2.1) ne donne que peu d'informations sur $\bar{A}^{\underline{\lambda}}$ car le morphisme $\bar{\pi}^{\underline{\lambda}}_{(1^n)} : \operatorname{Coh}^{\underline{\lambda}}_{X/k,(1^n)} \to \operatorname{Coh}^\ell_{X/k,n}$ n'est pas en général "small" ni même "semi-small" sur son image au sens de Goresky-MacPherson. Cependant, pour certains $\underline{\lambda}$, on a beaucoup mieux en utilisant une autre résolution des singularités de $\bar{\mathcal{S}}^{\underline{\lambda}}$.

PROPOSITION 5.2.3. *Fixons $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ avec $\lambda_1 + \lambda_2 = \ell$ et $\lambda_2 \leq (n-1)(\lambda_1 - 1)$. Alors le complexe*

$$K = \bar{K}^{(\lambda_1,\lambda_2))}_{(1,n-1)}(\bar{\mathbb{Q}}_\ell)|\operatorname{Fib}^\ell_{X/k,n}$$

sur $\operatorname{Fib}^\ell_{X/k,n}$ défini en (4.2.5) est pervers et admet la décomposition en somme directe de faisceaux pervers simples suivantes:

a) si $n = 2$,

$$K = \bigoplus_{\substack{(\mu_1,\mu_2)\in\mathbb{Z}^2 \\ \mu_1+\mu_2=\ell \\ \mu_1\geq\lambda_1}} A^{\underline{\mu}}(-\mu_1 + \lambda_1)$$

et en fait

$$A^{\underline{\mu}} = \bar{\mathbb{Q}}_{\ell,\bar{\mathcal{S}}^{(\mu_1,\mu_2)}}$$

dès que $\mu_1 > \mu_2$;

b) si $n > 2$ et $\lambda_2 < (n-1)(\lambda_1 - 1)$,

$$K = A^{\underline{\mu}}$$

où

$$\underline{\mu} = (\lambda_1, -[\frac{\lambda_1 - \ell}{n-1}], -[\frac{\lambda_1 - [\frac{\lambda_1-\ell}{n-1}] - \ell}{n-2}], \ldots),$$

i.e.

$$\underline{\mu} = (\mu_1, \ldots, \mu_n)$$

avec

$$\mu_1 = \lambda_1$$

et μ_i est le plus petit entier tel que

$$\mu_i \geq \frac{\ell - (\mu_1 + \cdots + \mu_{i-1})}{n - i + 1} \quad (i = 2, \ldots, n);$$

c) si $n > 2$ et $\lambda_2 = (n-1)(\lambda_1 - 1)$,

$$K = A^{\underline{\mu}} \oplus A^{\underline{\mu}'}(-1)$$

où

$$\underline{\mu} = (\lambda_1, \lambda_1 - 1, \lambda_1 - 1, \ldots, \lambda_1 - 1)$$

et

$$\underline{\mu}' = (\lambda_1, \lambda_1 - [\frac{2\lambda_1 - \ell}{n-2}], -[\frac{2\lambda_1 - [\frac{2\lambda_1 - \ell}{n-2}] - \ell}{n-3}], \ldots),$$

i.e.

$$\underline{\mu}' = (\mu_1', \ldots, \mu_n')$$

avec

$$\mu_1' = \mu_2' = \lambda_1$$

et μ_i' est le plus petit entier tel que

$$\mu_i' \geq \frac{\ell - (\mu_1' + \cdots + \mu_{i-1}')}{n - i + 1} \quad (i = 3, \ldots, n).$$

Preuve: D'après (4.3.5), le morphisme

$$\bar{\pi}_{\underline{\nu}}^{\lambda} : \overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1,\lambda_2,)} \to \bar{\pi}_{\underline{\nu}}^{\lambda}(\overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1,\lambda_2,)})$$

est "semi-small" si $\lambda_2 \leq (n-1)(\lambda_1 - 1)$ et même "small" si $\lambda_2 \leq (n-1)(\lambda_1 - 1)$ et $n > 2$. D'autre part, d'après Shatz ([Sh]), $\bar{\pi}_{\underline{\nu}}^{\lambda}(\overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1,\lambda_2,)})$ n'est autre que

$$\bar{\mathcal{S}}^{\underline{\mu}} = \bar{\mathcal{S}}^{(\lambda_1, -[\frac{\lambda_1 - \ell}{n-1}], -[\frac{\lambda_1 - [\frac{\lambda_1 - \ell}{n-1}] - \ell}{n-2}], \ldots)}$$

et d'après Harder-Narasimhan ([Ha-Na]) $\bar{\pi}_{\underline{\nu}}^{\lambda}$ est un isomorphisme au-dessus de $\mathcal{S}^{\underline{\mu}}$.

L'assertion b) résulte aussitôt de ces considérations.

Pour l'assertion c), on remarque de plus que, pour $n > 2$ et $\lambda_2 = (n-1)\lambda_1, \bar{\pi}_{\underline{\nu}}^{\underline{\lambda}}$ n'est pas "small" uniquement à cause de la strate $\mathcal{Z}_w$, $w = ((\nu_{ij})_{1\leq i,j\leq 2}, (\lambda_{ij})_{1\leq i,j\leq 2}), \nu_{11} = 0, \nu_{12} = \nu_{21} = 1, \nu_{22} = n-2$ et donc $\lambda_{11} = 0, \lambda_{12} = \lambda_{21} = \lambda_1$ et $\lambda_{22} = \lambda_2 - \lambda_1$, du produit fibré

$$\mathcal{Z} = \overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1,\lambda_2)} \times_{\mathrm{Fib}_{X/k,n}^{\ell}} \overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1,\lambda_2)} :$$

la dimension de cette strate est

$$-n^2 = \dim \overline{\mathrm{Fib}}_{X/k,(1,n-1)}^{(\lambda_1,\lambda_2)} = \dim \mathrm{Fib}_{X/k,n}^{\ell} .$$

Or, l'image de cette strate $\mathcal{Z}_w$ dans $\mathrm{Fib}_{X/k,n}^{\ell}$ a pour adhérence

$$\bar{\mathcal{S}}^{\underline{\mu}} = \bar{\mathcal{S}}^{(\lambda_1,\lambda_1,-[\frac{2\lambda_1-\ell}{n-2}],-[\frac{2\lambda_1-[\frac{2\lambda_1-\ell}{n-2}]-\ell}{n-3}],\dots)}$$

d'après Shatz ([Sh]) et $\bar{\pi}_{\underline{\nu}}^{\underline{\lambda}}$ est un fibré en $\mathbb{P}^1$ sur $\mathcal{S}^{\underline{\mu}'}$, de sorte que l'assertion c) résulte maintenant du théorème de décomposition ([B-B-D] (6.2.5).

L'assertion a) résulte elle aussi du théorème de décomposition car, pour $\lambda_1 > \lambda_2$ et $\mu = (\mu_1, \mu_2)$ avec $\mu_1 \geq \lambda_1$ et $\mu_1 + \mu_2 = \lambda_1 + \lambda_2 = \ell$, la restriction de $\bar{\pi}_{\underline{\nu}}^{\underline{\lambda}}$ à $\mathcal{S}^{\underline{\mu}}$ est un fibré en espaces projectifs de dimension $\mu_1 - \lambda_1$ (et $\mathrm{codim}_{\bar{\mathcal{S}}^{\underline{\lambda}}}(\mathcal{S}^{\underline{\mu}}) = 2(\mu_1 - \lambda_1)$).

(5.3) On va maintenant faire le lien entre les complexes d'intersection $A^{\underline{\lambda}}$ sur $\mathrm{Fib}_{X/k,n}^{\ell}$ et les polynômes de Kazhdan-Lusztig $Q_{y,w}(q)$ pour le groupe de Weyl affine associé à $GL_n(k[t,t^{-1}])$ quand k est une clôture algébrique du corps fini $\mathbb{F}_q$ (cf. [Ka-Lu] §5).

Ensemblistement, on peut décrire l'ensemble des classes d'isomorphie de fibrés vectoriels de rang n sur $X = \mathbb{P}_k^1$ (pour k un corps arbitraire) comme l'ensemble de doubles classes

$$GL_n(k[t])\backslash GL_n(k[t,t^{-1}])/GL_n(k[t^{-1}]) :$$

si $\mathcal{L}$ est un fibré vectoriel de rang n sur X, $\mathcal{L}_{|X-\{\infty\}}$ et $\mathcal{L}_{|X-\{0\}}$ sont non canoniquement des fibrés vectoriels triviaux et donc $\mathcal{L}$ s'obtient par recollement de $\mathcal{O}_{X-\{\infty\}}^n$ et de $\mathcal{O}_{X-\{0\}}^n$ le long de $X - \{0,\infty\}$ au moyen d'une fonction de transition appartenant à $GL_n(k[t,t^{-1}])$. Le degré de $\mathcal{L}$ n'est autre que le degré du déterminant de g si $\mathcal{L}$ correspond à la double classe $GL_n(k[t])g\,GL_n(k[t^{-1}])$, $g \in GL_n(k[t,t^{-1}])$, et le groupe des automorphismes de $\mathcal{L}$ est isomorphe à

$$g^{-1}GL_n(k[t])g \cap GL_n(k[t^{-1}]).$$

Cette description donne lieu en fait à une présentation du champ algébrique sur k

$$\mathrm{Fib}_{X/k,n} = \amalg_{\ell\in\mathbb{Z}} \mathrm{Fib}^{\ell}_{X/k,n}\,.$$

On a des k-ind-schémas réduits en groupes G, G^+ et G^- tels que

$$G(A) = GL_n(A[t,t^{-1}])$$
$$G^+(A) = GL_n(A[t])$$

et

$$G^-(A) = GL_n(A[t^{-1}])$$

pour tout k-algèbre réduite A;G^+ et G^- sont connexes alors que $\pi_0(G)$ est isomorphe à $\mathbb{Z}$ par le degré du déterminant, i.e.

$$G = \coprod_{\ell\in\mathbb{Z}} G^{\ell}.$$

Le quotient $GL_n(k[t])\backslash GL_n(k[t,t^{-1}])$ est donc l'ensemble des k-points du k-ind-schéma

$$F = G^+\backslash G = \amalg_{\ell\in\mathbb{Z}} G^+\backslash G^{\ell} = \amalg_{\ell\in\mathbb{Z}} F_{\ell}.$$

Plus précisément, pour chaque $\ell \in \mathbb{Z}$,

$$F^{\ell} = \text{"}\varinjlim_{N\geq|\ell|/n}\text{"}\; F^{\ell}_N$$

où F^{ℓ}_N est le k-schéma projectif des $\mathcal{O}_X$-Modules localement libres de rang n et de degré ℓ tels que

$$\mathcal{O}_X(-N.0)^n \subset \mathcal{L} \subset \mathcal{O}_X(N.0)^n$$

et où la flèche de transition $F^{\ell}_N \hookrightarrow F^{\ell}_{N+1}$ est l'inclusion évidente et est une immersion fermée (soient V_N le k-espace vectoriel de dimension $2nN$

$$V_N = (\mathcal{O}_X(N.0)/\mathcal{O}_X(-N.0))^n = (t^{-N}k[t]/t^{N}k[t])^n$$

et $t_N : V_N \to V_N$ l'endomorphisme nilpotent induit par la multiplication par t, alors F_N^ℓ est le fermé de la grassmannienne des Nn-plans W de V_N formé des W tels que $t_N(W) \subset W$). En d'autres termes F^ℓ est le k-ind-schéma des $\mathcal{O}_X$-Modules localement libres $\mathcal{L}$ de rang n et de degré ℓ munis d'un isomorphisme

$$\alpha : (\mathcal{O}_{X-\{0\}})^n \tilde{\to} \mathcal{L}_{|X-\{0\}}$$

de $\mathcal{O}_{X-\{0\}}$-Modules. On a un morphisme évident de k-ind-champs algébriques pour tout $\ell \in \mathbb{Z}$

$$P^\ell : F^\ell \to \mathrm{Fib}^\ell_{X/k,n}$$

(oubli de la trivialisation de $\mathcal{L}_{|X-\{0\}}$) et

$$P = \amalg_{\ell\in\mathbb{Z}} P^\ell : F \to Fib_{X/k,n}$$

est la présentation de $Fib_{X/k,n}$ cherchée.

Lemme 5.3.1. *Le morphisme P de k-ind-champs algébriques est formellement lisse et en fait un G^--torseur pour l'action par translation à droite de G^- sur $G^+\backslash G$.*

Remarque 5.3.2. Pour chaque $\ell \in \mathbb{Z}$ et chaque $N \geq |\ell|/n$, on a un morphisme de k-champs algébriques

$$P_N^\ell : F_N^\ell \to \mathrm{Fib}^\ell_{X/k,n}$$

par composition de P avec le morphisme canonique $F_N^\ell \to F^\ell$; bien que

$$P = \text{"} \varinjlim_{N\geq \ell|/n} \text{"} P_N^\ell$$

soit formellement lisse, aucun des P_N^ℓ ne l'est dès que $n \geq 2$ (en fait les k-schémas projectifs ne sont même pas lisses en général).

On a deux stratifications de F, toutes deux indexées par les $\underline{\lambda} \in \mathbb{Z}^n$ tels que $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, l'une, $(S_{\underline{\lambda}})$, par des strates de dimension finie, l'autre, $(S^{\underline{\lambda}})$, par des strates de codimension finie (cf. [Pr-Se] (8.4)). Plus précisément, pour chaque $\underline{\lambda} \in \mathbb{Z}^n$ avec $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, soit

$$f_{\underline{\lambda}} = (\mathcal{L}_{\underline{\lambda}}, \alpha_{\underline{\lambda}})$$

le point de F défini par

$$\mathcal{L}_{\underline{\lambda}} = \mathcal{O}_X(\lambda_1.0) \oplus \cdots \oplus \mathcal{O}_X(\lambda_1.0)$$

et

$$\alpha_{\underline{\lambda}} = \mathrm{can}_1 \oplus \cdots \oplus \mathrm{can}_n$$

où

$$\mathrm{can}_i : \mathcal{O}_{X-\{0\}} \tilde{\rightarrow} \mathcal{O}_X(\lambda_i.0)_{|X-\{0\}} \quad (i = 1, \ldots, n)$$

est la trivialisation canonique. Les k-ind-schémas en groupes $G^+ etG^-$ opèrent par translation à droite sur $F = G^+\backslash G$ et $S_{\underline{\lambda}}$ et $S^{\underline{\lambda}}$ sont les orbites de $f_{\underline{\lambda}}$ pour G^+ et G^- respectivement. Si l'on note $t^{\underline{\lambda}}$ la matrice diagonale

$$\begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix} \in G(k),$$

$f_{\underline{\lambda}}$ est la classe $G^+ t^{\underline{\lambda}}$ et $S_{\underline{\lambda}}$ et $S^{\underline{\lambda}}$ sont canoniquement isomorphes aux k-schémas

$$G^+/t^{-\underline{\lambda}} G^+ t^{\underline{\lambda}} \cap G^+$$

et

$$G^-/t^{-\underline{\lambda}} G^+ t^{\underline{\lambda}} \cap G^-$$

respectivement. On a

$$\bar{S}_{\underline{\lambda}} = \bigcup_{\underline{\mu} \leq \underline{\lambda}} S_{\underline{\mu}}, \quad \bar{S}^{\underline{\lambda}} = \bigcup_{\underline{\mu} \leq \underline{\lambda}} S^{\underline{\mu}}$$

et

$$S_{\underline{\mu}} \cap S^{\underline{\lambda}} = \emptyset \Leftrightarrow \underline{\lambda} \leq \underline{\mu};$$

de plus, $S_{\underline{\lambda}}$ rencontre transversalement $S^{\underline{\lambda}}$ et l'intersection est isomorphe à

$$GL_{n,k}/P(\underline{\lambda})$$

où $P(\underline{\lambda}) = t^{-\underline{\lambda}} G^+ t^{\underline{\lambda}} \cap GL_{n,k}$ n'est autre que la parabolique standard défini en (5.1). Pour tout $\underline{\lambda}$, on a

$$\dim S_{\underline{\lambda}} = \sum_{i<j} (\lambda_i - \lambda_j),$$

$$\mathrm{codim}_F(S^{\underline{\lambda}}) = \sum_{i<j} (\lambda_i - \lambda_j) - \#\{(i,j) | i > j \text{ et } \lambda_i \neq \lambda_j\}$$

et
$$S^{\underline{\lambda}} = P^{-1}(\mathcal{S}^{\underline{\lambda}}).$$

Soit
$$I^+ \subset G^+ (\text{resp. } I^- \subset G^-)$$
l'image réciproque par le k-morphisme
$$G^+ \to GL_{n,k}, g^+(t) \mapsto g^+(0)$$
(resp.
$$G^- \to GL_{n,k}, g^-(t) \mapsto g^-(0))$$
de $B_{n,k}$ (resp. $B^-_{n,k}$, le sous-groupe de Borel des matrices triangulaires inférieures). On a un morphisme naturel de k-ind-schémas
$$R : \tilde{F} = I^+ \backslash G \to G^+ \backslash G$$
qui est un fibré de fibre type la variété de drapeaux
$$I^+ \backslash G^+ = B_{n,k} \backslash GL_{n,k}.$$
Plus précisément, $\tilde{F}$ est le k-ind-schémas des triples
$$(\mathcal{L}, \alpha, V.)$$
où $(\mathcal{L}, \alpha) \in F$ et $V. = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathcal{L}_{(0)})$ est un drapeau complet de sous-k-espaces vectoriels de la fibre en 0 de $\mathcal{L}$; le morphisme R est l'oubli de $V.$.

Suivant [Ka-Lu] §5, on a deux stratifications de $\tilde{F}$ indexées par le groupe de Weyl affine
$$W = \mathfrak{S}_n \ltimes \mathbb{Z}^n$$
où le groupe symétrique $\mathfrak{S}_n$ agit par permutation sur les coordonnées des éléments de $\mathbb{Z}^n$. Si l'on note $T_n \subset GL_n$ le tore des matrices diagonales et $N_n \subset GL_n$ le normalisateur de T_n dans GL_n, on a les identifications suivantes
$$\begin{array}{rl}
\mathfrak{S} \simeq & T_n(k) \backslash N_n(k) \\
\downarrow \quad & \qquad \downarrow \\
W \simeq & T_n(k) \backslash N_n(k[t, t^{-1}]); \\
\uparrow \quad & \qquad \uparrow \\
\mathbb{Z}^n \simeq & T_n(k) \backslash T_n(k[t, t^{-1}])
\end{array}$$

en particulier, on a une injection évidente

$$W \hookrightarrow I^+\backslash G = \tilde{F}$$

puisque $T_n(k) = I^+ \cap N_n(k[t,t^{-1}])$. Les k-ind-schémas en groupes I^+ et I^- opère par translation à droite sur $\tilde{F} = I^+\backslash G$ et les strates $\tilde{S}_w$ et $\tilde{S}^w$ sont les orbites de l'image $\tilde{f}_w$ de $w \in W$ par l'injection ci-dessus. On a toujours

$$\overline{\tilde{S}}_w = \bigcup_{y\leq w} \tilde{S}_y$$
$$\overline{\tilde{S}}^w = \bigcup_{y\leq w} \tilde{S}^y$$

et

$$\tilde{S}_y \cup \tilde{S}^w \neq \emptyset \Leftrightarrow w \leq y$$

où $\leq$ est la relation d'ordre de Bruhat sur W. Pour tout $w \in W$, on a

$$dim\tilde{S}_w = \ell(w)$$

et

$$\operatorname{codim}_{\tilde{F}}(\tilde{S}^w) = \ell(w)$$

où $\ell : W \to \mathbb{N}$ est la fonction longueur définie par

$$\ell(\sigma.\underline{\lambda}) = \sum_{\substack{i<j\\ \sigma(i)<\sigma(j)}} |\lambda_i - \lambda_j| + \sum_{\substack{i<j\\ \sigma(i)>\sigma(j)}} |1 + \lambda_i - \lambda_j|.$$

L'ensemble des doubles classes

$$\mathfrak{S}_n\backslash W/\mathfrak{S}_n$$

s'identifie canoniquement à

$$\{\underline{\lambda} \in \mathbb{Z}^n | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$$

par

$$\underline{\lambda} \mapsto W_{\underline{\lambda}} = \{\sigma_1.\sigma_2(\underline{\lambda}) | \sigma_1.\sigma_2 \in \mathfrak{S}_n\}$$

et chaque double classe $W_{\underline{\lambda}}$ contient un élément et un seul de longueur maximale, à savoir

$$w_{\underline{\lambda}} = \sigma_0.\underline{\lambda}$$

où σ_0 est l'élément le plus long de $\mathfrak{S}_n(\sigma_0(1,\ldots,n) = (n,\ldots,1))$, et un élément et un seul de longueur minimale $w^{\underline{\lambda}}$ plus compliqué à décrire (si $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, on a $w^{\underline{\lambda}} = \underline{\lambda}.\sigma_0 = \sigma_0.\sigma_0^{-1}(\underline{\lambda})$ et, si $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, on a $w^{\underline{\lambda}} = 1$).

LEMME 5.3.3. *Pour tout* $\underline{\lambda} \in \mathbb{Z}^n$ *avec* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, on a

$$R^{-1}(S_{\underline{\lambda}}) = \bigcup_{w \in W_{\underline{\lambda}}} \tilde{S}_w (\text{resp. } R^{-1}(S^{\underline{\lambda}}) = \bigcup_{w \in W_{\underline{\lambda}}} \tilde{S}^w)$$

et en particulier $\tilde{S}_{w_{\underline{\lambda}}}$ *(resp.* $\tilde{S}^{w^{\underline{\lambda}}}$*) est un ouvert dense de* $R^{-1}(S_{\underline{\lambda}})$ *(resp.* $R^{-1}(S^{\underline{\lambda}})$*).*

Toujours suivant [Ka-Lu] §5, on introduit les complexes d'intersections

$$\tilde{A}_w = \underline{IC}(\overline{\tilde{S}}_w, \bar{\mathbb{Q}}_\ell) = \tilde{j}_{w,!\star}\bar{\mathbb{Q}}_\ell$$

et

$$\tilde{A}^y = \underline{IC}(\overline{\tilde{S}}^y, \bar{\mathbb{Q}}_\ell) = \tilde{j}^y_{!\star}\bar{\mathbb{Q}}_\ell$$

où

$$\tilde{S}_w \overset{\tilde{j}_w}{\hookrightarrow} \tilde{F} \overset{\tilde{j}^y}{\hookleftarrow} \tilde{S}^y$$

sont les inclusions ($w, y \in W$).

Les résultats suivants sont annoncés sans démonstrations dans [Ka-Lu] (Théorème 5.5, Proposition 5.7):

"THÉORÈME 5.3.4". *Soient* $y \leq w$ *dans* W. *Alors:*

(1) *pour tout entier* i, *on a*

$$\mathcal{H}^{2i+1}(\tilde{A}_w | \tilde{S}_y) = 0;$$

(2) *pour tout entier* i, *le* $\bar{\mathbb{Q}}_\ell$*-faisceau* $\mathcal{H}^{2i}(\tilde{A}_w | \tilde{S}_y)$ *est constant et canoniquement isomorphe à*

$$(\bar{\mathbb{Q}}_{\ell, \tilde{S}_y}(-i))^{\alpha_{y,w}(i)}$$

où

$$P_{y,w}(q) = \sum_{i \in \mathbb{Z}} \alpha_{y,w}(i) q^i$$

est le polynôme de Kazhdan-Lusztig de W *pour* $y \leq w$.

En particulier, pour $k = \bar{\mathbb{F}}_q$, *les* $\tilde{A}_w$ *sont naturellement définis sur* $\mathbb{F}_q$ *et la trace de Frobenius sur* $\tilde{A}_w$ *en tout point de* $\tilde{S}_y(\mathbb{F}_q)$ *est égale à* $P_{y,w}(q)$.

"THÉORÈME 5.3.5". *Soient $y \leq w$ dans W. Alors:*

(1) *pour tout entier i, on a*

$$\mathcal{H}^{2i+1}(\tilde{A}^y|\tilde{S}^w) = 0;$$

(2) *pour tout entier i, le $\bar{\mathbb{Q}}_\ell$-faisceau $\mathcal{H}^{2i}(\tilde{A}^y|\tilde{S}^w)$ est constant et canoniquement isomorphe à*

$$(\bar{\mathbb{Q}}_{\ell,\tilde{S}^w}(-i))^{\beta_{y,w}(i)}$$

où

$$Q_{y,w}(q) = \sum_{i\in\mathbb{Z}} \beta_{y,w}(i)q^i$$

est le polynôme de Kazhdan-Lusztig inverse de W pour $y \leq w$.

En particulier, pour $k = \bar{\mathbb{F}}_q$, les $\tilde{A}^y$ sont naturellement définis sur $\mathbb{F}_q$ et la trace de Frobenius sur $\tilde{A}^y$ en tout point de $\tilde{S}^w(\mathbb{F}_q)$ est égale à $Q_{y,w}(q)$.

Remarque 5.3.6.1. On a $\alpha_{y,w}(i) = \beta_{y,w}(i) = 0$ si $i \notin [0, \frac{1}{2}(\ell(w) - \ell(y) - 1)]$ pour $y < w$ et

$$P_{w,w}(q) = Q_{y,y}(q) = 1 \quad (\forall w, y \in W),$$

ce qui est bien compatible à la caractérisation de Goresky-MacPherson de $\tilde{A}_w$ et $\tilde{A}^y$ (on a les égalités de dimensions

$$\dim S_w - \dim S_y = \operatorname{codim} \overline{\tilde{S}}^y(\overline{\tilde{S}}^w) = \ell(w) - \ell(y)).$$

(5.3.6.2). Je n'ai pas su écrire de démonstration complète de (5.3.4) et (5.3.5); même la définition de $\tilde{A}^y$ pose problème. La difficulté vient de ce que $\tilde{F}$ est un ind-schéma, limite inductive de schémas **non lisses** sur k.

Compte-tenu de (5.3.3), le "théorème (5.3.5)" admet le corollaire.

"COROLLAIRE 5.3.6". *Soient $\underline{\lambda} \leq \underline{\mu}$ dans $\mathbb{Z}^n$ avec $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ et $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ (en particulier $\lambda_1+\cdots+\lambda_n = \mu_1+\cdots+\mu_n = \ell$). Alors*

(1) *pour tout entier i, on a*

$$\mathcal{H}^{2i+1}(A^{\underline{\lambda}}|S^{\underline{\mu}}) = 0$$

(2) *pour tout entier i, le $\bar{\mathbb{Q}}_\ell$-faisceau $\mathcal{H}^{2i}(A^{\underline{\lambda}}|\mathcal{S}^{\underline{\mu}})$ est constant et canoniquement isomorphe à*

$$(\bar{\mathbb{Q}}_{\ell,S^{\underline{\mu}}}(-i))^{\beta_{\underline{\lambda},\underline{\mu}}(i)}$$

où

$$Q_{\underline{\lambda},\underline{\mu}}(q) = \sum_{i\in\mathbb{Z}} \beta_{\underline{\lambda},\underline{\mu}}(i)q^i$$

est le polynôme de Kazhdan-Lusztig inverse de W pour $w^{\underline{\lambda}} \leq w^{\underline{\mu}}$.

PREUVE: On a $R^*P^*A^{\underline{\lambda}} = \tilde{A}^{w^{\underline{\lambda}}}$ d'après (5.3.3).

Remarque 5.3.7.1. Dans toute la discussion précédente, on peut remplacer GL_n par un groupe réductif sur k,G, arbitraire (on remplace $\mathrm{Fib}_{X/k,n}$ par le champ des G-torseurs sur $X, \ldots,$ cf. [Ra]).

(5.3.7.2) Pour $k = \bar{\mathbb{F}}_q$, l'ensemble des classes d'isomorphie d'objet de $Fib_{X/k,n}(\mathbb{F}_q)$ s'identifie à $\{\underline{\mu} \in \mathbb{Z}^n | \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\}$ et est muni d'une mesure naturelle:

$$m(\underline{\mu}) = 1/\#G(\underline{\mu})(\mathbb{F}_q)$$

(à $\underline{\mu}$ on associe la classe d'isomorphie de

$$\mathcal{O}(\underline{\mu}) = \mathcal{O}_X(\mu_1) \oplus \cdots \oplus \mathcal{O}_X(\mu_n)$$

et $G(\underline{\mu}) = \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{O}(\underline{\mu}))$). Pour chaque $\underline{\lambda}$ dans cet ensemble on a une fonction

$$a^{\underline{\lambda}} : \{\underline{\mu} \in \mathbb{Z}^n | \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n\} \to \bar{\mathbb{Q}}_\ell$$

définie par

$$a^{\underline{\lambda}}(\underline{\mu}) = tr(\mathrm{Frob}^*_{\mathcal{O}(\underline{\mu})}, A^{\underline{\lambda}})$$

et donc, d'après (5.3.6), par

$$a^{\underline{\lambda}}(\underline{\mu}) = \begin{cases} Q_{\underline{\lambda},\underline{\mu}}(q), & \text{si } \underline{\lambda} \leq \underline{\mu} \\ 0, & \text{sinon.} \end{cases}$$

On peut se poser la question de savoir si $a^{\underline{\lambda}}$ est de carré intégrable pour la mesure m, i.e. si

$$\sum_{\underline{\lambda}\leq\underline{\mu}} (Q_{\underline{\lambda},\underline{\mu}}(q))^2 / \# G(\underline{\mu})(\mathbb{F}_q) < +\infty$$

pour $\underline{\lambda}$ fixé. Pour GL_2 la réponse est positive car $Q_{\underline{\lambda},\underline{\mu}}(q) = 1$ si $\underline{\lambda} \leq \underline{\mu}$ et

$$\# G(\underline{\mu})(\mathbb{F}_q) = \begin{cases} (q-1)^2 q(q+1), & \text{si } \mu_1 = \mu_2 \\ q^{\mu_1 - \mu_2 + 1}(q-1)^2, & \text{si } \mu_1 = \mu_2. \end{cases}$$

Une question étroitement reliée à la précédente est de savoir si, pour tout $y \in W$,

$$(\ell(w) - \ell(y)) - 2 \deg Q_{y,w}(q)$$

tend vers $+\infty$ quand $\ell(w)$ tend vers $+\infty$ avec $y \leq w$ (cf.[Lu 3] 6.2): on a

$$\ell(w^{\underline{\mu}}) - \ell(w^{\underline{\lambda}}) = \dim G(\underline{\mu}) - \dim G(\underline{\lambda})$$

et donc $G(\underline{\mu})(\mathbb{F}_q)$ est un polynôme en q de degré $\ell(w^{\underline{\mu}})$ à une constante près.

6. Exemples de Faisceaux Automorphes sur une Courbe Arbitraire.

(6.0) Dans ce numéro X est de nouveau une courbe projective, lisse et connexe sur un corps algébriquement clos k et on note g le genre de X.

(6.1) On peut généraliser comme suit la discussion du numéro précédent.

D'après [Ha-Na] et [Sh], le champ lisse $\mathrm{Fib}_{X/n,k}$ admet une stratification indexée par les paires

$$(\underline{\nu}, \underline{\lambda})$$

où $\underline{\nu} = (\nu_1, \ldots, \nu_s)$ et $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s)$ sont deux suites d'entiers de même longueur s, où $\nu_1, \ldots, \nu_s \geq 0$ et $\nu_1 + \cdots + \nu_s = n$ et où

$$\frac{\lambda_1}{\nu_1} > \frac{\lambda_1}{\nu_2} > \cdots > \frac{\lambda_s}{\nu_s}.$$

Si l'on note $\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$ la strate correspondant à $(\underline{\nu}, \underline{\lambda})$, $\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$ est un sous-champ localement fermé de $\mathrm{Fib}_{X/k,n}$ et en fait de la composante connexe $\mathrm{Fib}_{X/k,n}^{\ell}$ de $\mathrm{Fib}_{X/k,n}$ avec $\ell = \lambda_1 + \cdots + \lambda_s$, $\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$ est lisse sur k, connexe et de dimension

$$\sum_{1 \leq i \leq j \leq s} (\nu_i \nu_j (g-1) + \nu_i \lambda_j - \nu_j \lambda_i).$$

En fait, les morphismes $\pi_{\underline{\nu}}^{\underline{\lambda}}$ et $\bar{\pi}_{\underline{\nu}}^{\underline{\lambda}}$,

$$\begin{array}{ccc} \mathrm{Fib}^{\underline{\lambda}}_{X/k,\underline{\nu}} & \hookrightarrow & \overline{\mathrm{Fib}}^{\underline{\lambda}}_{X/k,\underline{\nu}} \\ \pi^{\underline{\lambda}}_{\underline{\nu}} \searrow & & \swarrow \bar{\pi}^{\underline{\lambda}}_{\underline{\nu}} \\ & \mathrm{Fib}^{\ell}_{X/k,n} & \end{array}$$

(cf. (4.1.2) et (4.2.1) ont leurs images contenues dans l'adhérence de $\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$ et sont des isomorphismes au-dessus de $\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$ (unicitédu drapeau de Harder-Narasimhan). A chaque $(\underline{\nu},\underline{\lambda})$ on associe le polygone concave dans $\mathbb{R}^2$ d'origine (0,0), d'extrémité (n,ℓ) et ayant pour sommets successifs $(\nu_1,\lambda_1),(\nu_1+\nu_2,\lambda_1+\lambda_2),\dots,(\nu_1+\dots+\nu_s,\lambda_1+\dots+\lambda_s)=(n,\ell)$. On munit l'ensemble des paires $(\underline{\nu},\underline{\lambda})$ comme ci-dessus de la relation d'ordre suivante: $(\underline{\nu},\underline{\lambda})\leq(\underline{\nu}',\underline{\lambda}')$ si et seulement si le polygone associé à $(\underline{\nu},\underline{\lambda})$ est au-dessous du polygone associé à $(\underline{\nu}',\underline{\lambda}')$ mais a même extrémité que celui-ci. On a

$$\bar{\mathcal{S}}_{\underline{\nu}}^{\underline{\lambda}} \subset \bigcup_{(\underline{\nu},\underline{\lambda})\leq(\underline{\nu}',\underline{\lambda}')} \mathcal{S}_{\underline{\nu}'}^{\underline{\lambda}'}$$

pour toute paire $(\underline{\nu},\underline{\lambda})$.

Les faisceaux automorphes qui généralisent le plus directement ceux considérés au numéro 5 sont définis comme suit. Pour chaque paire $(\underline{\nu},\underline{\lambda})$, on a un morphisme de champs algébriques sur k

$$\mathcal{S}_{\underline{\lambda}}^{\underline{\nu}} \xrightarrow{\rho_{\underline{\nu}}^{\underline{\lambda}}\phi(\pi_{\underline{\nu}}^{\underline{\lambda}})^{-1}} \prod_{i=1}^{s}\mathrm{Fib}^{\lambda_i}_{X/k,\nu_i} \xrightarrow{\prod_{i=1}^{s}\det_i} \prod_{i=1}^{s}\mathrm{Pic}^{\lambda_i}_{X/k}$$

où $\rho_{\underline{\nu}}^{\underline{\lambda}}$ est défini en (4.1.2) et où $\det_i : \mathrm{Fib}^{\lambda_i}_{X/k,\nu_i} \to \mathrm{Pic}^{\lambda_i}_{X/k}$ est le morphisme "déterminant" $(i=1,\dots,s)$. Si $\mathcal{G}_1,\dots,\mathcal{G}_s$ sont des $\bar{\mathbb{Q}}_\ell$-faisceaux lisses de rang 1 sur $\mathrm{Pic}^{\lambda_1}_{X/k},\dots,\mathrm{Pic}^{\lambda_s}_{X/k}$ respectivement, par image réciproque on obtient un $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1

$$(\boxtimes_{i=1}^{s}\mathcal{G}_i)|\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$$

sur $\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}$ et donc par prolongement intermédiaire via l'inclusion $j_{\underline{\nu}}^{\underline{\lambda}} : \mathcal{S}_{\underline{\nu}}^{\underline{\lambda}} \hookrightarrow \mathrm{Fib}_{X/k,n}$ d'un $\bar{\mathbb{Q}}_\ell$-faisceau pervers irréductible (à un décalage près) sur $Fib_{X/k,n}$ supporté par $\bar{\mathcal{S}}_{\underline{\nu}}^{\underline{\lambda}} \subset \mathrm{Fib}^{\ell}_{X/k,n}(\ell=\lambda_1+\dots+\lambda_s)$

(6.1.1) $$A_{\underline{\nu}}^{\underline{\lambda}}(\underline{\mathcal{G}}) = j^{\underline{\lambda}}_{\underline{\nu},!*}((\boxtimes_{i=1}^{s}\mathcal{G}_i)|\mathcal{S}_{\underline{\nu}}^{\underline{\lambda}}).$$

LEMME 6.1.2. *Supposons que les $\mathcal{G}_i$ sont d'ordre fini (il existe un entier $N \geq 1$ tel que $\mathcal{G}_i^{\otimes N} \simeq \bar{\mathbb{Q}}_\ell$ pour $i = 1, \ldots, s$). Alors $A^{\underline{\lambda}}_{\underline{\nu}}(\underline{\mathcal{G}})$ est facteur direct dans $D^b_c(\mathrm{Fib}^\ell_{X/k,n}, \bar{\mathbb{Q}}_\ell)$ de*

$$K^{\underline{\lambda}}_{\underline{\nu}}(\underline{\det}^*\underline{\mathcal{G}})|\,\mathrm{Fib}^\ell_{X/k,n} = R(\pi^{\underline{\lambda}}_{\underline{\nu}})_*(\rho^{\underline{\lambda}}_{\underline{\nu}})^* \boxtimes_{i=1}^{s} \det{}_i^* \mathcal{G}_i$$

avec les notations de (4.2.1) et (4.2.5). En particulier, $A^{\underline{\lambda}}_{\underline{\nu}}(\underline{\mathcal{G}})$ s'obtient par induction parabolique à partir du $\bar{\mathbb{Q}}_\ell$-faisceau lisse de rang 1, $\boxtimes_{i=1}^s \det_i^* \mathcal{G}_i$, sur $\prod_{i=1} \mathrm{Coh}^{\lambda_i}_{X/k,\nu_i}$ (on a noté encore, pour $i = 1, \ldots, s$,

$$\det{}_i : \mathrm{Coh}^{\lambda_i}_{X/k,\nu_i} \to \mathrm{Pic}^{\lambda_i}_{X/k}$$

le morphisme déterminant, cf. [Kn-Mu]).

PREUVE: C'est une conséquence immédiate du théorème de décomposition ([B-B-D] (6.2.5)).

(6.2) Un autre type de faisceaux automorphes a été découvert par Drinfeld ([Dr 1], [Dr 2], [Dr 3] et [Lau 2]): il s'agit des faisceaux automorphes cuspidaux.

Conjecturalement les classes d'isomorphie de faisceaux automorphes cuspidaux sur $\mathrm{Fib}^\ell_{X/k,n}$ pour k algébriquement clos devraient être en bijection avec les classes d'isomorphie de $\bar{\mathbb{Q}}_\ell$–faisceaux lisses irréductibles de rang n sur X (cf. [Lau 2] (2.1.1)) et c'est un théorème pour $n = 2$ ([Dr 1], [Dr 2], [Dr 3]). Les faisceaux automorphes cuspidaux sur $\mathrm{Fib}^\ell_{X/k,n}$ devraient être supportés par $\mathrm{Fib}^\ell_{X/k,n}$ tout entier ; en particulier, ils devraient être de la forme

$$j_{!*}\mathcal{F}[n^2(g-1)]$$

où

$$j : \mathrm{Fib}^\ell_{X/k,n,ts} \hookrightarrow \mathrm{Fib}^\ell_{X/k,n}$$

est l'ouvert dense des fibrés vectoriels très stables de rang n et de degré ℓ sur X (cf. [Dr 3] et [Lau 1] (3.5)) et où $\mathcal{F}$ est un $\bar{\mathbb{Q}}_\ell$–faisceau lisse irréductible sur $\mathrm{Fib}^\ell_{X/k,n,ts}$.

Pour $n = 2$, Deligne a montré que pour tous les faisceaux automorphes cuspidaux (construits par Drinfeld) le rang du $\bar{\mathbb{Q}}_\ell$–faisceau lisse irréductible ci-dessus est égal à 2^{3g-3}.

CONJECTURE 6.2.1. *Pour tout faisceau automorphe cuspidal A sur $\mathrm{Fib}^{\ell}_{X/k,n}$ la restriction de A à l'ouvert dense $\mathrm{Fib}^{\ell}_{X/k,n,ts}$ des fibrés vectoriels très stables de rang n et de degré ℓ est (àun décalage près) un $\bar{\mathbb{Q}}_{\ell}$-faisceau lisse irréductible de rang*

$$1^{g-1}2^{3g-3}3^{5g-5}\ldots n^{(2n-1)(g-1)}$$

Si $k = \mathbb{C}$, on peut espérer qu'il existe une bijection naturelle entre les systèmes locaux de $\mathbb{C}$ -espaces vectoriels irréductibles de rang n sur X et les systèmes locaux de $\mathbb{C}$-espaces vectoriels irréductibles de rang $1^g 2^{3g-3}\ldots n^{(2n-1)(g-1)}\mathcal{F}$ sur $\mathrm{Fib}^{\ell}_{X/k,n,ts}$ tels que la variété caractéristique de $j_{!*}\mathcal{F}$ soit contenue dans le cône nilpotent $\Lambda^{\ell}_{X,n} \subset T^* \mathrm{Fib}^{\ell}_{X/k,n,ts}$, où $j : \mathrm{Fib}^{\ell}_{X/k,n,ts} \hookrightarrow \mathrm{Fib}^{\ell}_{X/k,n}$ est l'inclusion (cf. [Lau 1] (1.14) et [Lau 2]) et ceci, $\forall \ell \in \mathbb{Z}$.

La conjecture ci-dessus est motivée par les cas particuliers $n = 1$ et 2 déjà connus et par le résultat suivant.

Soit $\mathcal{L}$ un fibré vectoriel de rang n et de degré ℓ sur X. Hitchin ([Hi]) a défini une application polynomiale

$$c_{\mathcal{L}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^1_X) \to \bigoplus_{i=1}^{n} H^0(X, (\Omega^1_X)^{\otimes i})$$

en associant à $u : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^1_X$ son "polynôme caractéristique"

$$c_{\mathcal{L}}(u) = (tr(u), tr(\Lambda^2 u), \ldots, tr(\Lambda^n u))$$

(on a $\Lambda^i u : \Lambda^i \mathcal{L} \to (\Lambda^i \mathcal{L}) \otimes_{\mathcal{O}_X} (\Omega^1_X)^{\otimes i}$); de plus il a remarqué que, pour $\mathcal{L}$ simple, i.e. pour

$$\mathrm{End}_{\mathcal{O}_X}(\mathcal{L}) = k$$

la source et le but de $c_{\mathcal{L}}$ ont même dimension

$$n^2(g-1)+1.$$

Cette application est multi-homogène au sens suivant

$$c_{\mathcal{L}}(\lambda u) = (\lambda tr(u), \lambda^2 tr(\Lambda^2 u), \ldots, \lambda^n tr(\Lambda^n u))$$

et donc $c_{\mathcal{L}}$ est un morphisme fini (ramifié) dès que $c_{\mathcal{L}}^{-1}(0) = \{0\}$. Or dire que $c_{\mathcal{L}}^{-1}(0) = \{0\}$ c'est dire que $\mathcal{L}$ est très stable (si $tr(\Lambda^i u) = 0$ pour $i = 1, \ldots, n, u$ est nilpotent). Par suite si $\mathcal{L}$ est très stable, on peut parler du degré du morphisme fini $c_{\mathcal{L}}$. Comme l'a remarqué Beauville [B-N-R] (5.4), il résulte aussitôt du théorème de Bezout que

LEMME 6.2.2. *Si $\mathcal{L}$ est un fibré vectoriel de rang n et de degré ℓ sur X très stable, le degré de $c_{\mathcal{L}}$ est égal à*

$$1^{g-1}2^{3g-3}\dots n^{(2n-1)(g-1)}.$$

Remarque 6.2.3. L'entier ci-dessus est un analogue global de $n!$. En fait pour chaque groupe réductif sur k on a de même un analogue global de l'ordre du groupe de Weyl W. Pour les groupes classiques (presque simples), on a le tableau suivant

Type	$\lvert W\rvert$	Analogue global de $\lvert W\rvert$
A_n	$n!$	$1^{g-1}2^{3g-3}\dots n^{(2n-1)(g-1)}$
B_m ou C_m	$2^m m!$	$2^{3g-3}4^{7g-7}\dots(2m)^{(4m-1)(g-1)}$
D_m	$2^{m-1}m!$	$2^{3g-3}4^{7g-7}\dots(2m)^{(4m-5)(g-1)}m^{(2m-1)(g-1)}$

ces analogues globaux de $|W|$ s'interprétant encore comme les degrés génériques des applications de Hitchin et devant être liés aux faisceaux automorphes cuspidaux pour les groupes considérés. En général, si les p_i $(i = 1,\dots,\ell)$ sont les degrés des invariants (générateurs de $(\mathrm{Sym}\mathcal{G}^*)^G$ où $\mathcal{G}$ est l'algèbre de Lie de G), l'analogue global de $|W|$ est

$$\prod_{i=1}^{\ell} p_i^{(2p_i-1)(g-1)}$$

comme me l'a fait remarquer le referee.

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Canonical Models of (Mixed) Shimura Varieties and Automorphic Vector Bundles

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Introduction.

This article surveys what is known to be true, or is conjectured, concerning the rationality properties over $\mathbb{Q}$ of automorphic functions, holomorphic automorphic forms, and the Fourier-Jacobi series of automorphic forms.

The first chapter reviews the theory of abelian varieties with potential complex multiplication over $\mathbb{Q}$ and the motives that are built out of them. The constructions and results in this chapter are the basis of the statements in the succeeding chapters.

The second chapter reviews the definition and basic properties of Shimura varieties, and then states the main results: every Shimura variety has a canonical model over its reflex field, and the conjugate of the canonical model by an element of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is again the canonical model of a Shimura variety.

Holomorphic automorphic forms can be interpreted as the sections of certain vector bundles, called automorphic vector bundles, on a Shimura variety. These bundles are defined in the Chapter III, and the main theorems for them, which parallel those for Shimura varieties, are stated. In particular, every automorphic vector bundle has a canonical model over a specific number field, and we can define a holomorphic automorphic form to be rational over a field if it is a section of the canonical model of the vector bundle over that field.

As one approaches the boundary of a Hermitian symmetric domain, Hodge structures degenerate into mixed Hodge structures, and as one approaches the boundary of a Shimura variety, abelian varieties degenerate into one-motives. The theories of mixed Hodge structures and of one-motives are reviewed in Chapter IV.

In contrast to the Baily-Borel compactification of a Shimura variety, the method of toroidal compactification provides smooth compactifications of Shimura varieties. In Chapter V we describe these compactifications, and suggest how the various isomorphisms constructed in Chapters II and III should extend to the compactified varieties.

The study of the boundary of a Shimura variety suggests the introduction of a new object, generalizing that of a Shimura variety, which we here call a mixed Shimura variety. These varieties are defined in Chapter VI, and we indicate there how the results in Chapters II and III should extend to them. To give the reader some idea of how the notion of a mixed Shimura variety relates to that of a Shimura variety, we list some of the objects attached to a Shimura variety and the corresponding object attached to a mixed Shimura variety:

SHIMURA VARIETY	MIXED SHIMURA VARIETY
bounded symmetric domain	Siegel domain (of the third kind)
Hodge structure	mixed Hodge structure
reductive group	algebraic group with a 3-step filtration
abelian variety	one-motive
motive	mixed motive.

Roughly speaking, everything that is true for Shimura varieties should also be true for mixed Shimura varieties. For example, it will probably turn out to be most natural to study Hasse-Weil zeta functions in the context of mixed Shimura varieties rather than Shimura varieties. Lest the reader fear an unending hierachy, I mention that the study of the boundary of a mixed Shimura variety leads only to mixed Shimura varieties, not to some higher order object.

In the last chapter, we give a formal-algebraic definition of Fourier-Jacobi series, and suggest a theory for them also over $\mathbb{Q}$.

The contents of the second and third chapters will eventually be part of a book that I am currently writing on Shimura varieties. Once the theory outlined in the last four chapters is complete, a second book will be appropriate. Lest the reader think that that will then be the end of the subject, I point out that the theory for a general Shimura variety will then be in roughly the same happy state as the theory for elliptic modular curves was at the time of the publication of Shimura's book, Shimura (1971b), and that 1971 was the start of an explosion of interest in elliptic modular curves that continues to this day.

One of my goals in this article has been to write out the implications of Deligne's vision that Shimura varieties should be thought of as moduli varieties of motives and mixed Shimura varieties as the moduli varieties of mixed motives. I wish to thank Deligne for his patient explanation of his ideas to me over the years, and I mention specifically that the definition of a mixed Shimura variety in Chapter VI and the formal-algebraic definition of Fourier-Jacobi series in Chapter VII were suggested to me by him.

In this article, I have not attempted to describe in detail the origins of theorems, but have largely confined myself to listing the most recent work. Thus it is appropriate to mention that most of the questions discussed in this article first arose in the work of Shimura, and were often answered by him (or his students) in key cases. See in particular his talks to the International Congresses (Shimura 1968, 1971a, 1978a).

Finally I wish to thank Don Blasius and Michael Harris for many enjoyable and illuminating discussions on these questions; also I would like to thank them, Greg Anderson, and Pierre Deligne for their comments on parts of earlier drafts of this manuscript.

Conventions. All vector spaces and locally free sheaves are of finite rank. We use the same letter for a vector bundle and its associated locally free sheaf of sections.

A variety Y is a geometrically reduced scheme of finite-type over a field (it is not necessarily connected). For a variety Y over a field k and a homomorphism $\sigma : k \hookrightarrow k'$, we write σY for $Y \times_{\mathrm{Spec}(k),\sigma} \mathrm{Spec}(k')$ (the polynomials defining σY are obtained from those defining Y by applying σ to their coefficients). When it is not necessary to mention σ, we write $Y_{k'}$ for σY.

The following construction will be often used: let G be an algebraic group over $\mathbb{Q}$ acting on a variety Y on the left, and let P be a right principal homogeneous space for G; then $P \times^G Y$, the variety obtained from Y "by twisting by P", is the variety over $\mathbb{Q}$ such that, as a $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$-set,

$$(P \times^G Y)(\mathbb{Q}^{\mathrm{al}}) = P(\mathbb{Q}^{\mathrm{al}}) \times Y(\mathbb{Q}^{\mathrm{al}})/\sim,\ (pg, y) \sim (p, gy),\ g \in G(\mathbb{Q}^{\mathrm{al}}).$$

For an algebraic group G over $\mathbb{R}$, $G(\mathbb{R})^+$ is the identity component of the topological group $G(\mathbb{R})$ and $G(\mathbb{R})_+$ is the inverse image of $G^{\mathrm{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$; also $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$ and $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. An algebraic group is said to be simple when all its proper normal closed subgroups are finite. When an algebraic group G is defined over a field k, then all statements are relative to k; for example, "simple" means "k-simple", subgroups are defined over k, and representations take values in k-vector spaces.

When k is a field, k^{al} is an algebraic closure of k, k^{sep} is a separable algebraic closure, and k^{ab} is the maximal abelian extension of k. We always take $\mathbb{Q}^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

For a number field E, $\mathbb{A}_E$ is the ring of adèles of E and $\hat{E}$ the ring of finite adèles. We write $\mathbb{A}$ for $\mathbb{A}_{\mathbb{Q}}$, $\mathbb{A}_f$ for $\hat{\mathbb{Q}}$, and $\mathbb{A}'$ for $\mathbb{C} \times \mathbb{A}_f$. The reciprocity law $\mathrm{rec}_E : \mathbb{A}_E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$ is normalized so that a local uniformizing element maps to the inverse of the usual (number-theorists) Frobenius automorphism. Complex conjugation is denoted by ι or by $a \mapsto \bar{a}$, and $[*\ ]$ is the equivalence class of $*$.

Except in Chapter V, the symbol T^F denotes the restriction of scalars (in the sense of Weil) of $\mathbb{G}_m$ from F to $\mathbb{Q}$.

When V is a vector space over a field k, and k' is an extension of k, we sometimes denote $V \otimes_k k'$ by $V(k')$ or $V_{k'}$.

I. Abelian varieties with complex multiplication

In this chapter we review the theory of abelian varieties with potential complex multiplication over $\mathbb{Q}$, the category of motives they generate, and their periods.

1. Tannakian categories.

The Pontryagin duality theorem allows one to recover a locally compact abelian group from its character group. Tannaka (1938) showed that a compact group can be recovered from the category of continuous finite-dimensional real representations of the group. The theory of Tannakian categories allows one to recover an affine group scheme from its category of finite-dimensional representations, and it gives an axiomatic characterization of the categories that arise in this fashion. It therefore provides a way of realizing certain abstractly defined categories as the category of representations of an affine group scheme.

A *tensor category* $(\mathbf{C}, \otimes)$ is a category $\mathbf{C}$ together with a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and sufficient constraints so that the tensor product of any finite unordered set of objects is well-defined up to a unique isomorphism. In particular, there is an *identity object* $\mathbf{1}$, defined to be the tensor product of the empty set of objects, which has the property that

$$X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$$

for all objects X of $\mathbf{C}$.

A tensor category $(\mathbf{C}, \otimes)$ is said to be *abelian* when $\mathbf{C}$ is abelian and $\otimes$ is bi-additive. Then $k =_{\mathrm{df}} \mathrm{End}(\mathbf{1})$ is a commutative ring which acts on all objects of $\mathbf{C}$ in such a way that all morphisms of $\mathbf{C}$ are k-linear and $\otimes$ is bilinear; we call $(\mathbf{C}, \otimes)$ a *k-linear abelian tensor category* (in an alternative terminology, $(\mathbf{C}, \otimes)$ is called an *abelian tensor category with coefficents in k*). For example, $\mathbf{Vec}_k$ is a k-linear abelian tensor category.

A tensor category is said to be *rigid* if every object X of $\mathbf{C}$ has a dual $\check{X}$ and these duals have certain natural properties, for example, $\mathrm{Hom}(T \otimes \check{X}, Y) = \mathrm{Hom}(T, X \otimes Y)$.

A functor from one tensor category to a second is called a *tensor functor* if it carries tensor products into tensor products (including the identity object to the identity object). A *morphism of tensor functors* $c : F \to F'$ is a morphism of functors commuting with tensor products, i.e., such that the diagrams

$$\begin{array}{ccc} \mathbf{1}' & \xrightarrow{\approx} & F(\mathbf{1}) \\ \| & & \downarrow c_{\mathbf{1}} \\ \mathbf{1}' & \xrightarrow{\approx} & F'(\mathbf{1}) \end{array}, \qquad \begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\approx} & F(X) \otimes F(Y) \\ \downarrow c_{X \otimes Y} & & \downarrow c_X \otimes c_Y \\ F'(X \otimes Y) & \xrightarrow{\approx} & F'(X) \otimes F'(Y) \end{array}$$

commute. (The horizontal isomorphisms are part of the data that F and F' are tensor functors.)

Let k be a field. A *k-linear neutral Tannakian category* is a rigid k-linear abelian tensor category for which there exists an exact k-linear tensor functor $\omega : \mathbf{C} \to \mathbf{Vec}_k$. Such a functor is called a *fibre functor* for $(\mathbf{C}, \otimes)$. Since we shall never need to consider non-neutral Tannakian categories, from now "Tannakian category" means "neutral Tannakian category".

Example 1.1. For any affine group scheme G over a field k, the category $\mathbf{Rep}_k(G)$ of finite-dimensional representations of G on k-vector spaces is a k-linear Tannakian category with an obvious fibre functor, namely $(V, \xi) \mapsto V$. (An affine group scheme over k is an affine scheme G over k together with morphisms $G \times G \to G$ (multiplication), $G \to G$ (inverse), $\operatorname{Spec} k \to G$ (identity element) satisfying the usual axioms. Thus G is an algebraic group if it is of finite-type. Every affine group scheme is a projective limit of algebraic groups, and conversely every projective system of affine algebraic groups has an affine group scheme as limit.)

If ω is a fibre functor for the k-linear Tannakian category $(\mathbf{C}, \otimes)$ and R is a k-algebra, we define ω_R to be the tensor functor $X \mapsto \omega(X) \otimes_k R$ from $(\mathbf{C}, \otimes)$ to the category of R-modules. When ω' is a second fibre functor, $\mathit{Isom}^{\otimes}(\omega, \omega')$ denotes the functor from the category of k-algebras to that of sets,

$$R \mapsto \mathit{Isom}^{\otimes}(\omega_R, \omega'_R) \qquad \text{(isomorphisms of tensor functors).}$$

Also $\mathit{Aut}^{\otimes}(\omega)$ denotes $\mathit{Isom}^{\otimes}(\omega, \omega)$.

THEOREM 1.2. *Let $(\mathbf{C}, \otimes)$ be a Tannakian category with fibre functor ω. The functor $\mathit{Aut}^{\otimes}(\omega)$ is represented by an affine group scheme G over k, and ω defines an equivalence of tensor categories*

$$(\mathbf{C}, \otimes) \to (\mathbf{Rep}_k(G), \otimes).$$

If ω' is a second fibre functor, then the functor $\mathit{Isom}^{\otimes}(\omega, \omega')$ is represented by an affine scheme $P(\omega, \omega')$ which is a principal homogeneous space for G. The affine group scheme G' representing $\mathit{Aut}^{\otimes}(\omega')$ is the inner form of G obtained from G by twisting by $P(\omega, \omega')$.

PROOF: See for example Deligne and Milne (1982), 2.11, 3.2.

The picture to keep in mind when thinking of Tannakian categories is the following. Let X be a connected topological manifold, and let $\mathbf{C}$ be the category of local systems of $\mathbb{Q}$-vector spaces on X (= locally constant sheaves of $\mathbb{Q}$-vector spaces). When endowed with its usual tensor structure, this category is Tannakian. The choice of a point x of X determines a fibre functor $\omega_x : \mathcal{V} \mapsto \mathcal{V}_x$ (stalk of $\mathcal{V}$ at x) for $\mathbf{C}$, and the fundamental group $\pi_1(X, x)$ acts on $\mathcal{V}_x$; moreover ω_x defines an equivalence from $(\mathbf{C}, \otimes)$ to the tensor category of rational representations of the abstract group $\pi_1(X, x)$. If y is a second point, then the set $P(x, y)$ of paths from x to y (taken up to homotopy), is a principal homogeneous space for $\pi_1(X, x)$, and $\pi_1(Y, y)$ is the inner form of $\pi_1(X, x)$ obtained from $\pi_1(X, x)$ by twisting by $P(x, y)$.

Example 1.3. To give a grading on a vector space is the same as to give a representation of $\mathbb{G}_m$ on V: the grading $V = \oplus V^n$ corresponds to the representation for which $\mathbb{G}_m$ acts on V^n through the character $\chi_n = (t \mapsto t^n)$. The category of graded vector spaces over k has an obvious k-linear Tannakian structure, and our observation shows that the associated affine group scheme is $\mathbb{G}_m$.

Example 1.4. Let $\mathbf{C}$ be the category of continuous representations of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on vector spaces over $\mathbb{Q}$. This is a $\mathbb{Q}$-linear Tannakian category with the forgetful functor as fibre functor. Write $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ as a limit $\varprojlim \mathrm{Gal}(K/k)$ of finite Galois groups, and give each group $\mathrm{Gal}(K/k)$ the structure of a constant algebraic group of dimension zero. Then $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acquires the structure of a pro-algebraic group, and this is the affine group scheme attached to $\mathbf{C}$.

Remark 1.5. (a) A homomorphism $f : G \to G'$ of affine group schemes over k defines a tensor functor $F : \mathbf{Rep}_k(G') \to \mathbf{Rep}_k(G)$. Conversely, a tensor functor of k-linear Tannakian categories $F : (\mathbf{C}, \otimes) \to (\mathbf{C}', \otimes)$ carrying a fibre functor ω' into a fibre functor ω defines a homomorphism of affine group schemes $f : \mathit{Aut}^{\otimes}(\omega') \to \mathit{Aut}^{\otimes}(\omega)$. Moreover, f is injective if and only if the image of F generates $\mathbf{Rep}_k(G)$ as a Tannakian category[1], and f is surjective if and only if F is fully faithful and the essential image is stable under the formation of subquotients.

[1] We say that a set of objects S in a Tannakian category $\mathbf{C}$ generates $\mathbf{C}$ if there is no full Tannakian subcategory of $\mathbf{C}$ containing all objects of S and their subquotients other than $\mathbf{C}$ itself.

(b) Let $(\mathbf{C}, \otimes)$ be a k-linear Tannakian category, and let k' be a finite separable extension of k. The category $\mathbf{C}_{k'}$ is defined to be the pseudo-abelian envelope[2] of the category whose objects are those of $\mathbf{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{C}_{k'}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y) \otimes_k k'.$$

It is a k'- linear Tannakian category. Any fibre functor ω of $\mathbf{C}$ extends in a natural way to a fibre functor ω' of $\mathbf{C}_{k'}$, and the affine group scheme attached to $(\mathbf{C}_{k'}, \omega')$ is $G_{k'}$.

Graded Tannakian categories.

Definition 1.6. A *grading* of a k-linear Tannakian category $\mathbf{C}$ can be described as either:

(a) a grading $X = \otimes_{m \in \mathbb{Z}} X^m$ on each object of $\mathbf{C}$ that depends functorially on X and is compatible with tensor products in the sense that $(X \otimes Y)^m = \oplus_{r+s=m} X^r \otimes Y^s$; or

(b) a central homomorphism $w : \mathbb{G}_m \to G$, $G = Aut^{\otimes}(\omega)$, for some fibre functor ω. Central means that the image is contained in the centre of G. Note that, by (1.2), the centre of G is independent of the choice of ω. A grading of $\mathbf{C}$ defines a grading on $\omega(X)$ for each object X and fibre functor ω; we have $\omega(X)^n = \omega(X^n)$, which is the subspace of $\omega(X)$ on which $w(z)$ acts as z^n.

Filtrations of $\mathbf{Rep}_k(G)$. Let V be a vector space. A homomorphism $\mu : \mathbb{G}_m \to GL(V)$ defines a filtration

$$\cdots \supset F^p V \supset F^{p+1} V \supset \cdots, \qquad F^p V = \oplus_{i \geq p} V^i,$$

of V, where $V = \oplus V^i$ is the grading defined by μ.

Let G be an algebraic group over a field k of characteristic zero. A homomorphism $\mu : \mathbb{G}_m \to G$ defines a filtration $F^{\cdot}$ on V for each representation (V, ξ) of G, namely, that corresponding to $\xi \circ \mu$. These filtrations are compatible with the formation of tensor products and duals, and they are exact in the sense that $V \mapsto Gr^{\cdot}_F(V)$ is exact.

[2] An additive category is *pseudo-abelian* or (*Karoubian*) if, for every morphism $p : X \to X$ such that $p^2 = p$, the kernel of $p - 1$ exists. For any additive category $\mathbf{C}$, there is a pseudo-abelian category $\mathbf{PC}$ and a functor $\mathbf{C} \to \mathbf{PC}$ that is universal among functors from $\mathbf{C}$ into pseudo-abelian categories. The objects of $\mathbf{PC}$ are pairs (X, p) with p as above, and the morphisms are defined so as to make (X, p) the image of p in the enlarged category.

Conversely, any functor $(V,\xi) \mapsto (V,F^{\cdot})$ from representations of G to filtered vector spaces compatible with tensor products and duals which is exact in this sense arises from a (nonunique) homomorphism $\mu : \mathbb{G}_m \to G$. We call such a functor a *filtration* $F^{\cdot}$ of $\mathbf{Rep}_k(G)$, and a homomorphism $\mu : \mathbb{G}_m \to G$ defining $F^{\cdot}$ is said to *split* $F^{\cdot}$. We write $Filt(\mu)$ for the filtration defined by μ.

For each p, we define F^pG to be the subgroup of G of elements acting as the identity map on $\oplus_i F^iV/F^{i+p}V$ for all representations V of G. Clearly F^pG is unipotent for $p \geq 1$, and F^0G is the semi-direct product of F^1G with the centralizer $Z(\mu)$ of any μ splitting $F^{\cdot}$.

PROPOSITION 1.7. *Let G be a reductive group over a field k of characteristic zero, and let $F^{\cdot}$ be a filtration of $\mathbf{Rep}_k(G)$. From the adjoint action of G on $\mathfrak{g} =_{\mathrm{df}} \mathrm{Lie}(G)$, we acquire a filtration of $\mathfrak{g}$.*

(a) F^0G is the subgroup of G respecting the filtration on each representation of G; it is a parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$.

(b) F^1G is the subgroup of F^0G acting trivially on the graded module $\oplus(F^pV/F^{p+1}V)$ associated with each representation of G; it is the unipotent radical of F^0G, and $\mathrm{Lie}(F^1G) = F^1\mathfrak{g}$.

(c) The centralizer $Z(\mu)$ of any μ splitting $F^{\cdot}$ is a Levi subgroup of F^0G; therefore, $Z(\mu) \xrightarrow{\approx} F^0G/F^1G$, and the composite $\bar{\mu}$ of μ with $F^0G \to F^0G/F^1G$ is central.

(d) Two cocharacters μ and μ' of G define the same filtration of G if and only if they define the same group F^0G and $\bar{\mu} = \bar{\mu}'$; μ and μ' are then conjugate under F^1G.

PROOF: See Saavedra (1972), especially IV.2.2.5.

Remark 1.8. It is sometimes more convenient to work with ascending filtrations. To turn a descending filtration $F^{\cdot}$ into an ascending filtration $W_{\cdot}$, set $W_i = F^{-i}$; if μ splits $F^{\cdot}$ then $z \mapsto \mu(z)^{-1}$ splits $W_{\cdot}$. With this terminology, we have $W_0G = W_{-1}G \rtimes Z(\mu)$.

Notes. The essentials of the theory of Tannakian categories are due to Grothendieck. A full account of the theory can be found in Saavedra (1972) and a more succinct account in Deligne and Milne (1982). The paper Deligne (1989) fills an important gap in the theory of non-neutral Tannakian categories.

2. Hodge structures.

A *real Hodge structure* is a real vector space V together with a decomposition
$$V \otimes \mathbb{C} = \oplus V^{p,q}$$
such that the complex conjugate of $V^{p,q}$ is $V^{q,p}$, all p, q. The category of such structures has a natural Tannakian structure, and the affine group scheme attached to the category and the forgetful fibre functor is $\mathbb{S} =_{\mathrm{df}} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. According to Deligne's convention, $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{p,q}$ as multiplication by $z^{-p}\bar{z}^{-q}$. A Hodge structure is said to be of *weight* n if $p + q = n$ for all (p, q) with $V^{p,q} \neq 0$. The *type* of a Hodge structure is the set of pairs (p, q) for which $V^{p,q} \neq 0$.

The *Hodge filtration* defined by a Hodge structure is
$$\cdots \supset F^p \supset F^{p+1} \supset \cdots, \qquad F^p = \oplus_{r \geq p} V^{r,s}.$$
If V has weight n, then
$$\bar{F}^q = (\oplus_{s \geq q} \overline{V}^{s,r}) = \oplus_{s \geq q} V^{r,s} = \otimes_{r \leq n-q} V^{r,s},$$
and so $V_{\mathbb{C}}$ is the direct sum of F^p and $\bar{F}^q$ whenever $p + q = n + 1$. Conversely, if $F^{\cdot}$ is a finite descending filtration of $V_{\mathbb{C}}$ such that $V_{\mathbb{C}} = F^p \oplus \bar{F}^q$ whenever $p + q = n + 1$, then $F^{\cdot}$ defines a Hodge structure of weight n on $V_{\mathbb{C}}$ by the rule $V^{p,q} = F^p \cap \bar{F}^q$.

From now on, we shall regard a real Hodge structure as being a pair (V, h) consisting of a real vector space V and a homomorphism $h : \mathbb{S} \to GL(V)$. We identify $\mathbb{S}_{\mathbb{C}}$ with $\mathbb{G}_m \times \mathbb{G}_m$ in such a way that $\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ becomes $z \mapsto (z, \iota z)$. The Hodge filtration on V is then the descending filtration defined by $\mu_h : \mathbb{G}_m \to GL(V_{\mathbb{C}})$, $\mu_h(z) = h_{\mathbb{C}}(z, 1)$, and the weight grading is defined by $w_h : \mathbb{G}_m \to GL(V)$, $w_h(r) = h(r^{-1})$, $r \in \mathbb{R}^\times$.

For any $k \subset \mathbb{R}$, a *Hodge k-structure* is a vector space V over k together with a Hodge structure on $V \otimes_k \mathbb{R}$ such that the weight grading is defined over k. The category of such structures is a k-linear Tannakian category $\mathbf{Hdg}_k$. A Hodge $\mathbb{Q}$-structure will simply be called a *Hodge structure*. The *Mumford-Tate group* $MT(V, h)$ of a Hodge structure is the smallest $\mathbb{Q}$-rational algebraic subgroup of $GL(V) \times \mathbb{G}_m$ such that $MT(V, h)_{\mathbb{C}}$ contains the image of $(\mu_h, 1) : \mathbb{G}_m \to GL(V) \times \mathbb{G}_m$. It is a connected subgroup of $GL(V) \times \mathbb{G}_m$.

Example 2.1. (a) For any smooth projective variety X over $\mathbb{C}$, Hodge theory provides $H^n(X(\mathbb{C}), \mathbb{Q})$ with a Hodge structure of weight

n. Since $H_n(X(\mathbb{C}),\mathbb{Q})$ is dual to $H^n(X(\mathbb{C}),\mathbb{Q})$, it acquires a Hodge structure of weight $-n$.

(b) Giving a Hodge structure of type $\{(-1,0),(0,-1)\}$ on a real vector space V corresponds to giving a complex structure on V: given the complex structure, define $h(z)$ to be multiplication by z; given the Hodge structure, define the complex structure by the isomorphism $V \to V_{\mathbb{C}}/F^0$.

(c) For each integer n, $\mathbb{Q}(n)$ denotes the vector space $(2\pi\mathrm{i})^n\mathbb{Q}$ with the Hodge structure of type $\{(-n,-n)\}$.

A *polarization* of a Hodge k-structure (V,h) of weight n is a morphism of Hodge structures $\psi : V(\mathbb{R}) \otimes V(\mathbb{R}) \to \mathbb{R}(-n)$ such that the real-valued form $(x,y) \mapsto (2\pi\mathrm{i})^n\psi(x,h(i)y)$ is symmetric and positive-definite. The Mumford-Tate group of a polarizable Hodge structure is reductive.

Example 2.2. For an abelian variety A over $\mathbb{C}$, $H_1(A,\mathbb{Q})$ is a polarizable Hodge structure of type $\{(0,-1),(-1,0)\}$, and $A \mapsto H_1(A,\mathbb{Q})$ defines an equivalence between the category of abelian varieties over $\mathbb{C}$, considered up to isogeny, and the category of polarizable Hodge structures of type $\{(0,-1),(-1,0)\}$. The *Mumford-Tate* group MT^A of A is defined to be the Mumford-Tate group of $H_1(A,\mathbb{Q})$.

Hodge structures of CM-type. A Hodge structure is said to be of *CM-type* if it is polarizable and its Mumford-Tate group is commutative (and hence a torus).

Example 2.3. A field E of finite degree over $\mathbb{Q}$ is said to be a *CM-field* if there is a nontrivial involution ι of E that becomes complex conjugation under every embedding $E \hookrightarrow \mathbb{C}$. A finite product of CM-fields is called a *CM-algebra*. An abelian variety A is said to have *complex multiplication* (or be of *CM-type*) if there is a faithful homomorphism $E \to \mathrm{End}(A) \otimes \mathbb{Q}$ (mapping 1 to 1) with E a CM-algebra of degree $[E:\mathbb{Q}] = 2\dim(A)$, and it is said to have *potential complex multiplication* if it acquires complex multiplication over some extension of the ground field. With these definitions, an abelian variety over $\mathbb{C}$ is of CM-type if and only if the Hodge structure $H_1(A,\mathbb{Q})$ is of CM-type.

The category of Hodge structures of CM-type is Tannakian. Let $\mathfrak{S}$ be the affine group scheme attached to it and the forgetful fibre functor. The functor sending a Hodge structure (V,h) to the real

Hodge structure $(V \otimes \mathbb{R}, h)$ defines a homomorphism $h_{\text{can}} : \mathbb{S} \to \mathfrak{S}_{\mathbb{R}}$, and hence a cocharacter μ_{can} of $\mathfrak{S}_{\mathbb{C}}$.

PROPOSITION 2.4. *(a) The group scheme $\mathfrak{S}$ is a pro-torus. The map*

$$\xi \mapsto n_\chi, \qquad n_\chi(\tau) = \langle \chi, \tau\mu_{\text{can}} \rangle,$$

identifies the character group of $\mathfrak{S}$ with the group of all functions $n : \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}) \to \mathbb{Z}$ which factor through $\mathrm{Gal}(F/\mathbb{Q})$ for some CM-field F and which have the property that

$$n(\iota\sigma) + n(\sigma) = \textit{constant}.$$

(b) The pair $(\mathfrak{S}, \mu_{\text{can}})$ has the following universal property: for any torus T over $\mathbb{Q}$ and $\mu \in X_*(T)$ satisfying

$$(*) \quad (\tau - 1)(\iota + 1)\mu = 0 = (\iota + 1)(\tau - 1)\mu, \qquad \textit{all } \tau \in \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}),$$

there is a unique homomorphism $\rho_\mu : \mathfrak{S} \to T$ (defined over $\mathbb{Q}$) such that $(\rho_\mu)_{\mathbb{C}} \circ \mu_{\text{can}} = \mu$.

The pro-torus $\mathfrak{S}$ is called the *Serre group*, and the condition (*) is called the *Serre condition.*

Remark 2.5. (a) For a field F of finite degree over $\mathbb{Q}$, define $\mathfrak{S}^F$ to be the quotient of $T^F =_{\text{df}} \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ whose character group $X^*(\mathfrak{S}^F)$ is the subgroup of $X^*(T^F)$ of elements satisfying the Serre condition. The norm map induces a homomorphism $\mathfrak{S}^{F'} \to \mathfrak{S}^F$ for any F' containing F, and it is easily seen that $\mathfrak{S} = \varprojlim \mathfrak{S}^F$ (limit over $F \subset \mathbb{Q}^{\text{al}}$). In fact, it suffices to take the limit over all CM-fields $F \subset \mathbb{Q}^{\text{al}}$.

(b) Let $F \subset \mathbb{Q}^{\text{al}}$ be a finite Galois extension of $\mathbb{Q}$. The action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on T^F defined by its action on F induces an action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}^F$. In the limit we obtain an action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}$ (rational over $\mathbb{Q}$). There are therefore two distinct actions of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}(\mathbb{Q}^{\text{al}})$: the first arises from the action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}$, and the second from its action on $\mathbb{Q}^{\text{al}}$.

Example 2.6. Let E be a CM-algebra. A *CM-type* for E is a subset Φ of $\mathrm{Hom}(E, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \cup \iota\Phi$ (disjoint union). Let A be an abelian variety over $\mathbb{C}$ with complex multiplication $i : E \to \mathrm{End}(A) \otimes \mathbb{Q}$ by E. For $\sigma \in \mathrm{Hom}(E, \mathbb{C})$, write $\mathbb{C}_\sigma$ for $\mathbb{C}$ with E acting through σ. Then $\mathrm{Tgt}_0(A) \approx \prod_{\varphi \in \Phi} \mathbb{C}_\varphi$ with Φ a CM-type for E, and

(A, i) is said to be of *CM-type* (E, Φ). By assumption, $V = H_1(A, \mathbb{Q})$ is a free E-module of rank one, and we can regard T^E as a subtorus of $GL(V)$. Define $\mu_\Phi : \mathbb{G}_m \to (T^E)_\mathbb{C}$ to be the cocharacter such that

$$\sigma \circ \mu_\Phi = \begin{cases} 1 & \text{if } \sigma \in \Phi \\ 0 & \text{otherwise,} \end{cases}$$

and let h_Φ be the associated homomorphism $h_\Phi : \mathbb{S} \to (T^E)_\mathbb{R}$. When regarded as a homomorphism $\mathbb{S} \to GL(V_\mathbb{R})$, h_Φ is the representation of $\mathbb{S}$ defined by the Hodge structure on $H_1(A, \mathbb{Q})$.

Since μ_Φ satisfies the Serre condition, it determines a homomorphism $\rho_\Phi : \mathfrak{S} \to T^E \subset GL(V)$; ρ_Φ is the representation of $\mathfrak{S}$ defined by the CM-Hodge structure $H_1(A, \mathbb{Q})$.

The field of definition of μ_Φ (contained in $\mathbb{C}$) is called the *reflex field* $E^*(\Phi)$ of (E, Φ). For any number field $F \supset E^*(\Phi)$, μ_Φ defines a homomorphism N_Φ

$$T^F \xrightarrow{\mathrm{Res}_{F/\mathbb{Q}}(\mu_\Phi)} \mathrm{Res}_{F/\mathbb{Q}}(T^E) \xrightarrow{\mathrm{Norm}_{F/\mathbb{Q}}} T^E$$

called the *reflex norm.*

For any isomorphism $\sigma : E \to E'$ of CM-fields and automorphism τ of $\mathbb{Q}^{\mathrm{al}}$, $\tau\Phi\sigma^{-1}$ denotes the CM-type $\{\tau\phi\sigma^{-1} | \phi \in \Phi\}$ of E'; for any CM-field $E' \supset E$, Φ extends to a CM-type $\Phi' = \{\phi \in \mathrm{Hom}(E', \mathbb{Q}^{\mathrm{al}}) | \ \phi | E \in \Phi\}$. We shall need the following formulas:

$$\rho_\Phi \circ \tau = \rho_{\tau^{-1}\Phi}, \qquad \sigma \circ \rho_\Phi = \rho_{\Phi\sigma^{-1}}, \qquad N_{E'/E} \circ \rho_{\Phi'} = \rho_\Phi.$$

Hodge tensors. Let V be a Hodge structure. A *Hodge element* in V is an element of type $(0,0)$ in V. For example, the Hodge elements in $\check{V} \otimes W$ are precisely the elements corresponding to homomorphisms $V \to W$ that are morphisms of Hodge structures. According to the Hodge conjecture, the Hodge elements of $H^{2p}(X, \mathbb{Q}(p))$ should be linear combinations of the classes of algebraic cycles. A *Hodge tensor* of V is an element of type $(0,0)$ in

$$\mathcal{T}V =_{\mathrm{df}} \oplus V^{\otimes r} \otimes \check{V}^{\otimes s} \otimes \mathbb{Q}(m) \qquad \text{(sum over } (r,s,m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}).$$

We let $GL(V)$ act on $\mathcal{T}V$ through its actions on V and $\check{V}$, and we let $\mathbb{G}_m$ act on $\mathcal{T}V$ through its action on $\mathbb{Q}(1)$.

PROPOSITION 2.7. *The Mumford-Tate group of a Hodge structure (V, h) is the subgroup of $GL(V) \times \mathbb{G}_m$ of elements fixing all the Hodge tensors of V.*

PROOF: See Deligne (1982a), pp40-45.

COROLLARY 2.8. *Let* **C** *be the Tannakian subcategory of* $\mathbf{Hdg}_{\mathbb{Q}}$ *generated by V and $\mathbb{Q}(1)$. The affine group scheme attached by (1.2) to* **C** *and the forgetful fibre functor is $MT(V, h)$.*

PROOF: Since V and $\mathbb{Q}(1)$ generate **C**, the affine group scheme is a subgroup of $GL(V)\times\mathbb{G}_m$, and it consists of those elements of $GL(V)\times \mathbb{G}_m$ that commute with all morphisms of Hodge structures. But every morphism of Hodge structures in **C** can be interpreted as a Hodge tensor of V.

Notes. The Mumford-Tate group was introduced in Mumford (1966), and the Serre group in Serre (1968), pII-2. They are discussed in more detail in Deligne (1982a), §3, and Milne and Shih (1982a).

3. Hodge cycles.

A theorem of Deligne shows that Hodge cycles on an abelian variety have some of the properties of algebraic cycles; in particular, it will enable us to define Hodge cycles on an abelian variety over any field of characteristic zero.

We review the first homology groups attached to an abelian variety A over a field k of characteristic zero.

When $k = \mathbb{C}$, we have the usual "Betti" homology group $H_B(A) =_{\mathrm{df}} H_1(A(\mathbb{C}), \mathbb{Q})$. This is a vector space of dimension 2dim$A$ over $\mathbb{Q}$, and, as we noted in §2, it has a Hodge structure of type $\{(-1,0),(0,-1)\}$. For any field k and embedding $\tau : k \hookrightarrow \mathbb{C}$, we set $H_\tau(A) = H_B(\tau A)$. When k is a subfield of $\mathbb{C}$, we sometimes write $H_B(A)$ for $H_B(A_{\mathbb{C}})$.

For any choice of an algebraic closure k^{al} of k, we define the ℓ-adic homology group $H_\ell(A)$ to be the dual of the étale cohomology group $H^1_{\mathrm{et}}(A_{k^{\mathrm{al}}}, \mathbb{Q}_\ell)$. This is a vector space of dimension 2dim$A$ over $\mathbb{Q}_\ell$. In more down-to-earth terms, we could set $H_\ell(A) = T_\ell(A) \otimes \mathbb{Q}$, where $T_\ell(A)$ is the Tate module $\varprojlim A(k^{\mathrm{al}})_{\ell^n}$ of A. An embedding of k^{al} into an algebraically closed field K defines an isomorphism $H_\ell(A_{k^{\mathrm{al}}}) \to H_\ell(A_K)$; in particular, Gal$(k^{\mathrm{al}}/k)$ acts on $H_\ell(A)$. We set $\mathbb{Q}_\ell(1) = T_\ell(\mathbb{G}_m) \otimes \mathbb{Q}$, and $\mathbb{Q}_\ell(n) = \mathbb{Q}_\ell(1)^{\otimes n}$, $n \in \mathbb{Z}$.

We define $H_{\mathrm{dR}}(A)$ to be the dual of the de Rham cohomology group $H^1_{\mathrm{dR}}(A) =_{\mathrm{df}} \mathbb{H}^1(A, \Omega^{\cdot}_{A/k})$. It is a vector space of dimension 2dim$A$

over k, and if $K \supset k$, then $H_{\mathrm{dR}}(A_K) = H_{\mathrm{dR}}(A) \otimes_k K$. We sometimes write $H_\infty(A)$ for $H_{\mathrm{dR}}(A)$.

When $k = \mathbb{C}$, there are canonical *comparison isomorphisms*

$$H_B(A) \otimes \mathbb{Q}_\ell \to H_\ell(A), \qquad H_B(A) \otimes \mathbb{C} \to H_{\mathrm{dR}}(A).$$

The second of these can be obtained as follows: the map

$$\gamma \mapsto (\omega \mapsto \int_\gamma \omega),$$

identifies $H_B(A) \otimes \mathbb{C}$ with the dual of the space of differential forms of the first or second kind on A, which equals $\check{H}^1_{\mathrm{dR}}(A) = H_{\mathrm{dR}}(A)$. Thus the map is defined by the periods of A.

We extend these notations as follows:

$$\begin{aligned}
\mathcal{T}_B(A) &= \mathcal{T}(H_B(A)) \qquad \text{(case that } k = \mathbb{C}\text{);}\\
\mathcal{T}_\tau(A) &= \mathcal{T}(H_B(\tau A)) \qquad \text{(where } \tau \text{ is an embedding of } k \text{ into } \mathbb{C}\text{);}\\
\mathcal{T}_\ell(A) &= \oplus H_\ell(A)^{\otimes r} \otimes \check{H}_\ell(A)^{\otimes s} \otimes \mathbb{Q}_\ell(m);\\
\mathcal{T}_\infty(A) &= \mathcal{T}_{\mathrm{dR}}(A) = \otimes H_{\mathrm{dR}}(A)^{\otimes r} \otimes \check{H}_{\mathrm{dR}}(A)^{\otimes s}\\
\mathcal{T}_f(A) &= \Pi' \mathcal{T}_\ell(A) \qquad \text{(restricted product over finite primes } \ell\text{).}
\end{aligned}$$

When $k = \mathbb{C}$, the comparison isomorphisms extend to canonical isomorphisms

$$\mathcal{T}_B(A) \otimes \mathbb{Q}_\ell \to \mathcal{T}_\ell(A), \qquad \mathcal{T}_B(A) \otimes \mathbb{C} \to \mathcal{T}_{\mathrm{dR}}(A).$$

Thus, for any abelian variety A over k and inclusion $\tau : k^{\mathrm{al}} \hookrightarrow \mathbb{C}$, there are canonical maps

$$\mathcal{T}_B(\tau A) \to \mathcal{T}_\ell(\tau A) \leftarrow \mathcal{T}_\ell(A)$$

for each ℓ (including $\ell = \infty$).

When A is an abelian variety over $\mathbb{C}$, a Hodge tensor s for the Hodge structure $H_B(A)$ is called a *Hodge cycle* on A; thus s is an element of type $(0,0)$ in $\mathcal{T}_B(A)$. The images of s under the comparison isomorphisms are called the *local components* s_ℓ of s for each ℓ (including ∞).

Let A be an abelian variety over an algebraically closed field k. A family $(s_\ell)_\ell$ with $s_\ell \in \mathcal{T}_\ell(A)$ ($\ell = \infty$ included) is called a *Hodge* *cycle on A relative to $\tau : k \hookrightarrow \mathbb{C}$* if there is a Hodge cycle s on τA whose local components are the images of the s_ℓ in $\mathcal{T}_\ell(\tau A)$ for all ℓ. Equivalently, we can say that (s_ℓ) is a Hodge cycle on A relative to τ if

(a) $s_\infty \in F^0\mathcal{T}_\infty$;

(b) the image of (s_ℓ) in $\mathcal{T}_f(\tau A) \times \mathcal{T}_\infty(\tau A)$ lies in the $\mathbb{Q}$-subspace $\mathcal{T}_B(\tau A)$.

THEOREM 3.1. *Let A be an abelian variety over an algebraically closed field k of characteristic zero. If s is a Hodge cycle on A relative to one embedding $\tau : k \hookrightarrow \mathbb{C}$, then it is a Hodge cycle relative to every such embedding.*

PROOF: This is the main theorem of Deligne (1982a).

Of course, the theorem says nothing if there are no embeddings of k into $\mathbb{C}$. When k is an algebraically closed field of finite transcendence degree over $\mathbb{Q}$, we write $C_H(A)$ for the subspace of $\mathcal{T}_f(A) \times \mathcal{T}_\infty(A)$ of elements that are Hodge cycles relative to some embedding of k into $\mathbb{C}$. It is a vector space over $\mathbb{Q}$, and an inclusion $k \hookrightarrow K$ of algebraically closed fields of finite transcendence degree over $\mathbb{Q}$ induces an isomorphism $C_H(A) \to C_H(A_K)$. This remark allows us to define $C_H(A)$ for an abelian variety over any algebraically closed field K of characteristic zero: choose an algebraically closed subfield k of K of finite transcendence degree over Q such that A has a model A_k over k and set $C_H(A) = C_H(A_k)$.

An embedding $k \hookrightarrow K$ of algebraically closed fields defines a map $C_H(A) \to C_H(A_K)$. In particular, when A has a model A_0 over subfield k_0 of k, $\mathrm{Gal}(k/k_0)$ acts on $C_H(A)$. In this case, we define $C_H(A_0)$ to be the subspace of $C_H(A)$ of elements fixed by $\mathrm{Gal}(k/k_0)$.

Much of the above discussion extends to arbitrary smooth projective varieties X. In particular, it is possible to define the notion of a Hodge cycle on X relative to an embedding $\tau : k \hookrightarrow \mathbb{C}$ (see Deligne 1982a, §2), and it is reasonable to expect that (3.1) will hold also for X.

CONJECTURE 3.2. *For any smooth projective variety X over an algebraically closed field k of characteristic zero, a cycle s that is a Hodge cycle relative to one embedding $\tau : k \hookrightarrow \mathbb{C}$ will be a Hodge cycle relative to every such embedding.*

This conjecture is implied by the Hodge conjecture. In the absence of a proof of (3.2), Deligne makes the following definition: when X

is defined over an algebraically closed field k of finite transcendence degree over $\mathbb{Q}$, an *absolute Hodge cycle* on X is a cycle that is Hodge relative to *every* embedding $k \hookrightarrow \mathbb{C}$. The definition is extended to other ground fields by the same procedure as for Hodge cycles on abelian varieties. This gives a notion of an absolute Hodge cycle on any smooth projective variety over a field of characteristic zero, which, when the variety is an abelian variety, coincides with that of a Hodge cycle.

Remark 3.3. Let A be an abelian variety over $\mathbb{C}$. Proposition 2.7 provides the following description of MT^A: for any $\mathbb{Q}$-algebra R, $MT^A(R)$ is equal to the group of automorphisms $H_B(A) \otimes R$ fixing all elements of $C_H(A)$.

Notes. This section summarizes part of Deligne (1982a).

4. Motives.

Let k be a field of characteristic zero, and let $\mathbf{V}/k$ be a category of smooth projective varieties over k. The aim of the theory of motives is to attach to $\mathbf{V}/k$ a $\mathbb{Q}$-linear Tannakian category $\mathbf{Mot}/k$ and a "universal cohomology functor" $h : \mathbf{V}/k \to \mathbf{Mot}/k$ (see Saavedra (1972), VI.4).

Example 4.1. Let $\mathbf{V}_0/k$ be the category of varieties of dimension zero over k. For a variety $X = \operatorname{Spec} R$ of dimension zero and $\tau : k \hookrightarrow \mathbb{C}$, we have the (zero$^{\text{th}}$) cohomology groups,

$$H_\tau(X) = \operatorname{Hom}(X(\mathbb{C}), \mathbb{Q}),\ H_\ell(X) = \operatorname{Hom}(X(k^{\text{al}}), \mathbb{Q}_\ell),\ H_{\text{dR}}(X) = R.$$

Fix an algebraic closure k^{al} of k, and let $\mathbf{Art}/k$ be the Tannakian category defined in (1.4). For a representation $M = (V, \xi)$ of $\operatorname{Gal}(k^{\text{al}}/k)$, define

$$H_\tau(M) = V, \quad H_\ell(M) = V \otimes \mathbb{Q}_\ell, \quad H_{\text{dR}}(M) = (V \otimes k^{\text{al}})^{\operatorname{Gal}(k^{\text{al}}/k)}$$

(diagonal action). Set $hX = \operatorname{Hom}(X(k^{\text{al}}), \mathbb{Q})$ for X in $\mathbf{V}_0$; then $\mathbf{Art}/k$ is generated (as a Tannakian category) by the objects hX, and $H_*(hX) = H_*(X)$ for $* = \tau$, ℓ, or dR. Thus $h : \mathbf{V}_0 \to \mathbf{Art}/k$ is the universal cohomology functor for $\mathbf{V}_0/k$. The objects of $\mathbf{Art}/k$ are called *Artin motives*.

Unfortunately, not enough is known about algebraic cycles to construct a Tannakian category of motives for all varieties using them[3].

[3] Surprisingly, the difficulty is in adjusting the commutativity constraint (the functorial isomorphism $X \otimes Y \approx Y \otimes X$). For this one needs to use Grothendieck's "standard conjectures"–see Saavedra (1972), VI.4.

Instead, we use Hodge cycles. Assume k is algebraically closed, and let $\mathbf{V}/k$ be the category of abelian varieties over k. If A and B are objects of $\mathbf{V}/k$, define $\mathrm{Hom}(hA, hB)$ to the set of families $(f_\ell : H_\ell(A) \to H_\ell(B))_\ell$ ($\ell = \infty$ included) such that, when we regard f_ℓ as an element of $\check{H}_\ell(A) \otimes H_\ell(B) \subset \mathcal{T}_\ell(A \times B)$, then $(f_\ell)_\ell$ is a Hodge cycle on $A \times B$. Define $\mathbf{CV}/k$ to be the category with objects hA, one for each $A \in \mathrm{Ob}(\mathbf{V}/k)$, and the morphisms just defined. Adjoin the images of projectors p to the set of objects of $\mathbf{CV}/k$, and so embed $\mathbf{CV}/k$ into its pseudo-abelian envelope $\mathbf{CV}^+/k$ (cf. 1.5a). Next adjoin to $\mathbf{CV}^+/k$ all powers of the Tate motive $\mathbb{Q}(1)$. Finally modify the commutativity constraint (the identification of $M \otimes N$ with $N \otimes M$) to obtain the category $\mathbf{AV}/k$ of motives of abelian varieties over k (for the details, see Deligne and Milne 1982, §6).

THEOREM 4.2. *The category $\mathbf{AV}/k$ is a semisimple $\mathbb{Q}$-linear Tannakian category. It is generated (as a Tannakian category) by the motives hA with A an abelian variety over K. The functors H_τ, H_ℓ, and H_{dR} on $\mathbf{V}/k$ extend to $\mathbf{Mot}/k$, as do the comparison isomorphisms.*

Variants 4.3. (a) Drop the condition that k is algebraically closed, and take $\mathbf{V}/k$ to be the category of abelian varieties and varieties of dimension zero over k. We then obtain a semisimple $\mathbb{Q}$-linear Tannakian category $\mathbf{AV}/k$ with the properties in (4.2) except that $\mathbf{AV}/k$ is now generated by the motives of abelian varieties and the Artin motives.

(b) Drop the condition that k is algebraically closed, and take $\mathbf{V}/k$ to be the category of all smooth projective varieties over k. Replace "Hodge cycle" with "absolute Hodge cycle" in the definition of $\mathbf{CV}/k$. We then obtain a semisimple $\mathbb{Q}$-linear Tannakian category $\mathbf{Mot}/k$, the *category of motives* over k, with the properties in (4.2), except that it is now generated by the motives of smooth projective varieties.

PROPOSITION 4.4. *The functor $H_B : \mathbf{AV}/\mathbb{C} \to \mathbf{Hdg}_{\mathbb{Q}}$ is fully faithful.*

PROOF: In this case, $\mathrm{Hom}(hA, hB)$ consists of the maps $H_B(A) \to H_B(B)$ given by Hodge tensors. These are morphisms of Hodge structures.

Motives of *CM*-type. Define $\mathbf{CM}/k$ to be the Tannakian subcategory of $\mathbf{AV}/k$ generated by the motives of abelian varieties of potential CM-type over k and the Artin motives. Objects of $\mathbf{CM}/k$ will be called *motives of CM-type* or *CM-motives* over k.

PROPOSITION 4.5. *The functor $H_B : \mathbf{CM}/\mathbb{C} \to \mathbf{Hdg}_{\mathbb{Q}}$ is fully faithful, with essential image the category of Hodge structures of CM-type. Therefore the affine group scheme attached to the Tannakian category $\mathbf{CM}/\mathbb{C}$ and the Betti fibre functor is the Serre group $\mathfrak{S}$.*

PROOF: That H_B is fully faithful follows from (4.4). If we let $\mathfrak{S}'$ be the affine group scheme attached to $\mathbf{CM}/\mathbb{C}$, then (1.5a) shows that there is a surjective homomorphism $\mathfrak{S} \to \mathfrak{S}'$. To prove that this homomorphism is injective, it suffices to show that the intersection of the kernels of the homomorphisms $\rho_A : \mathfrak{S} \to GL(H_B(A))$, A an abelian variety A of CM-type over $\mathbb{C}$, is trivial. This follows from the next lemma.

LEMMA 4.6. *Let $F \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field, Galois over $\mathbb{Q}$. The intersection of the kernels of the homomorphisms (see 2.6) $\rho_\Phi : \mathfrak{S}^F \to T^F$ defined by the CM-types Φ on F is trivial.*

PROOF: It suffices to show that $X^*(\mathfrak{S}^F)$ is generated by the images of the maps $X^*(\rho_\Phi) : X^*(T^F) \to X^*(\mathfrak{S}^F)$. But, by definition, $X^*(\mathfrak{S}^F)$ consists of the sums $\sum n(\sigma)\,\sigma$, $\sigma \in \mathrm{Hom}(F, \mathbb{C})$, with $n(\sigma) + n(\iota\sigma)$ constant, and one sees easily that the image of $X^*(\rho_\Phi)$ contains $\sum_{\varphi\in\Phi} \varphi$. Thus the proof is an easy combinatorial exercise (see Lang 1983, p175).

The functor $A \mapsto A_{\mathbb{C}}$ defines an equivalence between the category of abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$ and the corresponding category over $\mathbb{C}$. Thus the base-change functor $\mathbf{CM}/\mathbb{Q}^{\mathrm{al}} \to \mathbf{CM}/\mathbb{C}$ is an equivalence of categories, and the affine group scheme attached to $\mathbf{CM}_{\mathbb{Q}^{\mathrm{al}}}$ and the Betti fibre functor is again the Serre group $\mathfrak{S}$.

Notes. The concept of a motive is due to Grothendieck. The definition adopted in this article is a variant of his. Most of the material in this section is from Deligne and Milne (1982), §6.

5. The main theorem of complex multiplication.

Let (A, i) be an abelian variety with complex multiplication over $\mathbb{Q}^{\mathrm{al}}$. The theorem of Shimura and Taniyama (Lang 1983, p84) describes how those automorphisms of $\mathbb{Q}^{\mathrm{al}}$ fixing the reflex field of (A, i) act on the torsion points of A. Work of Deligne and Langlands extends the result to the full Galois group of $\mathbb{Q}^{\mathrm{al}}$ over $\mathbb{Q}$. In this section, we give a statement and proof of this result in terms of abelian varieties, and in the next section, we re-interpret it in terms of motives.

Definition of the Taniyama element $f_\Phi(\tau)$. Let E be a CM-field. For each $\sigma \in \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})$, choose an element $v_\sigma \in \mathrm{Hom}(E^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{al}})$ in such a way that $v_\sigma|E = \sigma$ and $v_{\iota\sigma} = \iota v_\sigma$. For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau \circ v_\sigma$ and $v_{\tau\sigma}$ have the same action on elements of E, and so differ by an element of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. For a CM-type Φ for E, define

$$V_\Phi(\tau) = \prod_{\varphi \in \Phi} v_{\tau\varphi}^{-1} \cdot \tau \circ v_\varphi \in \mathrm{Gal}(E^{\mathrm{ab}}/E).$$

It is easily checked that $V_\Phi(\tau)$ is independent of the choice of the elements v_σ.

The *cyclotomic character* $\chi_{\mathrm{cyc}} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \hat{\mathbb{Z}}^\times$ is defined by the condition that $\sigma\zeta = \zeta^{\chi_{\mathrm{cyc}}(\sigma)}$ for every root of unity ζ in $\mathbb{Q}^{\mathrm{al}}$. With our conventions, $\mathrm{rec}_{\mathbb{Q}}(\chi_{\mathrm{cyc}}(\sigma)) = \sigma|\mathbb{Q}^{\mathrm{ab}}$.

PROPOSITION 5.1. *There is a unique element $F_\Phi(\tau) \in \hat{E}^\times/E^\times$ such that*

(a) $\mathrm{rec}_E(f_\Phi(\tau)) = V_\Phi(\tau)$, *and*

(b) $f_\Phi(\tau) \cdot \iota f_\Phi(\tau) = \chi_{\mathrm{cyc}}(\tau) E^\times$.

PROOF: See Tate (1981) (also Lang 1983, p168).

We call $f_\Phi(\tau)$ the *Taniyama element* for (E, Φ) and τ. With the notations of (2.6), we we have the following result.

PROPOSITION 5.2. *(a)* $f_\Phi(\sigma\tau) = f_{\tau\Phi}(\sigma) \cdot f_\Phi(\tau)$, $\sigma, \tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(b) $\sigma f_\Phi(\tau) = f_{\Phi\sigma^{-1}}(\tau)$, *$\sigma$ an isomorphism* $E \to E'$, $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(c) $f_\Phi(\iota) = 1$.

(d) If Φ' is the extension of Φ to $E' \supset E$, then $f_\Phi(\tau) = f_{\Phi'}(\tau)$ (in $\hat{E}'^\times/E'^\times$).

(e) If τ fixes E^*, then $f_\Phi(\tau) = N_\Phi(s) \cdot E^\times$ for any $s \in \hat{E}^*$ such that $\mathrm{rec}_{E^*}(s) = \tau|E^{\mathrm{ab}}$.

PROOF: See Tate (1981) (also Lang 1983, VII).

First statement of the main theorem. Let (A, i) be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type (E, Φ), and let $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. Define τi to be the map $E \to \mathrm{End}(\tau A) \otimes \mathbb{Q}$, $a \mapsto \tau(\mathrm{i}(a))$. Then $(\tau A, \tau \mathrm{i})$ is an abelian variety of CM-type $(E, \tau\Phi)$.

THEOREM 5.3. *(Main theorem, first form). Let (A, i) be of CM-type (E, Φ). For each $f \in \hat{E}^{\times}$ representing $f_{\Phi}(\tau)$, there is a unique E-linear isomorphism $\alpha : H_B(A) \to H_B(\tau A)$ such that $\tau x = \alpha(fx)$ for all $x \in H_f(A)$.*

PROOF: We explain in (5.10) below how to obtain a stronger result.

Remark 5.4. (a) It is obvious that α is uniquely determined by the choice of f representing $f_{\Phi}(\tau)$, and that if f is replaced by af $(a \in E^{\times})$, then α must be replaced by αa^{-1}.

(b) Let α be as in the theorem, and let ψ be a polarization of (A, i), that is, ψ is a polarization of $H_B(A)$ such that $\psi(ax, y) = \psi(x, \bar{a}y)$ for $a \in E$. Then, for x, $y \in H_f(A)$

$$(\tau\psi)(\tau x, \tau y) = \tau(\psi(x, y)) = \chi_{\text{cyc}}(\tau) \cdot \psi(x, y)$$

because $\psi(x, y) \in \mathbb{A}_f(1)$. Thus if α is as in the theorem, then

$$\chi_{\text{cyc}}(\tau) \cdot \psi(x, y) = (\tau\psi)(f\alpha(x), f\alpha(y)) = (\tau\psi)(f\bar{f}\alpha(x), \alpha(y))$$

and so

$$\psi(cx, y) = (\tau\psi)(\alpha x, \alpha y),$$

with $c = \chi_{\text{cyc}}(\tau)/f\bar{f} \in E^{\times}$.

Now assume that A has complex multiplication by the full ring of integers $\mathcal{O}_E$ of E. The choice of a basis element e_0 for $H_B(A)$ determines an isomorphism $E \to H_B(A)$, and hence an isomorphism $\mathbb{C}^{\Phi} = E \otimes_{\mathbb{Q}} \mathbb{R} \to H_B(A) \otimes \mathbb{R} = \mathrm{Tgt}_0(A)$ (see 2.6). On composing this with the exponential map $\mathrm{Tgt}_0(A) \to A(\mathbb{C})$, we obtain an $\mathcal{O}_E$-linear isomorphism $\theta : \mathbb{C}^{\Phi}/\mathfrak{a} \to A(\mathbb{C})$ for some ideal $\mathfrak{a}$ in E. Moreover, the choice of e_0 allows us to write a polarization ψ of (A, i) in the form

$$\psi(xe_0, ye_0) = 2\pi \mathrm{i} \operatorname{Tr}_{E/\mathbb{Q}}(tx\bar{y})$$

for some $t \in E$. The triple (A, i, ψ) is then said to be of *type* $(E, \Phi; \mathfrak{a}, t)$ with respect to the parametrization θ. The type determines (A, i, ψ) up to isomorphism. If e_0 is replaced by $a^{-1}e_0$, then θ is replaced by θa^{-1}, and (A, i, ψ) is of type $(E, \Phi; a\mathfrak{a}, t/a\bar{a})$ with respect to θa^{-1}.

COROLLARY 5.5. *Let (A, i, ψ) be a polarized abelian variety over $\mathbb{C}$ of CM-type $(E, \Phi; \mathfrak{a}, t)$ with respect to a parametrization $\theta : \mathbb{C}^{\Phi} \to A(\mathbb{C})$, and let τ be an automorphism of $\mathbb{C}$. For each $f \in \hat{E}^{\times}$ representing $f_{\Phi}(\tau|\mathbb{Q}^{\text{al}})$, there is a unique parametrization $\theta' : \mathbb{C}^{\tau\Phi} \to (\tau A)(\mathbb{C})$ of τA such that:*

(a) $\tau(A, i, \psi)$ *has type* $(E, \tau\phi; f\mathfrak{a}, t\chi_{\mathrm{cyc}}(\tau)/f\bar{f})$ *with respect to* θ';
(b) *the diagram*

$$\begin{array}{ccc} E\mathfrak{a} & \xrightarrow{\theta} & A(\mathbb{C})_{\mathrm{tors}} \\ \downarrow f & & \downarrow \tau \\ E/f\mathfrak{a} & \xrightarrow{\theta'} & (\tau A)(\mathbb{C})_{\mathrm{tors}}, \end{array}$$

commutes.

PROOF: If θ is defined by $e_0 \in H_B(A)$, take θ' to be the parametrization of τA defined by $\alpha(e_0) \in H_B(\tau A)$, where α is the map in the theorem.

Remark 5.6. If τ fixes the reflex field and $s \in \hat{E}$ is such that $\mathrm{rec}_E(s) = \tau|E^{\mathrm{ab}}$, then $N_\Phi(s) \in f_\Phi(\tau)$ by (5.2e) and (5.5) becomes the theorem of Shimura and Taniyama referred to earlier.

Definition of the universal Taniyama element $f(\tau)$. Let T be a torus over $\mathbb{Q}$. For any Galois splitting field L of T, we set

$$\wp(T) = (T(\hat{L})/T(L))^{\mathrm{Gal}(L/\mathbb{Q})}.$$

This is easily seen to be independent of the choice of L. Moreover, if

$$H^1(\mathbb{Q}, T) \to \prod_{\ell \text{ finite}} H^1(\mathbb{Q}_\ell, T)$$

is injective, then $\wp(T) = T(\hat{\mathbb{Q}})/T(\mathbb{Q})$. In particular, $\wp(T^E) = \hat{E}^\times/E^\times$. Define

$$\wp(\mathfrak{S}) = \varprojlim \wp(\mathfrak{S}^F).$$

PROPOSITION 5.7. *There is a unique element* $f(\tau) \in \wp(\mathfrak{S})$ *such that for each CM-field* E *and type* Φ, $\rho_\Phi(f(\tau)) = f_\Phi(\tau)$ *in* $\wp(T^E) = \hat{E}^\times/E^\times$. *The map* $\tau \mapsto f(\tau)$ *is a continuous reversed one-cocycle for* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ *with values in* $\wp(\mathfrak{S})$, *that is,* $f^F(\sigma\tau) = \tau^{-1}f^F(\sigma)\cdot f^F(\tau)$.

PROOF: The uniqueness follows from (4.6). It is possible to prove the existence of $f(\tau)$ by verifying compatibilities between the $f_\Phi(\tau)$ for different Φ, but I prefer use Langlands's original construction of $f(\tau)$.

Let F be a finite Galois extension of $\mathbb{Q}$ contained in $\mathbb{Q}^{\mathrm{al}}$. The Weil group $W_{F/\mathbb{Q}}$ of F fits into an exact commutative diagram,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{A}_F^\times/F^\times & \longrightarrow & W_{F/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{rec}_F} & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1
\end{array}$$

in which all the vertical arrows are surjective (see Tate 1979). If we assume further that F is a totally imaginary, then $(F \otimes \mathbb{R})^\times$ is contained in the kernel of rec_F, and so we can divide out by it and its image in $W_{F/\mathbb{Q}}$ to obtain an exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \hat{F}^\times/F^\times & \longrightarrow & W^f_{F/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{rec}_F} & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1.
\end{array}$$

For each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, choose an element $\tilde{\tau} \in W^f_{F/\mathbb{Q}}$ whose image in $\mathrm{Gal}(F^{\mathrm{ab}}/\mathbb{Q})$ is $\tau|F^{\mathrm{ab}}$. Choose elements $w_\sigma \in W^f_{F/\mathbb{Q}}$, one for each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, such that $w_\sigma \mapsto \sigma$ and $w_{\iota\sigma} = \tilde{\iota} w_\sigma$. Then $w_{\tau\sigma}$ and $\tilde{\tau} w_\sigma$ have the same image in $\mathrm{Gal}(F/\mathbb{Q})$, and so $w_{\tau\sigma}^{-1} \cdot \tilde{\tau} w_\sigma \in \hat{F}^\times/F^\times$.

LEMMA 5.8. *If F is a CM-field and Φ is a CM-type for F, then*

$$f_\Phi(\tau) = \prod_{\varphi \in \Phi} w_{\tau\varphi}^{-1} \cdot \tilde{\tau} w_\varphi.$$

PROOF: Write $f'_\phi(\tau)$ for the right hand side of the equation. It is obvious that the image of $f'_\Phi(\tau)$ in $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is $V_\Phi(\tau)$. Moreover, $f'_\Phi(\tau) \cdot \iota f'_\Phi(\tau) = f'_\Phi(\tau) \cdot f'_{\Phi\iota}(\tau) = \mathrm{Ver}(\tilde{\tau})$ where Ver is the transfer map $(W^f_{F/\mathbb{Q}})^{\mathrm{ab}} \to \hat{F}^\times/F^\times$ defined by the inclusion at top-left of the above diagram. But $\mathrm{Ver}(\tilde{\tau}) = \chi_{\mathrm{cyc}}(\tau) \cdot F^\times$, and so $f'_\Phi(\tau)$ has the properties characterizing $f_\Phi(\tau)$.

The canonical cocharacter μ^F of $\mathfrak{S}^F$ is defined over F, and therefore gives rise to a homomorphism $R^\times \to \mathfrak{S}^F(R)$ for any F-algebra R. Define

$$f^F(\tau) = \prod_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} (\sigma^{-1}\mu^F)(w_{\tau\sigma}^{-1}\tilde{\tau} w_\sigma) \in \mathfrak{S}^F(\hat{F})/\mathfrak{S}^F(F).$$

LEMMA 5.9. *Let E be a CM-field and Φ a CM-type for E. Assume that F is large enough to contain all conjugates of E in $\mathbb{C}$. Then $\rho_\Phi(f^F(\tau)) = f_\Phi(\tau)$ as elements of $T^E(\hat{F})/T^E(F) \supset T^E(\hat{\mathbb{Q}})/T^E(\mathbb{Q}) = \hat{E}^\times/E^\times$.*

PROOF: Let $\rho : E \hookrightarrow F \subset \mathbb{Q}^{\mathrm{al}}$ be an embedding of E. Then ρ defines a character $\boldsymbol{\rho}$ of T^E, and it suffices to show that $\boldsymbol{\rho}(\rho_\Phi(f^F(\tau))) = \boldsymbol{\rho}(f_\Phi(\tau))$ in $\hat{F}^\times/F^\times$. First note that, by (5.2),

$$\boldsymbol{\rho}(f_\Phi(\tau)) = \rho(f_\Phi(\tau)) = f_{\Phi\rho^{-1}}(\tau) = f_{\Phi'}(\tau),$$

where Φ' is the CM-type on F extending the CM-type $\Phi\rho^{-1}$ on $\rho E \subset F$. Next

$$\begin{aligned}
\boldsymbol{\rho}(\rho_\Phi(f^F(\tau)) &= \boldsymbol{\rho}(\prod_\sigma \rho_\Phi(\sigma^{-1}\mu^F)(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)) && \text{(definition of } f^F(\tau)) \\
&= \boldsymbol{\rho}(\prod_\sigma \sigma^{-1}(\rho_\Phi \circ \mu^F)(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)) && (\rho_\Phi \text{ defined over } \mathbb{Q}) \\
&= \boldsymbol{\rho}(\prod_\sigma \sigma^{-1}(\mu_\Phi)(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)) && \text{(definition of } \rho_\Phi) \\
&= \prod_\sigma (\boldsymbol{\rho} \circ \sigma^{-1}(\mu_\Phi))(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma) \\
&= \prod_\sigma (w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)^{\langle \boldsymbol{\rho}, \sigma^{-1}(\mu_\Phi)\rangle}
\end{aligned}$$

where $\langle\, ,\, \rangle$ is the usual pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$. But we have $\langle \boldsymbol{\rho}, \sigma^{-1}\mu_\Phi\rangle = \langle \boldsymbol{\sigma} \circ \boldsymbol{\rho}, \mu_\Phi\rangle$, and from the definition of μ_ϕ in (2.6), we see that $\langle \boldsymbol{\sigma} \circ \boldsymbol{\rho}, \mu_\Phi\rangle = 1$ if $\sigma\rho \in \Phi$, and is 0 otherwise. Therefore the last product is $\prod_{\sigma\in\Phi'} w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma$, which (5.8) shows to equal $f_{\Phi'}(\tau)$.

We now complete the proof of (5.7). The elements $f^F(\tau)$ have the following properties:

(a) $f^F(\tau)$ is independent of all choices;
(b) f^F is a reversed one-cocycle;
(c) $\sigma f^F(\tau) = f^F(\tau)$, all $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$;
(d) if $F' \supset F$, then $N_{F'/F}(f^{F'}(\tau)) = f^F(\tau)$.

Statement (a) follows from (4.6). The remainder can be proved by applying ρ_Φ to both sides and using (5.2) and the formulas in (2.6). Statements (a), (c), and (d) show that $f(\tau) =_{\mathrm{df}} (f^F(\tau))$ is a well-defined element of $\wp(\mathfrak{S})$. As $\rho_\Phi(f(\tau)) = f_\Phi(\tau)$, this completes the proof of the first statement in (5.7). The second statement follows from (b).

We call $f(\tau)$ the *(universal) Taniyama element.*

Statement of the main theorem. Let A be an abelian variety of CM-type over $\mathbb{Q}$. On applying the homomorphism $\rho_A : \mathfrak{S} \to MT^A$ to $f(\tau)$, we obtain an element $f_A(\tau) \in \wp(MT^A)$.

THEOREM 5.10. *(Main theorem of complex multiplication) Let F be a splitting field of MT^A. For each $f \in MT^A(\hat{F})$ representing $f_A(\tau)$, there is a unique F- linear isomorphism $\alpha : H_B(A) \otimes F \to H_B(\tau A) \otimes F$ such that*

(a) $\alpha(t) = \tau t$ for all Hodge cycles on A;

(b) $\tau x = \alpha(fx)$ for all $x \in H_f(A) \otimes F$.

Remark 5.11. (a) It is possible to replace A in the theorem with any CM-motive over $\mathbb{Q}$ – it makes sense to speak of Hodge cycles on a CM-motive, and we can define the Mumford-Tate group of a CM-motive to be the image of $\mathfrak{S}$ in $GL(H_B(M)) \times \mathbb{G}_m$. The proof we describe below also applies to this more general case.

(b) Endomorphisms of A are Hodge cycles on A, and so (a) implies that α commutes with the action of all endomorphisms of A.

(c) It is again obvious that α is uniquely determined by the choice of f representing $f_A(\tau)$, and that if f is replaced by af $(a \in MT^A(F))$, then α must be replaced with αa^{-1}.

(d) To see that (5.10) implies (5.3), let (A, i) be as in (5.3), and let f' represent $f_\Phi(\tau)$. Note that $MT^A \subset T^E$. The definition of $f(\tau)$ shows that there is an element $a \in T^E(F)$ such that $f' = af$. Then $\alpha' = \alpha \circ a^{-1}$ satisfies the conditions of (5.3).

Proof of the main theorem of complex multiplication. We first define an element $g(\tau)$ such that Theorem 5.10 holds (tautologically) with f replaced by g.

LEMMA 5.12. *Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type, and let F be a splitting field for MT^A. There exists an F-linear isomorphism $\alpha : H_B(A) \otimes F \to H_B(\tau A) \otimes F$ such that $\alpha(t) = \tau t$ for all Hodge cycles t on A.*

PROOF: For any $\mathbb{Q}$-algebra R, let

$$P(R) = \{\alpha : H_B(A) \otimes R \to H_B(\tau A) \otimes R \mid \alpha(t) = \tau t, \text{ all } t \in C_H(A)\}.$$

From (3.3) it is obvious that P is a torsor for MT^A unless it is empty. The comparison isomorphisms show that $P(\mathbb{C}) \neq 0$. Because MT^A is a torus split by F, the cohomology class of P in $H^1(\mathbb{Q}, MT^A)$ becomes trivial in $H^1(F, MT^A)$, which means that $P(F)$ is nonempty.

Let (A, i) be of CM-type (E, Φ), and choose an element $\alpha \in P(\hat{F})$. We can regard α as an isomorphism $\alpha : H_f(A) \otimes F \to H_f(\tau A) \otimes F$ sending t to τt, for all Hodge cycles t. The map $x \mapsto \alpha^{-1}(\tau x)$ is an automorphism of $H_f(A) \otimes F$ fixing all Hodge cycles, and so (3.3) shows that it is multiplication by an element $g \in MT^A(\hat{F})$. Write $g_A(\tau)$ for the image of g in $MT^A(\hat{F})/MT^A(F)$. Then $g_A(\tau)$ is independent of the choice of α, and it is fixed under the action of $\mathrm{Gal}(F/\mathbb{Q})$. It therefore lies in $\wp(MT^A)$. For varying A, the elements $g_A(\tau)$ form a projective system. As $\mathfrak{S} = \varprojlim MT^A$, they define an element $g(\tau) \in \wp(MT^A)$. Obviously (5.10) becomes true when $f(\tau)$ is replace by $g(\tau)$, and so, to prove (5.10), it suffices to show that $f(\tau) = g(\tau)$. Let $e(\tau) = g(\tau)/f(\tau)$ and, for each CM-type (E, Φ), let $e_\Phi(\tau) = \rho_\Phi(e(\tau))$. The next two lemmas prove that $e_\Phi(\tau) = 1$.

LEMMA 5.13. *The elements $e_\Phi(\tau)$ have the following properties:*

(a) $e_\Phi(\sigma\tau) = e_{\tau\Phi}(\sigma) \cdot e_\Phi(\tau)$, $\tau_1, \tau_2 \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(b) $\sigma e_\Phi(\tau) = e_{\Phi\sigma^{-1}}(\tau)$, σ an isomorphism $E \to E'$, $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(c) $e_\Phi(\iota) = 1$.

(d) If $E' \supset E$ and Φ' is the extension of Φ to E', then $e_\Phi(\tau) = e_{\Phi'}(\tau)$.

(e) If $\tau\Phi = \Phi$, then $e_\Phi(\tau) = 1$.

(f) If $\sum n_i \Phi_i = 0$, then $\prod e_{\Phi_i}(\tau)^{n_i} = 1$.

PROOF: Parts (b), (d), and (f) are automatic consequences of the fact that $e_\Phi(\tau) = \rho_\Phi(e(\tau))$ for an $e(\tau)$ in $\wp(\mathfrak{S})$. Part (a) follows from the fact that $f(\tau)$ and $g(\tau)$, and hence $e(\tau)$, are reversed one-cocycles. Part (c) holds for both f_Φ and g_Φ. For (e) note that $\tau\Phi = \Phi$ if and only if τ fixes the reflex field, and so the theorem of Shimura and Taniyama (see 5.6) shows that in this case $g_\Phi(\tau) = N_\Phi(s) \cdot E^\times$ where s is such that $\mathrm{rec}_E(s) = \tau | E^{\mathrm{ab}}$. Therefore (5.2e) implies (e).

LEMMA 5.14. *Let $(e_\Phi(\tau))$ be a family of elements satisfying the conditions of (5.13). Then $e_\Phi(\tau) = 1$ for all Φ and τ.*

PROOF: See Deligne 1981 (also Lang 1983, VII.4).

Remark 5.15. If $f(\tau)$ is a reversed one-cocycle, then $\tau \mapsto \tau f(\tau)$ and $\tau \mapsto f(\tau^{-1})^{-1}$ are both one-cocycles. It would have been possible to work throughout with one-cocycles rather than reversed one-cocycles, but the reversed one-cocycles are more consistent with the notations used in the literature.

Notes. See the end of the next section.

6. CM-motives over $\mathbb{Q}$; the Taniyama group.

In this section we study $\mathbf{CM}/\mathbb{Q}$, the category of CM-motives over $\mathbb{Q}$. It is a semisimple $\mathbb{Q}$-linear Tannakian category with additional structure, to which the Tannakian formalism attaches certain objects.

(6.1a) To $\mathbf{CM}/\mathbb{Q}$ and the Betti fibre functor H_B, Theorem 1.2 attaches an affine group scheme $\mathfrak{T}$.

(6.1b) To the fully faithful tensor functor $\mathbf{Art}/\mathbb{Q} \hookrightarrow \mathbf{CM}/\mathbb{Q}$, (1.5a) attaches a surjective homomorphism $\pi : \mathfrak{T} \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(6.1c) H_B is an essentially surjective functor from $\mathbf{CM}/\mathbb{Q}$ to the category of Hodge structures of CM-type; it therefore defines a injective homomorphism $i : \mathfrak{S} \to \mathfrak{T}$.

(6.1d) The action of $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $H_\ell(M)$ sends s_ℓ to τs_ℓ for each Hodge cycle s. Therefore, each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ defines an automorphism $sp_\ell(\tau)$ of the fibre functor $H_B \otimes \mathbb{Q}_\ell$ whose image in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is τ. The map sp_ℓ is a homomorphism $sp_\ell : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{T}(\mathbb{Q}_\ell)$ which is continuous for the Krull and ℓ-adic topologies, and the product of the sp_ℓ's defines a homomorphism

$$sp : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{T}(\mathbb{A}_f).$$

PROPOSITION 6.2. *The sequence of affine group schemes*

$$1 \to \mathfrak{S} \xrightarrow{i} \mathfrak{T} \xrightarrow{\pi} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to 1$$

is exact. In particular, i identifies $\mathfrak{S}$ with the identity component of $\mathfrak{T}$. Moreover, the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ *on $\mathfrak{S}$ defined by the sequence is that described in (2.5b).*

PROOF: See Deligne (1982b).

Symbolically, we have a diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{S} \arrow[r, "i"] & \mathfrak{T} \arrow[r, "\pi"] \arrow[d] & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \arrow[r] \arrow[dl, "sp"] & 1. \\ & & \mathfrak{T}(\mathbb{A}_f) & & \end{tikzcd}$$

The group $\mathfrak{T}$, together with the structure (π, i, sp), is called the *Taniyama group.* A CM-motive M over $\mathbb{Q}$ corresponds to a representation $\rho : \mathfrak{T} \to GL(V)$; then $H_B(M) = V$, and its Hodge structure of

CM-type is determined by $\rho \circ i$; the ℓ-adic cohomology group $H_\ell(M)$ is $V \otimes \mathbb{Q}_\ell$ with $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acting through $\rho \circ sp_\ell$; and M is an Artin motive if and only if ρ factors through π. The Taniyama group does not enable us to construct $H_{\mathrm{dR}}(M)$ from (V, ρ) (we discuss what is needed for this in the next section).

Remark 6.3. (a) It is possible to interpret the exact sequence in (6.2) in the following way: a representation ρ of $\mathfrak{S}$ determines a CM-motive M over $\mathbb{Q}^{\mathrm{al}}$; extending ρ to $\mathfrak{T}$ corresponds to giving a descent datum on M, and descent is effective for CM-motives.

(b) For each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $M \mapsto H_\tau(M) = H_B(\tau M)$ is a fibre functor for $\mathbf{CM}/\mathbb{Q}^{\mathrm{al}}$ with values in $\mathbf{Vec}_{\mathbb{Q}}$. Therefore $\mathit{Isom}(H_B, H_\tau)$ is a torsor for $\mathfrak{S}$. It is represented by ${}^\tau\mathfrak{S} =_{\mathrm{df}} \pi^{-1}(\tau)$.

An explicit description of $(\mathfrak{T}, \pi, i, sp)$. In this subsection, we let $(\mathfrak{T}, \pi, i, sp)$ denote any quadruple for which (6.2) is true. Let $\mathfrak{S}'$ be a quotient of $\mathfrak{S}$ of finite-type over $\mathbb{Q}$, and let $\mathfrak{T}'$ be the quotient of $\mathfrak{T}$ by the kernel of $\mathfrak{S} \to \mathfrak{S}'$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathfrak{S} & \xrightarrow{i} & \mathfrak{T} & \xrightarrow{\pi} & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathfrak{S}' & \xrightarrow{i'} & \mathfrak{T}' & \xrightarrow{\pi'} & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) & \longrightarrow & 1.
\end{array}$$

If L is a finite Galois extension of $\mathbb{Q}$ (contained in $\mathbb{Q}^{\mathrm{al}}$) splitting $\mathfrak{S}'$, then $H^1(L, \mathfrak{S}') = 0$, and so each of the $\mathfrak{S}'$-torsors $\pi'^{-1}(\tau)$ has a point in L. Therefore, we can choose a section $a : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{T}'(L)$ to π'. Identify $\mathfrak{T}'(L)$ and $\mathfrak{T}'(\hat{\mathbb{Q}})$ with subgroups of $\mathfrak{T}'(\hat{L})$, and write

$$sp(\tau) = a(\tau) \cdot h'(\tau), \qquad h'(\tau) \text{ in } \mathfrak{S}'(\hat{L}).$$

The class of $h'(\tau)$ in $\mathfrak{S}'(\hat{L})/\mathfrak{S}'(L)$ is independent of the choice of $a(\tau)$.

LEMMA 6.4. *The map $h' : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{S}'(\hat{L})/\mathfrak{S}'(L)$ has the following properties:*

(a) h' is a reversed one-cocycle;

(b) $\sigma h'(\tau) = h'(\tau)$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$; thus $h'(\tau) \in \wp(\mathfrak{S}')$.

PROOF: Straightforward.

Recall that $\wp(\mathfrak{S}) = \varprojlim \wp(\mathfrak{S}')$. The h''s therefore define a continuous reversed one-cocycle $h : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \wp(\mathfrak{S})$.

PROPOSITION 6.5. *Every quadruple $(\mathfrak{T}, \pi, i, sp)$ satisfying the conditions of (6.2) defines a continuous reversed one-cocycle*

$$h : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \wp(\mathfrak{S})$$

and h determines the quadruple $(\mathfrak{T}, \pi, i, sp)$ uniquely up to a unique isomorphism; moreover every reversed one-cocycle arises from a quadruple $(\mathfrak{T}, \pi, i, sp)$ satisfying the conditions of (6.2).

PROOF: We have already shown how to derive h from the quadruple. Obviously h determines the isomorphism class of $(\mathfrak{T}, \pi, i, sp)$, but such a quadruple is rigid: any automorphism of $\mathfrak{T}$ compatible with (π, i, sp) is the identity map. Finally, it is straightforward to construct the quadruple out of h (see for example Milne and Shih 1982a, §2).

The next result provides an explicit description of the Taniyama group.

THEOREM 6.6. *The reversed one-cocycle corresponding to the Taniyama group is $\tau \mapsto f(\tau)$, where $f(\tau)$ is the universal Taniyama element defined in §5.*

PROOF: Let h be the reversed one-cocycle corresponding to the Taniyama group. After the main theorem of complex multiplication (5.10) (more specifically, 5.14), we know that $f = g$, and so we have to prove that $h = g$. Let A be an abelian variety of CM-type over $\mathbb{Q}$, and let $h_A(\tau) = \rho_A(h(\tau))$. One sees immediately from their constructions that $h_A(\tau) = g_A(\tau)$ in $\wp(MT^A)$. Since $\mathfrak{S} = \varprojlim MT^A$, this proves the theorem.

Application to the zeta functions of CM-motives. It is possible to attach an L-series $L(\rho, s)$ to a complex representation $\rho : W_{\mathbb{Q}} \to GL(V)$ of the Weil group. Moreover, it is known that $L(\rho, s)$ extends to a meromorphic function on the whole complex plane and satisfies a functional equation (see Tate 1979). These L-series generalize both Hecke L-series and Artin L-series, and so are usually referred to as *Artin-Hecke* L-series.

PROPOSITION 6.7. *There is a homomorphism $W_{\mathbb{Q}} \to \mathfrak{T}(\mathbb{C})$ making the following diagram commute:*

$$\begin{tikzcd}
 & & & W_{\mathbb{Q}} \arrow[dl] \arrow[d] & \\
1 \arrow[r] & \mathfrak{S}(\mathbb{C}) \arrow[r] & \mathfrak{T}(\mathbb{C}) \arrow[r] & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \arrow[r] & 1.
\end{tikzcd}$$

PROOF: See for example Milne and Shih (1982a), 3.17.

THEOREM 6.8. *For any CM-motive M, the system of ℓ-adic representations $H_\ell(M)$ is strictly compatible (in the sense of Serre 1968). Therefore the zeta function of M is defined, and it is an Artin-Hecke L-series.*

PROOF: This follows directly from (6.7) (see Schappacher 1988).

Remark 6.9. There is in fact a one-to-one correspondence between the set of isomorphism classes of CM-motives with coefficients in $\mathbb{Q}^{\text{al}}$ defined over $\mathbb{Q}$ and the set of isomorphism classes of representations of $W_{\mathbb{Q}}$ of type A_0.

Algebraic Hecke characters. Let F be a finite extension of $\mathbb{Q}$, and let ${}^F\mathfrak{T}$ be the inverse image of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/F)$ in $\mathfrak{T}$:

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{S} \arrow[r] \arrow[d, equal] & {}^F\mathfrak{T} \arrow[r] \arrow[d, "\cap"] & \mathrm{Gal}(\mathbb{Q}^{\text{al}}/F) \arrow[r] \arrow[d, "\cap"] & 1 \\ 1 \arrow[r] & \mathfrak{S} \arrow[r] & \mathfrak{T} \arrow[r] & \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}) \arrow[r] & 1. \end{tikzcd}$$

Then ${}^F\mathfrak{T}$ is the affine group scheme attached to the $\mathbf{CM}/F$. A homomorphism $\chi : {}^F\mathfrak{T} \to T^E$ is called an *algebraic Hecke character* for F with values in E. The restriction of χ to $\mathfrak{S}$ is the *infinity type* of χ, and for each prime ℓ,

$$sp_\ell \circ \chi : \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}) \to T^E(\mathbb{Q}_\ell) = (E \otimes \mathbb{Q}_\ell)^\times$$

is the ℓ-adic representation attached to χ.

Notes. The reversed one-cocycle f (the universal Taniyama element of §5) was defined by Langlands in order to be able to describe the conjugate of a Shimura variety (Langlands 1979). Deligne recognized that it should define the affine group scheme attached to $\mathbf{CM}/\mathbb{Q}$, and proved that this was the case in (Deligne 1982b). The implications of Langlands's construction for abelian varieties of CM-type were also made explicit in Milne and Shih (1981a). Tate gave the construction of $f_\Phi(\tau)$ described in the first subsection of §5 in (Tate 1981). The relation between the constructions of Langlands and Tate has not previously been elucidated in print.

Deligne first proved the main theorem of complex multiplication in the form (6.6), expressing it in terms of extensions (Deligne 1982b). He then re-expressed the proof in terms of the functions e_Φ, as we did in §5 (Deligne 1981). It is also possible to express the proof directly in terms of the function e (Milne 1981).

7. Periods of CM-motives.

After the last section, it remains to describe the de Rham fibre functor on **CM/Q**. This is again a $\mathbb{Q}$-linear fibre functor, and so (see 1.2) $\mathfrak{P} = \mathit{Isom}^{\otimes}(H_B, H_{\mathrm{dR}})$ is a principal homogeneous space for $\mathfrak{T}$ – we call it the *period torsor*. The comparison isomorphisms $H_B(M)\otimes\mathbb{C} \to H_{\mathrm{dR}}(M \otimes \mathbb{C})$ preserve Hodge cycles, and so define a canonical point $p \in \mathfrak{P}(\mathbb{C})$.

When M is the CM-motive corresponding to the representation $\rho : \mathfrak{T} \to GL(V)$ of $\mathfrak{T}$, $\mathfrak{P}$ enables us to construct the de Rham cohomology of M : $H_{\mathrm{dR}}(M) = \mathfrak{P} \times^{\mathfrak{T},\rho} V$. The point p gives us the comparison isomorphism $H_B(M) \otimes \mathbb{C} \to H_{\mathrm{dR}}(M_{\mathbb{C}})$.

The next conjecture, which is a variant of a conjecture of Grothendieck, predicts that the only restrictions on the transcendence of the periods of CM-motives come from Hodge cycles.

CONJECTURE 7.1. *The point p is generic in the sense that it is not contained in the set of complex points of any proper $\mathbb{Q}$-rational subscheme of $\mathfrak{P}$.*

Remark 7.2. Let $\mathbb{Q}[\mathfrak{P}]$ be the affine algebra of $\mathfrak{P}$. Then the point p corresponds to a homomorphism $\mathbb{Q}[\mathfrak{P}] \to \mathbb{C}$, and the conjecture is equivalent to this map being injective (because $\mathfrak{P}$ is irreducible).

Remark 7.3. Let $F \subset \mathbb{Q}^{\mathrm{al}}$ be a number field. On **CM**$/F$, H_{dR} is an F-linear fibre functor, and so the comparison isomorphism gives us a period torsor ${}^F\mathfrak{P}$ for $({}^F\mathfrak{T})_F =_{\mathrm{df}} {}^F\mathfrak{T} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,F$. One sees easily that ${}^F\mathfrak{P}$ is the inverse image of i under $\mathfrak{P}_F \to \mathrm{Hom}(F, \mathbb{Q}^{\mathrm{al}})$, where i is the given inclusion $F \hookrightarrow \mathbb{Q}^{\mathrm{al}}$. The canonical point p of $\mathfrak{P}(\mathbb{C})$ lies in ${}^F\mathfrak{P}(\mathbb{C})$.

Let $\chi : {}^F\mathfrak{T} \to T^E$ be an algebraic Hecke character for F with values in E. Then $\chi_*({}^F\mathfrak{P}) = \mathfrak{P}_\chi$ is a principal homogeneous space for $(T^E)_F$ with a distinguished complex point p_χ. As $H^1(F, T^E) = 0$, $\mathfrak{P}_\chi$ will have an F-rational point p_0, and any two such points differ by multiplication by an element of $T^E(F)$. Write $p_\chi = p_0 \cdot p(\chi)$; then $p(\chi)$ is a well-defined element of $(E\otimes F)^\times \backslash (E \otimes \mathbb{C})^\times$ called the *period* of χ. For example, if χ is the algebraic Hecke character attached to an abelian variety A over F with complex multiplication by E, then $p(\chi)$ is the family of periods attached to A in the usual sense. The period $p(\chi)$ determines $(\mathfrak{P}_\chi, p_\chi)$ up to isomorphism.

Since many of the results in the following chapters will be expressed in terms of the pair $(\mathfrak{P}, p)$, we would like to have a description of it that is as explicit as the description in §6 of the Taniyama group. Unfortunately, this is probably not possible since such a description would, in particular, include an explicit description of all periods of all abelian varieties with potential complex multiplication which, as (7.1) suggests, tend to be transcendental numbers. Thus the best we can hope for is an explicit characterization of the pair $(\mathfrak{P}, p)$ that does not involve CM-motives (or abelian varieties).

It is easy to describe the period torsor $\mathfrak{Q}$ attached to the category of Artin motives: $\mathfrak{Q}$ is $\operatorname{Spec}\mathbb{Q}^{\mathrm{al}}$ regarded as a principal homogeneous space for $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and its canonical $\mathbb{C}$-valued point q is that defined by the given inclusion of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{C}$. This follows from the description of $H_{\mathrm{dR}}(X)$ given in (4.1).

This suggests that we should consider the pair $(\mathfrak{P}, \varphi)$, with φ the equivariant map $\varphi : \mathfrak{P} \to \mathfrak{Q}$. Blasius has found a description of the isomorphism class of $(\mathfrak{P}, \varphi)$. Before explaining his result, we need to review a little of the theory of a Hodge-Tate modules. Write $\mathfrak{T}_\ell = \mathfrak{T} \times \operatorname{Spec}\mathbb{Q}_\ell$ and $\mathfrak{P}_\ell = \mathfrak{P} \times \operatorname{Spec}\mathbb{Q}_\ell = \mathit{Isom}(H_\ell, H_{\mathrm{dR}} \otimes \mathbb{Q}_\ell)$.

Fix a prime ℓ, and let $D_\ell = \operatorname{Gal}(\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell)$. The *$\ell$-adic cyclotomic character* is the map $\chi_{\mathrm{cyc}} : D_\ell \to \mathbb{Z}_\ell^\times$ such that $\sigma(\zeta) = \zeta^{\chi_{\mathrm{cyc}}(\sigma)}$ for each root of unity ζ in $\mathbb{Q}_\ell^{\mathrm{al}}$ of ℓ-power order. The action of D_ℓ on $\mathbb{Q}_\ell^{\mathrm{al}}$ extends by continuity to the completion $\mathbb{C}_\ell$ of $\mathbb{Q}_\ell^{\mathrm{al}}$. Let V be a $\mathbb{Q}_\ell$-vector space with a continuous action of D_ℓ. We extend the action of D_ℓ on V to $\mathbb{C}_\ell \otimes V$ by the rule:

$$\sigma(c \otimes v) = \sigma c \otimes \sigma v, \qquad \sigma \in D_\ell, \quad c \in \mathbb{C}_\ell, \quad v \in V.$$

For $m \in \mathbb{Z}$, write $V\{m\}$ for the set of $v \in \mathbb{C}_\ell \otimes V$ such that

$$\sigma(v) = \chi(\sigma)^m \cdot v.$$

It is a $\mathbb{Q}_\ell$-subspace of $\mathbb{C}_\ell \otimes V$. The inclusions of the $V\{m\}$ into $\mathbb{C}_\ell \otimes V$ define a $\mathbb{C}_\ell$-linear map

$$\mathbb{C}_\ell \otimes (\oplus_{m \in \mathbb{Z}} V\{m\}) \to \mathbb{C}_\ell \otimes V,$$

which a theorem of Tate (Serre 1967) shows to be injective. When this map is an isomorphism, the D_ℓ-module V is said to be *Hodge-Tate*.

Let B_{HT} be the ring $\mathbb{C}_\ell[T, T^{-1}]$ with D_ℓ acting according to the rule $\sigma(T) = \chi(\sigma)T$. It is an immediate consequence of the definitions that $\oplus V\{m\} = (V \otimes_{\mathbb{Q}_\ell} B_{HT})^{D_\ell}$.

The D_ℓ-module $H_\ell(A)$ is known to be Hodge-Tate for all abelian varieties, and it follows that $H_\ell(M)$ is Hodge-Tate for all CM-motives over $\mathbb{Q}$. Therefore we can define a new fibre functor H'_ℓ on $\mathbf{CM}/\mathbb{Q}$ with values in $\mathbf{Vec}_{\mathbb{Q}_\ell}$ by setting

$$H'_\ell(M) = (H_\ell(M) \otimes B_{HT})^{D_\ell}.$$

Let $\mathfrak{P}'_\ell$ be the $\mathfrak{T}_\ell$-torsor $Isom^\otimes(H_\ell, H'_\ell)$. It is represented by

$$\operatorname{Spec}(\mathbb{Q}_\ell[\mathfrak{T}_\ell] \otimes B_{HT})^{D_\ell} \qquad \text{(diagonal action of } D_\ell).$$

These definitions can be extended to $\ell = \infty$ by replacing B_{HT} with $\mathbb{C}$ and D_ℓ with $D_\infty = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$.

THEOREM 7.4. *(a) $\mathfrak{P}_\ell$ is (canonically) isomorphic to $\mathfrak{P}'_\ell$ for each prime ℓ (including ∞).*

(b) The isomorphisms in (a) uniquely determine the isomorphism class of $(\mathfrak{P}, \varphi)$.

PROOF: (a) Let $H_{\mathrm{Hg}}(M) = \operatorname{Gr}(H_{dR}(M))$. A Hodge cycle s on an abelian variety A has components s_ℓ in $H'_\ell(A)$ and s_{Hg} in $H_{\mathrm{Hg}}(A)$, and Blasius shows that the isomorphism of Tate-Faltings $H'_\ell(A) \to H_{\mathrm{Hg}}(A) \otimes \mathbb{Q}_\ell$ maps one component to the other, and so defines an isomorphism of fibre functors $H'_\ell \to H_{\mathrm{Hg}}$. Since there is a canonical isomorphism of fibre functors $H_{\mathrm{dR}} \otimes \mathbb{Q}_\ell \to H_{\mathrm{Hg}} \otimes \mathbb{Q}_\ell$, this shows that there is a canonical isomorphism

$$Isom^\otimes(H_\ell, H'_\ell) \approx Isom^\otimes(H_\ell, H_{\mathrm{dR}} \otimes \mathbb{Q}_\ell),$$

as required.

(b) Let $\mathfrak{S}'$ be the affine group scheme obtained from $\mathfrak{S}$ by twisting by $\mathfrak{Q}$ according to the action of $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $\mathfrak{S}$ defined in (2.5b). Thus $\mathfrak{S}'(\mathbb{Q}^{\mathrm{al}}) = \mathfrak{S}(\mathbb{Q}^{\mathrm{al}})$ with $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acting through its action on *both* $\mathfrak{S}$ and $\mathbb{Q}^{\mathrm{al}}$. There is a natural action of $\mathfrak{S}'$ on $(\mathfrak{P}, \varphi)$: if $s' \in \mathfrak{S}'(\mathbb{Q}^{\mathrm{al}})$ is represented by (s, q), then s' acts on the fibre over q by multiplication by s. Moreover, for a second pair $(\mathfrak{P}', \varphi')$, $Isom^\otimes((\mathfrak{P}, \varphi), (\mathfrak{P}', \varphi'))$ is a principal homogeneous space for $\mathfrak{S}'$. Thus the set of isomorphism classes of pairs $(\mathfrak{P}', \varphi')$ is a principal homogeneous space for $H^1(\mathbb{Q}, \mathfrak{S}')$, and Blasius shows that $H^1(\mathbb{Q}, \mathfrak{S}')$ satisfies the Hasse principle.

Theorem 7.4 satisfactorily characterizes $(\mathfrak{P}, \varphi)$. It remains to characterize the canonical complex point p. This can be done in terms of the periods of Hecke characters.

PROPOSITION 7.5. *Let p' be a point of $\mathfrak{P}(\mathbb{C})$ mapping to q. If p' maps to p_χ in $\mathfrak{P}_\chi(\mathbb{C})$ for all algebraic Hecke characters χ, then $p' = p$.*

PROOF: We can write $p' = p \cdot s$ with $s \in \mathfrak{S}(\mathbb{C})$, and the condition implies that $\chi(s) = 1$ for all characters χ of $\mathfrak{S}$.

Remark 7.6. (a) It suffices to assume that the condition in (7.5) holds for enough Hecke characters χ so that their infinity types generate $X^*(\mathfrak{S})$; for example, it suffices to take the Hecke characters arising from abelian varieties with complex multiplication. Thus the combination of (7.4) and (7.5) characterizes the periods of abelian varieties over $\mathbb{Q}$ of potential CM-type in terms of the periods of abelian varieties defined over a number field and with complex multiplication defined over that field.

(b) Blasius (1986) shows that certain products of the periods of the motives attached to Hecke characters are equal to critical values of the L-series of the Hecke character. If it could be shown that $(\mathfrak{P}, p)$ is characterized by the property in (7.4) and the critical values of Hecke L-series, this would be the characterization sought.

Notes. Theorem 7.4 is proved in Blasius (1989). The monograph (Schappacher 1988) provides a detailed introduction to the periods of motives of CM-type.

II. SHIMURA VARIETIES

In this chapter, we define Shimura varieties and state the main theorems on canonical models: every Shimura variety $\mathrm{Sh}(G, X)$ has a (unique) canonical model $\mathrm{Sh}(G, X)_E$ over its reflex field $E(G, X)$; for each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau\mathrm{Sh}(G, X)_E$ is the canonical model over $\tau E(G, X)$ of an explicitly determined Shimura variety $\mathrm{Sh}({}^\tau G, {}^\tau X)$.

1. Connected Shimura varieties over $\mathbb{C}$.

A *bounded symmetric domain* is a bounded open connected subset D of $\mathbb{C}^m$, some m, that is symmetric in the sense that, for each point $x \in D$, there is an involutive automorphism s_x of D (the *symmetry with respect to* x) having x as an isolated fixed point. The simplest bounded symmetric domain is the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

A complex manifold isomorphic to a bounded symmetric domain will be called a *symmetric Hermitian domain.* The simplest example of a symmetric Hermitian domain is the complex upper-half-plane, $H^+ = \{z \in \mathbb{C} \mid |\mathrm{Im}(z) > 0\}$. The Bergmann metric on a bounded symmetric domain provides it with a natural structure of a Hermitian manifold. Thus every symmetric Hermitian domain D has a

Hermitian structure which is invariant under all automorphisms; in particular, D is symmetric as a Hermitian manifold.

Let D be a symmetric Hermitian domain. The group $\mathrm{Aut}(D)$ of automorphisms of D (as a complex manifold) is a real semisimple Lie group with only finitely many connected components, and trivial centre. If G is a connected simple real algebraic group with trivial centre such that $D = G(\mathbb{R})^+/K$ for some maximal compact subgroup K of $G(\mathbb{R})^+$, then $\mathrm{Aut}(D) \cap G(\mathbb{R}) = G(\mathbb{R})^+$, and $G(\mathbb{R})$ has either one or two connected components.

Locally symmetric varieties. Let D be a symmetric Hermitian domain, and let G be a semisimple algebraic group over $\mathbb{Q}$ such that $D = G(\mathbb{R})^+/K$ with K a maximal compact subgroup of $G(\mathbb{R})^+$. Let Γ be an arithmetic subgroup in $G(\mathbb{Q})$, which we suppose to be torsion-free. Then $S =_{\mathrm{df}} \Gamma\backslash D$ will again be a complex manifold.

THEOREM 1.1. *The complex manifold S has a canonical structure of an algebraic variety. With this structure, every holomorphic map $V^{\mathrm{an}} \to S$ from a complex algebraic variety V (viewed as an analytic space) to S is a morphism of algebraic varieties.*

PROOF: The first statement is the theorem of Baily and Borel (1966). It can also be regarded as a special case of the more general theorem of Nadel and Tsuji (1988). The second statement is proved in Borel (1972), 3.10.

The second statement shows that the algebraic structure on S is not only canonical but is also unique. With this structure, S is called a *locally symmetric variety.*

Remark 1.2. If D is has no factors isomorphic to the unit disk, then the algebraic structure on S can be described as follows. Let Ω^1 be the sheaf of holomorphic differentials on S (regarded as a complex manifold), and let $\omega = \wedge^d\Omega^1$, $d = \dim S$. Then $A = \oplus_{n\geq 0}\Gamma(S, \omega^{\otimes n})$ is a graded ring, and there is a canonical map $S \to \mathrm{Proj}\, A$, which identifies S with an open subvariety of $\mathrm{Proj}\, A$. Since $\mathrm{Proj}\, A$ is a projective algebraic variety, this shows that S is a quasi-projective algebraic variety.

This description extends to the case where D has factors isomorphic to the unit disk provided $\Gamma(S, \omega^{\otimes n})$ is replaced with the group of sections of $\omega^{\otimes n}$ having at worst logarithmic poles along the boundary in some smooth compactification of S (see Iitaka 1982, XI, for the definitions).

Let $\bar{S}$ be the closure of S in $\operatorname{Proj} A$. Then Borel (1972) shows that $\bar{S}$ has the following property: for any nonsingular algebraic variety V containing S as an open subvariety and such that the complement of S in V has only normal crossings as singularities, there is a unique morphism $V \to \bar{S}$ whose restriction to S is the identity map. For this reason, $\bar{S}$ is called the *minimal compactification* of S (alternatively, the *Satake-Baily-Borel compactification* of S).

The axioms for a connected Shimura variety. A connected Shimura variety is a projective system of locally symmetric varieties. The datum needed to define it is a pair (G, X^+) comprising a semisimple group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(\mathbb{R})^+$ conjugacy class X^+ of homomorphisms $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the following conditions:

(1.3.1) when composed with $\mathbb{G}^{\mathrm{ad}}_{\mathbb{R}} \to GL(\mathfrak{g})$, each h in X^+ defines a Hodge structure on $\mathfrak{g}$; this Hodge structure is required to be of type $\{(-1,1),(0,0),(1,-1)\}$;

(1.3.2) for each h in X^+, ad $h(i)$ is a Cartan involution of $G_{\mathbb{R}}$;

(1.3.3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group.

Remark 1.4. (a) It suffices to check the conditions in (1.3.1) and (1.3.2) for a single $h \in X^+$.

(b) Axiom (1.3.1) implies that the Hodge structure on $\mathfrak{g}$ defined by h has weight zero. Hence the weight map w_h (see I.2) is trivial, and so h factors through $\mathbb{S} \to \mathbb{S}/\mathbb{G}_m$.

(c) Since $h(i)^2 = h(-1) = 1$, ad $h(i)$ is an involution of $G_{\mathbb{R}}$. To say that it is a Cartan involution means that the corresponding real form G' of G, with complex conjugation $g \mapsto h(i) \cdot \bar{g} \cdot h(i)^{-1}$, is compact. Equivalently, for every representation (V, ξ) of G, the Hodge structure $(V, \xi \circ h)$ admits a G-invariant polarization (see Deligne 1972, 2.8).

(d) Axiom (1.3.3) is included for the sake of convenience. It has the following consequence: let H be a simple factor of the simply connected covering group G^{sc} of G; then $H(\mathbb{R})$ is not compact, and so the strong approximation theorem shows that $H(\mathbb{R}) \cdot H(\mathbb{Q})$ is dense in $H(\mathbb{A})$. This implies that $H(\mathbb{Q})$ is dense in $H(\mathbb{A}_f)$. Thus $G^{\mathrm{sc}}(\mathbb{Q})$ is dense in $G^{\mathrm{sc}}(\mathbb{A}_f)$.

Example 1.5. Let $G = SL_2$, and let X^+ be the set of $PGL_2(\mathbb{R})^+$ conjugates of

$$h_0 : \mathbb{S} \to G_{\mathbb{R}}, \qquad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Then (G, X^+) satisfies the axioms (1.3). If we write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$, then $\mathrm{ad}(g) \circ h_0 \mapsto g \cdot i$ identifies X^+ with H^+, the complex upper-half-plane.

The complex structure on X^+. Let (G, X^+) satisfy the axioms (1.3). Fix a point $o \in X^+$, and let K_o be the subgroup of $G(\mathbb{R})^+$ fixing o. Then the action of $G(\mathbb{R})^+$ on X^+ defines a bijection

$$(*) \qquad G(\mathbb{R})^+/K_o \to X^+$$

Since K_o is fixed by ad $h_o(i)$, axiom (1.3.2) implies that it is compact; moreover

$$\mathfrak{g} = \mathfrak{k}_o + \mathfrak{p}_o, \quad \mathfrak{g} = \mathrm{Lie}\, G, \quad \mathfrak{k}_o = \mathrm{Lie}\, K_o,$$

where $\mathfrak{k}_o$ and $\mathfrak{p}_0$ are the +1 and -1 eigenspaces for ad $h(i)$ acting on $\mathfrak{g}$. When we use (*) to endow X^+ with a real analytic structure, then (*) identifies $\mathfrak{p}_o$ with $\mathrm{Tgt}_o(X^+)$. There is a unique homogeneous complex structure on X^+ such that the action of i on $\mathrm{Tgt}_o(X^+)$ corresponds to the action of $h(e^{2\pi i/8})$ on $\mathfrak{p}_o$, and relative to this structure, X^+ becomes a symmetric Hermitian domain.

Since I prefer to regard X^+ as a symmetric Hermitian domain rather than a conjugacy class of homomorphisms, I write x for a point of X^+ (thought of as a domain) and h_x for the corresponding homomorphism $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$; thus $h_{g \cdot x} = \mathrm{ad}(g) \circ h_x$ for $g \in G^{\mathrm{ad}}(\mathbb{R})^+$ and $x \in X^+$. Also μ_x denotes the cocharacter $z \mapsto h_{x,\mathbb{C}}(z, 1)$ attached to h_x (see I.2).

The connected Shimura variety. We now construct the connected Shimura variety associated with a pair (G, X^+). A *congruence subgroup* of $G(\mathbb{Q})$ is a subgroup of the form $\Gamma = K \cap G(\mathbb{Q})$ with K a compact open subgroup of $G(\mathbb{A}_f)$. Endow $G^{\mathrm{ad}}(\mathbb{Q})$ with the topology for which the images of the congruence subgroups in $G(\mathbb{Q})$ form a fundamental system of neighbourhoods of the identity element, and let $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ be the completion of $G^{\mathrm{ad}}(\mathbb{Q})^+$ relative to this topology. The connected Shimura variety $\mathrm{Sh}^0(G, X^+)$ will be a scheme with a continuous right action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ in the sense of §10 below.

Let $\Sigma(G)$ be the set of torsion-free arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ that contain the image of a congruence subgroup of $G(\mathbb{Q})$. For $\Gamma \in \Sigma(G)$, $\Gamma \backslash X^+$ is a locally symmetric algebraic variety. The group $G^{\mathrm{ad}}(\mathbb{Q})^+$ acts on the projective system $(\Gamma \backslash X^+)_{\Gamma \in \Sigma(G)}$ as follows: for each $\Gamma \in \Sigma(G)$ and $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$, g defines a map

$$\Gamma \backslash X^+ \to g^{-1}\Gamma g \backslash X^+, \qquad [x] \mapsto [g^{-1}x].$$

This map is holomorphic, and hence algebraic by (1.1). The action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $(\Gamma\backslash X^+)_{\Gamma\in\Sigma(G)}$ extends by continuity to $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$. The *connected Shimura variety* $\mathrm{Sh}^0(G,X)$ is defined to be the projective system $(\Gamma\backslash X^+)_{\Gamma\in\Sigma(G)}$ (or its limit) together with the continuous right action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ just defined.

When G is simply connected, some simplifications occur. Then $G(\mathbb{R})$ is connected, and (1.4d) shows that $G(\mathbb{Q})\cdot K = G(\mathbb{A}_f)$. For any congruence subgroup $\Gamma = G(\mathbb{Q})\cap K$ of $G(\mathbb{Q})$,

$$[x]\mapsto[x,1],\quad \Gamma\backslash X^+ \mapsto G(\mathbb{Q})\backslash X^+\times G(\mathbb{A}_f)/K$$

is an isomorphism (on the right, $[qx,qak]=[x,a]$, for $q\in G(\mathbb{Q})$, $k\in K$).

In the limit,

$$\mathrm{Sh}^0(G,X)(\mathbb{C}) = \varprojlim \Gamma\backslash X^+ = G(\mathbb{Q})\backslash X^+\times G(\mathbb{A}_f),$$

(apply 10.1 below). The semi-direct product $G(\mathbb{A}_f)\rtimes G^{\mathrm{ad}}(\mathbb{Q})^+$ acts on this scheme:

$$[x,a](g,q) = [q^{-1}x, ad(q^{-1})(ag)], x\in X^+, a,g\in G(\mathbb{A}_f), q\in G^{\mathrm{ad}}(\mathbb{Q})^+.$$

The homomorphism $q\mapsto(q^{-1},\mathrm{ad}\,q)$ identifies $G(\mathbb{Q})$ with a normal subgroup $G(\mathbb{A}_f)\rtimes G^{\mathrm{ad}}(\mathbb{Q})^+$, and the quotient group $G(\mathbb{A}_f)*_{G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+$ continues to act on $\mathrm{Sh}^0(G,X^+)$. In this case

$$G(\mathbb{A}_f)*_{G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+ = G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$$

(Deligne 1979, 2.1.6.2), and the action just described agrees with that defined in the preceding paragraph.

Example 1.6. If Γ is an arithmetic subgroup of $PGL_2(\mathbb{Q})$ containing the image of a congruence subgroup in $SL_2(\mathbb{Q})$, then $\Gamma\backslash H^+$ is (by definition) an *elliptic modular curve* . Thus $\mathrm{Sh}^0(SL_2,H^+)$ is the projective system of elliptic modular curves equipped with a continuous right action of $PGL_2(\mathbb{Q})^{+\wedge}$. This is the object of study of Shimura (1971b).

Etale coverings and automorphisms of connected Shimura varieties. Connected Shimura varieties behave as though they are simply connected: a finite étale equivariant morphism from one connected Shimura variety to a second is an isomorphism (Milne 1983, 2.1). It is possible to compute the group of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$-equivariant automorphisms of $\mathrm{Sh}^0(G,X^+)$; for example, if $G=G^{\mathrm{ad}}$, then this group is zero (ib., 2.4). The full group of (not necessarily equivariant) automorphisms of $\mathrm{Sh}^0(G,X^+)$ contains $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ as a subgroup of finite index (Milne and Shih 1981b, 1.3).

Notes. The axioms for a connected Shimura variety are those of Deligne (1979), 2.1.8.

2. Shimura varieties over $\mathbb{C}$. For many reasons, for example, in order to have models over number fields of finite degree, it is necessary to consider nonconnected Shimura varieties. They are defined by reductive groups rather than semisimple groups. The connected Shimura varieties occur as the connected components of Shimura varieties.

The axioms for a Shimura variety. The datum needed to define a Shimura variety is a pair (G,X) comprising a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$ satisfying the following conditions:

(2.1.1) for each $x\in X$, the Hodge structure on $\mathfrak{g}$ defined by h_x is of type $\{(-1,1),(0,0),(1,-1)\}$;

(2.1.2) for each $x\in X$, $\operatorname{ad} h_x(i)$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;

(2.1.3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group;

(2.1.4) the identity component $Z(G)^0$ of the centre of $Z(G)$ of G splits over a CM-field.

Simplifications occur when (2.1.2) is replaced by a stronger axiom:

(2.1.2*) let $Z_0(G)$ be the maximal subtorus of $Z(G)$ split over $\mathbb{Q}$; then $\operatorname{ad} h_x(i)$ is a Cartan involution on $G/Z_0(G)$.

We say that (G,X) satisfies (2.1) when it satisfies (2.1.1) - (2.1.4); when it also satisfies (2.1.2*), we say that it satisfies (2.1*).

Remark 2.2. (a) Again it suffices to check (2.1.1) and (2.1.2) for a single $x\in X$.

(b) Let X^+ be a connected component of X, and for each $x\in X^+$, let h'_x be the composite of h_x with $G_{\mathbb{R}}\to G^{\mathrm{ad}}_{\mathbb{R}}$. Then $x\mapsto h'_x$ identifies X^+ with a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S}\to G^{\mathrm{ad}}_{\mathbb{R}}$, which satisfies the axioms (1.3). Therefore X^+ acquires from §1 a natural structure of a symmetric Hermitian domain, and so X is a finite disjoint union of symmetric Hermitian domains (indexed by $G(\mathbb{R})/G(\mathbb{R})_+$).

(c) Axiom (2.1.1) implies that the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{ad} \circ h_x$ has weight zero. Hence the weight map w_x is central, and so it is independent of x — we write it w_X.

(d) Axiom (2.1.4) is not in Deligne's list of axioms (Deligne (1979), 2.1.1), but it is harmless to impose it since, in practice, all examples satisfy it, and it allows some simplifications; for example, it implies that w_X is defined over a totally real field.

(e) Axiom (2.1.2*) is very restrictive; it excludes many important Shimura varieties, for example, all Hilbert modular varieties of dimension greater than one.

Example 2.3. Let V be a vector space of dimension 2 over $\mathbb{Q}$. Let $G = GL(V)$, and let X be the set of complex structures on $V \otimes \mathbb{R}$. With each $x \in X$ we associate the homomorphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$ such that $h_x(z)$ acts on $V \otimes \mathbb{R}$ as z for all $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. Then $x \mapsto h_x$ identifies X with a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$, and the pair (G, X) satisfies the axioms (2.1). The choice of a basis for V identifies G with GL_2 and X with $\mathbb{C} - \mathbb{R} = \{z \in \mathbb{C} \mid \Im(z) \neq 0\}$, the union of the upper and lower half-planes.

The Shimura variety. Let (G, X) satisfy the conditions (2.1). For K a compact open subgroup in $G(\mathbb{A}_f)$, consider the double coset space

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where

$$q(x, a)k = (qx, qak), \quad q \in G(\mathbb{Q}), x \in X,\ a \in G(\mathbb{A}_f),\ k \in K.$$

Let $\mathcal{C}$ be a set of representatives for the finite set $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, and, for each $g \in \mathcal{C}$, let Γ_g be the image in $G^{\mathrm{ad}}(\mathbb{R})^+$ of the subgroup $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ of $G(\mathbb{Q})_+$. Then

$$\mathrm{Sh}_K(G, X) = \cup \Gamma_g \backslash X^+ \quad \text{(disjoint union over } g \in \mathcal{C})$$

for any connected component X^+ of X. When K is sufficiently small, Γ_g will be torsion-free, and we conclude from (1.1) that $\mathrm{Sh}_K(G, X)$ will then be a finite disjoint union of locally symmetric varieties. It therefore has a unique structure of an algebraic variety. Let

$$\mathrm{Sh}(G, X) = \varprojlim \mathrm{Sh}_K(G, X).$$

This is a scheme over $\mathbb{C}$ whose complex points are

$$\mathrm{Sh}(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{Q})^-,$$

where $Z(\mathbb{Q})^-$ is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$ (to prove this, apply (10.1) below with $E = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{Q})^-$). When the maximal $\mathbb{R}$-split subtorus of $Z(G)$ is $\mathbb{Q}$-split, $Z(\mathbb{Q})$ is closed in $Z(\mathbb{A}_f)$, and so

$$\mathrm{Sh}(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

There is a continuous action of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)$, given by

$$[x, a]g = [x, ag], \quad x \in X,\ a \in G(\mathbb{A}_f),\ g \in G(\mathbb{A}_f).$$

The scheme $\mathrm{Sh}(G, X)$ together with this continuous action of $G(\mathbb{A}_f)$ is called the *Shimura variety* defined by (G, X). We write (g) or $\mathcal{T}(g)$ for the operation of $g \in G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)$ — it is often called the *Hecke operator* defined by g.

Example 2.4. (a) A *symplectic space* over $\mathbb{Q}$ is a vector space V over $\mathbb{Q}$ together with a nondegenerate skew-symmetric form ψ on V. The group $G = GSp(V, \psi)$ of symplectic similitudes of (V, ψ) has rational points

$$G(\mathbb{Q}) = \{\alpha \in GL(V) \mid \exists q \in \mathbb{Q}^\times \text{s.t.} \psi(\alpha v, \alpha w) = q\psi(v, w), \forall v, w \in V\}.$$

Let $S^\pm$ be the set of all Hodge structures of type $\{(-1, 0), (0, -1)\}$ on V for which $\pm 2\pi i\psi$ is a polarization. Then $S^\pm$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$, and the pair $(G, S^\pm)$ satisfies the conditions (2.1). The space $S^\pm$, regarded as a disjoint union of two Hermitian symmetric domain, is the *Siegel double space* , and the variety $\mathrm{Sh}(G, S^\pm)$ is the *Siegel modular variety* .

(b) Let F be a totally real number field, and let $G = GL_{2,F}$, so that $G(\mathbb{R}) = \prod_{\mathrm{Hom}(F,\mathbb{R})} GL_2(\mathbb{R})$. Let X be the set of $G(\mathbb{R})$-conjugates of

$$h_0 : \mathbb{S} \to G_{\mathbb{R}}, \quad a + ib \mapsto \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \ldots, \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right).$$

Then X is a product of $[E : \mathbb{Q}]$ copies of $\mathbb{C} - \mathbb{R}$, and (G, X) satisfies the axioms (2.1). The variety $\mathrm{Sh}(G, X)$ is the *Hilbert modular variety*.

Remark 2.5. The semi-direct product $G(\mathbb{A}_f)/Z(\mathbb{Q})^- \rtimes G^{\mathrm{ad}}(\mathbb{Q})^+$ acts on $\mathrm{Sh}(G, X)$. Moreover, the quotient

$$\mathcal{G}(G) =_{\mathrm{df}} (G(\mathbb{A}_f)/Z(\mathbb{Q})^-) *_{G(\mathbb{Q})_+/Z(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+$$

of this group by its normal subgroup

$$\{(q^{-1}, \mathrm{ad}\, q) \mid q \in G(\mathbb{Q})_+/Z(\mathbb{Q})\}$$

continues to act. The Shimura variety $\mathrm{Sh}(G, X)$ is a scheme with a continuous action of $\mathcal{G}(G)$ in the sense of §10 below.

The reflex field. The reflex field is the natural field of definition of the Shimura variety. It is defined purely in terms of G and X.

For any field k of characteristic zero, let $\mathcal{M}\ (k)$ be the set of $G(k)$-conjugacy classes of homomorphisms $\mathbb{G}_m \to G_k$. The map $\mathcal{M}(k_1) \to \mathcal{M}(k_2)$ defined by an inclusion $k_1 \hookrightarrow k_2$ of algebraically closed fields is bijective. In particular, $\mathcal{M}(\mathbb{Q}^{\mathrm{al}}) \approx \mathcal{M}(\mathbb{C})$.

The cocharacters μ_x for x in X lie in a single class $M_X \in \mathcal{M}(\mathbb{C})$, which we can regard as an element of $\mathcal{M}(\mathbb{Q}^{\mathrm{al}})$. The *reflex field* $E(G, X)$ is the fixed field of the subgroup $\{\sigma \mid \sigma M_X = M_X\}$ of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$; it is therefore the field of definition of the conjugacy class M_X. With our axiom (2.1.4), $E(G, X)$ will be contained in a CM-field (see Deligne 1971c, 3.8), which means that it is either a CM-field or a totally real field.

Special points. A point $x \in X$ is *special* if there is a maximal $\mathbb{Q}$-rational torus $T \subset G$ such that h_x factors through $T_{\mathbb{R}}$ (equivalently, $T(\mathbb{R})$ fixes x). Then

$$\mu_x^{\mathrm{ad}} =_{\mathrm{df}} (\mathbb{G}_m \xrightarrow{\mu_x} T \to T/Z(G) \subset G^{\mathrm{ad}})$$

satisfies the Serre condition, and so there is a unique homomorphism $\rho_x^{\mathrm{ad}} : \mathfrak{S} \to G^{\mathrm{ad}}$ such that $\mu_{\mathrm{can}} \circ (\rho_x^{\mathrm{ad}})_{\mathbb{C}} = \mu_x^{\mathrm{ad}}$ (see I.2.4b). There always exist many special points in X (Deligne 1971c, 5.1).

When μ_x itself satisfies the Serre condition, we call x a *CM-point*. In this case there exists a unique $\mathbb{Q}$-rational homomorphism $\rho_x : \mathfrak{S} \to G$ such that $\mu_{\mathrm{can}} \circ (\rho_x)_{\mathbb{C}} = \mu_x$. A $\mathbb{Q}$-linear representation (V, ξ) of G attaches a CM-motive over $\mathbb{Q}^{\mathrm{al}}$ to each CM-point x, namely, that corresponding to the representation $(V, \xi \circ \rho_x)$ of $\mathfrak{S}$ (see I.4). The existence of a single CM-point implies that the weight w_X is defined over $\mathbb{Q}$, and conversely, if w_X is defined over $\mathbb{Q}$, then every special point is CM (under our axiom (2.1.4); see Milne (1988), A.3).

A pair (T, x) comprising a point x of X (necessarily special) and a maximal torus $T \subset G$ such that h_x factors through $T_{\mathbb{R}}$ will be called a *special pair* in (G, X). When x is a CM-point, we refer to a *CM-pair.*

A point $[x, g]$ of $\mathrm{Sh}(G, X)$ is said to be *special* (or CM) if x is special (or CM) in X. There is always a special point in X, and for any special point x, $[x, 1] \cdot G(\mathbb{A}_f)$ is dense in $\mathrm{Sh}(G, X)$ for the Zariski topology (Deligne 1971c, 5.1).

Shimura varieties defined by tori. Let T be a torus over $\mathbb{Q}$ split by a CM-field. A pair (T, x), $h_x : \mathbb{S} \to T_{\mathbb{R}}$, automatically satisfies the axioms (2.1). The associated Shimura variety

$$\mathrm{Sh}(T, x) = \varprojlim T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K = T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

has dimension zero. The reflex field $E(T, x)$ of (T, x) is the field of definition of μ_x.

For example, let E be a CM-field and Φ a CM-type for E. Then (T^E, h_Φ) defines a Shimura variety whose reflex field is $E^*(\Phi)$, the reflex field of (E, Φ). (Notations as in I.2.6.)

Morphisms of Shimura varieties. Let (G, X) and (G', X') be pairs satisfying (2.1). By a *morphism* $f : (G, X) \to (G', X')$, we mean a homomorphism $f : G \to G'$ mapping X into X'. Such an f defines a morphism of schemes

$$\mathrm{Sh}(f) : \mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X'), \quad [x, a] \mapsto [f(x), f(a)]$$

which is equivariant for $f : G(\mathbb{A}_f) \to G'(\mathbb{A}_f)$, that is,

$$\mathrm{Sh}(f) \circ \mathcal{T}(g) = \mathcal{T}(f(g)) \circ \mathrm{Sh}(f), \quad \text{for } g \in G(\mathbb{A}_f).$$

If $f : G \to G'$ is a closed immersion, then so also is $\mathrm{Sh}(f)$ (Deligne 1971c, 1.15).

PROPOSITION 2.6. *Let (G, X) and (G', X') be two pairs satisfying (2.1), and suppose given*

(i) a morphism $f_1 : (G, X) \to (G', X')$;
(ii) a continuous homomorphism $f_2 : G(\mathbb{A}_f) \to G'(\mathbb{A}_f)$;
(iii) an element $a \in G_1(\mathbb{A}_f)$ such that $f_1 \circ \mathrm{ad}\, a^{-1} = f_2$.

Then the morphism $\varphi =_{\mathrm{df}} \mathrm{Sh}(f_1) \circ \mathcal{T}(a) : \mathrm{Sh}(G_1, X_1) \to \mathrm{Sh}(G_2, X_2)$ *maps* $[x, a^{-1}]$ *to* $[f_1(x), 1]$ *for all* $x \in X_1$, *and is equivariant:*

$$\varphi \circ \mathcal{T}(g) = \mathcal{T}(f_2(g)) \circ \varphi \text{ for all } g \in G_1(\mathbb{A}_f).$$

Moreover, φ *is unchanged when* f_1 *is replaced with* $f_1 \circ \mathrm{ad}\, q$, $q \in G(\mathbb{Q})$, *and* a *with* aq.

PROOF: Straightforward.

The relation between connected and nonconnected Shimura varieties. Let X^+ be a connected component of X, and let $\mathrm{Sh}(G, X)^0$ be the connected component of $\mathrm{Sh}(G, X)$ containing the image of X^+. As we observed in (2.2b), X^+ can be identified with a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$. It is an important observation of Deligne that $\mathrm{Sh}(G, X)^0$ can be described solely in terms of G^{der} and X^+; in particular, it is independent of the centre of G (except for $Z(G) \cap G^{\mathrm{der}}$).

PROPOSITION 2.7. *Let* (G, X) *be a pair satisfying (2.1), and let* X^+ *be a connected component of* X. *When* X^+ *is regarded as a conjugacy class of maps* $\mathbb{S} \to G^{\mathrm{ad}}(\mathbb{R})^+$, *the pair* (G^{der}, X^+) *satisfies the axioms (1.3), and*

$$[x] \mapsto [x, 1] : \mathrm{Sh}^0(G, X^+) \to \mathrm{Sh}(G, X)$$

defines an equivariant isomorphism of $\mathrm{Sh}^0(G, X^+)$ *onto* $\mathrm{Sh}(G, X)^0$. *The stabilizer of* $\mathrm{Sh}^0(G, X)$ *in* $\mathcal{G}(G)$ *is* $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$.

PROOF: Deligne (1979), 2.1.16.

In the language of §10 below, the proposition says that $\mathrm{Sh}(G, X)$ is obtained from $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ by induction from $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ to $\mathcal{G}(G)$. This result will enable us to relate statements about connected Shimura varieties to statements about nonconnected Shimura varieties. To this end, the following result, which shows that each connected Shimura variety occurs as a connected component of a particularly good Shimura variety, is useful.

PROPOSITION 2.8. *For any pair* (G, X^+) *defining a connected Shimura variety, there is a pair* (G_1, X_1) *defining a Shimura variety and such that:*

(a) $(G_1^{\mathrm{der}}, X_1^+) = (G, X^+)$;

(b) *the weight* w_{X_1} *is defined over* $\mathbb{Q}$.

Moreover, G_1 *can be chosen so that either:*

(c) $H^1(k, Z(G_1)) = 0$ for all fields $k \supset \mathbb{Q}$; or

(d) $\operatorname{ad} h(i)$ is a Cartan involution on $G_1/w_{X_1}(\mathbb{G}_m)$ (hence (2.1.2) holds).*

PROOF: See the Appendix to Milne (1988).

The minimal compactification of $\mathrm{Sh}(G,X)$. Assume that G^{ad} has no factors of dimension 3, and let

$$A = \oplus_{n\geq 0}\Gamma(\mathrm{Sh}(G,X), \omega^{\otimes n}), \quad \omega = \Lambda^d\Omega^1, \quad d = \dim X.$$

There is a canonical inclusion $\mathrm{Sh}(G,X) \to \operatorname{Proj} A$, the closure of whose image, $\overline{\mathrm{Sh}(G,X)}$, is called the *minimal (or Satake-Baily-Borel) compactification* of $\mathrm{Sh}(G,X)$. When G^{ad} has factors of dimension 3, we must replace $\Gamma(S, \omega^{\otimes n})$ with the group of sections having at worst logarithmic singularities along the boundary of some smooth compactification of $\mathrm{Sh}(G,X)$ (cf. 1.2).

Automorphisms of Shimura varieties. It is possible to use the results in §1 on automorphisms of connected Shimura varieties to compute the group of $G(\mathbb{A}_f)$-equivariant automorphisms of a Shimura variety. Clearly the Hecke operator $\mathcal{T}(g)$ associated with any $g \in Z(\mathbb{A}_f)$ is such an automorphism of $\mathrm{Sh}(G,X)$, and conversely one can show that when $Z(G)$ satisfies the Hasse principle for finite primes, that is, $H^1(\mathbb{Q}, Z(G)) \hookrightarrow \prod_{\text{finite primes}} H^1(\mathbb{Q}_\ell, Z(G))$, then all $G(\mathbb{A}_f)$-automorphisms of $\mathrm{Sh}(G,X)$ are of this form. Thus, in this case,

$$\mathrm{Aut}_{G(\mathbb{A}_f)}\mathrm{Sh}(G,X) = Z(\mathbb{A}_f)/Z(\mathbb{Q})^-.$$

See Milne (1983), 2.7.

Notes. The axioms for a Shimura variety were introduced in Deligne (1971c) and, in slightly revised form, in Deligne (1979). They were suggested by the work of Shimura. This section summarizes parts of the two articles of Deligne.

3. Shimura varieties as moduli varieties for motives.

In this section, we explain how the choice of a representation $\xi : G \to GL(V)$, V a $\mathbb{Q}$-vector space, endows $\mathrm{Sh}(G,X)$ with all the additional structure that a family of motives over $\mathrm{Sh}(G,X)$ would give. This suggests that, under some restrictions on (G,X), $\mathrm{Sh}(G,X)$ should be a fine moduli space for motives.

Review of local systems and flat vector bundles. Let S be an algebraic variety over k, and let $\mathcal{V}$ be a vector bundle on S. A *connection* on $\mathcal{V}$ is a k-linear homomorphism

$$\nabla : \mathcal{V} \to \Omega^1_S \otimes \mathcal{V} \qquad (\mathcal{V} \text{ regarded as a sheaf })$$

satisfying the Leibniz identity,

$$\nabla(fv) = df \cdot v + f \cdot \nabla v$$

for all local sections f of $\mathcal{O}_S$ and v of $\mathcal{V}$. A vector field Z on S defines a mapping $\nabla_Z : \mathcal{V} \to \mathcal{V}$ by the rule: for a section v of $\mathcal{V}$ on an open subset U of S,

$$\nabla_Z(v) = \langle \nabla v, Z \rangle \in \Gamma(U, \mathcal{V}).$$

A connection is said to be *flat* (or *integrable*) if its curvature tensor is zero, that is,

$$\nabla_Y \cdot \nabla_Z - \nabla_Z \cdot \nabla_Y = \nabla_{[Y,Z]}, \quad \text{all } Y \text{ and } Z.$$

A local section v of $\mathcal{V}$ is said to be *horizontal* for ∇ if $\nabla v = 0$. A vector bundle with a flat connection can be regarded as a $\mathcal{D}$-module, where $\mathcal{D}$ is the ring of differential operators — see Borel et al. (1987), Chapter VI.

These definition carry over *mutatis mutandis* to a complex manifold S. Let $\pi_1(S, s)$ be the fundamental group of S regarded as the group of covering transformations of the universal covering space $\tilde{S}$ of S (acting on the right). A complex representation $\xi : \pi_1(S, s) \to GL(V)$ defines a vector bundle on S

$$\mathcal{V}(\xi) = \tilde{S} \times V/\sim, \quad (s\gamma, v) = (s, \gamma v),\ s \in \tilde{S},\ \gamma \in \pi_1(S, s),\ v \in V,$$

having a canonical flat connection $\nabla(\xi)$. Conversely, if $\mathcal{V}$ is a vector bundle on S with a flat connection ∇, then $V =_{\mathrm{df}} \mathcal{V}^{\nabla}$ is a local system of $\mathbb{C}$-vector spaces on S, and for any such system, there is an natural representation of $\pi_1(S, s)$ on the stalk V_s of V at $s \in S$.

We refer to Borel et al. (1987), Chapter IV, for the notion of a flat connection being *regular at infinity.*

PROPOSITION 3.1. *Let S be a complex manifold. The above constructions define equivalences between:*

(a) the category of vector bundles with flat connection $(\mathcal{V},\nabla)$ on S;

(b) the category of local systems of $\mathbb{C}$-vector spaces;

(c) the category of complex representations of $\pi_1(S,s)$.

When X is a smooth algebraic variety, the functor $(\mathcal{V},\nabla)\mapsto(\mathcal{V}^{an},\nabla^{an})$ is an equivalence from the category of algebraic vector bundles with a flat connection regular at infinity to that of analytic vector bundles with a flat connection.

PROOF: Except for the last statement, this is a standard result. The last statement can be found in (Deligne 1970) and (Borel et al. 1987, Chapter IV).

Variations of Hodge structures. A *variation of Hodge structures* on a complex manifold S is a local system of $\mathbb{Q}$-vector spaces V on S together with a continuously varying family of Hodge structures on the stalks V_s of V such that

(a) the Hodge filtration on $(\mathbb{C}\otimes_{\mathbb{Q}}V)_s$ varies holomorphically with s, that is, it defines a filtration of the vector bundle $\mathcal{V}=_{\mathrm{df}}\mathcal{O}_S\otimes_{\mathbb{Q}}V$;

(b) (axiom of transversality): $\nabla(F^p\mathcal{V})\subset\Omega^1_S\otimes F^{p-1}\mathcal{V}$.

When $\mathbb{Q}$ is replaced by $k\subset\mathbb{R}$, we speak of a variation of Hodge k-structures. All families of Hodge structures arising naturally in algebraic geometry are variations of Hodge structures.

X as a parameter space for Hodge structures. As a first step to realizing $\mathrm{Sh}(G,X)$ as a moduli variety for motives, we show how to realize X as a parameter space for Hodge structures; in fact, the axioms (2.1) are virtually forced on us by our wish that this be so.

Let G be a connected algebraic group over $\mathbb{R}$, and let X^+ be a connected component of the space of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$. Then X^+ is a $G(\mathbb{R})^+$-conjugacy class of homomorphisms. Choose a faithful representation (V,ξ) of G. For each $x\in X^+$, we obtain a real Hodge structure $\xi\circ h_x$ on V. We assume that the corresponding weight gradation is independent of x (equivalently, $\xi\circ h_x(\mathbb{R}^\times)$ is contained in the centre of $G(\mathbb{R})^+$ for all x).

PROPOSITION 3.2. *Let $V(\xi)$ be the constant sheaf of $\mathbb{R}$-vector spaces on X^+ defined by V.*

(a) There is a unique complex structure on X^+ such that the Hodge filtrations on the stalks of $\mathbb{C}\otimes V(\xi)$ vary holomorphically.

(b) The Hodge structures $\xi\circ h_x$ make $V(\xi)$ into a variation of real Hodge structures if and only if the Hodge structure on $\mathfrak{g}$ defined by h_x is of type $\{(-1,1),(0,0),(1,-1)\}$ for all $x\in X^+$.

(c) Let G_1 be the smallest algebraic subgroup of G through which all the h_x, $x \in X^+$, factor, and let V_n be the component of V of weight n. There exists a bilinear form $\psi : V_n \otimes V_n \to \mathbb{R}(-n)$ that is a polarization of $(V_n, \xi \circ h_x)$ for all $x \in X^+$ if and only if G_1 is reductive and $\operatorname{ad} h_x(i)$ is a Cartan involution on G_1^{ad}, all x.

PROOF: This is proved in Deligne (1979), 1.1.14. We merely note that the Hodge filtrations on the stalks of $\mathbb{C} \otimes V(\xi)$ define a map from X into a Grassman manifold, and (a) is equivalent to this map being holomorphic. Moreover, that if Z is a vector field on X corresponding to an element of $F_x^r(\operatorname{Lie} G)$ then $\nabla_Z(F^s\mathcal{V}_x) \subset F^{r+s}\mathcal{V}_x$; the condition implies that $\operatorname{Lie} G = F_x^{-1}(\operatorname{Lie} G)$. Finally, the result noted in (1.4c) implies the existence of ψ.

Now assume that (G, X) is a pair satisfying (2.1). The structure on X that we defined in §2 is the unique complex structure such that every real representation (V, ξ) of G defines a variation of real polarizable Hodge structures on X. If the weight w_X is defined over $\mathbb{Q}$, then every rational representation (V, ξ) of G defines a rational polarizable variation of Hodge structures on X. We can extend $V(\xi)$ to $X \times G(\mathbb{A}_f)$, and when (2.1.2*) holds we can pass to the quotient to obtain a polarizable variation of Hodge structures (rational or real) on $\operatorname{Sh}(G, X)$. In the rational case, this variation of Hodge structures is a candidate to be the family of Betti cohomology groups of a family of motives over $\operatorname{Sh}(G, X)$.

Local systems of $\mathbb{Q}_\ell$-adic vector spaces. Let S be a scheme. By a *local system of $\mathbb{Q}_\ell$-vector spaces* on S_{et} I mean a twisted-constant constructible (or smooth) $\mathbb{Q}_\ell$-sheaf; see for example Milne (1980), p165. When S is connected and s is a geometric point of S, the map $V \mapsto V_s$ (stalk of V at s) defines an equivalence from the category of local systems of $\mathbb{Q}_\ell$-vector spaces on S to that of continuous representations of $\pi_1^{\mathrm{et}}(S, s)$ on $\mathbb{Q}_\ell$-vector spaces. More generally, if $X \to S$ is a Galois covering of S with Galois group $\mathcal{G}$ (see 10.2), then

$$V \mapsto \varinjlim V(X^{\mathcal{H}}) \quad \text{(limit over the open subgroups } \mathcal{H} \text{ of } \mathcal{G}\text{)},$$

defines an equivalence from the category of local systems of $\mathbb{Q}_\ell$-vector spaces on S whose pull-back to X is constant to that of continuous representations of $\mathcal{G}$.

Now take S to be a smooth connected variety over $\mathbb{C}$, and let $s \in S(\mathbb{C})$. In this case, s is also a geometric point of S, and $\pi_1^{\mathrm{et}}(S, s)$

is the profinite completion of $\pi_1(S,s)$. A local system of $\mathbb{Q}$-vector spaces V on $S(\mathbb{C})$ defines a representation $\xi : \pi_1(S,s) \to GL(V_s)$, which extends to a representation $\xi_\ell : \pi_1^{\text{et}}(S,s) \to GL(V_s \otimes \mathbb{Q}_\ell)$ if and only if it is continuous relative to the ℓ-adic topology on V and the profinite topology on $\pi_1(S,s)$. In this case, we abuse notation, and write $V \otimes \mathbb{Q}_\ell$ for the local system of $\mathbb{Q}_\ell$-vector spaces on S_{et} associated with ξ_ℓ.

The systems attached to a rational representation of G.

PROPOSITION 3.3. *Assume that (G,X) satisfies (2.1*). A representation (V,ξ) of G defines (in a natural way):*

(a) a local system of $\mathbb{Q}$-vector spaces $V(\xi)$ on $\mathrm{Sh}(G,X)$*;*

(b) a local system of $\mathbb{Q}_\ell$-vector spaces $V_\ell(\xi)$ on $\mathrm{Sh}(G,X)_{\text{et}}$*, each ℓ;*

(c) a vector bundle $\mathcal{V}(\xi)$ on $\mathrm{Sh}(G,X)$ *together with a (regular) flat connection $\nabla(\xi)$.*

These are related by canonical isomorphisms:

(i) $V(\xi) \otimes \mathbb{Q}_\ell \to V_\ell(\xi)$;

(ii) $V(\xi) \otimes \mathbb{C} \to \mathcal{V}(\xi)^{\nabla(\xi)}$.

When the weight w_X is defined over $\mathbb{Q}$, the maps $\xi \circ h_x$ define on $V(\xi)$ the structure of a variation of polarizable Hodge structures.

PROOF: Let K be compact open subgroup of $G(\mathbb{A}_f)$. Then (see §2) $\mathrm{Sh}_K(G,X)$ is a finite union $\cup \Gamma_g \backslash X^+$, where Γ_g is the image of $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ in $G^{\text{ad}}(\mathbb{Q})_+$. When K is sufficiently small, Γ_g will be the fundamental group of $\Gamma_g \backslash X^+$. The condition (2.1.2*) implies that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$, and so we can take K to be sufficiently small so that $K \cap Z(\mathbb{Q}) = \{1\}$. Since the kernel of $\Gamma'_g \to \Gamma_g$ is contained in $Z(\mathbb{Q})$, this shows that we can assume that $\Gamma'_g = \Gamma_g$. Now each of $V(\xi)$ and $(V(\xi), \nabla(\xi))$ is defined on $\Gamma_g \backslash X^+$ by the restriction of ξ to Γ'_g. The sheaf $V_\ell(\xi)$ can be defined to be $V(\xi) \otimes \mathbb{Q}_\ell$ or, better, we can proceed as follows. The above discussion shows that when K is sufficiently small, Γ'_g will act without fixed points on X^+. Under the same hypothesis, K will act without fixed points on $\mathrm{Sh}(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)$. Then $\mathrm{Sh}(G,X)$ will be a Galois covering of $\mathrm{Sh}_K(G,X)$, and we can take $\mathcal{V}_\ell(\xi)$ to be sheaf associated with the representation of K on $V \otimes \mathbb{Q}_\ell$ defined by ξ.

The motives attached to the points of $\mathrm{Sh}(G,X)$**.** Our discussion in this and the next subsection is predicated on the assumption of (I.3.2), so that there is a theory of motives over any field of characteristic zero, and the Betti fibre functor $\mathbf{Mot}/\mathbb{C} \to \mathbf{Hdg}_{\mathbb{Q}}$ is fully faithful (see I.4). Let (G, X) be a pair satisfying (2.1*), and assume that w_X is defined over $\mathbb{Q}$. To simplify the discussion, we assume there is a homomorphism $t : G \to \mathbb{G}_m$ such that $t \circ w_X(z) = z^{-2}$. Fix a faithful representation (V, ξ) of G.

HOPE 3.4. *Each $(V, \xi \circ h_x)$ is the rational Hodge structure attached to a motive M_x over $\mathbb{C}$ (uniquely determined, because of our assumption of I.3.2).*

As we noted in §2, when x is a CM-point we know that M_x exists, and it is a motive of CM-type. Let $\mathfrak{t} = (t_\alpha)_{\alpha \in I}$ be a family of tensors for V such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the t_α.

Consider the set of triples $(M, \mathfrak{s}, \eta)$ consisting of a motive M over $\mathbb{C}$, a family $\mathfrak{s} = (s_\alpha)_{\alpha \in I}$ of Hodge cycles on M, and an isomorphism $\eta : V(\mathbb{A}_f) \to H_f(M)$ such that:

(3.5a) there exists an isomorphism $i : H_B(M) \to V$ mapping each s_α to t_α and such that $(z \mapsto i \circ h_M(z) \circ i^{-1}) \in X$;

(3.5b) η maps each s_α to t_α.

An isomorphism from one such triple $(M, \mathfrak{s}, \eta)$ to a second $(M', \mathfrak{s}', \eta')$ is an isomorphism $\gamma : M \to M'$ sending each s_α to s'_α and such that $\gamma \circ \eta = \eta'$. Write $\mathcal{M}(G, X, \xi)$ for the set of isomorphism classes of such triples.

PROPOSITION 3.6. *Under the above assumptions, there is a canonical bijection*

$$\Phi_\xi : \mathcal{M}(G, X, \xi) \to \mathrm{Sh}(G, X).$$

PROOF: Given $(M, \mathfrak{s}, \eta)$, choose an isomorphism $i : H_B(M) \to V$ as in (3.5a), and let $x \in X$ be such that $h_x(z) = i \circ h_M(z) \circ i^{-1}$. Because $i_{\mathbb{A}_f} \circ \eta : V(\mathbb{A}_f) \to V(\mathbb{A}_f)$ preserves Hodge cycles, it is multiplication by $\xi(a)$, some $a \in G(\mathbb{A}_f)$. The map i is uniquely determined up to an element of $G(\mathbb{Q})$, and so the class of (x, a) in $\mathrm{Sh}(G, X)$ is well-defined: we set $\Phi_\xi(M, \mathfrak{s}, \eta) = [x, a]$. Conversely, given $(x, a) \in X \times G(\mathbb{A}_f)$, let M_x be the motive determined by (3.4), and define t_α to be s_α and η to be multiplication by $\xi(a)$.

Remark 3.7. It is possible to recover (G, X) from the triple $(M, \mathfrak{s}, \eta)$ attached to a single point of $\mathrm{Sh}(G, X)$: by definition G is the subgroup of $GL(H_B(M)) \times \mathbb{G}_m$ fixing the s_α; because s_α is a Hodge cycle, $h_M(\mathbb{S})$ fixes it, and so h_M factors through $G_{\mathbb{R}}$; X is the $G(\mathbb{R})$-conjugacy class of h_M.

Families of motives. We define a family of motives over a scheme S to be a motive over the generic point "with good reduction everywhere".

Definition 3.8. Let S be a smooth connected variety over $\mathbb{C}$ with generic point η, and let $\bar{\eta}$ be a geometric point lying over η. A *motive* $\mathcal{M}$ over S is a motive M_η over $\mathbb{C}(\eta)$ such that the action of $\mathrm{Gal}(\mathbb{C}(\eta)^{\mathrm{al}}/\mathbb{C}(\eta))$ on $H_\ell(\mathcal{M}_\eta)$ factors through $\pi_1^{\mathrm{et}}(S, \bar{\eta})$, all ℓ.

Write $\mathcal{H}_\ell(\mathcal{M})$ for the local system of $\mathbb{Q}_\ell$-vector spaces on S_{et} defined by the representation of $\pi_1^{\mathrm{et}}(S, \bar{\eta})$ on $H_\ell(M_\eta)$. Let S_0 be a model of S over a subfield k_0 of $\mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$, and let η_0 be the generic point of S_0; assume k_0 is sufficiently large that M_η has a model M_0 over η_0. For any sufficiently general closed point t of S, there will be a k-morphism $\eta_0 \to S$ with image t, and M_0 will define a motive M_t over t. There is a local system of $\mathbb{Q}$-vector spaces $\mathcal{H}_B(\mathcal{M})$ on S such that $\mathcal{H}_B(\mathcal{M})_t = H_B(M_t) \subset \mathcal{H}_\ell(\mathcal{M})_t$ for every such t. From (3.1) we then obtain a pair $(\mathcal{H}_{\mathrm{dR}}(\mathcal{M}), \nabla)$ such that $\mathcal{H}_{\mathrm{dR}}(\mathcal{M})^\nabla = \mathbb{C} \otimes \mathcal{H}_B(\mathcal{M})$.

A motive on a nonconnected smooth scheme S over $\mathbb{C}$ is defined to be a motive on each of the connected components of S.

HOPE 3.9. *For any representation* (V, ξ) *of* G, *there exists a motive* $\mathcal{M}$ on $\mathrm{Sh}(G, X)$ such that

$$\mathcal{H}_B(\mathcal{M}) = V(\xi), \quad \mathcal{H}_\ell(\mathcal{M}) = V_\ell(\xi) \text{ each } \ell, \quad \mathcal{H}_{\mathrm{dR}}(\mathcal{M}) = \mathcal{V}(\xi).$$

Take ξ to be faithful, and let $\mathcal{M}$ be the family of motives given by (3.9). There will be a family $\mathfrak{t} = (t_\alpha)$ of tensors for V such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the t_α. For each α, t_α defines a global section s_α of $\mathcal{H}_B(\mathcal{M})$, and we let $\mathfrak{s} = (s_\alpha)$. By construction, there is an isomorphism $\eta : V_f(\xi) \to \mathcal{H}_f(\mathcal{M})$ sending t_α to s_α.

HOPE 3.10. *The triple* $(\mathcal{M}, \mathfrak{s}, \eta)$ *is universal: let* S *be a smooth* $\mathbb{C}$-*scheme with a continuous action of* $G(\mathbb{A}_f)$, *and let* $(\mathcal{M}', \mathfrak{s}', \eta')$ *be a triple over* S *such that* $(\mathcal{M}', \mathfrak{s}', \eta')_s \in \mathcal{M}(G, X, \xi)$ *for all closed points* s *of* S; *then there is a unique* $G(\mathbb{A}_f)$-*morphism* $\Psi : S \to \mathrm{Sh}(G, X)$ *such that* $\Psi^*(\mathcal{M}, \mathfrak{s}, \eta) = (\mathcal{M}', \mathfrak{s}', \eta')$.

Shimura varieties as moduli varieties for abelian varieties. We now drop all assumptions on motives. Let (G, X) be a pair satisfying (2.1), and assume that there is an inclusion $\xi : (G, X) \hookrightarrow (GSp, S^\pm)$, where GSp and $S^\pm$ are as in (2.4a). In this case, (3.4) is true; in fact, $(V, \xi \circ h_x)$ is the Hodge structure of an abelian variety A over $\mathbb{C}$, uniquely determined up isogeny. Thus $\mathcal{M}(G, X, \xi)$ consists of isogeny classes of triples $(A, \mathfrak{s}, \eta)$ satisfying (3.5), with A an abelian variety. (We say $(A, \mathfrak{s}, \eta)$ and $(A', \mathfrak{s}', \eta')$ are *isogenous* if there is an isogeny $\gamma : A \to A'$ sending s_α to s'_α, each α, and such that $\gamma \circ \eta = \eta'$.)

THEOREM 3.11. *(a) The map $\Phi_\xi : \mathcal{M}(G, X, \xi) \to \mathrm{Sh}(G, X)$ realizes $\mathrm{Sh}(G, X)$ as the coarse moduli scheme for the set $\mathcal{M}(G, X, \xi)$ of isogeny classes of triples $(A, \mathfrak{s}, \eta)$.*

(b) When (G, X) satisfies (2.1.2), $Sh(G, X)$ is a fine moduli scheme; in particular, it carries a universal family $(\mathcal{A}, \mathfrak{s}, \eta)$.*

PROOF: This follows from the main theorem of Mumford (1965).

A Shimura variety $\mathrm{Sh}(G, X)$ is said to be of *Hodge type* when there is an embedding $(G, X) \hookrightarrow (GSp(V, \psi), S^{\pm})$. As we have just seen, every such Shimura variety is a (coarse) moduli scheme for abelian varieties with Hodge-cycle and level structure. When each of the Hodge cycles defining the moduli problem is an endomorphism or a polarization then the Shimura variety is said to be of *PEL-type*.

Notes. This section makes more explicit the philosophy underlying Deligne (1979).

4. Conjugates of Shimura varieties.

Let τ be an automorphism of $\mathbb{C}$. We want to identify $\tau\mathrm{Sh}(G, X)$ with the Shimura variety defined by a possibly different pair (G', X'). Fix a faithful representation (V, ξ) of G, and assume (3.4), so that attached to each point s of $\mathrm{Sh}(G, X)$, there is a triple $(M, \mathfrak{s}, \eta)_s$ satisfying the conditions (3.5). The triple attached to $\tau s \in \tau\mathrm{Sh}(G, X) = \mathrm{Sh}(G', X')$ should be $\tau(M, \mathfrak{s}, \eta)_s$. As we noted in (3.7) it is possible to recover (G', X') from $\tau(M, \mathfrak{s}, \eta)_s$. This gives us a description of (G', X'), but only in terms of a conjectural theory of motives. A key observation in Langlands (1979) is that, when we take s to be a CM-point, M_s becomes a CM-motive, and so we can apply the theory of the Taniyama group to define (G', X').

Now drop all assumptions, and choose a special point $x \in X$. Then x defines a homomorphism $\rho_x^{\mathrm{ad}} : \mathfrak{S} \to G^{\mathrm{ad}}$ (see §2), and hence an action of $\mathfrak{S}$ on G. Write ${}^{\tau,x}G$ for the inner twist of G defined by ${}^{\tau}\mathfrak{S}$: ${}^{\tau,x}G = {}^{\tau}\mathfrak{S} \times^{\mathfrak{S}} G$. The point $\mathrm{sp}(\tau) \in {}^{\tau}\mathfrak{S}(\mathbb{A}_f)$ defines an isomorphism

$$G(\mathbb{A}_f) \to {}^{\tau,x}G(\mathbb{A}_f), \quad g \mapsto {}^{\tau,x}g =_{\mathrm{df}} sp(\tau) \cdot g.$$

Let $T \subset G$ be a maximal torus such that $T(\mathbb{R})$ fixes x. The action of $\mathfrak{S}$ on T is trivial, and so $T = {}^{\tau,x}T \subset {}^{\tau,x}G$. Thus $\tau\mu_x$ can be regarded as a homomorphism

$$\mathbb{G}_m \to T = {}^{\tau,x}T \hookrightarrow {}^{\tau,x}G.$$

Since $\tau\mu_x$ commutes with its complex conjugate, it defines a homomorphism $h_{\tau X} : \mathbb{S} \to {}^{\tau,x}G$, and when we take ${}^{\tau,x}X$ to be the set of $G(\mathbb{R})$-conjugates of ${}^{\tau,x}h$, the pair $({}^{\tau,x}G, {}^{\tau,x}X)$ satisfies the axioms (2.1).

Remark 4.1. (a) If x is a CM-point and (V, ξ) is a faithful representation of G, then, as we observed in §2, $(V, h_x \circ \xi)$ is $H_B(M)$ for a well-defined CM-motive M over $\mathbb{C}$. Let $\mathfrak{t}$ be a family of Hodge tensors for V such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the elements of $\mathfrak{t}$. Then ${}^{\tau,x}G$ is the subgroup of $GL(H_B(\tau M)) \times \mathbb{G}_m$ fixing τt for each $t \in \mathfrak{t}$. Moreover, $h_{^\tau x} = h_{\tau M}$, and $g \mapsto {}^{\tau,x}g$ is defined by $H_f(M) \xrightarrow{\tau} H_f(\tau M)$.

(b) The group ${}^{\tau,x}G$ is obtained from G by twisting at infinity. For example, if $G = GL_1(B)$ with B a quaternion algebra over a totally real field F, then ${}^{\tau,x}G = GL_1(B')$ where $B' \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \approx B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, all ℓ, and $B' \otimes_{F,\sigma} \mathbb{R} \approx B \otimes_{F,\tau\circ\sigma} \mathbb{R}$, all $\sigma : F \hookrightarrow \mathbb{R}$.

The next result is the main theorem of the chapter: it shows that the choice of a special point x determines a realization of $\tau\mathrm{Sh}(G,X)$ as the Shimura variety of $({}^{\tau,x}G, {}^{\tau,x}X)$; the following Theorem 4.4 then shows that the realization is essentially independent of the choice of x.

THEOREM 4.2. *For each $\tau \in Aut(\mathbb{C})$ and special point $x \in X$, there is a unique isomorphism*

$$\varphi_{\tau,x} : \tau\mathrm{Sh}(G,X) \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$$

such that

(a) $\tau[x,1] \mapsto [{}^\tau x, 1]$, and

(b) $\varphi_{\tau,x} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^{\tau,x}g) \circ \varphi_{\tau,x}$, all $g \in G(\mathbb{A}_f)$.

PROOF: The uniqueness is obvious from the fact that $[x,1] \cdot G(\mathbb{A}_f)$ is dense in $\mathrm{Sh}(G,X)$. We discuss the proof of the existence in §9 below. (If we knew (3.10), $\varphi_{\tau,x}$ would be the map given by the family of motives $\tau\mathcal{M}$ over $\tau\mathrm{Sh}(G,X)$ and the universality of $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$.)

Let x and x' be CM points of X (supposed to exist). A calculation shows that $\rho_{x*}({}^{\tau}\mathfrak{S})$ and $\rho_{x'*}({}^{\tau}\mathfrak{S})$ have the same class in $H^1(\mathbb{Q}, G)$. The choice of an isomorphism $f : \rho_{x*}({}^{\tau}\mathfrak{S}) \to \rho_{x'*}({}^{\tau}\mathfrak{S})$ determines an isomorphism $f_1 : {}^{\tau,x}G \to {}^{\tau,x'}G$, and there is an $a \in {}^{\tau,x}G(\mathbb{A}_f)$ such that $f_1(a^{-1} \cdot {}^{\tau,x}g) = {}^{\tau,x'}g$. If f is replaced by $f \circ q$, $q \in {}^{\tau,x}G(\mathbb{Q})$, then f_1 is replaced by $f_1 \circ \operatorname{ad} q$ and a with aq. Therefore (see 2.6), there is a well-defined isomorphism

$$\varphi(\tau; x', x) : \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X) \to \mathrm{Sh}({}^{\tau,x'}G, {}^{\tau,x'}X).$$

PROPOSITION 4.3. *Let $\tau \in Aut(\mathbb{C})$. For each pair (G, X) defining a Shimura variety and special points x and x' of X there is an isomorphism*

$$\varphi(\tau; x', x) : \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X) \to \mathrm{Sh}({}^{\tau,x'}G, {}^{\tau,x'}X)$$

such that $\varphi(\tau; x', x) \circ \mathcal{T}({}^{\tau,x}g) = \mathcal{T}({}^{\tau,x'}g) \circ \varphi(\tau; x', x)$, all $g \in G(\mathbb{A}_f)$. These isomorphisms are uniquely determined by the following properties:

(a) *when x and x' are CM-points, $\varphi(\tau; x', x)$ is as defined above;*
(b) *if $(G, X)^+ = (G', X')^+$ and x and $x' \in X^+$ $(= X'^+)$, then*

$$\varphi(\tau; x', x)|\mathrm{Sh}(G, X)^0 = \varphi(\tau; x', x)|\mathrm{Sh}(G', X')^0.$$

PROOF: When the weight w_X is defined over $\mathbb{Q}$, every special point is CM and the map is as above. Next check that $\varphi(\tau; x', x)|\mathrm{Sh}(G, X)^0 = \varphi(\tau; x', x)|\mathrm{Sh}(G', X')^0$ when $(G^{\mathrm{der}}, X^+) = (G'^{\mathrm{der}}, X'^+)$, x and x' both lie in X^+, and w_X and $w_{X'}$ are defined over $\mathbb{Q}$. In the general case, after possibly replacing x' by gx' with $g \in G(\mathbb{Q})$, we can assume that x and x' lie in the same connected component X^+ of X. Now (2.8) provides us with a pair (G', X') such that $(G', X')^+ = (G, X)^+$ and $w_{X'}$ is defined over $\mathbb{Q}$. Take $\varphi(\tau; x', x)$ to be the unique equivariant map whose restriction to $\mathrm{Sh}(G, X)^0$ is $\varphi(\tau; x', x)|\mathrm{Sh}(G', X')^0$.

THEOREM 4.4. *For any pair of special points x and x', we have $\varphi(\tau; x', x) \circ \varphi_{\tau,x} = \varphi_{\tau,x'}$:*

$$\begin{tikzcd} & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X) \arrow[dd, "\varphi(\tau; x', x)"] \\ \tau\mathrm{Sh}(G, X) \arrow[ur, "\varphi_{\tau,x}"] \arrow[dr, "\varphi_{\tau,x'}"'] & \\ & \mathrm{Sh}({}^{\tau,x'}G, {}^{\tau,x'}X). \end{tikzcd}$$

PROOF: We discuss the proof in §9.

Remark 4.5. Let I be an index set. To give a family of objects $(S_i)_{i\in I}$ and isomorphisms $\varphi_{ji} : S_i \to S_j$, one for each pair (i, j), such that $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ for all i, j, k, is essentially the same as to give a single object: the inverse limit of the family is an object S together with isomorphisms $\varphi_i : S \to S_i$ such that $\varphi_{ji} \circ \varphi_i = \varphi_j$. From this point of view, Theorems 4.2 and 4.4 realize $\tau\mathrm{Sh}(G, X)$ as the inverse limit of the Shimura varieties $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$, x running over the special points of X.

Remark 4.6. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let K' be the image of K in ${}^{\tau,x}G(\mathbb{A}_f)$ under $g \mapsto {}^{\tau,x}g$. Then $\varphi_{\tau,x}$ induces an isomorphism

$$\tau\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}({}^{\tau,x}G, {}^{\tau,x}X).$$

Let f be a rational function on $\mathrm{Sh}_K(G, X)$ that is defined at the special point $[x, 1]$. Then (4.2) associates with f a function ${}^{\tau}f =_{\mathrm{df}} \tau \circ f \circ \tau^{-1} \circ \varphi_{\tau,x}^{-1}$ on $S_{K'}({}^{\tau,x}G, {}^{\tau,x}X)$ such that

(i) ${}^{\tau}f([{}^{\tau}x, 1]) = \tau(f([x, 1]))$

(ii) $f \mapsto {}^{\tau}f$ commutes with the Hecke operators.

This leads to a reciprocity law, which can be made more explicit (see Milne and Shih 1981b, §5).

Notes. Theorems (4.2) and (4.4) were conjectured by Langlands (Langlands 1979), who was motivated by the problem of computing the zeta function of a Shimura variety. For Shimura varieties of abelian type (see §9 for a definition of this class), they were proved in Milne and Shih (1982b), where also the proof of the general case was reduced to a statement about connected Shimura varieties defined by simply-connected simple groups. This statement was proved in Milne (1983) using a theorem of Kazhdan (1982) (whose proof is completed in Clozel (1986)) and theorems of Margulis (1977). See also Borovoi (1983/4) (completed in Borovoi (1987)) and the notes to §9 below.

5. Canonical models.

By a *model* of $\mathrm{Sh}(G, X)$ over a subfield E of $\mathbb{C}$, we mean a scheme S over E endowed with an action of $G(\mathbb{A}_f)$ (defined over E) and an equivariant isomorphism (over $\mathbb{C}$) $\psi : \mathrm{Sh}(G, X) \to S \otimes_E \mathbb{C}$. Note that ψ can also be regarded as morphism $\mathrm{Sh}(G, X) \to S$ over E inducing an isomorphism $\mathrm{Sh}(G, X) \to S \otimes_E \mathbb{C}$.

Let (T, x) be a special pair in (G, X). The field of definition of the cocharacter μ_x of T is the reflex field $E(T, x)$. As in (I.2.6), μ_x defines a $\mathbb{Q}$-rational homomorphism $N_x : T^E \to T$ for any field $E \supset E(T, x)$. The *reciprocity map*

$$r_E(T,x) : \mathrm{Gal}(E^{\mathrm{ab}}/E) \to T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

is defined as follows: let $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, and let $s \in \mathbb{A}_E^\times$ be such that $\mathrm{rec}_E(s) = \tau^{-1}$; write $s = s_\infty \cdot s_f$ with $s_\infty \in E_\infty$ and $s_f \in \hat{E}$; then $r_E(T,x)(\tau) = N_x(s_f) \pmod{T(\mathbb{Q})^-}$.

Definition 5.1. A model $\mathrm{Sh}(G,X)_E$ of $\mathrm{Sh}(G,X)$ over $E = E(G,X)$ is said to be *canonical* if each special point $[x,a]$ is rational over $E(T,x)^{\mathrm{ab}}$ and $\mathrm{Gal}(E(T,x)^{\mathrm{ab}}/E(T,x))$ acts on $[x,a]$ according to the rule:

$$\tau[x,a] = [x, r(\tau)\cdot a], \quad \text{where } r = r_E(T,x).$$

PROPOSITION 5.2. *Consider a morphism $f : (G,X) \to (G',X')$. If $\mathrm{Sh}(G,X)$ and $\mathrm{Sh}(G',X')$ have canonical models, then the morphism $\mathrm{Sh}(f) : \mathrm{Sh}(G,X) \to \mathrm{Sh}(G',X')$ is defined over any field E containing the reflex fields of (G,X) and (G',X'), that is, there exists a (unique) morphism $\mathrm{Sh}(f)_E : \mathrm{Sh}(G,X)_E \to \mathrm{Sh}(G',X')_E$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Sh}(f) : \mathrm{Sh}(G,X) \arrow[r] \arrow[d, "\psi"] & \mathrm{Sh}(G',X') \arrow[d, "\psi'"] \\
\mathrm{Sh}(f)_E : \mathrm{Sh}(G,X)_E \arrow[r] & \mathrm{Sh}(G',X')_E.
\end{tikzcd}$$

PROOF: See Deligne (1971c), 5.4.

COROLLARY 5.3. *The canonical model of $Sh(G,X)$ (if it exists) is uniquely determined up to a unique isomorphism.*

PROOF: This is an immediate consequence of the proposition.

Example 5.4. (a) Let T be a torus. Since $\mathrm{Sh}(T,x)$ is of dimension zero, it is completely described by its set of points (with the profinite topology), and so it has a unique model over $\mathbb{Q}^{\mathrm{al}}$. Giving a model of $\mathrm{Sh}(T,x)$ over $E = E(T,x)$ corresponds to giving an action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$ on $\mathrm{Sh}(T,x)(\mathbb{Q}^{\mathrm{al}}) = T(\mathbb{A}_f)/T(\mathbb{Q})^-$. If the model is to be the canonical model, this action must be that given by $r(T,x)$.

(b) When (G,X) is of Hodge type, it follows from the theorem of Shimura and Taniyama (see I.5.6) that a solution to the moduli problem over $E(G,X)$ will be a canonical model.

THEOREM 5.5. *Let (G,X) be a pair satisfying (2.1), and write $E = E(G,X)$.*

(a) The Shimura variety $\mathrm{Sh}(G,X)$ *has a canonical model* $\mathrm{Sh}(G,X)_E$.

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau E(G,X) = E({}^{\tau,x}G, {}^{\tau,x}X)$, and $\tau\mathrm{Sh}(G,X)_E$ is the canonical model of $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$.

PROOF: This follows from (4.2) and (4.4). Suppose first that w_X is defined over $\mathbb{Q}$. A calculation shows that if τ fixes $E(G,X)$, then the class of $\rho_{x*}({}^{\tau}\mathfrak{S})$ in $H^1(\mathbb{Q},G)$ is trivial. The choice of a point $p \in \rho_{x*}({}^{\tau}\mathfrak{S})$ determines an isomorphism $f_1 : G \to {}^{\tau,x}G$. Write $p = sp(\tau)\cdot\beta$. Then (2.6) give us a well-defined equivariant isomorphism

$$\varphi_x : \mathrm{Sh}(G,X) \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X).$$

A similar argument to that in the proof of (4.3) allows us to extend the definition of φ_x to any Shimura variety. For each τ, let

$$f_\tau = \varphi_x^{-1} \circ \varphi_{\tau,x} : \tau\mathrm{Sh}(G,X) \to \mathrm{Sh}(G,X).$$

Then $f_{\sigma\tau} = f_\sigma \circ \sigma(f_\tau)$, and so the f_τ define a descent datum for $\mathrm{Sh}(G,X)$ which gives us a model $\mathrm{Sh}(G,X)_E$ over $E(G,X)$. When applied to a pair (T,x), this procedure leads directly to the canonical model of $\mathrm{Sh}(T,x)$; thus $[x,a]$ is rational over $E(T,x)^{\mathrm{ab}}$, and the action of the Galois group on it is as required. Now (4.4) can be used to show that the model obtained is independent of the special point x, and so it fulfills the condition for every special point. This completes the proof of (a). The statement in (b) about the reflex fields is obvious from the definitions. Moreover, it is straightforward to check that

$$(\tau\mathrm{Sh}(G,X)_E) \otimes_{\tau E} \mathbb{C} = \tau\mathrm{Sh}(G,X) \xrightarrow{\varphi_{\tau,x}} \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$$

realizes $\tau\mathrm{Sh}(G,X)_E$ as the canonical model of $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$.

COROLLARY 5.6. *Let $E = E(G,X)$. Then*

$$\prod_{\tau\in\mathrm{Hom}(E,\mathbb{C})} \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$$

has a canonical model over $\mathbb{Q}$.

PROOF: In fact, the maps $\varphi_{\tau,x}$ define an isomorphism

$$(\mathrm{Res}_{E/\mathbb{Q}}\mathrm{Sh}(G,X)_E)_{\mathbb{C}} \to \prod \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X).$$

For any field L containing $E(G,X)$, $\mathrm{Sh}(G,X)_E$ gives rise to a model $\mathrm{Sh}(G,X)_L$ of $\mathrm{Sh}(G,X)$ over L. This model will be referred to as the *canonical model of* $\mathrm{Sh}(G,X)$ *over* L.

Notes. Canonical models (in the above sense) were introduced, and shown to be unique in Deligne (1971c). Again, the notion was suggested by a similar notion introduced by Shimura (see the next section). They were shown to exist for Shimura varieties of abelian type (see §9) in Deligne (1979). That (4.2) and (4.4) imply the existence of canonical models was already noted in Langlands (1979).

6. Canonical models in the sense of Shimura.

According to Shimura's original definition, the canonical model of a Shimura variety should be a projective system of *connected* varieties. We explain how such models can be constructed from the canonical models of the preceding section.

Let (G, X) be a pair satisfying (2.1), and choose a connected component X^+ of X. The canonical model (in the sense of Shimura) will be defined in terms of the pair (G, X^+) — note that this is not a pair satisfying (1.3) — G is a *reductive* group. Write $\mathrm{Sh}(G, X)^0$ for the connected component of $\mathrm{Sh}(G, X)$ containing the image of X^+, and let E be the reflex field of (G, X). Since $\mathrm{Sh}(G, X)$ has a canonical model over E, there is a homomorphism $\ell : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E) \to \pi_0(\mathrm{Sh}(G, X))$ giving the action of the Galois group on the set of connected components of $\mathrm{Sh}(G, X)$ (see Deligne (1979), 2.6.2.1, for an explicit description of ℓ). According to (2.7), there is an exact sequence

$$1 \to G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \to \mathcal{G}(G) \to \pi_0(\mathrm{Sh}(G, X)) \to 1.$$

On pulling back by ℓ, we obtain a sequence

$$1 \to G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \to \mathcal{E}(G, X) \xrightarrow{\sigma} \mathrm{Gal}(\mathfrak{k}/E) \to 1$$

with $\mathrm{Gal}(\mathfrak{k}/E)$ the image of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$ in $\pi_0(\mathrm{Sh}(G, X))$ and $\mathcal{E}(G, X)$ the subgroup of $\mathcal{G}(G)$ of elements mapping to $\mathrm{Gal}(\mathfrak{k}/E)$). From $\mathrm{Sh}(G, X)_E$ we obtain a canonical model $\mathrm{Sh}(G, X)^0_{\mathfrak{k}}$ of $\mathrm{Sh}(G, X)^0$ over $\mathfrak{k}$.

Let $\mathfrak{z}$ be the set of compact open subgroups of $\mathcal{E}(G, X)$. For any S in $\mathfrak{z}$, set

$\Gamma_S = S \cap G^{\mathrm{ad}}(\mathbb{Q})^+$;

$k_S =$ the subfield of $\mathfrak{k}$ fixed by $\sigma(S)$;

$V_S = S\backslash \mathrm{Sh}(G, X)^0$; it is defined over k_S, and there is an isomorphism $\varphi_S : \Gamma_S\backslash X^+ \to (V_S)_{\mathbb{C}}$. Let $\alpha \in \mathcal{E}(G, X)$; if $\alpha S \alpha^{-1} \subset T$, then the action of α on $\mathrm{Sh}(G, X)^0$ induces a map $J_{TS}(\alpha) : V_S \to \sigma(\alpha)^{-1} V_T$.

THEOREM 6.1. *(a) For each $S \in \mathfrak{z}$, (V_S, φ_S) is a model of $\Gamma_S \backslash X^+$ over k_S.*

(b) Let $\alpha \in \mathcal{E}(G, X)$; for any S, $T \in \mathfrak{k}$ such that $\alpha S \alpha^{-1} \subset T$, $J_{TS}(\alpha) : \Gamma_S \backslash X^+ \to \sigma(\alpha)^{-1}\Gamma_T \backslash X^+$ is defined over k_S. Moreover

$J_{SS}(\alpha)$ is the identity map if $\alpha \in S$;

$(\sigma(\alpha)^{-1} J_{TS}(\beta)) \circ J_{SR}(\alpha) = J_{TR}(\beta\alpha)$;

$J_{TS}(\alpha) \circ \varphi_S = \varphi_T \circ \alpha$ for all $\alpha \in G(\mathbb{Q})_+$ such that $\alpha S \alpha^{-1} \subset T$.

(c) Let $x \in X^+$ be special; for each $S \in \mathfrak{z}$, $\varphi_S(z)$ is rational over $E(x)^{\mathrm{ab}}$, and for every $\nu \in \hat{E}(x)^\times$,

$$\mathrm{rec}_E(\nu)(\varphi_S(x)) = J_{ST}(N_x(\nu))\varphi_T(x), \quad T = N_x(\nu)^{-1} \cdot S \cdot N_x(\nu)$$

where $N_x : T^{E(x)} \to G$ is defined by μ_x.

PROOF: This can be deduced from (5.5a), using results about the automorphism groups of $\mathrm{Sh}(G, X)$ and its function field. See Milne and Shih (1981b).

Notes. Theorem 6.1 says that canonical models exist in the sense of Shimura (1971a). It was proved in various cases in Shimura (1970), Miyake (1971), and Shih (1978). It was shown to follow from Theorem 5.5 in Milne and Shih (1981b) (the restriction to classical groups in that paper is unnecessary).

7. The action of complex conjugation on a Shimura variety with a real canonical model.

Let $\mathrm{Sh}(G, X)$ be a Shimura variety whose reflex $E(G, X)$ is real. Then $\mathrm{Sh}(G, X)$ has a canonical model $\mathrm{Sh}(G, X)_{\mathbb{R}}$ over $\mathbb{R}$, and so complex conjugation defines an involution of $\mathrm{Sh}(G, X)$. In order to be able to compute the factor of the zeta function of $\mathrm{Sh}(G, X)$ corresponding to the (given) infinite prime of $E(G, X)$, it is necessary to have an explicit description of this involution.

LEMMA 7.1. *Let x be a special point of X. There is a unique $G(\mathbb{R})$-equivariant antiholomorphic map $X \to X$ such that $\eta(x) = {}^\iota x$, where ${}^\iota x$ is the point in X such that $\mu_{{}^\iota x} = \iota\mu_x$.*

PROOF: The uniqueness is obvious. Let T be a maximal torus in G such that $T(\mathbb{R})$ fixes x, and let N be the normalizer of T in G. There is an $n \in N(\mathbb{R})$ such that $n \cdot x = {}^\iota x$ (Milne and Shih 1982b, 4.3), and we can define η to be $g \cdot x \mapsto gn \cdot x$.

THEOREM 7.2. *Let* $\mathrm{Sh}(G,X)$ *be a Shimura variety whose reflex field is real. The involution of* $\mathrm{Sh}(G,X)$ *defined by complex conjugation is* $[x,g] \mapsto [\eta(x),g]$.

PROOF: Since both maps are continuous and equivariant, it suffices to show that they agree at the single point $[x,1]$. The action of ι on $\mathrm{Sh}(G,X)$ (relative to $\mathrm{Sh}(G,X)_E$ is

$$\mathrm{Sh}(G,X) \xrightarrow{\iota} \iota\mathrm{Sh}(G,X) \xrightarrow{\varphi_{\iota,x}} \mathrm{Sh}({}^{\iota,x}G, {}^{\iota,x}X) \xrightarrow{\varphi_x^{-1}} \mathrm{Sh}(G,X).$$

From §5 and §6, we see that $[x,1] \mapsto \iota[x,1] \mapsto [{}^{\iota}x,1] \mapsto [\eta(x),1]$ under these maps.

Notes. Theorem 7.2 was conjectured in Langlands (1979). An equivalent statement for connected Shimura varieties defined by groups G of type C was proved in Shih (1976), and this result was extended to all Shimura varieties of abelian type in Milne and Shih (1981a). That Theorem 7.2 follows from Theorems 4.2 and 4.4 was noted in Langlands (1979).

8. The minimal compactification.

Let $\mathrm{Sh}(G,X)^-$ be the minimal compactification of $\mathrm{Sh}(G,X)$. Because $\mathrm{Sh}(G,X)^-$ can be constructed out of $\mathrm{Sh}(G,X)$ by a canonical algebraic method (see §2), all the maps $\varphi_{\tau,x}$, $\varphi(\tau;x',x)$ and φ_x have unique extensions to $\mathrm{Sh}(G,X)^-$. In particular, we see that all the theorems in this chapter remain valid when the Shimura varieties are replaced by their minimal compactifications. (We shall discuss the boundary components of $\mathrm{Sh}(G,X)^-$in more detail in Chapter V.)

9. The strategy for proving the main theorems.

The proofs of Theorems 4.2 and 4.4 are too long to describe in detail. Instead I outline the strategy for proving them, and other theorems, on Shimura varieties. Recall that in §3 we defined the notion of a Shimura variety of Hodge type and noted that the choice of a faithful representation of G realizes such a variety as a moduli variety (over $\mathbb{C}$) for abelian varieties with Hodge cycle and level structure.

The class of connected Shimura varieties of *abelian type* is the smallest containing:

(a) the connected component of every Shimura variety of Hodge type;

(b) a product of connected Shimura varieties if it contains the factors;

(c) $\mathrm{Sh}(G,X^+)$ if it contains $\mathrm{Sh}(G',X^+)$ with G' a finite covering group of G.
Deligne (1979) gives a classification of connected Shimura varieties of abelian type based on Satake's classification of symplectic embeddings (Satake 1965). A (nonconnected) Shimura variety is of *abelian type* if a connected component of it is of abelian type. Note that a Shimura variety of abelian type will *not* in general be a moduli variety for abelian varieties, contrary to some assertions in the literature.

Let $P(G,X)$ be a statement about the Shimura variety $\mathrm{Sh}(G,X)$. The first step in proving P for all Shimura varieties is to prove it for those of Hodge type by identifying the Shimura variety with a moduli variety for abelian varieties. The second step is to find a statement $P^+(G,X^+)$ for connected Shimura varieties, and to prove that

$$P(G,X) \text{ is true } \Leftrightarrow P^+(G^{\mathrm{der}},X^+) \text{ is true.}$$

As a consequence, one finds that if $P(G',X')$ is true and $(G^{\mathrm{der}},X^+) = (G'^{\mathrm{der}},X'^+)$, then $P(G,X)$ is true. The third step (usually easy) is to prove:

$P^+(G_i,X_i^+)$ true for all $i \Rightarrow P^+(\prod G_i, \prod X_i^+)$ true;

$P^+(G',X^+)$ true for G' a finite covering of $G \Rightarrow P^+(G,X^+)$ is true.

This then implies that P^+ is true for all connected Shimura varieties of abelian type, and hence (by the previous step) that P is true for all Shimura varieties of abelian type. Moreover, it shows that in order to prove P for all Shimura varieties, it suffices to prove $P^+(G,X^+)$ in the case that G is a simply connected $\mathbb{Q}$-simple group. Then G is of the form $G = \mathrm{Res}_{F/\mathbb{Q}}G'$ for some absolutely simple group G' over a totally real field F. For a totally real field F' containing F, set $G_* = \mathrm{Res}_{F'/\mathbb{Q}}G$ and define X_*^+ so that $(G,X^+) \subset (G_*,X_*^+)$. When F' is chosen sufficiently large, there will be many embeddings $(G_\alpha,X_\alpha^+) \hookrightarrow (G,X^+)$ with G_α a group of type A_1 (thus G_α is an algebraic group associated with a quaternion algebra, possibly split, over a totally real field). We have

$$\mathrm{Sh}(G,X^+) \hookrightarrow \mathrm{Sh}(G_*,X_*^+) \hookleftarrow \mathrm{Sh}(G_\alpha,X_\alpha^+).$$

The final step is to exploit these inclusions, and the fact that the statement $P^+(G_\alpha,X_\alpha^+)$ is known (the associated Shimura variety is of abelian type), to prove $P^+(G,X^+)$.

One final note: several authors have criticized the above approach for its dependence on abelian varieties and their moduli. In defence

I point out that, in the case that the weight is defined over $\mathbb{Q}$, all of the results in this and the next chapter would be an immediate consquence of the existence of a sufficiently strong theory of motives and their moduli; moreover, this is the only heuristic argument I know for them. Also, the approach does not use the classification of semisimple algebraic groups (at present, the only place where this is used is in Kazhdan (1982), but the author has shown that it is unnecessary there). Finally, this is the *only* approach that gives strong results.

Notes. For the existence of canonical models, the first three steps were carried out in Deligne (1979). For Langlands's conjecture (theorems 4.2 and 4.4) they were carried out in Milne and Shih (1982b). The embedding of $\mathrm{Sh}(G, X)$ into $\mathrm{Sh}(G_*, X_*)$ was used in Piatetski-Shapiro (1971) in the case the group G is of type A_n to obtain a pair (G_*, X_*) for which $G_*(\mathbb{Q}_\ell)$ has no compact factors. Borovoi suggested (in 1981) using the embeddings $(G_\alpha, X_\alpha) \hookrightarrow (G_*, X_*)$ to prove the existence of canonical models for Shimura varieties not of abelian type. (Obtaining canonical models using embeddings of Shimura varieties of type A_1 was also an unstated object of Garrett (1982, 1984).)

10. Appendix: Schemes with a continuous action of a locally profinite group.

A *locally profinite group* is a locally compact totally disconnected group. In such a group $\mathcal{G}$, the compact open subgroups K form a fundamental system of neighbourhoods of the identity element, and $\cap K = 1$.

LEMMA 10.1. *Let $\mathcal{G}$ be a locally profinite group, and let E be a separated topological space with a continuous action $E \times \mathcal{G} \to E$ of $\mathcal{G}$. For each compact open subgroup K in $\mathcal{G}$, set $E_K = E/K$. Then (E_K) is a projective system, and $E = \varprojlim E/K$.*

PROOF: Apply Bourbaki (1960), III.7.2, Cor 1 to the groups K acting on E, and observe that $\varprojlim K = \cap K = 1$.

To give E together with the action of $\mathcal{G}$ is the same as to give the family (E_K) together with the maps

$$x \mapsto xg : E_K \to E_L, \quad L \supset g^{-1}Kg.$$

These remarks motivate the following definitions.

For the remainder of this section, "scheme" will mean "quasi-projective scheme over a field k", or a projective limit of such schemes.

Let $\mathcal{G}$ be a locally profinite group, and consider a family (S_K) of schemes, indexed by the open compact subgroups K of $\mathcal{G}$. Suppose that for each $g \in \mathcal{G}$ and each K and L with $L \supset g^{-1}Kg$, there is given a morphism

$$\rho_{L,K}(g) : S_K \to S_L$$

satisfying the conditions:

(i) $\rho_{K,K}(k) = id$ if $k \in K$;

(ii) $\rho_{M,L}(g) \circ \rho_{L,K}(h) = \rho_{M,L}(gh)$;

(iii) whenever K is normal in L, so that $\rho_{K,K}$ defines an action of the finite group L/K on S_K, S_L is isomorphic to the quotient of S_K by the finite group L/K.

We then call the family $(S_K, \rho_{L,K})$ a *scheme with a continuous right action* of $\mathcal{G}$.

For each $K \subset L$, there is a map $\rho_{L,K}(1) : S_K \to S_L$. In this way we get a projective system of schemes whose limit S has a right action by $\mathcal{G}$ such that $S_K = S/K$ for all compact open subgroups K of $\mathcal{G}$. We shall also refer to S as a *scheme with a continuous right action* of $\mathcal{G}$.

Example 10.2. Suppose $\mathcal{G}$ is compact and S is smooth. If $\mathcal{G}$ acts continuously on S in such a way that the isotropy group of each geometric point of S is trivial, then $S \to S/\mathcal{G}$ is a Galois covering with Galois group $\mathcal{G}$. Conversely, if $S \to S_0$ is a Galois covering with Galois group $\mathcal{G}$, then $\mathcal{G}$ acts on S in such a way that the isotropy group of each geometric point is trivial.

For example, let S_0 be a connected scheme, and fix a geometric point $s \to S_0$. Take S to be the projective system of commutative triangles

$$\begin{tikzcd} s \arrow[r] \arrow[dr] & S' \arrow[d, "\pi"] \\ & S_0 \end{tikzcd}$$

with π finite and étale. Then S is a Galois covering of S_0 with Galois group the étale fundamental group $\pi_1^{\text{et}}(S_0, s)$.

Let S be a scheme over k with a continuous action of $\mathcal{G}$. For a scheme Y over k, we set $\text{Hom}(Y, S) = \varinjlim \text{Hom}(Y, S_K)$. To give a scheme Y over S is the same as to give a scheme Y_K over S_K, each K sufficiently small, such that $Y_K = Y_L \times_{S_L} S_K$, for $K \subset L$.

Fix a locally profinite group $\mathcal{G}$ and a profinite set π with a continuous right action of $\mathcal{G}$. Assume that the action is transitive, and that the orbits of a compact open subgroup are open: for any $e \in \pi$, the bijection $\mathcal{G}/\mathcal{G}_e \to \pi$ ($\mathcal{G}_e$ = isotropy group at e) is a homeomorphism. Consider systems consisting of a scheme S with a continuous right action of $\mathcal{G}$ together with a continuous equivariant map $S \to \pi$. For $e \in \pi$, the fibre S_e over e is endowed with a continuous action of $\mathcal{G}_e$.

PROPOSITION 10.3. *The functor $S \mapsto S_e$ is an equivalence from the category of schemes S, endowed with a continuous action of $\mathcal{G}$ and a continuous equivariant map $S \to \pi$, to the category of schemes S_e endowed with a continuous action of $\mathcal{G}_e$.*

PROOF: See Deligne 1979, 2.7.3.

In particular, there is a reverse functor, $S_e \mapsto S$. The scheme S will be said to have been obtained from S_e by *induction* from $\mathcal{G}_e$ to $\mathcal{G}$.

III. AUTOMORPHIC VECTOR BUNDLES

Just as automorphic functions are sections of the sheaf of germs of functions on a Shimura variety, holomorphic automorphic forms are sections of certain vector bundles, called automorphic vector bundles, on a Shimura variety. The main theorems for automorphic vector bundles parallel those for Shimura varieties: every automorphic vector bundle $\mathcal{V}(\mathcal{J})$ has a canonical model $\mathcal{V}(\mathcal{J})_E$ over its reflex field E, and for each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau\mathcal{V}(\mathcal{J})_E$ is the canonical model over τE of an explicitly determined automorphic vector bundle $\mathcal{V}({}^{\tau}\mathcal{J})$. In particular, this allows us to define, in complete generality, the notion of a holomorphic automorphic form being rational over a number field.

Throughout this chapter (G, X) is a pair satisfying (II.2.1). We write $Z_s(G)$ for the maximal subtorus of $Z(G)$ that is split over $\mathbb{R}$ but which has no subtorus split over $\mathbb{Q}$; thus $Z_s(G)$ is the largest subtorus of $Z(G)$ such that

$$\begin{cases} X_*(Z_s)^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})} = 0 \\ \iota \text{ acts as } +1 \text{ on } X_*(Z_s). \end{cases}$$

We write G^c for $G/Z_s(G)$. Note that (G, X) satisfies (II.2.1.2*) if and only if $G = G^c$.

1. The compact dual symmetric Hermitian space $\check{X}$.
For each $x \in X$, μ_x defines a decreasing filtration $\mathrm{Filt}(\mu_x)$ of $\mathbf{Rep}_{\mathbb{C}}(G)$ (see I.1), and we define $\check{X}$ to be the $G(\mathbb{C})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(G)$ containing $\mathrm{Filt}(\mu_x)$. If (V, ξ) is a faithful representation of $G_{\mathbb{C}}$, then $\check{X}$ can be identified with a $G(\mathbb{C})$-conjugacy class of filtrations of V.

Fix a point o of X, and let P_o be the subgroup of $G_{\mathbb{C}}$ fixing $\mathrm{Filt}(\mu_o)$. Then P_o is a parabolic subgroup of $G_{\mathbb{C}}$ (see I.1.7) and there is a bijection

$$G(\mathbb{C})/P_o(\mathbb{C}) \to \check{X},$$

which endows $\check{X}$ with the structure of a smooth projective variety over $\mathbb{C}$. We call $\check{X}$ the *compact dual symmetric Hermitian space* of X. For any connected component X^+ of X, $\check{X}$ is the dual of X^+ in the sense of Helgason (1978), V.2.

Interpretation of $\check{X}$ as a classifying space. Let $\mathcal{V}$ be a vector bundle on a connected complex variety S. The *type* of a filtration

$$\mathcal{V} \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n = 0$$

is the sequence of numbers, $d_i = \operatorname{rank} \mathcal{S}_i$. Fix a vector space V over $\mathbb{C}$ and a filtration $F_0^{\cdot}$ of V of type $\mathbf{d} = (d_1, \ldots, d_n)$. Then the functor of complex varieties

$$\mathcal{F}(S) = \{\text{filtrations of } \mathcal{V}_S =_{\mathrm{df}} S \times V \text{ of type } \mathbf{d}\}$$

is represented by the Grassman variety $GL(V)/Q_0$, where Q_0 is the subgroup of $GL(V)$ stabilizing $F_0^{\cdot}$. When V is defined over $\mathbb{Q}$, so also is the Grassman variety.

Fix a family of tensors $\mathfrak{t} = (t_\alpha)_{\alpha \in I}$ for V, and let G be the subgroup of $GL(V)$ fixing the t_α. Then each t_α defines a global tensor of $\mathcal{V}_S$, and the functor

$$\mathcal{F}_0(S) = \{\text{filtrations } F^{\cdot} \text{ of } \mathcal{V}_S \text{ s.t. } (\mathcal{V}_s, F_s^{\cdot}, \mathfrak{t}) \approx (V, F_0^{\cdot}, \mathfrak{t}) \text{ all } s \in S\}$$

is represented by the subvariety G/P_0 of the Grassman variety, where P_0 is now the stabilizer of $F_0^{\cdot}$ in G.

We apply these remarks to (G, X). Choose a faithful representation $\xi : G \to GL(V)$ of G, and let $\mathfrak{t} = (t_\alpha)$ be a family of tensors of V such that G is the subgroup of $GL(V)$ fixing the t_α. Choose a point $o \in X$, and let $F_o^{\cdot}$ be the corresponding Hodge filtration of $V(\mathbb{C})$. Then $\check{X}$ represents the functor $\mathcal{F}_0$ described above: the $F_x^{\cdot}$ for $x \in \check{X}$ define a filtration of the vector bundle $\mathcal{V} =_{\text{df}} \check{X} \times V(\mathbb{C})$ and the triple $(\mathcal{V}, F^{\cdot}, \mathfrak{t})$ is universal.

In particular, we see that $\check{X}$ is realized as a subvariety of a Grassman variety $GL(V(\mathbb{C}))/Q_o$. As in (II.2), we let M_X be the $G(\mathbb{C})$-conjugacy class of homomorphisms $\mathbb{G}_m \to G_{\mathbb{C}}$ containing μ_x for $x \in X$, and we let $E(G, X)$ be the field of definition of M_X. The map

$$\mu_x \mapsto \mathrm{Filt}(\mu_x) : M_X \to \check{X}$$

is surjective, from which it is clear that $\check{X}$, regarded as a subvariety of $GL(V)/Q_0$, is stable under the action of any automorphism τ fixing $E(G, X)$. Therefore $\check{X}$ is defined over $E(G, X)$.

The Borel embedding.

PROPOSITION 1.1. *The map*

$$\beta : X \to \check{X}, \quad x \mapsto \mathit{Filt}(\mu_x)$$

embeds X as an open complex submanifold of $\check{X}$. For $o \in X$, let K_o be the isotropy group at o in $G(\mathbb{R})$, and let P_o be the isotropy group at $o \in \check{X}$ in $G(\mathbb{C})$; then $K_o = P_o \cap G(\mathbb{R})$, and the inclusion of K_o into P_o identifies $(K_o)_{\mathbb{C}}$ with a Levi subgroup of P_o; we have

$$\begin{array}{ccc} G(\mathbb{R})/K_o & \hookrightarrow & G(\mathbb{C})/P_o(\mathbb{C}) \\ \downarrow \approx & & \downarrow \approx \\ X & \hookrightarrow & \check{X}. \end{array}$$

PROOF: The fact that β is holomorphic is a restatement of (II.3.2a). For the rest, we merely note that the injectivity of $X \to \check{X}$ follows from the fact that the Hodge filtration determines the Hodge decomposition (I.2). (See Helgason 1978, VIII.7 for the details.)

The map β is the *Borel embedding* of X into $\check{X}$.

Example 1.2. Let (V, ψ) be a symplectic space, and let $(G, S^{\pm})$ be as in (II.2.4a). Thus $G = GSp(V, \psi)$ and $S^{\pm}$ is the space of Hodge structures on V of type $\{(-1,0),(0,-1)\}$ for which $\pm(2\pi i)\psi$ is a polarization. In this case, $\check{X}$ can be identified with the set of maximal isotropic subspaces of $V(\mathbb{C})$ and β with the map $x \mapsto F_x^0 V$.

Conjugates of $\check{X}$. As $\check{X}$ is an algebraic variety, $\tau\check{X}$ is defined for any $\tau \in \mathrm{Aut}(\mathbb{C})$. Recall from (I.7) that the period torsor $\mathfrak{P}$ is a torsor for $\mathfrak{T}$ having a canonical point $p \in \mathfrak{P}(\mathbb{C})$. Define $z_\infty(\tau) \in \mathfrak{T}(\mathbb{C})$ by:

$$\tau p = p \cdot z_\infty(\tau).$$

Then $z_\infty(\tau) \in {}^\tau\mathfrak{S}(\mathbb{C})$, and so it defines an isomorphism

$$g \mapsto {}^{\tau,x}g =_{\mathrm{df}} [z_\infty(\tau) \cdot g] : G(\mathbb{C}) \to {}^{\tau,x}G(\mathbb{C}).$$

PROPOSITION 1.3. *(a) Let x be a special point of X, and let ${}^{\tau,x}\check{X}$ be the dual Hermitian symmetric space associated with $({}^{\tau,x}G, {}^{\tau,x}X)$. There is a unique isomorphism*

$$\check{\varphi}_{\tau,x} : \tau\check{X} \to {}^{\tau,x}\check{X}$$

such that

(i) the point τx is mapped to ${}^\tau x$, and
(ii) $\check{\varphi}_{\tau,x} \circ (\tau g) = ({}^{\tau,x}g) \circ \check{\varphi}_{\tau,x}$, for all g in $G(\mathbb{C})$.
(b) Let x' be a second special point; then the isomorphism

$${}^{\tau,x}g \mapsto {}^{\tau,x'}g : {}^{\tau,x}G(\mathbb{C}) \to {}^{\tau,x'}G(\mathbb{C})$$

induces an isomorphism $\check{\varphi}(\tau; x', x) : {}^{\tau,x}\check{X} \to {}^{\tau,x'}\check{X}$ such that

$$\check{\varphi}(\tau; x', x) \circ \check{\varphi}_{\tau,x} = \check{\varphi}_{\tau,x'}.$$

PROOF: Straightforward.

Remark 1.4. Let x be a CM-point of X, and let (V, ξ) be a faithful representation of G. Then $(V, \xi \circ \rho_x)$ defines a CM-motive M over $\mathbb{Q}^{\mathrm{al}}$ with $V = H_B(M)$. There are Hodge cycles t_α on M such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the t_α, and we noted in (II.4.1) that ${}^{\tau,x}G$ is the subgroup of $GL(H_B(\tau M)) \times \mathbb{G}_m$ fixing the tensors τt_α. The comparison isomorphisms between Betti and de Rham cohomology allow us to interprete $G_{\mathbb{C}}$ and ${}^{\tau,x}G_{\mathbb{C}}$ as subgroups of $GL(H_{\mathrm{dR}}(M_{\mathbb{C}})) \times \mathbb{G}_m$ and $GL(H_{\mathrm{dR}}(\tau M_{\mathbb{C}})) \times \mathbb{G}_m$ respectively. If we regard τ as an embedding of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{C}$, then the map $G(\mathbb{C}) \to {}^{\tau,x}G(\mathbb{C})$ is induced by the isomorphism

$$H_{\mathrm{dR}}(M) \otimes_{\mathbb{Q}^{\mathrm{al}},\tau} \mathbb{C} \to H_{\mathrm{dR}}(\tau M).$$

2. Automorphic vector bundles.
Let S be an algebraic variety over a field k with an action $G \times S \to S$ of an algebraic group. By a *G-vector bundle* on S we mean a vector bundle $(\mathcal{V}, p)$ on S together with an action of G on $\mathcal{V}$ (as an algebraic variety) such that

(a) $p(g \cdot v) = g \cdot p(v)$ for all $g \in G$, $v \in \mathcal{V}$;
(b) the maps $g : \mathcal{V}_s \to \mathcal{V}_{gs}$ are linear for all $s \in S$.

We shall be interested in $G_{\mathbb{C}}$ vector bundles $\mathcal{J}$ on $\check{X}$. As we saw in (1.1), the map $\beta : X \hookrightarrow \check{X}$ embeds X as an open submanifold of $\check{X}$, and the action of $G(\mathbb{C})$ on $\check{X}$ extends that of $G(\mathbb{R})$ on X. Therefore such a vector bundle $\mathcal{J}$ restricts to a $G(\mathbb{R})$-vector bundle $\beta^{-1}(\mathcal{J})$ on X. If the action of $G_{\mathbb{C}}$ on $\mathcal{J}$ factors through $G^c_{\mathbb{C}}$, and K is sufficiently small, then, as in the proof of (II.3.3), we can pass to the quotient and obtain a vector bundle

$$\mathcal{V}_K(\mathcal{J}) = G(\mathbb{Q})\backslash\beta^{-1}(\mathcal{J}) \times G(\mathbb{A}_f)/K$$

on $\mathrm{Sh}(G, X)$. (In §8 we discuss what happens when we no longer require that the action factors through G^c.) For each $g \in G(\mathbb{A}_f)$ and pair of open compact subgroups K and L of $G(\mathbb{A}_f)$ such that $L \supset g^{-1}Kg$, there is a morphism

$$\rho_{L,K}(g) : \mathcal{V}_K(\mathcal{J}) \to \mathcal{V}_L(\mathcal{J}), \quad [x, a] \mapsto [x, ag].$$

PROPOSITION 2.1. *(a) The vector bundles $\mathcal{V}_K(\mathcal{J})$, and the maps $\rho_{L,K}(g) : \mathcal{V}_K(\mathcal{J}) \to \mathcal{V}_L(\mathcal{J})$, are algebraic.*

(b) If X^+ has no factors isomorphic to the unit disk, then every analytic section of $\mathcal{V}_K(\mathcal{J})$ is algebraic, and the space of such sections is finite-dimensional over $\mathbb{C}$.

PROOF: (a) When the boundary of $\mathrm{Sh}(G, X)$ in its minimal compactification has codimension ≥ 3, the proposition is a consequence of the following general lemma. We omit the proof in the remaining case (but see (3.6) below).

(b) The hypothesis implies that the codimension of the boundary is ≥ 2, and so the next lemma applies.

LEMMA 2.2. *Let S be a nonsingular algebraic variety over $\mathbb{C}$, embedded as an open subvariety of a complete algebraic variety $\bar{S}$. If $\bar{S} - S$ has codimension ≥ 2, then the functor $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{an}}$ taking an algebraic vector bundle on S to its associated analytic vector bundle* *is fully faithful; moreover* $\Gamma(S,\mathcal{V}) = \Gamma(S,\mathcal{V}^{\mathrm{an}})$ *and these spaces are finite-dimensional. If* $\bar{S} - S$ *has codimension* ≥ 3, *then* $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{an}}$ *is an equivalence of categories.*

PROOF: This follows from theorems of Serre, Grothendieck, Siu, and Trautmann; see Hartshorne (1970), p222.

The family $\mathcal{V}(\mathcal{J}) = (\mathcal{V}_K(\mathcal{J}))_K$ is a scheme with a right action of $G(\mathbb{A}_f)$, in the sense of (II.10). A vector bundle of the form $\mathcal{V}_K(\mathcal{J})$, $\mathcal{J}$ a $G_{\mathbb{C}}$-vector bundle on $\check{X}$, will be called an *automorphic vector bundle*, and a section f of $\mathcal{V}(\mathcal{J})_K$ over $\mathrm{Sh}_K(G,X)$ will be called a *(holomorphic) automorphic form* of type $\mathcal{J}$ and level K. (When the boundary of $\mathrm{Sh}(G,X)$ in its minimal compactification has codimension one we must also require that f be "holomorphic at infinity".)

Remark 2.3. (a) Fix a point $o \in \check{X}$, and let P_o be the (parabolic) subgroup of $G_{\mathbb{C}}$ fixing o. For any $G_{\mathbb{C}}$-vector bundle $\mathcal{J}$ on $\check{X}$, P_o acts on the fibre $\mathcal{J}_o$, and the map $\mathcal{J} \mapsto \mathcal{J}_o$ defines an equivalence from the category of $G_{\mathbb{C}}$-vector bundles on $\check{X}$ to $\mathbf{Rep}_{\mathbb{C}}(P_o)$.

(b) From (a) we see that, in particular, every complex representation (V,ξ) of G^c defines a G^c-vector bundle on $\check{X}$, and hence an automorphic vector bundle $\mathcal{V}(\xi)$. There is a local system $V(\xi)$ of $\mathbb{C}$-vector spaces underlying $\mathcal{V}(\xi)$, which can be described as follows: for K sufficiently small, the fundamental group of $\Gamma_g\backslash X^+$ is the image Γ_g^c of $gKg^{-1} \cap G(\mathbb{Q})_+$ in $G^c(\mathbb{Q})_+$ (notation as in II.2); the restriction of $V_K(\xi)$ to $\Gamma_g\backslash X^+$ is defined by the representation of Γ_g^c on V given by ξ. It follows from (II.3.1) that $\mathcal{V}(\xi)$ in this case has a natural flat connection and that it is algebraic. (Note that a representation (V,ξ) of G^c defined over a subfield L of $\mathbb{C}$ gives rise, in the same way, to an L-local system on $\mathrm{Sh}(G,X)$ contained in $\mathcal{V}(\xi)^{\nabla(\xi)}$.)

(c) There is an infinite-dimensional version of the above construction: $(\mathfrak{g},P_o)$-modules (not necessarily finite-dimensional) correspond to $G_{\mathbb{C}}$-equivariant quasi-coherent $\mathcal{D}$-modules on $\check{X}$, and the same construction as above defines a functor from the category of $G^c_{\mathbb{C}}$-equivariant quasi-coherent $\mathcal{D}$-modules on $\check{X}$ to the category of $G(\mathbb{A}_f)$-equivariant quasi-coherent $\mathcal{D}$-modules on $\mathrm{Sh}(G,X)$. Recall that a $(\mathfrak{g},P_o)$-module is a P_o-module with an action of $\mathfrak{g}$ whose restriction to $\mathfrak{p}_o$ coincides with the differential of the P_o-action. In the case that the module is finite-dimensional, the action of $\mathfrak{g}$ can be integrated to an action of G extending that of P, and the corresponding D-module is coherent; it is therefore locally free (Borel et al. 1987, p211), and the $\mathcal{D}$-module structure on the module corresponds to a flat connection. Thus this case reverts to that discussed in (b).

Example 2.4. Let (G, X) be the pair, as in (II.2.4), associated with a symplectic space (V, ψ). There is a naturally defined abelian scheme $\mathcal{A}$ over $\mathrm{Sh}(G, X)$ (cf. II.3.11). A point $o \in \check{X}$ corresponds to a maximal isotropic subspace W of $V(\mathbb{C})$, and P_o is the subgroup of G stabilizing W. Write S for $\mathrm{Sh}(G, X)$, and $2g$ for the dimension of V.

(a) The automorphic vector bundle associated with the natural representation of P_o on V/W is the tangent space of $\mathcal{A}/S$.

(b) The line bundle $\omega(\mathcal{A}/S)$ is the dual of the automorphic vector bundle associated with the determinant of the natural representation of P_o on V/W.

(c) The canonical line bundle on S is the automorphic vector bundle associated with the $(g+1)^{\mathrm{st}}$-power of the determinant of the natural representation of P_o on V/W.

(d) The automorphic vector bundle associated with the standard representation of G on V is $\mathcal{H}_{\mathrm{dR}}(\mathcal{A})$, and the flat connection on it is the Gauss-Manin connection.

Relation to automorphic forms in the classical sense. The above discussion also makes sense for connected Shimura varieties $\mathrm{Sh}^0(G, X^+)$: β defines an embedding $X^+ \hookrightarrow \check{X}$, and a $G_{\mathbb{C}}$-vector bundle $\mathcal{J}$ on $\check{X}$ defines an *automorphic vector bundle* $\mathcal{V}^0(\mathcal{J})$ on $\mathrm{Sh}^0(G, X^+)$. We now explain how to interprete sections of such bundles as holomorphic automorphic forms in the classical sense.

Let Γ be a discrete subgroup of $\mathrm{Aut}(X^+)$. Classically, one defines an *automorphy factor* for (Γ, X^+) with values in a complex vector space V to be a mapping $j : \Gamma \times X^+ \to GL(V)$ such that:

(a) for each $\gamma \in \Gamma$, $x \mapsto j(\gamma, x)$ is holomorphic on X^+;

(b) $j(\gamma\gamma', x) = j(\gamma, \gamma' x) \cdot j(\gamma', x)$, all γ, $\gamma' \in \Gamma$, $x \in X^+$.

An *automorphic form* for Γ of type j is then a function $f : X^+ \to V$ such that

(a) f is holomorphic;

(b) $f(\gamma x) = j(\gamma, x) f(x)$;

(c) f is "holomorphic at infinity".

Let $\mathcal{J}$ be a $G_{\mathbb{C}}$-vector bundle on $\check{X}$; choose a point $o \in X^+$, and let $V = \mathcal{J}_{\beta(o)}$. Because X^+ is simply connected, the isomorphism $V \to \beta^{-1}(\mathcal{J})_o$ extends to an isomorphism $X^+ \times V \approx \beta^{-1}(\mathcal{J})$, and we can transfer the action of $G(\mathbb{R})^+$ on $\beta^{-1}(\mathcal{J})$ to $X^+ \times V$. Write

$$\gamma(x, v) = (\gamma x, j(\gamma, x)v) \text{ for } \gamma \in G(\mathbb{R})^+,\ x \in X^+, \text{ and } v \in V.$$

Then $j : G(\mathbb{R})^+ \times V \to V$ satisfies the conditions (a) and (b), and so its restriction to $\Gamma_K \times V$ is an automorphy factor. A section of $\mathcal{V}^0(\mathcal{J})_K$ on $\mathrm{Sh}^0_K(G, X^+)$ can then be identified with an automorphic form for Γ_K of type j.

Example 2.5. Let $G = SL_2$, and let X^+ be the complex upper-half-plane (see II.1.5). The map $z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$ is an isomorphism from X^+ to $D = \{z \in \mathbb{C} \mid |z| < 1\}$. In this case $\check{X}$ is the Riemann sphere, and $X \hookrightarrow \check{X}$ is an isomorphism of X with the upper hemisphere. If we take $o = \mathrm{i}$ (in the upper-half-plane), then $P_0 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$. If χ_k is the $2k^{\text{th}}$ power of the obvious character of P_0 and $\mathcal{V}_k$ is the corresponding automorphic vector bundle, then the sections of $\mathcal{V}_k$ holomorphic at infinity are elliptic modular forms of weight k.

3. The standard principal bundle.

The functor $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})$ takes one algebraic object to a second, but passes through the intermediary of the non-algebraic object X. In order to understand the rationality properties of the functor, we need to replace X by an algebraic object — this we call the standard principal bundle.

Review of principal bundles. Let S a complex manifold, and let G be a complex Lie group. A *flat structure* on a principal G-bundle P is given by a covering U_α of S for which the transition maps are constant.

Assume S is connected, and let $\tilde{S}$ be the universal covering space of S. A homomorphism $\xi : \pi_1(S, s) \to G$ defines a principal G-bundle

$$P(\xi) = \tilde{S} \times G/\sim, \ (s\gamma, g) \sim (s, \xi(\gamma)g), \ \ s \in \tilde{S}, \ \gamma \in \pi_1(S, s), \ g \in G,$$

on S, and there is a canonical flat structure on $P(\xi)$. Every principal G-bundle P over S admitting a flat structure arises in this way. In the case that $G = GL(V)$, V a $\mathbb{C}$-vector space, $P(\xi)$ is the frame bundle of $\mathcal{V}(\xi)$: the sections of $P(\xi)$ over an open subset U of S can be identified with the isomorphisms $a : U \times V \xrightarrow{\approx} \mathcal{V}|U$ (trivializations of $\mathcal{V}$ over U). Now suppose that ξ factors through a reductive algebraic subgroup G of $GL(V)$. Then $P(\xi)$ can be interpreted as the bundle of frames of $\mathcal{V}(\xi)$ respecting certain tensors. When S is a complex algebraic variety and $\mathcal{V}(\xi)$ and the tensors are algebraic, then $P(\xi)$ is also algebraic: it is a G-torsor over S.

LEMMA 3.1. *Let G be an algebraic group over a field k, and let $\pi : P \to S$ be a torsor for G over an algebraic k-variety S.*

(a) The functor $\mathcal{V} \mapsto \pi^{-1}\mathcal{V}$ defines an equivalence between the category of vector bundles on S and the category of G-vector bundles on P.

(b) If P has a flat structure, then to give a (flat) connection on $\mathcal{V}$ is the same as to give a (flat) connection in $\pi^{-1}(\mathcal{V})$.

PROOF: This is a standard consequence of descent theory.

Define

$$P(G,X) = G(\mathbb{Q})\backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f)/Z(\mathbb{Q})^-,$$

$$q(x,c,a)z = (qx, qc, qaz),\ q \in G(\mathbb{Q}),\ z \in Z(\mathbb{Q})^-.$$

Then $P(G,X)$ is a principal $G^c(\mathbb{C})$-bundle on $\mathrm{Sh}(G,X)^{\mathrm{an}}$, which we call the *standard principal bundle*. The group $G(\mathbb{A}') =_{\mathrm{df}} G(\mathbb{C}) \times G(\mathbb{A}_f)$ acts on $P(G,X)$ according to the rule

$$[x,z,a](c,g) = [x, zc, ag],\ x \in X,\ z, c \in G(\mathbb{C}),\ a, g \in G(\mathbb{A}_f).$$

Write π for the projection map $P(G,X) \to \mathrm{Sh}(G,X)$.

PROPOSITION 3.2. *The bundle $P(G,X)$ is algebraic, and the action of $G(\mathbb{A}')$ is algebraic.*

PROOF: For any faithful representation (V,ξ) of $G^c_{\mathbb{C}}$, $P(G,X)$ is the bundle of frames, respecting certain tensors, of the vector bundle $\mathcal{V}(\xi)$. Now apply (II.3.1).

Remark 3.3. Let ξ be as in the above proof. The functor represented by $P(G,X)$ can be described as follows: for any morphism $\gamma : T \to \mathrm{Sh}(G,X)$, the liftings of γ to $P(G,X)$ correspond to the trivializations $T \times V \to \gamma^{-1}(\mathcal{V}(\xi))$ of $\gamma^{-1}(\mathcal{V}(\xi))$ respecting certain tensors.

For example, suppose (G,X) satisfies (II.2.1*) and is of Hodge type. Corresponding to a symplectic representation $\xi : G \hookrightarrow GSp(V,\psi)$ there is an abelian scheme $\mathcal{A}$ over $\mathrm{Sh}(G,X)$ such that $\mathcal{H}_B(\mathcal{A}) = V(\xi)$. For any point $s \in \mathrm{Sh}(G,X)$, $\pi^{-1}(s)$ is equal to the set of morphisms $H_B(\mathcal{A}_s) \otimes \mathbb{C} \to H_{\mathrm{dR}}(\mathcal{A}_s)$ respecting certain Hodge cycles on $\mathcal{A}_s$.

PROPOSITION 3.4. *There is a canonical $G(\mathbb{C})$-equivariant map $\gamma : P(G,X) \to \check{X}$.*

PROOF: Choose a faithful representation $\xi : G^c_{\mathbb{C}} \hookrightarrow GL(V)$, as before. The last remark shows that a complex point p of $P(G,X)$ corresponds to an isomorphism $V \longrightarrow \mathcal{V}(\xi)_{\pi(p)}$ respecting certain tensors. The Hodge filtration on $\mathcal{V}(\xi)_{\pi(p)}$ pulls back to a filtration on V, and we can map p to the corresponding point of $\check{X}$. That this is a morphism of algebraic varieties follows from the universal property of $\check{X}$ described in §1.

PROPOSITION 3.5. *Let $\mathcal{J}$ be a $G_{\mathbb{C}}$- vector bundle on $\check{X}$. Then $\mathcal{V}(\mathcal{J})$ is the unique vector bundle on* $\mathrm{Sh}(G,X)$ *such that* $\pi^{-1}(\mathcal{V}(\mathcal{J})) = \gamma^{-1}(\mathcal{J})$ *(as a G^c-vector bundle).*

PROOF: This follows directly from (3.3) and the definitions.

The following diagram summarizes the situation:

$$\begin{tikzcd}
\mathcal{V}(\mathcal{J}) \arrow[r, dash] & \pi^{-1}(\mathcal{V}(\mathcal{J})) = \gamma^{-1}(\mathcal{J}) \arrow[r, dash] & \mathcal{J} \\
\mathrm{Sh}(G,X) & P(G,X) \arrow[l, "\pi"'] \arrow[r, "\gamma"] & \check{X} \\
& X \arrow[ul] \arrow[u, leftrightarrow] \arrow[ur, dashed] &
\end{tikzcd}$$

Remark 3.6. Proposition 3.5 provides an alternative proof that the vector bundles $\mathcal{V}(\mathcal{J})$ are algebraic.

4. Canonical models of standard principal bundles. The key result that allows us to construct canonical models is the following.

THEOREM 4.1. *Let $\tau \in Aut(\mathbb{C})$.*

(a) For any special point $x \in X$, $\varphi_{\tau,x}$ lifts canonically to an equivariant morphism

$$\varphi^P_{\tau,x} : \tau P(G,X) \to P({}^{\tau,x}G, {}^{\tau,x}X).$$

(b) If x' is a second special point, then $\varphi(\tau; x', x)$ lifts canonically to an equivariant morphism

$$\varphi^P(\tau; x', x) : P({}^{\tau,x}G, {}^{\tau,x}X) \to P({}^{\tau,x'}G, {}^{\tau,x'}X),$$

and

$$\varphi^P(\tau; x', x) \circ \varphi^P_{\tau,x} = \varphi^P_{\tau,x'}.$$

PROOF: The strategy is that outlined in (II.9); see the notes at the end of the Chapter.

Example 4.2. (a) Suppose that (G, X) is of Hodge type, and that it satisfies (II.2.1.2*). Then the choice of a faithful representation (V, ξ) of G defines an abelian scheme $\mathcal{A}$ (with additional structure) on $\mathrm{Sh}(G, X)$. From a CM-point x, we obtain a representation of $({}^{\tau,x}V, {}^{\tau,x}\xi)$ of ${}^{\tau,x}G$, and therefore an abelian scheme (with additional structure) ${}^{\tau,x}\mathcal{A}$ on $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$. Under our hypotheses, $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$ is a fine moduli variety and ${}^{\tau,x}\mathcal{A}$ is the universal abelian scheme over it. The universality implies the existence of a commutative diagram:

$$\begin{array}{ccc} \tau\mathcal{A} & \longrightarrow & {}^{\tau,x}\mathcal{A} \\ \downarrow & & \downarrow \\ \tau\mathrm{Sh}(G,X) & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X). \end{array}$$

Since $\mathcal{V}(\xi) = \mathcal{H}_{\mathrm{dR}}(\mathcal{A})$ and $\mathcal{V}({}^{\tau,x}\xi) = \mathcal{H}_{\mathrm{dR}}({}^{\tau,x}\mathcal{A})$, and $P(G, X)$ and $P({}^{\tau,x}G, {}^{\tau,x}X)$ are the frame bundles of $V(\xi)$ and $\mathcal{V}({}^{\tau,x}\xi)$, the diagram gives $\varphi^P_{\tau,x}$.

(b) For the Shimura variety defined by a CM-pair (T, x), it is possible to give an explicit description of $\varphi^P_{\tau,x}$ in terms of the period torsor.

THEOREM 4.3. *(a) The standard principal bundle $P(G, X)$ has a canonical model $P(G, X)_E$ over $E = E(G, X)$.*

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau P(G, X)_E$ is a canonical model of $P({}^{\tau,x}G, {}^{\tau,x}X)$.

PROOF: This can be deduced from (4.1) in the same way as (II.5.5) is deduced from (II.4.2) and (II.4.4).

Example 4.4. (a) In the situation of (4.2a), $\mathcal{A}$ is defined over the canonical model $\mathrm{Sh}(G, X)_E$, and for any point $s \in \mathrm{Sh}(G, X)_E$, $\pi^{-1}(s)$ is equal to the set of morphisms $H_B(\mathcal{A}_s) \otimes E \to H_{\mathrm{dR}}(\mathcal{A}_s/E)$ respecting certain Hodge cycles on $\mathcal{A}_s$.

(b) In the situation of (4.2b), it is possible to give an explicit description of $P(T, x)_E$ in terms of the period torsor.

Remark 4.5. The following properties of $\varphi^P_{\tau,x}$ provide justification for calling it canonical.

(i) A morphism $(G, X) \to (G', X')$ and a special point $x \in X$ give rise to a commutative diagram,

$$\begin{array}{ccc}
\varphi^P_{\tau,x} : \tau P(G,X) & \longrightarrow & P({}^{\tau,x}G, {}^{\tau,x}X) \\
\downarrow & & \downarrow \\
\varphi^P_{\tau,x} : \tau P(G',X') & \longrightarrow & P({}^{\tau,x'}G', {}^{\tau,x'}X).
\end{array}$$

Here x' is the image of x in X'.

(ii) Consider two pairs (G,X) and (G',X') together with an identification $(G^{\mathrm{der}}, X^+) = (G'^{\mathrm{der}}, X'^+)$. Let x be a special point of X^+, and let x' be the corresponding point of X'^+. Then there is an equivariant commutative diagram:

$$\begin{array}{ccccc}
\tau P(G,X) & \longleftarrow & \tau P^0(G^{\mathrm{der}}, X^+) & \longrightarrow & \tau P(G',X') \\
\downarrow{\scriptstyle \varphi^P_{\tau,x}} & & \downarrow & & \downarrow{\scriptstyle \varphi^P_{\tau,x'}} \\
P({}^{\tau,x}G, {}^{\tau,x}X) & \longleftarrow & P^0({}^{\tau,x}G^{\mathrm{der}}, {}^{\tau,x}X^+) & \longrightarrow & P({}^{\tau,x'}G', {}^{\tau,x'}X')
\end{array}$$

where $P^0(G^{\mathrm{der}}, X^+)$ is a principal bundle for G^{der} on $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ and $P^0({}^{\tau,x}G^{\mathrm{der}}, {}^{\tau,x}X^+)$ is a certain principal bundle for ${}^{\tau,x}G^{\mathrm{der}}$ on $\mathrm{Sh}^0({}^{\tau,x}G^{\mathrm{der}}, {}^{\tau,x}X^+)$.

The family of maps $(\varphi^P_{\tau,x})$ is uniquely determined by the properties (i) and (ii) and that mentioned in (4.2b).

So far as the canonical model of $P(G,X)$ is concerned all one can say in general is that it is constructed in a canonical fashion using the (canonical) maps $\varphi^P_{\tau,x}$. However, if one is prepared to confine one's attention to Shimura varieties whose weight is defined over $\mathbb{Q}$, it is possible to give a characterization similar to that for canonical models of Shimura varieties: the map $P(G,X) \to P(G',X')$ defined by a morphism $(G,X) \to (G',X')$ is defined over any field containing the reflex fields of (G,X) and (G',X'); for a pair (T,x) as in (4.2b), there is an explicit description of the canonical model of $P(T,x)$ in terms of the period torsor; the canonical model of $P(G,X)$ is uniquely determined by the condition that, for each CM-pair $(T,x) \subset (G,X)$, $P(T,x) \to P(G,X)$ is defined over $E(T,x)$.

THEOREM 4.6. *(a) The map $\gamma : P(G,X) \to \check{X}$ is rational over $E(G,X)$.*

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, the diagram

$$\begin{array}{ccc} \tau P(G,X)_E & \longrightarrow & \tau \check{X}_E \\ \downarrow & & \downarrow \\ P({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E} & \longrightarrow & {}^{\tau,x}X_{\tau E} \end{array}$$

commutes.

PROOF: See the notes at the end of the Chapter.

5. Canonical models of automorphic vector bundles.

From the results in §4 on the standard principal bundle, it is possible to read off similar results for automorphic bundles.

THEOREM 5.1. *Let $\mathcal{J}$ be a G^c-vector bundle on $\check{X}$, and assume that $\mathcal{J}$ is defined over a number field $E \supset E(G,X)$.*

(a) The automorphic vector bundle $\mathcal{V}(\mathcal{J})$ has a canonical model $\mathcal{V}(\mathcal{J})_E$ over E.

(b) Let τ be an automorphism of $\mathbb{C}$, and let ${}^{\tau,x}\mathcal{J}$ be the vector bundle on ${}^{\tau,x}\check{X}$ corresponding to $\tau\mathcal{J}$ under the isomorphism of (1.3). There is a canonical commutative diagram

$$\begin{array}{ccc} \tau\mathcal{V}(\mathcal{J})_E & \longrightarrow & \mathcal{V}({}^{\tau,x}\mathcal{J})_{\tau E} \\ \downarrow & & \downarrow \\ \tau \mathrm{Sh}(G,X)_E & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E}; \end{array}$$

that is, $\tau\mathcal{V}(\mathcal{J})_E$ is isomorphic to the canonical model of $\mathcal{V}({}^{\tau,x}\mathcal{J})$.

When $\mathcal{J}$ is defined by a representation (V,ξ) of G^c, then the flat connection $\nabla(\xi)$ descends to the canonical model $\mathcal{V}(\xi)_E$, and the isomorphism in (b) respects the flat connections on $\tau\mathcal{V}(\mathcal{J})_E$ and $\mathcal{V}({}^{\tau,x}\mathcal{J})_{\tau E}$.

PROOF: According to (4.3) and (4.5), the maps

$$\mathrm{Sh}(G,X) \xleftarrow{\pi} P(G,X) \xrightarrow{\gamma} \check{X}$$

are defined over E. We define $\mathcal{V}(\mathcal{J})_E$ to be the vector bundle on $\mathrm{Sh}(G,X)_E$ such that $\pi^{-1}(\mathcal{V}(\mathcal{J})_E) = \gamma^{-1}(\mathcal{J}_E)$ (see 4.4). Part (b) can be proved using the diagram:

$$\begin{array}{ccccc} \tau \mathrm{Sh}(G,X)_E & \longleftarrow & \tau P(G,X)_E & \longrightarrow & \tau \check{X}_E \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E} & \longleftarrow & P({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E} & \longrightarrow & {}^{\tau,x}X_{\tau E}. \end{array}$$

Remark 5.2. Any equivariant differential operator $D : \mathcal{J} \to \mathcal{J}'$ between $G_{\mathbb{C}}$-vector bundles on $\check{X}$ induces a differential operator $\mathcal{V}(D) : \mathcal{V}(\mathcal{J}) \to \mathcal{V}(\mathcal{J}')$ between the $G(\mathbb{A}_f)$-vector bundles on $\mathrm{Sh}(G, X)$. If D, $\mathcal{J}$, and $\mathcal{J}'$ are defined over $E \supset E(G, X)$, then so also is $\mathcal{V}(D)$.

Remark 5.3. It would be of interest to re-interprete the above results in the context of (II.6), and to extend (II.7.2) to the standard principal bundle.

6. The local systems defined by a rational representation. We examine in more detail the various local systems defined by a representation (V, ξ) of G^c. As is explained above and in Chapter II, attached to such a representation we have:

(a) a local system of $\mathbb{Q}$-vector spaces $V(\xi)$ on $\mathrm{Sh}(G, X)$;

(b) a local system of $\mathbb{Q}_\ell$-vector spaces $V_\ell(\xi)$ on $\mathrm{Sh}(G, X)$;

(c) a vector bundle $\mathcal{V}(\xi)$ with a flat connection $\nabla(\xi)$ on $\mathrm{Sh}(G, X)$.

These are related by canonical comparison isomorphisms:

(i) $V(\xi) \otimes \mathbb{Q}_\ell \to V_\ell(\xi)$;

(ii) $V(\xi) \otimes \mathbb{C} \to \mathcal{V}(\xi)^{\nabla(\xi)}$.

All these objects have an action of $G(\mathbb{A}_f)$, and the comparison isomorphisms are compatible with the actions.

Remark 6.1. It is an elementary result that $V_\ell(\xi)$ has a canonical model over $E(G, X)$. For K sufficiently small, $\mathrm{Sh}(G, X)$ is Galois over $\mathrm{Sh}_K(G, X)$ with Galois group the image K^c of K in $G^c(\mathbb{A}_f)$, and $V_\ell(\xi)$ is the sheaf on $\mathrm{Sh}_K(G, X)$ corresponding to the representation of K^c on $V \otimes \mathbb{Q}_\ell$ defined by ξ. This construction works over $E(G, X)$, and gives us the canonical model of $V_\ell(\xi)$. Moreover, when the weight w_X is defined over $\mathbb{Q}$, the local system $\tau V_\ell(\xi)$ on $\tau\mathrm{Sh}(G, X)$ corresponds under $\varphi_{\tau,x}$ to $V_\ell({}^{\tau,x}\xi)$, where ${}^{\tau,x}\xi$ is the representation of ${}^{\tau,x}G$ obtained from ξ by twisting by ${}^{\tau}\mathfrak{S}$.

The objects in (b) and (c) are algebraic, and we can think of $V(\xi)$ as providing a rational structure to the family $(\mathcal{V}_\ell(\xi), (\mathcal{V}(\xi), \nabla(\xi))$. The next result shows that the family $(\tau V_\ell(\xi), \tau(\mathcal{V}(\xi), \nabla(\xi))$ on $\tau\mathrm{Sh}(G, X)$ also has a canonical rational structure.

THEOREM 6.2. *Let τ be an automorphism of $\mathbb{C}$, and let (V, ξ) be a representation of G^c. Assume that the composite of the weight map w_X with $G \to G^c$ is defined over $\mathbb{Q}$. Then there is a canonical local system ${}^{\tau}V(\xi)$ of $\mathbb{Q}$-vector spaces on $\tau\mathrm{Sh}(G, X)$ such that*

(a) ${}^{\tau}V(\xi) \otimes \mathbb{Q}_\ell = \tau V_\ell(\xi)$, for all primes ℓ;

(b) ${}^\tau V(\xi)\otimes\mathbb{C}=(\tau\mathcal{V}(\xi))^{\tau\nabla(\xi)}$.

PROOF: We can use ${}^\tau\mathfrak{S}$ and the map $\rho_x:\mathfrak{S}\to G^c$ to twist the representation (V,ξ), and so obtain a representation $({}^{\tau,x}V,{}^{\tau,x}\xi)$ of ${}^{\tau,x}G^c$. Define ${}^\tau V(\xi)$ to be the local system of $\mathbb{Q}$-vector spaces on $\tau\mathrm{Sh}(G,X)$ corresponding to $V({}^{\tau,x}\xi)$ under the isomorphism $\varphi_{\tau,x}$. Theorem 4.4 implies that ${}^\tau V(\xi)$ is independent of the choice of x, and it follows directly from its construction that ${}^\tau V$ satisfies (a) and (b).

If we assume (II.3.9, 3.10), then (V,ξ) defines a family of motives $\mathcal{M}$ on $\mathrm{Sh}(G,X)$, and we should have
${}^\tau V(\xi)=\mathcal{H}_B(\tau\mathcal{M})\quad(\tau\mathcal{M}$ on $\tau\mathrm{Sh}(G,X))$;
$V_\ell(\xi)_E=\mathcal{H}_\ell(\mathcal{M}_E)$, $\mathcal{M}_E$ the canonical model of $\mathcal{M}$ over $\mathrm{Sh}(G,X)_E$;
$(\mathcal{V}(\xi),\nabla(\xi))_E=\mathcal{H}_{\mathrm{dR}}(M_E)$ with its Gauss-Manin connection.

7. Automorphic forms rational over a subfield of $\mathbb{C}$.

Definition 7.1. Let $\mathcal{J}$ be a $G_{\mathbb{C}}$-vector bundle on $\check{X}$, rational over a number field E, with $E(G,X)\subset E\subset\mathbb{C}$. An automorphic form f of type $\mathcal{J}$ and level K is *rational over* E if it arises from a section of $\mathcal{V}_K(\mathcal{J})_E$ over $\mathrm{Sh}_K(G,X)_E$.

Write $A_K(\mathcal{J})_E=A_K(G,X,\mathcal{J})_E$ for the space of such forms; it is a vector space over E.

PROPOSITION 7.2. *With the above notations:*

$$A_K(\mathcal{J})_E\otimes_E\mathbb{C}=A_K(\mathcal{J})_{\mathbb{C}}.$$

PROOF: In general, if $\mathcal{V}$ is a vector bundle on a variety S over a field E, and C is an extension field of E, then $\Gamma(S,\mathcal{V})\otimes_E C=\Gamma(S_C,\mathcal{V}_C)$.

COROLLARY 7.3. *The vector space $A_k(\mathcal{J})_E$ is finite-dimensional over E.*

PROOF: This follows from (2.1b).

We now discuss rationality criteria in terms of special values. Assume that the weight w_X is defined over $\mathbb{Q}$ and that (G,X) satisfies (II.2.1*). Consider the automorphic vector bundle $\mathcal{V}(\xi)$ defined by a representation (V,ξ) of G. For each CM-pair $(T,x)\subset(G,X)$, there is a unique homomorphism $\rho_x:\mathfrak{S}\to T$ such that $\mu_{\mathrm{can}}\circ\rho_x=\mu_x$ (see II.2.4). From the representation $(\xi|T)\circ\rho_x$ we obtain a CM-motive M over $\mathbb{Q}^{\mathrm{al}}$ with $H_B(M)=V$, and from the model M_E of M over the canonical model of $\mathrm{Sh}(T,x)$, we obtain an $E(T,x)$-structure $V_{E,x}=_{\mathrm{df}}H_{\mathrm{dR}}(M_E)$ on $V(\mathbb{C})$. It is also possible to construct $V_{E,x}$

directly from the period torsor. There is a canonical identification of $V_{E,x}$ with the fibre $\mathcal{V}(\xi)_{E,x}$. Thus, if an automorphic form f is defined over $E(G,X)$, then $f(x)$, regarded as an element of $\mathcal{V}(\xi)_x = V(\mathbb{C})$ lies in the subspace $V_{E,x}$; conversely, when this condition holds for all CM-points, then f is defined over $E(G,X)$.

8. Automorphic stacks. Throughout this chapter we have insisted that the action of $G_{\mathbb{C}}$ on a vector bundle $\mathcal{J}$ on $\check{X}$ factor through $G^c_{\mathbb{C}}$, and that a representation ξ of G factor through G^c . We now explain why we have made these assumptions, and why it would be better to avoid them. Then we explain how to do this.

Consider the case of a representation $\xi : G \to GL(V)$, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. The connected components of $\mathrm{Sh}_K(G,X)$ are of the form $\Gamma_g\backslash X^+$ where Γ_g is the image of $\Gamma'_g =_{\mathrm{df}} gKg^{-1} \cap G(\mathbb{Q})_+$ in $G^{\mathrm{ad}}(\mathbb{Q})^+$; here $g \in G(\mathbb{A}_f)$ and X^+ is a connected component of X. When ξ factors through G^c we define $\mathcal{V}(\xi)$ to be the vector bundle whose restriction to $\Gamma_g\backslash X^+$ is $\Gamma^c_g\backslash X^+ \times V(\mathbb{C})$ where Γ^c_g is the image of Γ'_g in $G^c(\mathbb{Q})$. This makes sense because, when K is is sufficiently small, the map $\Gamma^c_g \to \Gamma_g$ is an isomorphism, the fibre of $\Gamma^c_g\backslash X^+ \times V(\mathbb{C}) \to \Gamma_g\backslash X^+$ over any point is isomorphic to $V(\mathbb{C})$, and $\mathcal{V}(\xi)$ is a vector bundle. When we drop this condition, $\mathcal{V}(\xi)$ will no longer be a vector bundle. Consider for example the pair (G,X) in (II.2.4b) defining the Hilbert modular variety, and assume $F \neq \mathbb{Q}$. The centre Z of G is $F^\times$. For $g = 1$, the kernel of $\Gamma'_g \to \Gamma_g$ is $K \cap Z(\mathbb{Q})$, which is equal to the set of elements of $F^\times$ congruent to 1 modulo some ideal. This will be of finite index in the group of units of $F^\times$, and so is never trivial. The fibre of $\Gamma'_g\backslash X \times V(\mathbb{C}) \to \Gamma_g\backslash X$ will be the quotient of $V(\mathbb{C})$ by the action of this kernel, and so we do not get a vector bundle by this construction. This same problem also occurs when trying to define the universal family of abelian varieties over $\mathrm{Sh}(G,X)$ (van der Geer 1988, Chapter X).

So why not simply do as have done in this chapter and exclude this them? Classically, one defines automorphic forms as functions on the universal covering space X transforming in certain ways relative to the group Γ. The reason we wish to consider them as sections of a vector bundle on $\mathrm{Sh}(G,X)$ is so that we can apply the methods of algebraic geometry. From the classical point of view, it is unnatural to exclude them.

So how do we handle them? Just as in the case of the universal abelian scheme over the Hilbert moduli variety, we should use stacks.

Briefly, the idea is to pass to a partial quotient of X which makes sense algebraically, and on which $\mathcal{V}(\xi)$ is an equivariant vector bundle. In this way we obtain the notion of an *automorphic stack.*

In the case that weight is defined over $\mathbb{Q}$, it is possible to consider a concrete realization of the stack. Let G' be the smallest subgroup of G such that all h factor through $G'_{\mathbb{R}}$. Then $Z_s(G') = 0$. Consider

$$\mathrm{Sh}'(G,X) =_{\mathrm{df}} G'(\mathbb{Q})\backslash X \times G(\mathbb{A}_f) = \varprojlim G'(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

It is a covering of $\mathrm{Sh}(G,X)$, and every $G_{\mathbb{C}}$-vector bundle on $\check{X}$ defines a vector bundle on $\mathrm{Sh}'(G,X)$.

All results in the chapter continue to hold *mutandis mutatis* for automorphic stacks. In fact, since the proofs proceed via connected Shimura varieties, where this problem doesn't arise, there is little extra difficulty in working with stacks rather than vector bundles.

Notes. The principal theme of this chapter has been the problem of making sense of what it means for an automorphic form to be defined over a number field. In the case of elliptic modular functions, there is no difficulty: a modular form is defined over a number field if and only if its Fourier coefficients lie in the field. Unfortunately, in higher dimension, Fourier-Jacobi series are much more difficult to work with (see Chapter VII); moreover this method can only apply to noncompact Shimura varieties.

There are basically four approaches to defining rationality of automorphic forms:

(a) using Fourier-Jacobi series (or their null-values...)

(b) in terms of the special values of the forms (that is, values at the special points);

(c) pulling-back to sub-Shimura varieties of type A_1;

(d) directly defining a canonical model of the automorphic vector bundle.

Of course these approaches are not independent, and all should give the same answer when they apply.

Shimura used special values (and periods) to define the notion of an automorphic form being rational over $\mathbb{Q}^{al}$—see Shimura (1979). For applications of his results, see Shimura (1980), (1981). He studies Fourier-Jacobi series in Shimura (1978a), (1978b). For certain Shimura varieties Garrett (1983) shows that (a), (b), and (c) lead to consistent notions of rationality.

Under the hypothesis that the weight w_X is defined over $\mathbb{Q}$ and (G, X) satisfies (2.1.2*), Harris (1985) defined a functor $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})_E$ from $G_{\mathbb{C}}$-vector bundles on $\check{X}$ to vector bundles on $\mathrm{Sh}(G, X)_E$, but did not show that the functor was canonical. This result was the inspiration for Milne (1988), which proves the major statements in this section in the context of connected Shimura varieties. They can be extended to (nonconnected) Shimura varieties by "induction" (in the sense of (II.10)). Full details will be given in the book mentioned in the introduction. See also Harris (1986) where the relation between (a) and (d) is investigated.

IV. One-motives

A mixed Hodge structure on a vector space is an increasing filtration of the vector space together with a Hodge structure on each of the quotients. Hodge structures degenerate into mixed Hodge structures. The cohomology groups of a complex algebraic variety (not necessarily smooth or complete) carry mixed Hodge structures.

Just as abelian varieties provide an algebro-geometric realization of certain Hodge structures, one-motives provide an algebro-geometric realization of certain mixed Hodge structures.

1. Mixed Hodge structures. A *mixed Hodge structure* is

(a) a finite-dimensional vector space V over $\mathbb{Q}$,

(b) a finite increasing (weight) filtration $W_{\cdot}$ on V, and

(c) a finite decreasing (Hodge) filtration $F^{\cdot}$ on $V \otimes \mathbb{C}$ such that, for each n, $F^{\cdot}$ induces a Hodge structure of weight n on

$$\mathrm{Gr}_n^W(V) =_{\mathrm{df}} W_n V / W_{n-1} V.$$

When $\mathbb{Q}$ in the definition is replaced by $k \subset \mathbb{R}$, we obtain the notion of a *mixed k-Hodge structure.*

Example 1.1. (a) A Hodge structure $(V, F^{\cdot})$ of weight n can be made into a mixed Hodge structure by setting $W_n V = V$ and $W_{n-1} V = 0$.

(b) The cohomology groups $H^n(X, \mathbb{Q})$ of any variety X over $\mathbb{C}$ (not necessarily nonsingular or complete) have natural mixed Hodge structures. This is the main theorem in Deligne (1975).

(c) Let (V, ψ) be a symplectic space over $\mathbb{Q}$, and endow $V \otimes \mathbb{R}$ with a Hodge structure of type $\{(-1,0),(0,-1)\}$ for which ψ is a Riemann form (i.e., such that $(2\pi i)\psi$ is a polarization of the Hodge structure). Write $F^{\cdot}$ for the corresponding filtration of $V \otimes \mathbb{C}$. Let W be a totally isotropic subspace of V, and let $W^\perp$ be the orthogonal complement of W in V. Then we have a filtration

$$\begin{array}{ccccccc} 0 & \subset & W & \subset & W^\perp & \subset & V \\ \| & & \| & & \| & & \| \\ W_{-3}V & & W_{-2}V & & W_{-1}V & & W_0V, \end{array}$$

and one can check that $(V, W_., F^{\cdot})$ is a mixed Hodge structure (see Brylinski 1983, 4.2.1).

The *level* of a mixed Hodge structure is the length of the shortest interval $[c,d]$ such that $F^p/F^{p+1} \neq 0 \Rightarrow c \leq p \leq d$. A *morphism* of mixed Hodge structures $F : V \to V'$ is a linear map $V \to V'$ respecting the weight filtrations on V and V' and the Hodge filtrations on $V \otimes \mathbb{C}$ and $V' \otimes \mathbb{C}$. The category of mixed Hodge structures has a natural structure of a Tannakian category. The *Mumford-Tate group* $MT(V)$ of a mixed Hodge structure V is defined to be the affine group scheme attached to the sub-Tannakian category generated by V and $\mathbb{Q}(1)$.

The canonical bigrading. Let V be a mixed Hodge structure. For integers p and q, set $\tilde{V}^{p,q}$ equal to

$$(W_n(V) \cap F^p(V)) \cap (W_n(V) \cap \bar{F}^q(V) + \sum_{2 \leq i}(W_{n-i}(V) \cap \bar{F}^{q-i+1}(V)),$$

where $n = -p-q$.

Then

(a) $V = \oplus_{p,q} \tilde{V}^{p,q}$;

(b) the projection of $W_n(V)$ onto $\mathrm{Gr}_n^W(V)$ induces an isomorphism

$$\tilde{V}^{p,q} \to H^{p,q}(\mathrm{Gr}_n^W(V))$$

for all p, q with $p+q=n$;

(c) $W_n(V) = \sum_{p+q \leq n} \tilde{V}^{p,q}$;

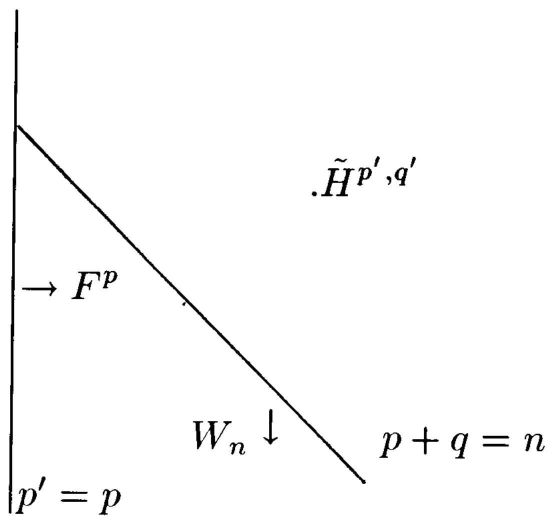
(d) $F^p(V) = \sum_{p' \geq p} \tilde{V}^{p',q}$.

(e) If W is a second mixed Hodge structure, then

$$(V \tilde{\otimes} W)^{m,n} = \otimes_{\substack{p+p'=m \\ q+q'=n}} \tilde{V}^{p,q} \otimes \tilde{W}^{p',q'}.$$

(f) A morphism of mixed Hodge structures respects the bigrading.

For the proof, see Deligne (1971a), 1.2.10, 1.2.11. We may visualize (c) and (d) as:



Clearly, an element of $V(\mathbb{R})$ is in $\tilde{V}^{0,0}$ if and and only if it is in both W_0V and F^0V. An element of a space $T = V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes \mathbb{Q}(r)$ lying in $\tilde{H}^{0,0}$ (or a sum of such elements) will be called a *Hodge tensor* of V. As before, we let $\mathbb{G}_m$ act on T through its action on $\mathbb{Q}(1)$. Define

$$\tilde{h} : \mathbb{S}_{\mathbb{C}} \to GL(V(\mathbb{C})), \quad \tilde{h}(z_1, z_2) \cdot v = z_1^{-p} z_2^{-q} \cdot v,\ v \in \tilde{V}^{p,q},$$

and define $\tilde{h}' : \mathbb{S}_{\mathbb{C}} \to GL(V(\mathbb{C})) \times \mathbb{G}_m$ to be $(z_1, z_2) \mapsto (\tilde{h}(z_1, z_2), z_1 z_2)$. Then $t \in T$ is a Hodge tensor if and only if it is fixed by $\mathrm{Im}(\tilde{h}')$.

PROPOSITION 1.2. *(a) The Mumford-Tate group of V is the subgroup of $GL(V) \times \mathbb{G}_m$ of elements fixing all Hodge tensors of V.*

(b) The Mumford-Tate group of V is the smallest subgroup of $GL(V) \times \mathbb{G}_m$ whose complex points contain the image of $\tilde{h}'$.

PROOF: (a) With any $t \in V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes \mathbb{Q}(r)$ we can associate an $\alpha(t) \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}(r))$, and t will be a Hodge cycle if and only if $\alpha(t)$ is a morphism of Hodge structures. From this it follows that Hodge tensors are fixed by $GL(V) \times \mathbb{G}_m$, and that $\mathrm{Im}(\tilde{h}') \subset MT(V)(\mathbb{C})$. Thus the tensors fixed by $MT(V)$ are precisely the Hodge tensors.

Let M' be the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the Hodge tensors. According to Deligne (1982a), 3.1c, in order to prove that $M' = MT(V)$, it suffices to show that every $\mathbb{Q}$-rational character of $MT(V)$ extends to $GL(V) \times \mathbb{G}_m$. Let $\chi : MT(V) \to GL(W)$ be such a character. Then W acquires a mixed Hodge structure, and since it has dimension one, we must have $W \approx \mathbb{Q}(r)$ for some r. It is now obvious that χ extends to $GL(V) \times \mathbb{G}_m$.

(b) Let H be the smallest subgroup of $GL(V) \times \mathbb{G}_m$ such that $H(\mathbb{C})$ contains the image of $\tilde{h}'$. Then an element of some subquotient S of $V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes \mathbb{Q}(r)$ is in $\tilde{S}^{0,0}$ if and only if it is fixed by H. Thus $MT(V)$ and H fix the same tensors in all such subquotients, and this shows that the two groups are equal (see Deligne (1982a), 3.2a).

PROPOSITION 1.3. *Let G be an algebraic group over $\mathbb{R}$, and let $W.$ and $F^{\cdot}$ be filtrations of* $\mathbf{Rep}(G)$. *Suppose that for some family (V_i, ξ_i) of representations of G such that $\cap \mathrm{Ker}(\xi_i)$ is finite, $(W., F^{\cdot})$ defines a mixed Hodge structure on V_i for all i; then $(W., V^{\cdot})$ defines a mixed Hodge structure on V for all representations (V, ξ) of G.*

PROOF: See Deligne (1973), III.1.11.

Variations of mixed Hodge structures. A *variation of mixed Hodge structures* on a complex manifold S is

(a) a local system of $\mathbb{Q}$-vector spaces V on S,

(b) a filtration $W.$ of V by local systems W_iV.

(c) a holomorphic filtration $F^{\cdot}$ of $\mathcal{V} =_{\mathrm{df}} \mathcal{O}_S \otimes V$

such that

(H_1) $\nabla(F^p\mathcal{V}) \subset \Omega^1 \otimes F^{p-1}\mathcal{V}$

(H_2) for all $s \in S$, $(V_s, W._s, F^{\cdot}_s)$ is a mixed Hodge structure.

When $\mathbb{Q}$ in the definition is replaced by $k \subset \mathbb{R}$, then we obtain the notion of a *variation of mixed k-Hodge structures.* The families of mixed Hodge structures arising naturally in algebraic geometry are variations of mixed Hodge structures.

Notes. Mixed Hodge structures were introduced by Deligne in order to be able to state the theorem quoted in (1.1b). See Deligne (1971a), (1971b), (1975).

2. One-motives.

A *semi-abelian variety* over a field k is an extension of an abelian variety by a torus:

$$0 \to T \to G \to A \to 0.$$

When k is algebraically closed, a character χ of T then defines (by pushout) an element of $\mathrm{Ext}^1(A, \mathbb{G}_m) = \check{A}(k)$; conversely, a homomorphism $X^*(T) \to \check{A}(k)$ defines an extension of A by T.

A *one-motive* M over an algebraically closed field k is a triple (G_M, X_M, u) comprising a semi-abelian variety G_M over k, a finitely generated torsion-free abelian group X_M, and a homomorphism $u : X_M \to G_M(k)$. The definition when k is not algebraically closed is the same except that X_M is a $\mathrm{Gal}(k^{\mathrm{al}}/k)$-module and u is required to be an equivariant homomorphism $X_M \to G_M(k^{\mathrm{al}})$. We often drop the subscripts M, and write $M = (X \xrightarrow{u} G)$. We regard it as a complex of length one. Thus a *morphism* of one-motives is a commutative square:

$$\begin{array}{ccc} X & \xrightarrow{u} & G \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{u'} & G'. \end{array}$$

A morphism (α,β) is an *isogeny* if the cokernel of α and the kernel of β are both finite. A one-motive has a filtration:

$$\begin{array}{rcccl} W_0M = & (X & \to & G) & \\ & \cup & & \cup & \mathrm{Gr}_0(M) = X \\ W_{-1}M = & (0 & \to & G) & \\ & \cup & & \cup & \mathrm{Gr}_{-1}(M) = A \\ W_{-2}M = & (0 & \to & T) & \\ & \cup & & \cup & \mathrm{Gr}_{-2}(M) = T \\ & 0 & \to & 0 & \end{array}$$

Betti homology. The Betti homology group of a one-motive M over $\mathbb{C}$ is a mixed Hodge structure $(H_B(M), F^{\cdot}, W_{\cdot})$ of type $\{(0,0);(0,-1),(-1,0);(-1,-1)\}$ such that

$$\begin{aligned} \mathrm{Gr}_0 H_B(M) &= X \otimes \mathbb{Q} \\ \mathrm{Gr}_{-1} H_B(M) &= H_1(A_M, \mathbb{Q}) \\ \mathrm{Gr}_{-2} H_B(M) &= H_1(T_M, \mathbb{Q}) \approx X_*(T) \otimes \mathbb{Q}. \end{aligned}$$

To construct it, pull-back the top row of the following diagram by $X \to G$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(G) & \longrightarrow & \mathrm{Lie}(G) & \xrightarrow{\exp} & G & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_1(G) & \longrightarrow & H_B(M, \mathbb{Z}) & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

and define $H_B(M) = H_B(M,\mathbb{Z}) \otimes \mathbb{Q}$.

THEOREM 2.1. *The functor $M \mapsto H_B(M)$ defines an equivalence between the category of one-motives over $\mathbb{C}$, considered up to isogeny, and the category of mixed Hodge structures of level ≤ 1 for which $Gr_{-1}H_B(M)$ is polarizable.*

PROOF: Deligne (1975), 10.1.3.

COROLLARY 2.2. *Let (V, h) be a Hodge structure of type $\{(-1,0), (0,-1)\}$, and let ψ be a polarization for (V, h). Let $W \subset V$ be a totally isotropic subspace. There is a unique one-motive M (up to isogeny) such that $H_B(M)$ is the mixed Hodge structure defined in (1.1c).*

Remark 2.3. The theorem explains the one in "one-motive". Note that one-motives are not motives but mixed motives (the Betti homology of a motive is a sum of (pure) Hodge structures).

The *Mumford-Tate group* MT^M of M is defined to be the Mumford-Tate group of the mixed Hodge structure $H_B(M)$.

de Rham homology. Let $M = (X \to G)$ be a one-motive over a field k. The exact sequence

$$0 \to X \to G \to M \to 0,$$

gives rise to an exact sequence of vector groups,

$$0 \to \mathrm{Hom}(X, \mathbb{G}_a) \to \mathrm{Ext}^1(G, \mathbb{G}_a) \to \mathrm{Ext}^1(M, \mathbb{G}_a) \to 0.$$

There is an extension $M^\natural = (X \to G^\natural)$ of M by $\mathrm{Ext}^1(M, \mathbb{G}_a)^\vee$, which fits into a diagram,

$$\begin{tikzcd} & & & X \arrow[dl, dashed] \arrow[d] & \\ 0 \arrow[r] & \mathrm{Ext}^1(M, \mathbb{G}_a)^\vee \arrow[r] & G^\natural \arrow[r] & G \arrow[r] & 0, \end{tikzcd}$$

and which is universal among extensions of M by vector groups (Deligne 1975, 10.1.7). Define $H_{\mathrm{dR}}(M) = \mathrm{Lie}(G^\natural)$. The map $M \mapsto H_{\mathrm{dR}}(M)$ is functorial in M, and so the weight filtration on M defines a filtration $W.$ on $H_{\mathrm{dR}}(M)$. The Hodge filtration is defined by

$F^{-1}H_{\mathrm{dR}}(M) = H_{\mathrm{dR}}(M)$,

$F^0 H_{\mathrm{dR}}(M) = \mathrm{Ext}^1(M, \mathbb{G}_a)^\vee = \mathrm{Ker}(\mathrm{Lie}\, G^\natural \to \mathrm{Lie}\, G)$,

$F^1 H_{\mathrm{dR}}(M) = 0$.

PROPOSITION 2.4. *When $k = \mathbb{C}$, there is a canonical isomorphism*

$$(H_{\mathrm{dR}}(M), F^{\cdot}, W_{\cdot}) \to (H_B(M) \otimes \mathbb{C}, F^{\cdot}, W_{\cdot}).$$

PROOF: See Deligne (1975), 10.1.8.

ℓ-adic homology. Let $M = (X \xrightarrow{u} G)$ be a one-motive over an algebraically closed field k, which, for simplicity, we take to be of characteristic zero. Define

$$M_m = H^0(M \otimes^{\mathbf{L}} (\mathbf{Z}/m\mathbf{Z})).$$

Thus M_m is the zero$^{\text{th}}$ cohomology group of the simple complex associated with the double complex:

$$\begin{array}{ccc} X & \xrightarrow{u} & G \\ \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle -m} \\ X & \xrightarrow{u} & G, \end{array}$$

so that

$$M_m = \{(x, g) \in X \times G(k) \mid u(g) = mg\} / \{(mx, u(x)) \mid x \in X\}.$$

Define

$$H_\ell(M) = (\varprojlim M_{\ell^n}) \otimes_{\mathbf{Z}_\ell} \mathbb{Q}_\ell,$$

$$H_f(M) = \prod{}' H_\ell(M) \text{ (restricted product)}.$$

When k is not algebraically closed, we set $H_\ell(M) = H_\ell(M \otimes_k k^{\mathrm{al}})$.

PROPOSITION 2.5. *When $k = \mathbb{C}$ there is a canonical isomorphism $H_B(M) \otimes \mathbb{Q}_\ell \to H_\ell(M)$.*

PROOF: This amounts to checking that $H_B(M, \mathbf{Z}) \otimes (\mathbf{Z}/m\mathbf{Z}) = M_m$.

The dual one-motive. There is a functor sending a one-motive M to its dual $\check{M}$. Set $\check{X} = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$,

$\check{A}$ = the dual abelian variety of A, $\mathrm{Ext}^1(A, \mathbb{G}_m)$,

$\check{T} = \mathrm{Hom}(X, \mathbb{G}_m)$.

Define $\check{G}$ to be $\mathrm{Ext}^1(M/W_{-2}M, \mathbb{G}_m)$. The sequence

$$0 \to X \to A \to M/W_{-2}M \to 0$$

gives rise to an exact sequence

$$0 \to \check{T} \to \check{G} \to \check{A} \to 0.$$

As M is an extension of $M/W_{-2}M$ by T, from each $x \in \check{X}$ we get an extension of $M/W_{-2}M$ by $\mathbb{G}_m$, and hence an element $\check{u}(k)$ of $\check{G}(k)$. This defines the map $\check{u}$, and completes the construction of $\check{M}$. There are the following formulas:

$H_B(\check{M}) = \mathrm{Hom}(H_B(M), \mathbb{Q}(1))$,
$H_\ell(\check{M}) = \mathrm{Hom}(H_\ell(M), \mathbb{Q}_\ell(1))$,
$H_{\mathrm{dR}}(\check{M}) = \mathrm{Hom}(H_{\mathrm{dR}}(M), k)$.

Symmetric one-motives. A *polarization* of a one-motive M is an isogeny $\lambda : M \to \check{M}$ such that $\mathrm{Gr}_{-1}(\lambda) : A \to \check{A}$ is a polarization of A. A one-motive together with a polarization, is called a *symmetric* one-motive.

PROPOSITION 2.6. *Giving a symmetric one-motive over k is equivalent to giving the following data:*

(a) a polarized abelian variety (A, λ) over k;

(b) a finitely generated torsion-free abelian group X with an action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$;

(c) a $\mathrm{Gal}(k^{\mathrm{al}}/k)$-homorphism $v : X \to A(k^{\mathrm{al}})$; let $\check{v} = \lambda \circ v$;

(d) a trivialization ψ of the inverse image by $(v, \check{v})$ of the Poincaré biextension of A; ψ is required to be symmetric, i.e., invariant under $(x, x') \mapsto (x', x) : X \times X \to X \times X$.

PROOF: In fact, $(M, \lambda) \mapsto (\mathrm{Gr}_{-1}(M), \mathrm{Gr}_{-1}(\lambda), v)$ can be made into an equivalence of categories; cf. (Deligne 1975, 10.2.14).

We explain (d). The Poincaré line bundle is the line bundle on $A \times \check{A}$ which expresses the duality between A and $\check{A}$ (Mumford 1970, § 13). The Poincaré biextension is the $\mathbb{G}_m$ torsor on $A \times \check{A}$ obtained by removing the zero section from the Poincaré line bundle. Its inverse image by $(v, \check{v})$ is a $\mathbb{G}_m$-torsor L on $X \times X$ regarded as a scheme of dimension zero. If ψ is one trivialization, then any other is of the form $\psi \circ g$, with g an element of $\mathbb{G}_m(X \times X)$ invariant under the symmetry $X \times X \to X \times X$. Consequently, we have the following result.

COROLLARY 2.7. *The symmetric one-motives with* $(A, \lambda, v : X \to A)$ *fixed form a torsor under* $Hom_{\mathrm{sym}}(X \times X, \mathbb{G}_m) = Hom(S^2(X), \mathbb{G}_m)$.

Hodge cycles. When M is a one-motive over $\mathbb{C}$, we define a *Hodge cycle* on M to be a Hodge tensor for the mixed Hodge structure $H_B(M)$. Propositions 2.4 and 2.5 show that such a cycle has realizations in the de Rham and ℓ-adic homology groups of M. When M is defined over an algebraically closed field k, we say that a family $s = (s_{\mathrm{dR}}, (s_\ell))$ is a *Hodge cycle relative to an embedding* $\tau : k \hookrightarrow \mathbb{C}$ if the components of s become the components of a Hodge cycle s_0 on τM.

PROPOSITION 2.8. *Let M be a one-motive over an algebraically closed field k. If s is a Hodge cycle on M relative to one embedding of k in $\mathbb{C}$, then it is a Hodge cycle for every embedding.*

PROOF: The proof of (I.3.1) can be extended to one-motives; see Brylinski (1983), 2.2.5.

The procedure of (I.3) now allows us to define the notion of a Hodge cycle for a one-motive over any field of characteristic zero.

One-motives of CM**-type.** A one-motive $M = (X \xrightarrow{u} G)$ over a field k is said to be *rationally decomposed* if the image of u is finite and the class of G in $\mathrm{Ext}^1(A, T)$ is of finite order. It is then isogenous to the one motive $X \xrightarrow{0} T \times A$. When $k = \mathbb{C}$, M is rationally decomposed if and only if the mixed Hodge structure $H_B(M)$ is isomorphic to the direct sum of the pure Hodge structures $H_B(T)$, $H_B(A)$, and $X \otimes \mathbb{Q}$ (these are of types $\{(-1,-1)\}$, $\{(-1,0),(0,-1)\}$, and $\{(0,0)\}$ respectively. To such a one-motive M, we attach a motive

$$hM = h(X_*(T) \otimes \mathbb{Q}) \oplus h(A) \oplus h(X \otimes \mathbb{Q})$$

in $\mathbf{AV}/k$ (the first and last summands are elements of $\mathbf{Art}/k$; see I.4.1).

A one-motive M is said to be of *CM-type* if it is rationally decomposed and A_M is of CM-type. Then hM lies in $\mathbf{CM}/k$. In particular, when M is defined over $\mathbb{C}$, its Mumford-Tate group is a quotient of $\mathfrak{S}$, and when M is defined over $\mathbb{Q}$, it corresponds to a representation of the Taniyama group.

Moduli of one-motives. Let M be a one-motive over $\mathbb{C}$, and write $(H, W_., F^.)$ for $H_B(M)$ with its mixed Hodge structure. As in (I.1), the Mumford-Tate group P of M acquires a filtration

$$1 = W_{-3}P \subset W_{-2}P \subset W_{-1}P \subset W_0 P = P$$

from the weight filtration on H:

$$W_{-i}P = \{p \in P \mid (id - p)(W_m H_B(M)) \subset W_{m-i}H_B(M), \text{ all } m\}.$$

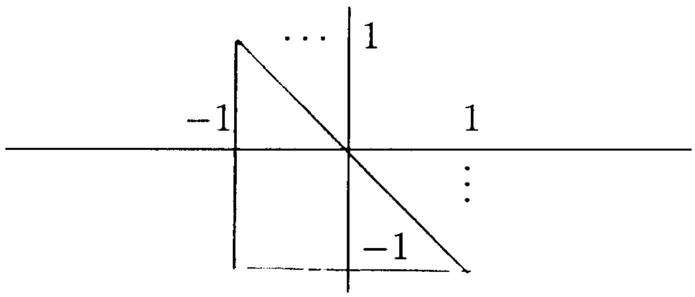
The group $W_{-1}P$ is unipotent, and the quotient $P/W_{-1}P$ is the Mumford-Tate group of $\mathrm{Gr}_{-1}(M) = A$. Therefore $P/W_{-1}P$ is reductive, and $W_{-1}P$ is the unipotent radical of P.

LEMMA 2.9. *(a) For all $p \in P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C})$, the filtration $p \cdot F^{\cdot}$ of $H \otimes \mathbb{C}$ defines a mixed Hodge structure on (H, W).*

(b) There exists a $p \in W_{-1}P(\mathbb{C})$ such that the mixed Hodge structure $(H, W_., p \cdot F^{\cdot})$ is rationally decomposed.

PROOF: Brylinski (1983), 2.2.8 (see also VI.1).

LEMMA 2.10. *The mixed Hodge structure on Lie P defined by $(W_., p \cdot F^{\cdot})$ is of type $\{(-1,-1); (-1,0), (0,-1); (-1,1), (0,0), (1,-1)\}$.*



It follows that $F^0P \cap W_{-1}P$ is commutative, because,

$$[F^0P \cap W_{-1}P, F^0P \cap W_{-1}P] \subset F^0P \cap W_{-2}P = 0.$$

Choose a lattice $H(\mathbb{Z})$ in H. The family of one-motives $p \cdot M$, $p \in P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C})$ is parametrized by the space

$$V = \Gamma \backslash P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C}) / F^0P(\mathbb{C})$$

where Γ is the subgroup of $P(\mathbb{Q})$ respecting the lattice.

THEOREM 2.11. *When Γ is replaced by a sufficiently small congruence subgroup, the variety V has a natural structure of an algebraic variety, and the analytic family of one-motives over it also has a natural structure of an algebraic variety.*

PROOF: Brylinski (1983), 2.3.2.1 (see also Chapter VI).

By introducing level structures and Hodge cycles, it is possible to strengthen the theorem in order to obtain a universal family of one-motives.

Notes. The concept of a one-motive is due to Deligne (1975).

3. Degenerating families of symmetric one-motives.
Understanding the boundaries of Shimura varieties of Hodge type is closely related to understanding the degeneration of abelian varieties and one-motives. The degeneration theorem we state below is an algebraic analogue of the following analytic statements. Let D be the unit disk and let $D' = D - \{0\}$. Consider functions $f_i : D \to \mathbb{C}$ such that $f_i(z) \neq 0$ for $z \neq 0$ and $f_i(0) = 0$ for $1 \leq i \leq r$. Let $\mathcal{T}$, $\mathcal{G}$, and $\mathcal{A}$ to be the complex manifolds over D whose fibres over $z \in D$ are:

$\mathcal{T}_z = \mathbb{C}^{\times r}$,

$\mathcal{G}_z = \mathbb{C}^{\times m} / \langle f_{r+1}(z), \ldots, f_m(z) \rangle$,

$\mathcal{A}_z = \mathbb{C}^{\times m-r} / \langle f_{r+1}(z), \ldots, f_m(z) \rangle$.

Here $\langle f_{r+1}(z), \ldots, f_m(z) \rangle$ is the abelian subgroup generated by $f_{r+1}(z), \ldots, f_m(z)$. There is an exact sequence

$$0 \to \mathcal{T} \to \mathcal{G} \to \mathcal{A} \to 1.$$

The functions $f_1, \ldots, f_r$ define a map $u : X \to \mathcal{G}$ where X is the constant local system $\mathbb{Z}^r$ on D. Let $A = \mathcal{G}/u(X)$. Then A is the complex-analytic analogue of a semi-abelian variety, the map $\mathcal{G} \to A$ is a local isomorphism, and the fibre of A over 0 is equal to the fibre of $\mathcal{G}$ over 0.

Let R a Noetherian, excellent, normal ring that is complete with respect to a radical ideal $\mathfrak{I}$; let

$S = \operatorname{Spec} R$;

$\eta =$ generic point of $S = \operatorname{Spec} K$;

$S_0 = \operatorname{Spec} A/\mathfrak{I}$.

Intuitively, a degenerating one-motive over S is a one-motive over $S - S_0$ whose period group degnerates totally along S_0. It is most convenient to state the definition in terms of the quadruple considered in (2.6).

Definition 3.1. A *degenerating family of symmetric one-motives* over S is:

(a) an abelian scheme $p : \mathcal{A} \to S$ and a polarization $\lambda : \mathcal{A} \to \check{\mathcal{A}}$;

(b) a morphism $v : X \to \mathcal{A}(S)$, where X is a free $\mathbb{Z}$-module of finite rank; let $\check{v} = \lambda \circ v$;

(c) a symmetric trivialisation ψ of the inverse image by $(v, \check{v})$ of the Poincaré biextension of $\mathcal{A}_K$ and $\check{\mathcal{A}}_K$ by $\mathbb{G}_m$.

There is also a degeneracy condition for whose statement we refer to Brylinski (1983), 3.1.1.

From the data in (a) and (b), we can construct a semi-abelian variety $\mathcal{G}$ over S: let $\mathcal{T}$ be the constant split torus over S with $X^*(\mathcal{T}) = X$; then $\mathcal{G}$ is an extension of $\mathcal{A}$ by $\mathcal{T}$, such that, for all characters χ of T, $\chi_*(\mathcal{G})$ is an element of $\mathrm{Ext}^1(\mathcal{A}, \mathbb{G}_m)$ representing $\check{v}(\chi)$.

THEOREM 3.2. *There exists a semi-abelian scheme A over S, arising in a natural way from a degenerating one-motive, such that*

(a) the formal completion of A is the quotient of the formal completion of $\mathcal{G}$ by the group of periods $u(X)$;

(b) the restrictions to S_0 of the semi-abelian schemes A and $\mathcal{G}$ are canonically isomorphic.

PROOF: In the case that $\mathcal{G} = \mathcal{T}$ this was proved by Mumford (1972). Apparently, he also proved the general case, but never published it. There is a sketch of a proof in Brylinski (1983) and a detailed proof in Chai (1985).

Remark 3.3. In Faltings (1985) there is an important converse to (3.2).

Notes. The theorems in this section are due to Mumford (1972), Brylinski (1983), Faltings (1985), and Chai (1985). The most complete account is in Chai and Faltings (1989).

V. TOROIDAL COMPACTIFICATION

We explain the how to construct (smooth) toroidal compactifications of Shimura varieties, and suggest how the isomorphisms of Chapters II and III extend to these compactifications.

1. Torus embeddings. We review (without proofs) the construction in algebraic geometry on which the method of toroidal compactifications is based. Throughout this section, k will be an algebraically closed field, and "variety" will mean a reduced irreducible separated scheme locally of finite-type over k. All semigroups have zero elements and a subsemigroup of a (semi-) group contains the zero element of the (semi-) group.

Definitions. Let T be an d-dimensional torus over a field k. Write $M = X^*(T) \subset \Gamma(T, \mathcal{O}_T)$ and $N = X_*(T)$. For $r \in M$ let χ^r be the corresponding element of $\Gamma(T, \mathcal{O}_T)$, and for $a \in N$, let $\mu_a : \mathbb{G}_m \to T$ be the corresponding cocharacter. We have a pairing

$$\langle , \rangle : M \times N \to \mathbb{Z}, \quad \chi^r(\mu_a(t)) = t^{\langle r,a \rangle}.$$

As a k-algebra, $\Gamma(T, \mathcal{O}_T)$ is generated by $\{\chi^r \mid r \in M\}$. Moreover, if $r_1, \ldots, r_d$ is a basis for M, then

$$\Gamma(T, \mathcal{O}_T) = k[\chi_{r_1}, \chi_{r_1}^{-1}, \ldots, \chi_{r_d}, \chi_{r_d}^{-1}].$$

A *torus embedding* of T is an open immersion $T \hookrightarrow X$ of varieties together with an action of T on X whose restriction to T is the multiplication map. A morphism of torus embeddings is a homomorphism $f : X \to X'$ whose restriction to T is a homomorphism $T \to T'$ and which makes

$$\begin{array}{ccc} T \times X & \longrightarrow & X \\ {\scriptstyle (F|T)\times f}\downarrow & & \downarrow{\scriptstyle f} \\ T' \times X' & \longrightarrow & X' \end{array}$$

commute. The torus embedding is said to be *affine* if X is affine.

Affine torus embeddings. Let $S \subset M$ be a finitely generated semigroup, and let $k[S]$ be the subalgebra of $\Gamma(T, \mathcal{O}_T)$ generated by $\{\chi^r \mid r \in S\}$. It is a finitely generated k-subalgebra of $\Gamma(T, \mathcal{O}_T)$, and its field of fractions is $k(T)$ if and only if S generates M (as a group). In this case, T acts on $X_S =_{\mathrm{df}} \operatorname{Spec} k[S]$, and $T \hookrightarrow X_S$ is an affine torus embedding. We have

$$X_S(k) = \operatorname{Hom}_*(S, k) =_{\mathrm{df}} \{x : S \to k \mid x(0) = 1, x(s+s') = x(s)x(s')\}.$$

Example 1.1. Let $T = \mathbb{G}_m^d$, so that $M = \mathbb{Z}^d$ and the coordinate ring of T, $k[T] = k[\chi_1, \chi_1^{-1}, \ldots]$. Let

$$S = \{(m_i, \ldots, m_d) \mid m_i \geq 0, i = 1, \ldots, s\}.$$

Then $\operatorname{Spec} k[S] = k^s \times (k^\times)^{d-s}$.

Let φ be a morphism $\mathbb{A}^1 - \{0\} \to X$; when φ extends to a morphism $\tilde{\varphi} : \mathbb{A}^1 \to X$, we say that $\lim_{t\to 0} \varphi(t)$ exists and equals $\tilde{\varphi}(a)$. With this definition, it is possible to describe X_S as the variety obtained from T by adding certain limit points: for each $a \in N$, $\lim_{t\to 0} \mu_a(t)$ exists in X_S if and only if $\langle a, S\rangle \geq 0$.

Proposition 1.2. *(a) The map $S \mapsto (T \hookrightarrow X_S)$ defines a one-to-one correspondence between the set of finitely generated semigroups S in M generating M as a group and the set of isomorphism classes of affine torus embeddings of T.*

(b) An inclusion $S \subset S'$ defines a morphism $X_{S'} \hookrightarrow X_S$

(c) X_S is a normal variety if and only if S is a saturated in M, i.e., $m \in S$ whenever $rm \in S$ for some $r \in \mathbb{N}$, $r \neq 0$.

We want to patch affine torus embeddings together; for this it is convenient use different combinatorial data, so that the functor attaching a torus embedding to the data is covariant. A subset $\sigma \subset N_{\mathbb{R}}$ is called a *convex polyhedral cone* if there exist vectors $n_1, \ldots, n_s$ in $N_{\mathbb{R}}$ such that

$$\sigma = \{\sum_{i \geq 1} a_i n_i \mid a_i \in \mathbb{R},\ a_i \geq 0\}.$$

It is *rational* if the n_i can be chosen in N, and it is *strongly convex* if further $\sigma \cap (-\sigma) = 0$ (equivalently, σ contains no nonzero subspace of $N_{\mathbb{R}}$). The dimension of the subspace generated by σ is called the *dimension* of σ.

Let $\sigma = \sum \mathbb{R}_{\geq 0} n_i$ be a strongly convex rational polyhedral cone. If we remove redundant n_i's and require each to be primitive (that is, such that $rn_i \notin N$, $r \in \mathbb{Z}$, $r > 1$), then the set $\{n_1, \ldots, n_r\}$ is uniquely determined. These n_i are called the *fundamental generators* of σ.

The *dual* of σ is the convex rational polyhedral cone $\check{\sigma}$ in $M_{\mathbb{R}}$:

$$\check{\sigma} = \{r = M_{\mathbb{R}} \mid \langle r, a \rangle \geq 0, \text{ all } a \in \sigma\}.$$

PROPOSITION 1.3. *The map $\sigma \mapsto \check{\sigma} \cap M$ defines a one-to-one correspondence between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of finitely generated semigroups $S \subset M$ generating M and saturated in M.*

For a convex rational polyhedral cone σ in $N_{\mathbb{R}}$, write X_σ for $\operatorname{Spec} k[\check{\sigma} \cap M]$. Note that for the cone $\sigma_0 = \{0\}$, $X_{\sigma_0} = T$. On combining the last two propositions, we obtain the following result.

COROLLARY 1.4. *The map $\sigma \mapsto X_\sigma$ defines a one-to-one correspondence between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of affine normal torus embeddings of T.*

Remark 1.5. The following criterion allows us to reconstruct σ from X_σ: an element a of N lies in $\sigma \Leftrightarrow \lim_{t \to 0} \mu_a(t)$ exists in X_σ.

PROPOSITION 1.6. *The variety X_σ is nonsingular if and only if the fundamental generators of σ form part of a $\mathbb{Z}$-basis of N.*

A strongly convex rational polyhedral cone satisfying the condition in the proposition is said to be *nonsingular*.

The intersection of a strongly convex rational polyhedral cone σ with a hyperplane that does not meet the interior of σ is called a *face*, $\tau \prec \sigma$, of σ. There is then an r_0 in $\check{\sigma} \cap M$ such that

$$\tau = \{x \in \sigma \mid \langle r_0, x\rangle = 0\},$$

and τ is again a strongly convex rational polyhedral cone. The semigroup $\check{\tau} \cap M$ associated with τ is $\check{\sigma} \cap M + \mathbb{N}(-r_0)$.

PROPOSITION 1.7. *If τ and σ are strictly convex rational polyhedral cones and $\tau \subset \sigma$, then there is a natural morphism $X_\tau \to X_\sigma$ of torus embeddings; the morphism is an open immersion if and only if τ is a face of σ.*

On points, the map is the natural inclusion $\mathrm{Hom}_*(\check{\tau} \cap M, k) \hookrightarrow \mathrm{Hom}_*(\check{\sigma} \cap M, k)$ induced by $\check{\sigma} \cap M \hookrightarrow \check{\tau} \cap M$.

General torus embeddings. The last result suggests how to patch together X_σ for different σ.

Definition 1.8. A *fan* (formerly, rational partial polyhedral decomposition) of $N_{\mathbb{R}}$ is a nonempty collection $\Delta = \{\sigma\}$ of strongly convex rational polyhedral cones such that:

(i) every face of a cone in Δ is also in Δ;

(ii) if σ and σ' are in Δ, then $\sigma \cap \sigma'$ is a face of both σ and σ' .

The set $|\Delta| = \cup_{\sigma \in \Delta} \sigma$ is called the *support* of Δ, and Δ is said to be *complete* if $|\Delta| = N_{\mathbb{R}}$.

For example, the set of all faces of a strongly convex rational polyhedral cone is a fan. Let Δ be a fan in $N_{\mathbb{R}}$, and let

$$X_\Delta = \{(\sigma, \pi) \mid \sigma \in \Delta,\ \pi \in \mathrm{Hom}_*(\check{\sigma} \cap M, k)\}/\sim,$$

where $(\sigma, \pi) \sim (\sigma', \pi')$ if and only if π and π' are restrictions of a single element of $\mathrm{Hom}_*((\sigma \cap \sigma')^\vee \cap M, k)$.

PROPOSITION 1.9. *The space X_Δ has a unique structure of an algebraic variety for which the maps $X_\sigma \hookrightarrow X_\Delta$ are open immersions for all $\sigma \in \Delta$. In particular, $T = X_{\sigma_0} \hookrightarrow X_\Delta$ is an open immersion. There is a unique action of T on X_Δ extending its action on each X_σ.*

To summarize: we have attached to each fan in $N_{\mathbb{R}}$ a normal torus embedding $T \subset X_\Delta$.

Example 1.10. Let $N = \mathbb{Z}$, $\sigma = \mathbb{R}_{\geq 0} \subset N_{\mathbb{R}}$, $\Delta = \{\sigma, -\sigma, \{0\}\}$; then $X_\Delta = \mathbb{P}^1$.

THEOREM 1.11. *(a) X_Δ is of finite-type if and only if Δ is finite.*

(b) X_Δ is nonsingular if and only each X_σ is nonsingular.

(c) X_Δ is complete if and only if Δ is a finite and complete fan.

(d) X_Δ is quasi-projective if and only if Δ is finite and there is a continuous real-valued convex function on the convex hull of $|\Delta|$ such that

(i) $f|\sigma$ is $\mathbb{R}$-linear, all $\sigma \in \Delta$;

(ii) f takes integer values on $N \cap |\Delta|$;

(iii) for each $\sigma \in \Delta$, there is an $r_\sigma \in M$ and an $n_\sigma > 0$ such that $n_\sigma f \geq r_\sigma$ on $|\Delta|$ and

$$\sigma = \{a \in N_{\mathbb{R}} \mid \langle r_\sigma, a\rangle = n_\sigma f(a)\}.$$

The function f in (iii) is called a *polar function.* It defines a T-equivariant ample invertible sheaf on X_Δ.

Remark 1.12. (a) The X_σ for $\sigma \in \Delta$ are the T-stable affine open subsets of X_Δ. In particular, X_Δ is affine if and only if there is a $\sigma \in \Delta$ such that Δ coincides with the set of faces of π.

(b) The description given above for the k-points of X_Δ extends to a description of the functor of k-schemes defined by X_Δ (see Ash et al. 1975, p10, except note that they forget to pass to the equivalence classes).

PROPOSITION 1.13. *Each torus embedding $T \subset X$ with X normal is isomorphic to the torus embedding defined by a fan Δ in $X_*(T) \otimes \mathbb{R}$, and Δ is uniquely determined.*

Equivariant maps. A *map of fans* $\varphi : (N', \Delta') \to (N, \Delta)$ is a homomorphism $\varphi : N' \to N$ such that the image under $\varphi_{\mathbb{R}}$ of each $\sigma' \in \Delta'$ is contained in a $\sigma \in \Delta$.

PROPOSITION 1.14. *Let $\varphi : (N', \Delta') \to (N, \Delta)$ be a map of fans; the map $T_{N'} \to T_N$ defined by φ extends uniquely to a morphism $\varphi_* : X_{\Delta'} \to X_\Delta$, and ϕ_* is equivariant. Each morphism of torus embeddings $X_{\Delta'} \to X_\Delta$ arises in this way from a unique map of fans.*

PROPOSITION 1.15. *The morphism φ_* is proper and birational if and only if $\varphi : N' \to N$ is an isomorphism and Δ' is a locally finite subdivision of Δ.*

Rationality of torus embeddings over subfields. Let $\tau : k \hookrightarrow k'$ be an inclusion of k into a second algebraically closed field k'. Then τ defines an isomorphism $X_*(T) \to X_*(\tau T)$, and a fan Δ in $X_*(T) \otimes \mathbb{R}$ is mapped to a fan $\tau\Delta$ in $X_*(\tau T) \otimes \mathbb{R}$. Clearly, $\tau(X_\Delta) = X_{\tau\Delta}$ as torus embeddings of τT.

Now suppose that T is defined over a subfield k_0 of k over which k is Galois. Then $\mathrm{Gal}(k/k_0)$ acts on N (through its action on T), and descent theory shows that a quasi-projective normal torus embedding $T \hookrightarrow X_\Delta$ is defined over k_0 if and only if Δ is stable under the action of $\mathrm{Gal}(k/k_0)$ on $N_{\mathbb{R}}$.

Toroidal embeddings. Let Y be a normal variety, and let U be a smooth open subset of Y. We say that $U \subset Y$ is a *toroidal embedding* if it is a torus embedding locally for the étale topology. We mean by this that for every closed point y of Y there is an open neighbourhood Y' of y, a normal affine torus embedding $T \subset X$, and an étale map $\pi : Y' \to X$ such that $\pi^{-1}(T) = U \cap Y'$:

$$\begin{array}{ccccc} Y & \overset{\text{open}}{\hookleftarrow} & Y' & \xrightarrow{\text{étale}} & X \\ \cup & & \cup & & \cup \\ U & \hookleftarrow & U \cap Y' & \longrightarrow & T. \end{array}$$

Compactification of torsors. Let V be a variety, and let P be a T-torsor over V. For any torus embedding $T \hookrightarrow X$ we can define:

$$P \times {}^T X = (P \times X)/\sim,\ (pt, x) \sim (p, tx),\ p \in P,\ x \in X,\ t \in T.$$

This is a variety over X. The choice of a point p in the fibre P_v of P over a closed point $v \in V$ defines an isomorphism

$$\begin{array}{ccc} T & \hookrightarrow & X \\ \downarrow\approx & & \downarrow\approx \\ P_v & \hookrightarrow & (P \times {}^T X)_v. \end{array}$$

A similar construction can be made when V is a complex manifold. In this case, $P \times {}^T X$ is a fibre bundle over V with standard fibre X (see Kobayashi and Nomizu, 1963).

Notes. Detailed proofs of the results in this section can be found in Kempf et al. (1972) and Oda (1978), (1987).

2. Study of the boundary of symmetric Hermitian domains. There is a very elaborate theory concerning the boundaries of Hermitian symmetric domains. We can include only a very brief sketch.

Rational boundary components. Let D be a symmetric Hermitian domain. Since we are interested in its boundary, we assume D to be noncompact. There then exists a semisimple group G over $\mathbb{Q}$ such that $G(\mathbb{R})^+ = \mathrm{Aut}(D)^+$.

As was explained in (III.1), there is a canonical embedding $\beta : D \hookrightarrow \check{D}$ of D into its compact dual. The closure $\bar{D}$ of D in $\check{D}$ is called the *natural compactification* of D. The action of $G(\mathbb{R})^+$ on D extends to a continuous action on $\bar{D}$. The space $\bar{D}$ can be decomposed according to the equivalence relation generated by the following relation: $x \sim y$ if there is a holomorphic map $\lambda : D_1 \to \check{D}$ from the unit disk D_1 into $\check{D}$ such that $\{x, y\} \subset \lambda(D_1) \subset \bar{D}$. The equivalence classes are called the *boundary components* of D. Note that this definition allows D itself to be an boundary component of $\bar{D}$ (called the *improper* boundary component).

The *normalizer* of a boundary component F is the subgroup $\mathcal{N}$ of $G(\mathbb{R})^+$ containing those g such that $gF = F$. The component F is said to be *rational* if there is a subgroup $N^F \subset G$ (defined over $\mathbb{Q}$ such that $\mathcal{N}^+ = N^F(\mathbb{R})^+$.

PROPOSITION 2.1. *(a) When G is simple, the map $F \mapsto N^F$ is a bijection from the set of proper rational boundary components of D to the set of maximal parabolic subgroups of G.*

(b) Suppose $G = G_1 \times \cdots \times G_m$ with each G_i simple, and let $D = D_1 \times \cdots \times D_m$ be the corresponding decomposition of X. The rational boundary components F of D are products $F_1 \times \cdots \times F_m$ with each F_i a rational boundary component of D_i, and the normalizer of such an F is the product of the normalizers of the F_i

PROOF: See Baily and Borel (1966), 3.7.

From now on, we assume G to be simple (over $\mathbb{Q}$).

Example 2.2. Let (V, ψ) be a symplectic space, let $G = \mathrm{Sp}(V, \psi)$, and let D be the corresponding Siegel upper-half-space. For any totally isotropic subspace W of V, the stabilizer N of W in V is a maximal parabolic subgroup of G, and all such subgroups are of this form. The boundary component F corresponding to N is isomorphic to the Siegel upper-half-space defined by the symplectic space $(W^{\perp}/W, \bar{\psi})$.

For example, if $\dim V = 2$, then the totally isotropic subspaces are the (rational) lines in V. They are in one-to-one correspondence with the points of $\mathbb{P}^1(\mathbb{Q})$. When D is realized as the open unit disk, then the rational boundary components are the points on the circle that lie on a line through the origin with rational slope.

Cayley filtrations. For each point $x \in D$, there is homomorphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$ such that $h_x(z)$ fixes x and acts on $\mathrm{Tgt}_x(D)$ as multiplication by z^2. The map $x \mapsto h_x$ identifies D with a $G(\mathbb{R})^+$-conjugacy class of maps. For a representation (V, ξ) of $G_{\mathbb{R}}$, $\xi \circ h_x$ defines a Hodge structure on V and a (decreasing) Hodge filtration $F_x^{\cdot}$ on $V(\mathbb{C})$.

Definition 2.3. A filtration $W_{\cdot}$ of $\mathbf{Rep}_{\mathbb{Q}}(G)$ is said to be *Cayley* if for all $x \in D$ and all representations $\xi : G \to GL(V)$, the filtrations $W_{\cdot}$ and $F_x^{\cdot}$ of V define a mixed Hodge structure on V.

PROPOSITION 2.4. *If $W_{\cdot}$ is a Cayley filtration, then W_0G is a maximal parabolic subgroup of G, and every maximal parabolic subgroup of G is associated in this way with a unique Cayley filtration.*

PROOF: See Deligne (1973), 3.1.13.

Thus each rational boundary component F defines a Cayley filtration $W_{\cdot}$ of $\mathbf{Rep}_{\mathbb{Q}}(G)$. Deligne (ibid. 3.1.14) shows that for each F, there is a unique cocharacter w_F of G splitting the corresponding Cayley filtration and such that $(\mathrm{ad}\, h(i)) \circ w_F = w_F^{-1}$.

THEOREM 2.5. *Fix a base point $o \in D$ and a rational boundary component F of $\bar{D}$. Then there exists a unique homomorphism*

$$\varphi_F : U^1 \times SL(2, \mathbb{R}) \to G(\mathbb{R})$$

such that

(i) $\varphi_F(e^{i\theta}, r(\theta)) = h_o(e^{i\theta})$, $r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,

(ii) $\varphi_F(1, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}) = w_F(\lambda)$, $\lambda \in U^1$.

PROOF: Deligne (1973), 3.1.14.

Remark 2.6. (a) Let H be the upper-half-plane. There is a holomorphic map $f_F : H \to D$ that is equivariant for φ_F and such that $f_F(i) = o$ and $f_F(\infty) \in F$ (Ash et al. 1975, p199).

(b) Since G is simple it can be written $G = \mathrm{Res}_{F/\mathbb{Q}} G'$ with G' an absolutely simple group over a totally real field F. Choose a point $o \in D$ such that h_o factors through $T(\mathbb{R})$ with T a maximal torus in G. If E is a CM-field splitting T', then φ_F is defined over the maximal totally real subfield of E (because both h_0 and w_F are).

The structure of N^F. Fix a base point $o \in D$ and a rational boundary component F. The Hodge structure on $\mathfrak{g}$ defined by h_0 is of type $\{(-1,1),(0,0),(1,-1)\}$ (cf. II.1). It follows that the nonzero Hodge numbers h^{pq} of the mixed Hodge structure $(\mathfrak{g}, W_\cdot, F_o^\cdot)$ satisfy $|p|$, $|q| \leq 1$. The action of w_F therefore defines a grading:

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2.$$

There are attached to F the following algebraic groups over $\mathbb{Q}$:

$N^F = W_0G$; $\operatorname{Lie} N^F = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$.

$W^F = W_{-1}G =$ unipotent radical of N^F; $\operatorname{Lie} W^F = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$.

$U^F = W_{-2}G =$ centre of W^F; this is an abelian group, which we can identify with its Lie algebra $\mathfrak{g}^{-2}$.

$Z(w_F) =$ centralizer of w_F in N^F; $\operatorname{Lie}(w_F) = \mathfrak{g}^0$, and $N^F = W^F \rtimes Z(w_F)$.

$V^F = W^F/U^F$; this is an abelian group, which we can identify with its Lie algebra $\mathfrak{g}^{-1}$. Write $\mathfrak{g}_\ell = [\mathfrak{g}^2, \mathfrak{g}^{-2}]$, and $\mathfrak{g}_h =$ orthogonal complement $[\mathfrak{g}^2, \mathfrak{g}^{-2}]$ in $\mathfrak{g}^0$. The decomposition $\mathfrak{g}^0 = \mathfrak{g}_\ell + \mathfrak{g}_h$ can be integrated to an isogeny $G_h \times G_\ell \to Z(w_F)$. In summary:

$$\begin{array}{rl}
W^F \rtimes Z(w_F) = & N^F \\
 & \mid \sim G_h \times G_\ell \\
 & W^F \\
 & \mid \quad V^F \\
 & U^F \\
 & \mid \\
 & \{1\}
\end{array}$$

PROPOSITION 2.7. *(a) F is a symmetric Hermitian domain; G_h is semisimple, and*

$$G_h(\mathbb{R})^+/(\textit{maximal compact subgroup}) = \mathit{Aut}(F)^+.$$

(b) The morphism φ_W sends U^1 into G_h, and it sends $SL_2(\mathbb{R})$ into G_ℓ; moreover, $\varphi_W|U^1 : U^1 \to G_h(\mathbb{R})$ defines the complex structure on F.

(c) G_ℓ is reductive without compact factors.

(d) The centralizer of F, $\mathcal{Z} = \{g \in G(\mathbb{R}) \mid gx = x \text{ all } x \in F\}$, has identity component $G_\ell \times W^F$.

(e) $G_h \cdot W^F$ centralizes U^F.

PROOF: Ash et al. (1975), III.3.

Example 2.8. With the notations of (2.2), $G_h = \mathrm{Sp}(W^{\perp}/W, \bar{\psi})$, $G_\ell = GL(W)$, and $M^F = 0$. Moreover, U^F is the space of symmetric bilinear forms on $V(\mathbb{R})$.

The canonical self-dual open cone in $U^F(\mathbb{R})$. In addition to the closed cones of §1, we shall need to consider *open cones* in real vector spaces. Such a cone C in a real space V is said to be *self-dual* if there exists a positive-definite inner form $\langle\, ,\rangle$ on V with the property that $x \in C$ if and only if $\langle x, y\rangle > 0$ whenever $0 \neq y \in \bar{C}$ (closure of C). The cone is said to be *homogeneous* if the group $\mathrm{Aut}(V, C)$ of automorphisms of V stabilizing C acts transitively on C.

Example 2.9. Every homogeneous self-adjoint cone can be written as a product of indecomposable cones. Apart from one family of semi-classical cones and one exceptional cone, every indecomposable homogeneous self-adjoint cone is isomorphic to a cone in the following list:

(i) the cone of positive-definite real symmetric matrices;
(ii) the cone of positive-definite Hermitian complex matrices;
(iii) the cone of positive-definite Hermitian quaternion matrices.

The Killing form B defines a Hermitian form on $\mathfrak{g}_{\mathbb{C}}$,

$$B'(x, y) = -B(x, \iota y), \quad x, y \in \mathfrak{g}_{\mathbb{C}},$$

which restricts to a positive-definite form on $\mathfrak{u}^F$. The isomorphism $\exp : \mathfrak{u}^F \to U^F$ allows us to transfer this to U^F.

Define Ω_F to be the point $\varphi_F(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ of U^F. Then the orbit of Ω_F in $U^F(\mathbb{R})$ under $G_\ell(\mathbb{R})$,

$$C(F) = \{g\Omega_F g^{-1} \mid g \in G_\ell(\mathbb{R})\},$$

is a homogeneous open cone in $U_F(\mathbb{R})$, which is self-dual relative to B'.

Example 2.10. In the situation of (2.2), $C(F)$ is the cone of all positive-definite bilinear forms on W.

Definition of Siegel domains.

Definition 2.11. Let U be a real vector space and let C be an open convex cone in U whose closure does not contain an entire straight line. Then

$$S = \{z \in U(\mathbb{C}) \mid \mathrm{Im}(z) \in C\} = U + \mathrm{i}C$$

is a *tube domain* (or Siegel domain of the first kind).

Let U be a real vector space and V a complex vector space; a real-bilinear map $V \times V \to U(\mathbb{C})$ is said to be *semi-Hermitian* if it can be written as the sum of a symmetric complex-bilinear map and a Hermitian map.

Definition 2.12. Let U be a real vector space, let V be a complex vector space, and let $\mathcal{D}$ be a bounded domain in some space $\mathbb{C}^k$; let $C \subset U$ be a cone satisfying the conditions of (2.11). Suppose that for each $t \in D$ there is given a nondegenerate semi-Hermitian form L_t on V with values in U. Then

$$S = \{w = (z, v, t) \in U \times V \times \mathcal{D} \mid \operatorname{Im}(z) - \operatorname{Re}(L_t(v, v)) \in C\}$$

is a *Siegel domain (of the third kind).* Thus a Siegel domain can be thought of as a family of tube domains parametrized by $V \times \mathcal{D}$.

Realization of D as a Siegel domain. We now describe the realization of D as a Siegel domain of the third kind, attached to the component F. Let

$$D(F) = U^F(\mathbb{C}) \cdot D = \cup_{g \in U^F(\mathbb{C})} gD \subset \check{D}.$$

Example 2.13. In the situation of 2.2, $D(F)$ is the set of maximal isotropic subspaces $F^0 \subset V$ such that $(V_., W_., F^\cdot)$ is a mixed Hodge structure and $\bar{\psi}$ is a polarization of $W/W^\perp$. Here $W.$ and $F^\cdot$ are the filtrations:

$$0 \subset W \subset W^\perp \subset V, \quad V = F^{-1}V \supset F^0 \supset F^1V = 0.$$

There is a $N^F(\mathbb{R}) \cdot U^F(\mathbb{C})$-equivariant map $\Phi_F : D(F) \to U^F(\mathbb{C})$ such that $\mathcal{D} = \Phi_F^{-1}(C)$. The space $D(F)$ can be decomposed by means of two successive fibrations:

$$\begin{array}{cl} & D(F) \\ & \big\downarrow \pi_1 \\ \pi_F \Bigg\downarrow & D(F)' = U^F(\mathbb{C})\backslash D(F) \\ & \big\downarrow \pi_2 \\ & F \end{array}$$

Moreover,

$D(F)$ is a fibre bundle over $D(F)'$ for the complex group $U^F(\mathbb{C})$;
$D(F)' \to F$ is a principal C^∞-fibration for the group $V^F(\mathbb{R})$.
Both fibrations can be trivialized,

$$D(F) \approx U^F(\mathbb{C}) \times D(F)' \approx U^F(\mathbb{C}) \times V^F(\mathbb{C}) \times F,$$

and with the choice of such a decomposition, Φ_F can be expressed

$$\Phi_F(z, v, t) = \mathrm{Im}(z) - h_t(v, v), \;\; z \in U^F(\mathbb{C}),\; v \in V^F(\mathbb{C}),\; z \in F$$

with h_t a real bilinear form $V^F(\mathbb{R}) \times V^F(\mathbb{R}) \to U^F$ depending real-analytically on t. Thus D is equal to

$$\{(z, v, t) \mid z \in U^F(\mathbb{C}),\; v \in V^F(\mathbb{C}),\; t \in F, \mathrm{Im}(z) - h_t(v, v) \in C(F)\},$$

which realizes it as a Siegel domain.

Algebraicity of the quotient of $D(F)$ by a discrete group. An arithmetic subgroup Γ of $G(\mathbb{R})$ is said to be *neat* if it consists of elements g such that, for one (hence every) faithful complex representation ξ of G, the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of $\xi(g)$ is torsion-free. In particular, a neat subgroup is torsion-free.

Choose a neat arithmetic subgroup Γ of $G(\mathbb{R})$ (every arithmetic subgroup contains a subgroup of finite index that is neat), and define:

$\Gamma(F) = \Gamma \cap \mathcal{N}$; it is a discrete subgroup of N^F;

$\Gamma'(F) =$ subgroup of $\Gamma(F)$ of elements acting trivially (by conjugation) on U^F;

$\Gamma_h(F) =$ image of $\Gamma(F)$ in $G_h(\mathbb{Q})$; it is a neat subgroup of $G_h(\mathbb{Q})$, and so $\Gamma_h(F)\backslash F$ is a locally symmetric variety.

The quotient $U^F(\mathbb{C})/(U^F(\mathbb{C})\cap\Gamma)$ is compact, and $U^F(\mathbb{C})\cap\Gamma$ is discrete in $U^F(\mathbb{C})$; therefore $U^F(\mathbb{C}) \cap \Gamma$ is a lattice in $U^F(\mathbb{C})$, and $T^F = U^F(\mathbb{C})/(U^F(\mathbb{C}) \cap \Gamma)$ is a complex torus.

THEOREM 2.14. *The quotient $\Gamma'(F)\backslash D(F)$ has a canonical structure of an algebraic variety for which the map $\Gamma'(F)\backslash D(F) \to \Gamma_h(F)\backslash F$ is a morphism of algebraic varieties. In fact, ${\Gamma'}^F\backslash D^F$ is a torus bundle (with fibres $T^F(\mathbb{C})$) over an abelian scheme over $\Gamma_h(F)\backslash F$.*

PROOF: This is proved in Brylinski (1979). (See also Brylinski (1983), 2.3.2.5, and Chapter VI below.)

Remark 2.15. The algebraic structure in (2.14) is canonical, but it is not unique: there is no analogue of the Borel extension theorem (cf. II.1.1).

Example: the Siegel case. We return to the situation of (2.2). Choose a lattice $V(\mathbb{Z})$ in V such that ψ takes integral values and has discriminant one on $V(\mathbb{Z})$, and let $\mathrm{Sp}(\mathbb{Z})$ be the subgroup of $\mathrm{Sp}(V, \psi)$ preserving $V(\mathbb{Z})$.The quotient $\mathrm{Sp}(\mathbb{Z})\backslash D$ is the moduli variety for polarized abelian varieties in the principal series. Fix an isotropic subspace W of V, and define the filtration $W.$ as in (2.13). The form ψ induces on $\mathrm{Gr}_{-1}(V(\mathbb{Z}))$ a skew-symmetric form $\bar{\psi}$ of discriminant 1. Set $\dim \mathrm{Gr}_{-1} V = 2g_0$. We have:

(a) F is the space of Hodge structures of type $\{(-1,0),(0,-1)\}$ on $\mathrm{Gr}_{-1}(V)$ for which $\bar{\psi}$ is a polarization.

(b) $D(F)$ is the space of maximal isotropic subspaces F^0 of $V(\mathbb{C})$ such that $(V, W., F^{\cdot})$ is a mixed Hodge structure and ψ is a polarization of $\mathrm{Gr}_{-1}(V)$.

Let Γ' be the subgroup of $\mathrm{Sp}(\mathbb{Z})$ of elements that respect the filtration and act trivially on $\mathrm{Gr}_0(V)$.

(c) The quotient $\Gamma'\backslash D(F)$ is the (coarse) moduli variety for symmetric one-motives $(A, \lambda, X, v, \delta)$ with (A, λ) a principally polarized abelian variety of dimension $2g_0$, X the abelian group $\mathrm{Gr}_0(V(\mathbb{Z}))$, v a homomorphism $X \to A(\mathbb{C})$, and δ a symmetric trivialization of the Poincaré biextension (see IV.2.6).

(d) The quotient $\Gamma'\backslash D(F)'$ is the (coarse) moduli variety for the quadruples (A, λ, X, v).

(e) The quotient $\Gamma'\backslash F$ is the (coarse) moduli variety for principally polarized abelian varieties of dimension g_0.

The maps

$$\Gamma'\backslash D(F) \to \Gamma'\backslash D'(F) \to \Gamma'\backslash F$$

correspond to

$$(A, \lambda, X, v, \delta) \mapsto (A, \lambda, X, v) \mapsto (A, \lambda).$$

Notes. Piatetski-Shapiro (1966) showed how to realize all the classical symmetric Hermitian domains as Siegel domains of the third kind. Wolf and Korányi (1965) gave a more uniform treatment that includes the nonclassical domains. There are expositions of (parts of) the material in this section in Baily and Borel (1966), Deligne (1973), Ash et al. (1975), Satake (1980), and Brylinski (1983).

3. Toroidal compactification of locally symmetric varieties. The results of the last two sections, allow us to construct toroidal compactifications of locally symmetric varieties.

We use the same notations as in §2 (except that we no longer require G to be $\mathbb{Q}$-simple). Thus D is a symmetric Hermitian domain, G is an algebraic group over $\mathbb{Q}$ with $G(\mathbb{R})^{+} = \mathrm{Aut}(D)^{+}$, F is a rational boundary component of D, and N^F, W^F, and U^F are certain subgroups of G attached to F. Recall that we have a canonical self-adjoint open cone $C(F)$ in $U^F(\mathbb{R})$. We choose a neat arithmetic subgroup Γ of $G(\mathbb{R})^{+}$, and define $\bar{\Gamma}(F)$ to be the image of $\Gamma(F)$ in $\mathrm{Aut}(U^F)$. As in §2, T^F is the torus over $\mathbb{C}$ with $X_*(T) = U^F(\mathbb{Z}) =_{\mathrm{df}} U^F(\mathbb{C}) \cap \Gamma$. Finally, we write S for the locally symmetric variety $\Gamma\backslash D$.

Definition 3.1. A fan Δ in $U^F(\mathbb{R})$ is said to be $\bar{\Gamma}(F)$-admissible if

(a) $\gamma \in \bar{\Gamma}(F)$, $\sigma \in \Delta \Rightarrow \gamma\sigma \in \Delta$;

(b) the number of classes of cones mod $\bar{\Gamma}(F)$ is finite;

(c) $C(F) \subset |\Delta| \subset C(F)^{-}$ (closure of $C(F)$).

Note that $X_*(T) \otimes \mathbb{R} = U^F(\mathbb{R})$. Therefore a $\bar{\Gamma}(F)$-admissible fan gives a torus embedding $T^F \subset X^F_\Delta$. As $U^F(\mathbb{Z})\backslash D(F)$ is a principal bundle for T^F over $D(F)'$, we can construct a partial compactification,

$$(U^F(\mathbb{Z})\backslash D(F))_\Delta = (U^F(\mathbb{Z})\backslash D(F)) \times^{T^F} X^F_\Delta,$$

as at the end of §1. This is a fibre bundle over $D(F)'$ with fibres X^F_Δ. Define $(U^F(\mathbb{Z})\backslash D)_\Delta$ to be the interior of the closure of $U^F(\mathbb{Z})\backslash D$ in $(U^F(\mathbb{Z})\backslash D(F))_\Delta$. Because Δ is invariant under $\bar{\Gamma}(F)$, $\Gamma(F)$ acts on $(U^F(\mathbb{Z})\backslash D(F))_\Delta$, and it can be shown that $\Gamma(F)$ acts properly discontinuously on $(U^F(\mathbb{Z})\backslash D)_\Delta$.

Definition 3.2. A family of fans $\Delta = (\Delta^F)$, F running over the rational boundary components of D, is Γ-*admissible* if

(a) each Δ^F is $\bar{\Gamma}(F)$-admissible;

(b) for $\gamma \in \Gamma$, $\gamma\Delta^F = \Delta^{\gamma F}$ (note that γ defines an isomorphism $\gamma : C(F) \to C(\gamma F)$);

(c) if $F \supset F'$, $\Delta^{F'} = \{\sigma \cap C(F') \mid \sigma \in \Delta\}$ (note that $C(F')^{-} = C(F)^{-} \cap U(F')$).

THEOREM 3.3. *For every Γ-admissible family of fans $\Delta = (\Delta^F)$, there is a unique normal separated complex analytic variety $(\Gamma\backslash D)_\Delta$ containing $\Gamma\backslash D$ as an open dense set and such that:*

(a) for every rational boundary component F of D, there is an open analytic morphism π_F making the following diagram commute:

$$\begin{array}{ccc} U^F(\mathbb{Z})\backslash D & \hookrightarrow & (U^F(\mathbb{Z})\backslash D)_{\Delta^F} \\ \Big| & & \Big|\pi_F \\ \Gamma\backslash D & \hookrightarrow & (\Gamma\backslash D)_\Delta; \end{array}$$

(b) $(\Gamma\backslash D)_\Delta = \cup \mathrm{Im}(\pi_F)$. *Moreover,* $(\Gamma\backslash D)_\Delta$ *has a unique structure of a complete algebraic space compatible with its analytic structure, and there is a natural morphism* $(\Gamma\backslash D)_\Delta \to (\Gamma\backslash D)^-$ *that restricts to the identity map on* $\Gamma\backslash D$.

PROOF: This is the main theorem of Ash et al. (1975) (ibid. p253).

The algebraic space $(\Gamma\backslash D)_\Delta$ in the theorem is called the *toroidal compactification* of $\Gamma\backslash D$ defined by Δ.

An algebraic space is the quotient of a scheme by an étale equivalence relation (see Knutson (1971) for a full account of the theory of algebraic spaces). In this article, the distinction between a scheme and an algebraic space will not be important, and we shall ignore it. The next two results show that Δ can be chosen so that the toroidal compactification is in fact a projective variety.

Let $U = \cup U^F$ and $C = \cup C(F)$ (unions over the rational boundary components of D).

Definition 3.4. Let $\Delta = (\Delta^F)_F$ be a Γ-admissible family of fans.

(a) Δ is *nonsingular* if every cone in every Δ_F is nonsingular (see 1.6);

(b) Δ is *projective* if there exists a Γ-invariant continuous convex piecewise linear function $f : C \to \mathbb{R}$ such that $f|U^F$ is a polar function for each F (see 1.11).

THEOREM 3.5. *(a) If* Δ *is nonsingular, then* $(\Gamma\backslash D)_\Delta$ *is nonsingular.*

(b) If Δ *is projective, then*$(\Gamma\backslash D)_\Delta \to (\Gamma\backslash D)^-$ *is the normalization of the blowing up of* $(\Gamma\backslash D)$ *along a sheaf of ideals* $\mathcal{I}$ *such that* $\mathcal{O}/\mathcal{I}$ *has support on* $(\Gamma\backslash D)^- - \Gamma\backslash D$. *In particular,* $(\Gamma\backslash D)_\Delta$ *is projective.*

PROOF: The first statement follows from (1.11). The second is a theorem of Tai (see Ash et al. (1975), IV.2.1).

PROPOSITION 3.6. *(a) There exist projective* Γ*-admissible families of fans.*

(b) Every Γ*-admissible fan has a refinement that is nonsingular.*

PROOF: (a) See Ash et al. (1975), p310.

(b) In Kempf et al. (1972), p32, this is proved for torus embeddings of finite type, but essentially the same proof works in the present context.

One can show, more precisely, that every toroidal compactification is dominated by a nonsingular toroidal compactification whose boundary is a divisor with normal crossings—we shall refer to such a compactification as a *smooth toroidal compactification.*

Remark 3.7. (a) The sheaf of ideals $\mathcal{I}$ in (3.5b) has a precise description in terms of the function f (see Ash et. al. 1975, p312).

(b) In Ash et. al (1975), p287, there is a more intrinsic statement of the main theorem.

Notes. Toroidal compactification were introduced independently by Mumford and Satake (see Mumford 1975 and Satake 1973). The theory was worked out in detail by Mumford and his collaborators, Ash, Kempf, Knudsen, Rapoport, Saint-Donat, and Tai, in (Kempf et al. 1972) and (Ash et al. 1975).

4. Toroidal compactification of Shimura varieties. We extend the results of the last section to Shimura varieties.

Toroidal compactification of connected Shimura varieties. Let (G, X^+) be a pair satisfying the axioms (II.1.3). The group G^{ad} plays the role of G in the previous section. Let Γ be a neat arithmetic subgroup of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})^+$. A Γ-admissible fan Δ will also be Γ'-admissible for any arithmetic subgroup $\Gamma' \subset \Gamma$, and the morphism $\Gamma'\backslash X^+ \to \Gamma\backslash X^+$ extends to a morphism $(\Gamma'\backslash X^+)_\Delta \to (\Gamma\backslash X^+)_\Delta$. We write $\mathrm{Sh}^0(G,X)_\Delta$ for the projective system $(\Gamma\backslash X^+)_\Delta$, where Γ runs over the neat arithmetic subgroups containing the image of a congruence subgroup.

Unfortunately, the action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ on $\mathrm{Sh}^0(G,X)$ does not extend to $\mathrm{Sh}^0(G,X)_\Delta$. However, we have the following observation of Faltings and Stuhler.

Lemma 4.1. *Let Γ and Γ' be neat arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup, and let $\gamma_1, \ldots, \gamma_n \in G(\mathbb{Q})^+$ be such that $\gamma_i^{-1}\Gamma\gamma_i \subset \Gamma'$; then for any pair of smooth toroidal compactifications $(\Gamma\backslash X^+)_\Delta$ and $(\Gamma'\backslash X^+)_{\Delta'}$ of $\Gamma\backslash X^+$ and $\Gamma'\backslash X^+$, there exists a smooth compactification $(\Gamma\backslash X^+)_{\Delta''}$ of $\Gamma\backslash X^+$ and maps:*

$$(\Gamma\backslash X^+)_{\Delta''} \to (\Gamma\backslash X^+)_\Delta \quad \text{restricting to id on } \Gamma\backslash X^+, \text{ and}$$
$$(\Gamma\backslash X^+)_{\Delta''} \to (\Gamma'\backslash X^+{}_{\Delta'} \quad \text{restricting to } \gamma_i \text{ on } \Gamma\backslash X^+.$$

Proof: Stated in Faltings (1984).

Thus, if we define $\mathrm{Sh}^0(G,X)^*$ to be the projective system $(\Gamma\backslash X^+)_\Delta$, where Γ runs over the neat arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})^+$ and (for each Γ) Δ runs over the Γ-admissible families of fans for which $(\Gamma\backslash X^+)_\Delta$ is a smooth toroidal compactification, then the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\mathrm{Sh}^0(G,X)$ extends to $\mathrm{Sh}^0(G,X)^*$. By continuity, we obtain an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\mathrm{Sh}^0(G,X)^*$.

Toroidal compactification of Shimura varieties. Let (G, X) be a pair defining a Shimura variety, and assume that the weight w_X is defined over $\mathbb{Q}$ (this is true for all naturally occurring Shimura varieties with boundary). Choose a connected component X^+ of X. Corresponding to a boundary component F of X^+, we obtain a Cayley filtration w^F of G^{ad}. It follows from results in Deligne (1973) that w^F lifts to a filtration w of $G_{\mathbb{C}}$, and that w can be normalized so that $(w_X \cdot w^{-1})(\mathbb{G}_m) \subset G^{\text{der}}$ (i.e., w and w_X become equal when composed with $G_{\mathbb{C}} \to (G/G^{\text{der}})_{\mathbb{C}}$). Because the map $G \to G^{\text{ad}} \times (G/G^{\text{der}})$ has finite kernel, w is uniquely determined, and because w^F and w_X are defined over $\mathbb{Q}$, so also is w. Moreover, for any representation (V, ξ) of G, the filtrations defined by w and $F_x^{\cdot}$ form a mixed Hodge structure on V (according to (IV.1.3), this has to be checked only for representations factoring through $G^{\text{ad}} \times (G/G^{\text{der}})$, and for these it is obvious). These remarks suggest the following definition.

Definition 4.2. A Cayley filtration $W.$ on G is *admissible* if the filtration on G/G^{der} is that defined by w_X.

Now fix an admissible Cayley filtration $W.^F$ of G. Here the F is simply an index. Set

$$N^F = W_0^F(G),\ W^F = W_{-1}^F(G),\ U^F = W_{-2}^F(G).$$

Note that $Z(G) \subset Z(w)$ for any w splitting $W.^F$, and so $Z(G) \cap W^F = \{1\}$. Therefore W^F and U^F are mapped isomorphically onto their images in G^{ad}.

Choose a connected component X^+ of X, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. For any $g \in G(\mathbb{A}_f)$, let Γ_g be the image in $G^{\text{ad}}(\mathbb{Q})^+$ of the group $gKg^{-1} \cap G(\mathbb{Q})_+$. As in §3, associated with Γ_g we have groups $\Gamma_g(F)$, $\Gamma'_g(F)$ and $\bar{\Gamma}_g(F)$, and we have a canonical cone $C(F) \subset U^F(\mathbb{C})$. Let $\mathcal{C}$ be a set of representatives for the finite set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ (see II.2).

Definition 4.3. A fan $\Delta \subset C(F)$ is said to be $\bar{\Gamma}(F)$-admissible if it is $\bar{\Gamma}_g(F)$-admissible for all $g \in \mathcal{C}$.

From such a fan, we obtain a partial toroidal compactification

$$\mathrm{Sh}_K(G, X)_\Delta = \cup (\Gamma_g \backslash X^+)_\Delta.$$

Definition 3.9. A family of fans (Δ^F), with w^F running over the admissible Cayley filtrations of G, is *K-admissible* if

(a) each Δ^F is $\bar{\Gamma}(F)$-admissible;

(b) for all $g \in \mathcal{C}$ and all $\gamma \in \Gamma_g$, we have $\gamma\Delta^F = \Delta^{\gamma F}$ where $W.^{\gamma F} =_{\mathrm{df}} \mathrm{ad}(\gamma) \cdot W.^F$;

(c) if $N^F \subset N^{F'}$, then $\Delta(F') = \{\sigma \cap C(F') \mid \sigma \in \Delta^F\}$.

A K-admissible family of fans $\Delta = (\Delta^F)$ defines a toroidal embedding $\mathrm{Sh}(G,X) \hookrightarrow \mathrm{Sh}(G,X)_\Delta$. We say that $\mathrm{Sh}(G,X)_\Delta$ is a *smooth toroidal compactification* if $\mathrm{Sh}(G,X)_\Delta$ is smooth and the boundary is a divisor with normal crossings, and we write $\mathrm{Sh}(G,X)^*$ for the projective system of smooth toroidal compactifications of $\mathrm{Sh}(G,X)$. The actions of $G(\mathbb{A}_f)$ and $\mathcal{G}(G)$ on $\mathrm{Sh}(G,X)$ extend to $\mathrm{Sh}(G,X)^*$.

Notes. There is a more detailed discussion, from a somewhat different point of view, of toroidal compactifications of nonconnected Shimura varieties in Harris (1989), §2.

5. Canonical models of toroidal compactifications.

Connected Shimura varieties. Let (G,X^+) be a pair defining a connected Shimura variety. Let x be a special point of X^+, and let τ be an automorphism of $\mathbb{C}$. Recall from (II.4.2) that there is a unique isomorphism

$$\varphi^0_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+) \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)$$

sending $\tau[x]$ to $[{}^\tau x]$ and such that $\varphi^0_{\tau,x} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^{\tau,x}g) \circ \varphi^0_{\tau,x}$. It would be very surprising if the following statement were not true:

CONJECTURE 5.1. *The isomorphism $\varphi^0_{\tau,x}$ extends uniquely to an isomorphism*

$$\varphi^{0\,*}_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+)^* \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^*.$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc}
\varphi^{0\,*}_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+)^* & \longrightarrow & \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^* \\
\downarrow & & \downarrow \\
\varphi^{0\,-}_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+)^- & \longrightarrow & \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^-.
\end{array}$$

(The vertical arrows are the natural maps from the toroidal compactification to the minimal compactification.)

Note that, because $\mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^*$ is separated and $\mathrm{Sh}^0(G,X^+)$ is dense in $\mathrm{Sh}^0(G,X^+)^*$, $\varphi^{0;*}_{\tau,x}$ will certainly be unique if it exists. It appears likely that the following argument will suffice to prove the existence of $\varphi^{0\,*}_{\tau,x}$. For connected Shimura varieties of Hodge type, the existence of $\varphi^{*}_{\tau,x}$ follows from the description of $\mathrm{Sh}^0(G,X^+)_\Delta$ as a moduli space of degenerating abelian varieties (see Chai and Faltings (1989) and Brylinski (1983), §4). To apply the strategy of II.9, the following statement will be needed:

(*) an inclusion $(G,X^+)\hookrightarrow(G',X'^+)$ induces a closed immersion

$$\mathrm{Sh}^0(G,X)^*\hookrightarrow\mathrm{Sh}^0(G',X'^+)^*.$$

Note that we already know that the map $\mathrm{Sh}^0(G,X^+)\hookrightarrow\mathrm{Sh}^0(G',X'^+)$ is a closed immersion (cf. Deligne 1971c, 1.15), and so (*) comes down to a combinatorial question about fans. Let $\Gamma\subset G^{\mathrm{ad}}(\mathbb{Q})^+$ and $\Gamma'\subset G'^{\mathrm{ad}}(\mathbb{Q})^+$ be such that $\Gamma\backslash X^+\hookrightarrow\Gamma'\backslash X'^+$ is a closed immersion; when Δ is a Γ-admissible fan, we wish to find a Γ'-admissible fan Δ' such that the preceding map extends to a closed immersion $(\Gamma\backslash X^+)_\Delta\hookrightarrow(\Gamma'\backslash X'^+)_{\Delta'}$ (after possibly replacing Δ by a refinement Δ''). For this we can take Δ' to be any Γ'-admissible fan refining the image of Δ, and apply (Harris 1989, §3) to obtain a Δ'' for which $(\Gamma\backslash X^+)\hookrightarrow(\Gamma'\backslash X'^+)$ extends to a map $(\Gamma\backslash X^+)_{\Delta''}\hookrightarrow(\Gamma'\backslash X'^+)_{\Delta'}$.

Now assume $G=\mathrm{Res}_{L/\mathbb{Q}}G'$ with G' absolutely simple. After extending L we can suppose that there is an inclusion $(G_\alpha,X^+_\alpha)\hookrightarrow(G,X^+)$ with G_α of type A_1 and such that a boundary point of $\mathrm{Sh}^0(G_\alpha,X_\alpha)$ maps into any particular boundary component of $\mathrm{Sh}^0(G,X^+)^-$ we choose (see 2.6b). Then the domain of definition of the rational map

$$\tau\mathrm{Sh}^0(G,X^+)^*\to\mathrm{Sh}^0({}^{\tau,x}G,{}^{\tau,x}X.)^*$$

includes at least one point of the boundary component in question, and the Hecke operators then allow us to show that it will contain all points.

In practice, conjecture (5.1) is probably all one will need—in most situations where toroidal compactifications are needed, exactly which toroidal compactification is being used is irrelevant. In fact, the usual procedure is to choose a toroidal compactification and then show that the statements or objects one arrives at are independent of the choice. Nevertheless, it would be interesting to have a more precise result than (5.1) where, starting from a fan Δ, one constructs a fan Δ' for which $\varphi^0_{\tau,x}$ extends to an isomorphism

$$\tau\mathrm{Sh}^0(G,X)_\Delta\to\mathrm{Sh}({}^{\tau,x}G,{}^{\tau,x}X)_{\Delta'}.$$

It is easy to guess what Δ' should be. For simplicity, assume G to be simply connected. Let F be a rational boundary component of X^+, and let Δ be a $\bar{\Gamma}(F)$-admissible fan. We wish to identify $\tau\mathrm{Sh}^0(G,X)_\Delta$ with a partial compactification of $\mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X)$. Choose a faithful representation (V,ξ) of G^{ad}. Associated with this data, we have a one-motive $M = (X_M \to G_M)$ such that $U^F = \mathrm{Hom}(S^2X_M, \mathbb{C})$. The fan Δ corresponds to a torus embedding $T \hookrightarrow X_\Delta$ of $T = \mathrm{Hom}(S^2X_M, \mathbb{G}_m)$. Then τM is the motive attached to ${}^\tau x \in {}^{\tau,x}X$, and we can choose $\Delta' \subset \mathrm{Hom}(S^2X_{\tau M}, \mathbb{C})$ to be the fan corresponding to the torus embedding $\tau T \hookrightarrow \tau X_\Delta$.

CONJECTURE 5.2. *The isomorphism $\varphi^0_{\tau,x}$ extends uniquely to an isomorphism*

$$(\varphi^0_{\tau,x})_\Delta : \tau\mathrm{Sh}^0(G,X)_\Delta \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X)_{\Delta'}.$$

compatible with the maps to the minimal compactification.

Shimura varieties. Let (G,X) be a pair defining a Shimura variety.

THEOREM 5.3. *Assume (5.1). For any $\tau \in Aut(\mathbb{C})$ and special point $x \in X$, the isomorphism $\varphi_{\tau,x} : \tau\mathrm{Sh}(G,X) \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$ of (II.4.2) extends uniquely to an isomorphism $\varphi^*_{\tau,x} : \tau\mathrm{Sh}(G,X)^* \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X^+)^*$; moreover, the diagram*

$$\begin{array}{ccc} \tau\mathrm{Sh}(G,X)^* & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)^* \\ \downarrow & & \downarrow \\ \tau\mathrm{Sh}(G,X)^- & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)^- \end{array}$$

commutes.

PROOF: This can be obtained by induction from (5.1).

COROLLARY 5.4. *(Assuming 5.1.)*

(a) $\mathrm{Sh}(G,X)^*$ *has a canonical model over* $E(G,X)$.

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau\mathrm{Sh}(G,X)^*$ *is canonically isomorphic to the canonical model of* $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)^*$ *over* $\tau E(G,X)$.

Conjecture 5.2 has an obvious analogue for nonconnected Shimura varieties.

Remark 5.5. So far we have not mentioned Eisenstein series. Briefly, Eisenstein series attach an automorphic form on the whole Shimura variety to a cusp form on a boundary component. This construction should be compatible with all the isomorphisms in this article. In particular, an Eisenstein series should be defined over a field E when the cusp form is.

Notes. As we noted (3.7a) a smooth projective toroidal compactification is obtained from the minimal compactification by blowing it up at certain ideals, described by the polarizing function f, and then normalizing. Brylinski (1983) uses this and Fourier-Jacobi series to prove the existence of canonical models of projective toroidal compactifications of Shimura varieties of Hodge type. Harris (1989), 2.8, suggests that the results in Harris (1986) can be used to generalize this result. (I understand that Richard Pink will also examine the question of the existence of canonical models of toroidal compactifications in his Bonn thesis.)

6. Canonical extensions of automorphic vector bundles. First we note that automorphic vector bundles extend to toroidal compactifications.

THEOREM 6.1. *Let (G, X) be a pair defining a Shimura variety, and let $\mathrm{Sh}(G, X)_\Delta$ be a smooth toroidal compactification of $\mathrm{Sh}(G, X)$. There is an exact faithful functor $\mathcal{J} \mapsto \mathcal{V}_\Delta(\mathcal{J})$ from the category of $G^c_{\mathbf{C}}$-vector bundles on $\check{X}$ to that of vector bundles on $\mathrm{Sh}(G, X)_\Delta$ such that*

(a) $\mathcal{V}(\mathcal{J})_\Delta|\mathrm{Sh}(G, X) = \mathcal{V}(\mathcal{J})$ (notation as in III.2);

(b) $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})_\Delta$ commutes with tensor products and duals (i.e., it is a morphism of tensor categories);

(c) equivariant differential operators between the $\mathcal{V}(\mathcal{J})$'s extend to the $\mathcal{V}_\Delta(\mathcal{J})$'s.

Moreover, $\mathcal{V}_\Delta(\mathcal{J})$ is uniquely determined by the properties (a), (b), and (c).

PROOF: Let $o \in \check{X}$. Then $G_{\mathbf{C}}$-vector bundles correspond to representations of P_o (see III.2.3a). For the $\mathcal{J}$ corresponding to irreducible representations of P_o, the result is essentially proved in Mumford (1977).

For Siegel modular varieties, the theorem is proved in Chai and Faltings (1989), VI.4. Briefly, their proof proceeds as follows. Using the structure of $\mathrm{Sh}(G, X)$ near the boundary, it is possible to construct the extension of $\mathcal{V}(\mathcal{J})_\Delta$ locally; the problem is to show that the local extensions patch. It is clear that the category of representations of P_o for which this is true is closed under tensor products, duals, and subquotients. It therefore suffices to construct $\mathcal{V}(\mathcal{J}_o)_\Delta$ for a single faithful representation of P_o. Chai and Faltings take $\mathcal{J}_o$ to be the standard representation G, and show that $\mathcal{V}(\mathcal{J}_o)_\Delta$ can be obtained from the de Rham cohomology of the universal semi-abelian scheme that they have already constructed. Harris (1989) shows that by applying Deligne's existence theorem (Deligne 1970) it is possible to avoid using the universal semi-abelian scheme.

THEOREM 6.2. *Let* $\mathrm{Sh}(G,X)_\Delta$ *be a smooth toroidal compactification of a Shimura variety having a canonical model over* $E \supset E(G,X)$. *If* $\mathcal{J}$ is defined over E, then so also is $\mathcal{V}(\mathcal{J})_\Delta$.

PROOF: The descent datum on $\mathcal{V}(\mathcal{J})$ extends to $\mathcal{V}(\mathcal{J})_\Delta$.

Once general results on canonical models have been obtained, essentially all the results in Chapter III will extend to vector bundles on the toroidal compactifications.

Notes. See the references in the proof of (6.1).

VI. Mixed Shimura varieties

In this chapter, we suggest how the results of Chapter II should generalize to mixed Shimura varieties.

1. Definition of a mixed Shimura variety.

Let P be a connected algebraic group over $\mathbb{Q}$. Recall from (I.1) that we have the notion of a filtration $W.$ of $\mathbf{Rep}_{\mathbb{Q}}(P)$. Moreover, $P = W_0P$ if and only if P preserves the filtration on each representation of P, and $W_{-1}P$ is the (unipotent) subgroup of W_0P acting trivially on $\mathrm{Gr}^W_{\cdot}(V)$ for all representations of P. For any cocharacter w of P splitting the filtration, $W_0P = W_{-1} \rtimes Z(w)$, where $Z(w)$ is centralizer of w.

The axioms for a mixed Shimura variety. The datum needed to define a mixed Shimura variety is a triple $(P, W., Y)$ comprising a connected algebraic group P over $\mathbb{Q}$, an ascending filtration $W.$ of $\mathbf{Rep}_{\mathbb{C}}(P)$, and a $P(\mathbb{R}) \cdot (W_{-2}P(\mathbb{C}))$-conjugacy class Y of descending filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$. For $y \in Y$, write $F^{\cdot}_y$ for the filtration defined by $y \in Y$. The filtration $W.$ is defined over some totally real number field, and the filtration it induces on $\mathbf{Rep}_{\mathbb{C}}(P/Z(P))$ is defined over $\mathbb{Q}$. The triple is required to satisfy the following conditions:

(1.1.0) for any representation (ξ, V) of P, $W.$ and $F^{\cdot}_y$ define a real mixed Hodge structure on $V(\mathbb{R})$, all $y \in Y$;

(1.1.1) $\mathrm{Lie}(P_{\mathbb{C}}) = W_0\mathrm{Lie}(P_{\mathbb{C}}) = F_y^{-1}\mathrm{Lie}(P_{\mathbb{C}})$ for each $y \in Y$;

(1.1.2) for any μ_y splitting the filtration $F_y^{\cdot}$, $\mu_y(i) \cdot \overline{\mu_y(\mathrm{i})}$ is a Cartan involution on $(\mathrm{Gr}_0^W P)^{\mathrm{ad}}$;

(1.1.3) $(\mathrm{Gr}_0^W P)^{\mathrm{ad}}$ has no $\mathbb{Q}$-rational factors that are anisotropic over $\mathbb{R}$;

(1.1.4) $Z(P)^0$ is a torus, splitting over a CM-field;

(1.1.5) the (adjoint) action of $\mathrm{Gr}_0^W P$ on $\mathrm{Gr}_{-1}^W \mathrm{Lie}\, P$ factors through $\mathrm{Gr}_0^W P)^c$ (notation as in the introduction to Chapter III).

Simplifications occur when we strengthen some of the axioms:

(1.1.0*) the filtration $W.$ is defined over $\mathbb{Q}$, and $W.$ and $F_y^{\cdot}$ define a rational mixed Hodge structure on V for any representation (V, ξ) of P;

(1.1.2*) for any μ_y splitting the filtration $F_y^{\cdot}$, $\mu_y(i) \cdot \overline{\mu_y(i)}$ is a Cartan involution on $P/(W_{-1}P \cdot w(\mathbb{G}_m))$;

(1.1.4*) (1.1.4) holds and there is a one-dimensional representation V_0 of P such that $(V_0, W., F_y^{\cdot})$ is the pure Hodge structure $\mathbb{Q}(1)$ for all y.

We usually drop the W from the notation Gr_r^W. For each $y \in Y$, there is a homomorphism $\tilde{h}_y : \mathbb{G}_m \times \mathbb{G}_m \to P_{\mathbb{C}}$ such that, for every representation (V, ξ) of P, $\xi \circ \tilde{h}_y$ provides $V(\mathbb{C})$ with the bigrading associated with the mixed Hodge structure (see IV.1). It is important to note, however, that in general $\tilde{h}_{py} \neq (\mathrm{ad}\, p) \circ \tilde{h}_y$ unless $p \in P(\mathbb{R})$.

Remark 1.2. (a) Axiom (1.1.5) has been imposed only so that the mixed Shimura variety exists as a scheme rather than a stack. Probably this condition should be dropped. In any case, the axioms should be viewed as tentative.

(b) Axiom (1.1.2) implies that $(\mathrm{Gr}_0 P)^{\mathrm{ad}}$ is semisimple, and (1.1.4) implies that the connected centre of $\mathrm{Gr}_0 P$ is a torus. Therefore $\mathrm{Gr}_0 P$ is a reductive group, $W_{-1}P_{\mathbb{C}}$ is the unipotent radical of $P_{\mathbb{C}}$, and, for any w splitting $W.$, $Z(w)$ is a Levi subgroup of $P_{\mathbb{C}}$. Note that if $\mathrm{Gr}_0 P = 0$, then $w(\mathbb{G}_m) = 0$, which implies that $W_{-1}P = 0$ and that $P = 0$.

(c) Let $\mathrm{Lie}(P)_{\mathbb{C}} = \oplus \tilde{H}^{p,q}$ be the decomposition of $\mathrm{Lie}(P)_{\mathbb{C}}$ corresponding to the mixed Hodge structure $(W., F_y^{\cdot})$ some $y \in Y$. Then (1.1.1) implies that $\tilde{H}^{p,q} = 0$ for $p + q > 0$ and $p < -1$. Hence

$\mathrm{Gr}_0(\mathrm{Lie}\, P)$ has a Hodge structure of type $\{(-1,1), (0,0), (1,-1)\}$;

$\mathrm{Gr}_{-1}(\mathrm{Lie}\, P)$ has a Hodge structure of type $\{(-1,0), (0,-1)\}$;

$\mathrm{Gr}_{-2}(\mathrm{Lie}\, P)$ has a Hodge structure of type $\{(0,0)\}$;

(see the picture in IV.2.10). Thus

(1.2.1) $$\operatorname{Lie} P_{\mathbb{C}} = \operatorname{Lie} P_{\mathbb{R}} + F_y^0 \operatorname{Lie} P_{\mathbb{C}} + W_{-2} \operatorname{Lie} P_{\mathbb{C}}.$$

From the last equality it follows that Y can also be regarded as a $P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C})$-conjugacy class.

(d) It suffices to check (1.1.0) for a single $y \in Y$ (cf. Brylinski 1983, 2.3.1.2), and for a finite family of representations (V_i, ξ_i) such that $\cap \operatorname{Ker}(\xi_i)$ is finite (see IV.1.3).

The complex structure on Y.

PROPOSITION 1.3. *Let $\check{Y}$ be the $P(\mathbb{C})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$ containing $F_y^{\cdot}$ for all $y \in Y$. Then $\check{Y}$ is a Grassman variety, and the map*

$$\beta : Y \hookrightarrow \check{Y},\ y \mapsto F_y^{\cdot},$$

identifies Y with an open complex submanifold of $\check{Y}$. The induced complex structure on Y is the unique structure such that, for all representations (V, ξ) of P, the filtrations $F_y^{\cdot}$ on $\mathcal{V}(\xi) =_{\mathrm{df}} Y \times V(\mathbb{C})$ vary homolomorphically.

PROOF: Fix a point $o \in Y$. Then $\check{Y} = P(\mathbb{C})/F_o^0 P(\mathbb{C})$, which is a Grassman variety, and β is the map

$$g \cdot o \mapsto g \pmod{F_o^0 P(\mathbb{C})} : Y \to P(\mathbb{C})/F_o^0 P(\mathbb{C}).$$

This is obviously injective, and (1.2.1) shows that it identifies Y with an open (almost) complex submanifold $\check{Y}$.

PROPOSITION 1.4. *Let $\xi : P \to GL(V)$ be a rational representation of P, and let $V_{\mathbb{R}} = \oplus V(i)$ be the decomposition of $V_{\mathbb{R}}$ under the action of $Z(P)^0_{\mathbb{R}}$; then $y \mapsto (V(i), W_{\cdot}, F_y^{\cdot})$ is a variation of real mixed Hodge structures on Y.*

PROOF: On $\mathrm{Gr}_n(V(i))$, we have a representation of $\mathrm{Gr}_0(P)$; apply (II.3.2) to see that it defines a variation of real Hodge structures. The transversality axiom (condition (H_1) of (IV.1)) follows from the fact that $\operatorname{Lie} P_{\mathbb{C}} = F_y^{-1}(\operatorname{Lie} P_{\mathbb{C}})$.

Define Y' to be the $(P/W_{-2}P)(\mathbb{R})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P/W_{-2}P)$ containing the image of Y, and define X to be the $(\mathrm{Gr}_0 P)(\mathbb{R})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(\mathrm{Gr}_0 P)$ containing the image of Y'. Proposition 1.3 shows that both Y' and X also have natural complex structures.

PROPOSITION 1.5. *The natural maps* $Y \xrightarrow{\pi_1} Y' \xrightarrow{\pi_2} X$ *are both holomorphic. Moreover,*

X is a symmetric Hermitian domain;
$Y' \to X$ is a fibre bundle with structure group $V(\mathbb{R})$, $V = Gr_{-1}(P)$;
$Y \to Y'$ is a fibre bundle with structure group $U(\mathbb{C})$, $U = Gr_{-2}(P)$.

PROOF: Straightforward from the definitions (and II.3.2).

Write π for the composite $Y \to X$.

The mixed Shimura variety. For any compact open subgroup K of $P(\mathbb{A}_f)$, define

$$M_K(P, W., Y) = P(\mathbb{Q})\backslash Y \times P(\mathbb{A}_f)/K.$$

It is a complex manifold if K is sufficiently small; in fact, it is a disjoint union of varieties of the form $\Gamma\backslash Y^+$ with Y^+ a connected component of Y and Γ a discrete subgroup of $P(\mathbb{R})^+$. Each $g \in P(\mathbb{A}_f)$ defines a holomorphic map,

$$\mathcal{T}(g) : M_K(P, W., Y) \to M_{g^{-1}Kg}(P, W., Y), \ [y, p] \mapsto [y, pg].$$

THEOREM 1.6. *(a) The complex manifold $M_K(P, W., Y)$ has a natural structure as an algebraic variety. More precisely, it is a torus bundle over a polarizable abelian scheme over a Shimura variety.*

(b) For each $g \in P(\mathbb{A}_f)$, $\mathcal{T}(g)$ is algebraic.

PROOF: For any quotient P' of P by a subgroup of $Z(P)$, we have a triple $(P', W.', Y')$ satisfying the axioms (1.1), and for each pair of open compact subgroups $K \subset P(\mathbb{A}_f)$ and $K' \subset P'(\mathbb{A}_f)$ such that K' contains the image of K, there is a morphism

$$M_K(P, W., Y) \to M_{K'}(P', W.', Y').$$

Each connected component of $M_K(P, W., Y)$ is a finite covering of a connected component of $M_{K'}(P', W.', Y')$. Thus, if we can prove (a) for $(P', W.', Y')$, then the Riemann existence theorem will show that it is also true for $(P, W., Y)$. A similar remark applies to (b). This allows us to assume that conditions (1.1.0*) and (1.1.2*) hold. Later in this section we outline a proof of the theorem in this case.

We obtain a scheme $M(P, W., Y)$ with a continuous action of $P(\mathbb{A}_f)$, which we call the *mixed Shimura variety* defined by $(P, W., Y)$.

Special points. A point $y \in Y$ is said to be *special* if for one faithful (hence every) representation (V,ξ) of P^{ad}, the mixed Hodge structure $(V, W_., F_y^{\cdot})$ decomposes into a sum of pure Hodge structures, each of CM-type. We say that y is a CM-point if the same condition holds for the representations of P itself. A mixed Hodge structure is said to be *rationally decomposed* if it is a direct sum of pure Hodge structures.

PROPOSITION 1.7. *(a) Let $x = \pi(y)$; then y is special if and only if x is special and $(V, W_., F_y^{\cdot})$ is rationally decomposed for each representation of P^{ad}.*

(b) For each special $x \in X$, there is a $y \in \pi^{-1}(X)$ such that $(V, W_., F_y^{\cdot})$ is rationally decomposed for each representation of P^{ad}.

PROOF: Part (a) is obvious. We outline a proof of (b) later in this section.

For each special point y, there is a unique homomorhism $\rho_y : \mathfrak{S} \to P^{\mathrm{ad}}$ such that $\rho_y \circ \mu_{\mathrm{can}} = \mu_y$. When y is a CM-point, ρ_y is a homomorphism $\mathfrak{S} \to P$.

Connected mixed Shimura varieties. Let $(P, W_., Y)$ define a mixed Shimura variety, and let $(G, X) = (\mathrm{Gr}_0 P, Y \pmod{W_{-1}P})$. The fibres of the map $M(P, W_., Y) \to \mathrm{Sh}(G, X)$ are connected, and so the inverse image of $\mathrm{Sh}(G, X)^0$ is connected. Let P' be the inverse image of G^{der} in P, let $W_.'$ be the filtration of $\mathbf{Rep}(P^{\mathrm{ad}})$ defined by $W_.$, and let Y^+ be a connected component of Y. Assume G^{der} to be simply connected. Then

$$M(P, W_., Y)^0 = \varprojlim P'(\mathbb{Q})\backslash Y^+ \times P'(\mathbb{A}_f)/K'.$$

In particular, $M(P, W_., Y)^0$ depends only on $(P', W_.', Y^+)$. Just as in the case of (pure) Shimura varieties, there is a theory of connected mixed Shimura varieties, which we will not discuss this further.

Examples. Mixed Shimura varieties abound.

Example 1.8. ($W_{-1}P = 0$; Shimura varieties). Let (G, X) be a pair satisfying (II.2.1). Set

$$P = G; \quad W_. = \mathrm{Filt}(w_X); \quad Y = \{\mathrm{Filt}(\mu_x) \mid x \in X\}.$$

The triple $(P, W_., Y)$ satisfies the axioms (1.1) (use II.3.2), and the variety $M(P, W_., Y) = \mathrm{Sh}(G, X)$. Conversely, if $(P, W_., Y)$ satisfies

(1.1) and $W_{-1}P = 0$, then P is a reductive group and the pair (P, X), $X = \{z \mapsto \tilde{h}_y(z, \bar{z}) \mid y \in Y\}$ satisfies the axioms (II.2.1). Thus mixed Shimura varieties defined by triples $(P, W., Y)$ with $W_{-1}P = 0$ are Shimura varieties, and every Shimura variety is of this form.

Example 1.9. ($\mathrm{Gr}_{-1}P = 0$; automorphic vector bundles). Consider a triple $(P, W., Y)$ satisfying (1.1) and (1.1.0*), and assume that $\mathrm{Gr}_{-1}P = 0$. Write $U = W_{-2}P$. It is commutative, and so the exponential map allows us to identify it with its Lie algebra. The adjoint action defines a representation ξ of P on U, factoring through $G =_{\mathrm{df}} \mathrm{Gr}_0 P$. Then $M_K(P, W., Y) = \mathcal{V}_K(\xi)/(\text{lattice})$, where $\mathcal{V}_K(\xi)$ is the automorphic vector bundle on $\mathrm{Sh}_K(G, X)$ defined by (V, ξ). The fibre of $M_K(P, W., Y)$ over a point of $\mathrm{Sh}_K(G, X)$ is $V(\mathbb{C})/\Lambda$ for some lattice Λ in V, and the exponential map shows that this is isomorphic to a product of copies of $\mathbb{C}^\times$. In particular, $M(P, W., Y)$ is algebraic (by III.2.1).

Conversely, let (G, X) be a pair satisfying (II.2.1*), and let (U, ξ) be a faithful representation of G. Define $P = U \rtimes G = \left\{\begin{pmatrix} 1 & 0 \\ u & g \end{pmatrix}\right\}$, and let it act on $V =_{\mathrm{df}} U \otimes U$ in the obvious way. Define a filtration of V by

$$0 = W_{-3}V \subset U \oplus 0 = W_{-2}V \subset V = W_0 V,$$

and give P the induced filtration $W.$. Define Y to be the set of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$ inducing on $\mathbf{Rep}_{\mathbb{C}}(G)$ the Hodge filtration corresponding to some $x \in X$. Then $(P, W., Y)$ defines a mixed Shimura variety, which is a quotient of the automorphic vector bundle $\mathcal{V}(\xi)$ on $\mathrm{Sh}(G, X)$.

Example 1.10. ($W_{-2}P = 0$; Kuga varieties). Consider a triple $(P, W., Y)$ satisfying (1.1) and (1.1.0*), and assume that $\mathrm{Gr}_{-2}P = 0$. Write $V = W_{-1}P$. It is a commutative algebraic group over $\mathbb{Q}$, and so the exponential map allows us to regard it as a vector space. The adjoint action defines a representation ξ of P on V, factoring through $G =_{df} Gr_0 P$. Each $y \in Y$ defines a Hodge structure on V of type $\{(-1, 0), (0, -1)\}$, which, according to (II.3.2), is polarizable. The choice of a compact open subgroup K' of $P(\mathbb{A}_f)$ defines a lattice in V, and consequently we obtain a family of abelian varieties $\mathcal{A}$ over $\mathrm{Sh}_K(G, X)$, where K is the image of K' in $G(\mathbb{A}_f)$ (cf. II.3.11). We have $M_{K'}(P, W., Y) = \mathcal{A}$. In particular, $M(P, W., Y)$ is algebraic.

The simplest example of such a mixed Shimura variety is the universal elliptic curve over $\mathrm{Sh}(GL_2, X)$. This (rather, a connected component of it) has been extensively studied; see for example Eichler and Zagier (1985) and Berndt (1983).

A more interesting case is that where the base Shimura variety is defined by a quaternion algebra over a totally real field (not necessarily totally indefinite, so the Shimura variety is not a moduli variety; see Deligne 1979, §6, Modèles étranges). These mixed Shimura varieties (rather, their connected components) have been extensively studied by students of Kuga; see for example Addington (1987) and Petri (1989).

We have noted that a connected component of a mixed Shimura varieties with $W_{-2}P = 0$ is a Kuga fibre variety, but the converse is not true: there are "nonrigid" Kuga fibre varieties that move in families and do not have models over number fields.

Example 1.11. (Mixed Shimura varieties arising from boundary components). Consider a Shimura variety $\mathrm{Sh}(G, X)$, and let $W.$ be an admissible Cayley filtration of G (see V.4.2). Define P to be the subgroup of W_0G acting trivially on $U =_{\mathrm{df}} W_{-2}G$. Then there is a natural way to attach to $W.$ a family Y of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$ so that $(P, W., Y)$ defines a mixed Shimura variety. The base Shimura variety is $\mathrm{Sh}(\mathrm{Gr}_0(P), F)$, where F is the rational boundary component of X corresponding to $W.$.

Example 1.12. (Mixed Shimura varieties of Hodge type). Let M be a one-motive over $\mathbb{Q}$, and let P be the Mumford-Tate group of M. The weight and Hodge filtrations on $H_B(M)$ define filtrations $W.$ and $F_o^{\cdot}$ on $\mathbf{Rep}_{\mathbb{C}}(P)$. Let Y be the $P(\mathbb{R}) \cdot W_{-2}P(\mathbb{C})$-conjugacy class of $F_o^{\cdot}$. Then $(P, W., Y)$ satisfies the stronger axioms (1.1*) (see IV.2.9). A mixed Shimura variety $M(P, W., Y)$ will be said to be of *Hodge type* if there is a one-motive M and a representation (V, ξ) of P such that

(a) for some $o \in Y$, $(H_B(M), W., F^{\cdot}) = (V, W., F_o^{\cdot})$;

(b) P is the subgroup of $GL(H_B(M)) \times \mathbb{G}_m$ fixing a family of Hodge tensors.

Such a mixed Shimura variety is a (coarse) moduli variety for a family of one-motives with Hodge cycle and level structures. Note that the total space of a fine moduli variety for abelian varieties is a moduli variety for one-motives of the form $(\mathbb{Z} \to A)$.

Outline for proofs of 1.6 and 1.7. Since it suffices to prove both statements for a triple $(P, W., Y)$ satisfying (1.1.0*) and (1.1.2*), we henceforth assume this. We have already verified the statements when:

(i) $P = \mathrm{Gr}_0(P)$; then $M(P, W_., Y)$ is a Shimura variety;

(ii) $W_{-2}P = 0$; then $M(P, W_., Y)$ is the total space of an abelian scheme over $\mathrm{Sh}(G, X)$;

(iii) $\mathrm{Gr}_{-1}(P) = 0$; then both statements reduce to statements about automorphic vector bundles.

The next lemma is slightly stronger than (1.7b).

LEMMA 1.13. *Let $(P, W_., Y)$ be as above, and let $x \in X$. For every representation (V, ξ) of P, there exists a $y \in \pi^{-1}(x)$ such that the mixed Hodge structure $(V, W_., F_y^{\cdot})$ is rationally decomposed.*

PROOF: Fix a $y \in \pi^{-1}(x)$. We have to show that there is a $p \in W_{-1}P(\mathbb{C})$ such that $(V, W_., pF_y^{\cdot})$ is rationally decomposed. The proof proceeds by induction on the length of the filtration $W_.$ of V (see Brylinski 1983, 2.3.1.5).

Under our assumptions, a representation (V, ξ) of P defines a variation of mixed Hodge structures $\mathcal{V}$ on $M(P, W_., Y)$. Let K be a compact open subgroup of $P(\mathbb{A}_f)$, and write K also for its image in $G(\mathbb{A}_f)$, $G = \mathrm{Gr}_0 P$.

LEMMA 1.14. *There exists a section $s : \mathrm{Sh}_K(G, X) \to M_K(P, W_., Y)$ to π such that $s^*(\mathcal{V})$ is rationally decomposed (after possibly replacing K by a subgroup).*

PROOF: See Brylinksi (1983), 2.3.1.7.

Thus we get a canonical section $s : \mathrm{Sh}(G, X) \to M(P, W_., Y)$ to π.

We now come to the proof of (1.6). First, the sheaf $R\pi_{1*}\mathbb{Z}$ is constant. Thus it splits up (analytically) under the characters of T, where T is the algebraic torus $W_{-2}P(\mathbb{C})/W_{-2}\Gamma$, $W_{-2}\Gamma = K \cap (W_{-2}P(\mathbb{C}))$. Let ρ be such a character.

LEMMA 1.15. *There exists on each $\mathcal{L}_\rho$ a unique algebraic structure such that*

(i) $\mathcal{L}_{2\rho}$ is isomorphic (algebraically) to $\sigma^*(\mathcal{L}_{2\rho})$ (σ is the map $x \mapsto -x$ on $\mathcal{A}$);

(ii) The restriction of $\mathcal{L}_\rho$ to the zero section of $\mathcal{A}$ is trivial.

Moreover, $\mathcal{L}_\rho|$(zero section) is canonically trivial.

PROOF: Brylinski (1983), 2.3.2.4.

LEMMA 1.16. *$M(P, W_., Y)$ has a unique algebraic structure such that*

(i) σ: $\mathcal{A} \to \mathcal{A}$ is algebraic;

(ii) the section $s : \mathrm{Sh}(G,X) \to M(P,W.,Y)$ is algebraic.

PROOF: See Brylinski (1983), 2.3.2.5.

Remark 1.17. If $M(P,W.,Y)$ is of Hodge type, then a representation of P defines an algebraic family of one-motives over $M(P,W.,Y)$, except that the family may only exist as a stack (cf. III.8).

Notes. The general notion of a mixed Shimura variety is due to Deligne. A slightly restricted form can be found in Brylinski's thesis (Brylinski 1983), where the varieties are called generalized Shimura varieties. The proofs of (1.6) and (1.7) are adapted from this source.

2. Canonical models of mixed Shimura varieties. Let y be a special point of Y. Then we get a homomorphism $\rho_y : \mathfrak{S} \to P^{\mathrm{ad}}$. Thus the $\mathfrak{S}$-torsor ${}^{\tau}\mathfrak{S}$ can be used to twist P to give a group ${}^{\tau,y}P$, and the canonical element $\mathrm{sp}(\tau)$ defines an isomorphism $g \mapsto {}^{\tau,y}g : P(\mathbb{A}_f) \to {}^{\tau,y}P(\mathbb{A}_f)$. Define ${}^{\tau,y}Y$ to be the conjugacy class containing $\tau F_y^{\cdot}$ for y a special point of Y. Then the triple $({}^{\tau,y}P, {}^{\tau,y}W., {}^{\tau,y}Y)$ satisfies the axioms for a mixed Shimura variety.

CONJECTURE 2.1. *For each $\tau \in Aut(\mathbb{C})$, there exists a unique isomorphism*

$$\varphi_{\tau,y} : \tau M(P,W.,Y) \to M({}^{\tau,y}P, {}^{\tau,y}W., {}^{\tau,y}Y)$$

such that

(i) $\varphi_{\tau,y}(\tau[y,1]) = [{}^{\tau}y, 1]$;

(ii) $\varphi_{\tau,y} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^{\tau,y}g) \circ \varphi_{\tau,y}$ *for all* $g \in G(\mathbb{A}_f)$.

Moreover, when y' is a second special point in Y, then there is a canonical map

$$\varphi(\tau; y', y) : M({}^{\tau,y}P, {}^{\tau,y}W., {}^{\tau,y}Y) \to M({}^{\tau,y'}P, {}^{\tau,y'}W., {}^{\tau,y'}Y),$$

and we have the identity

$$\varphi(\tau; y', y) \circ \varphi_{\tau,y} = \varphi_{\tau,y'}.$$

Remark 2.2. We know the above result in several cases:

(i) $W_{-1}P = 0$. Here the mixed Shimura variety is a (pure) Shimura variety, and the conjecture is (II.4.2) and (II.4.4).

(ii) $\mathrm{Gr}_{-1}P = 0$. Here the conjecture follows from the results on automorphic vector bundles in Chapter III.

(iii) $W_{-2}P = 0$; assume (1.1.0*). Here the mixed Shimura variety is an abelian scheme over a Shimura variety. To give an abelian scheme over $\mathrm{Sh}(G, X)$ is the same as to give a polarizable variation of integral Hodge structures on $\mathrm{Sh}(G, X)$. In this case the conjecture follows from (III.6.2).

(iv) Mixed Shimura varieties of Hodge type. Here the conjecture follows from the fact that the mixed Shimura variety is a moduli variety for one-motives (see Brylinski 1983, 2.3.3.1).

Thus to complete the proof of the conjecture, it remains

(i) to lift the isomorphism

$$\tau M(P/W_{-2}P, W_{\cdot}, Y') \to M({}^{\tau,y}(P/W_{-2}P, {}^{\tau,y}W_{\cdot}, {}^{\tau,y}Y')$$

to the covering $\tau M(P, W_{\cdot}, Y)$ (equivalently, to the sheaves $\tau\mathcal{L}_\rho$ on $\tau M(P, W_{\cdot}, Y)$) in the case that (1.1.0*) holds, and

(ii) to remove the condition (1.1.0*).

Probably the best approach to (i) will be to deduce it from an extension of the theorems in Chapter III to automorphic vector bundles on mixed Shimura varieties (see §4 below). It should be possible to prove (ii) by using connected mixed Shimura varieties.

Just as for Shimura varieties, the conjecture will imply that a mixed Shimura variety has a canonical model over a reflex field (suitably defined), and that the conjugate of a canonical model by $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is the canonical model of the mixed Shimura variety defined by the conjugate data.

3. Partial compactification of mixed Shimura varieties. Consider a mixed Shimura variety $M_K(P, W_{\cdot}, Y)$. Let $U = W_{-2}P$, and let T be the torus $U(\mathbb{C})/U(\mathbb{Z})$, where $U(\mathbb{Z}) = U(\mathbb{Q}) \cap K$. For a fan $\Delta \subset X_*(T) \otimes \mathbb{R} = U(\mathbb{R})$ satisfying suitable conditions, the construction in Chapter V can be mimicked to give a partial compactification

$$\pi_{1\Delta} : M(P, W_{\cdot}, Y)_\Delta \to M(P/W_{-2}P, W_{\cdot}, Y')$$

of the map $\pi_1 : M_K(P, W_{\cdot}, Y) \to M(P/W_{-2}P, W_{\cdot}, Y')$ (cf. Brylinski (1983), §4)). The isomorphism in (2.1) should extend to an isomorphism

$$\tau M(P, W_{\cdot}, Y)_\Delta \to M({}^{\tau,y}P, {}^{\tau,y}W_{\cdot}, {}^{\tau,y}Y)_{\Delta'}$$

for a suitable fan Δ' in ${}^{\tau,y}U(\mathbb{R})$.

4. Automorphic vector bundles.
As we saw in (1.3), there is an embedding $\beta : Y \hookrightarrow \check{Y}$ from Y into a variety of filtrations of $\mathbf{Rep}(P_{\mathbb{C}})$, and the action of $P_{\mathbb{C}}$ on $\check{Y}$ extends that of $P(\mathbb{R}) \cdot W_{-2}P(\mathbb{C})$ on Y. Let $\mathcal{J}$ be an $P_{\mathbb{C}}$-vector bundle on $\check{Y}$. If $\beta^*(\mathcal{J})$ defines a vector bundle $\mathcal{V}_K(\mathcal{J})$ on the quotient $M_K(P, W., Y)$ of Y, then we call $\mathcal{V}_K(\mathcal{J})$ an *automorphic vector bundle* . The theorems in Chapter III for automorphic vector bundles on Shimura varieties should extend to mixed Shimura varieties.

5. Toroidal compactification of mixed Shimura varieties.
Consider a mixed Shimura variety,

$$M(P, W., Y) \xrightarrow{\pi_1} M(P/W_{-2}P, W., Y') \xrightarrow{\pi_2} \mathrm{Sh}(G, X).$$

Form a toroidal compactification $\mathrm{Sh}(G, X)_{\Delta}$ of $\mathrm{Sh}(G, X)$. It should be possible to compactify successively the morphisms π_2 and π_1. The compactifications of the total space of the Siegel modular variety by Namikawa over $\mathbb{C}$ (Namikawa 1976, 1979) and Chai over $\mathbb{Z}$ (Chai and Faltings 1989), should serve as models for the compactification π_2.

VII. Fourier-Jacobi series

Fourier-Jacobi series play a central role in the theory of holomorphic automorphic forms. In this chapter, we briefly indicate how they fit into the schema described in the first six chapters.

For elliptic modular forms, there are three different approaches to defining Fourier series: the (classical) analytic approach; the modular approach, based on the moduli of elliptic curves; and the formal-algebraic approach, based on analyzing the structure of the elliptic modular curve at its cusps. The first is available for a general Shimura variety, but is badly adapted for studying rationality questions. The second applies only to Shimura varieties of Hodge type. Therefore, it is the third approach that will be most important.

The q-expansion principal asserts that an automorphic form is determined by (certain of) its Fourier-Jacobi series. Since there should be the notion of the conjugate of a Fourier-Jacobi series by an automorphism of $\mathbb{C}$, and hence the notion of a Fourier-Jacobi series being rational over a field, this means that it will be possible to read off the field of rationality of an automorphic form from the coefficients of its Fourier-Jacobi series. Since these live on lower dimensional (mixed Shimura) varieties, this will be a useful tool.

1. Elliptic modular forms.

An elliptic modular function f of level N satisfies

$$f(z+N)=f(z), \quad z \in H^{+}.$$

It therefore has a Fourier expansion

$$f(z)=\sum a_n q_N^n, \quad q_N=e^{2\pi iz/N}$$

corresponding to the cusp at infinity, and a similar expansion at the other cusps. It is known that f is rational over a subfield $L \subset \mathbb{C}$ (in the sense of Chapter III) if and only if the coefficients of these series lie in L.

We next explain the moduli definition (for details, see Katz 1973). Let

$$K_N=\{\alpha \in GL_2(\hat{\mathbb{Z}}) \mid \alpha \equiv I (\mathrm{mod} N)\}.$$

Write S_N for the corresponding modular curve $\mathrm{Sh}_{K(N)}(GL_2, H^{+})$, and $\mathcal{A}$ for the universal elliptic curve over S_N. On S_N we have the line bundle $\omega=\omega_{\mathcal{A}/S}$, and a modular form of weight k and level N is a section of $\omega^{\otimes k}$ holomorphic at the cusps. It is possible to re-write this definition so that it makes sense over any ring R containing $1/N$. Briefly, a modular form f of weight k and level N over R is a rule assigning to each triple (A, η, κ) consisting of an elliptic curve A over $\operatorname{Spec} R'$, a basis η for $\omega_{A/R'}$, and a level structure κ, an element of R'; here R' is an R-algebra. When we apply f to the Tate curve and its canonical differential over $R[[q]]$, then the element of $R[[q]]$ that we obtain is the Fourier series of f.

For the final approach, one computes the formal completion at a cusp of the compactification of S_N. It is the formal spectrum of a power series ring over $\mathbb{C}$ in one variable. By extending the modular form f to the compactification, and using the computation, one obtains the Fourier series of f.

2. The analytic definition of Fourier-Jacobi series.

Piatetski-Shapiro (1966) (especially §12, §15) associates a Fourier-Jacobi series with any automorphic form (or function) on a Siegel domain. In order to apply the construction to an automorphic form f on a bounded symmetric domain D, we use the realization of D as a Siegel domain of the third kind corresponding to a rational boundary component F of X (see V.2). The Fourier-Jacobi series attached to f and the boundary component F is then of the form

$$FJ^F(f) = \sum_\rho \psi_\rho(u,t)e^{2\pi i(\rho,z)}.$$

Here ρ runs over a finitely generated abelian group, t runs over the symmetric Hermitian domain F, and, for a fixed ρ and t, $\psi_\rho(u,t)$ is a theta function. Recall that a theta function can be regarded as a section of a line bundle on an abelian variety. Since a mixed Shimura variety is, roughly speaking, a sum of line bundles (with the zero sections removed) over an abelian scheme over a Shimura variety, a function on it can be written $(\psi_\rho(u,t))_\rho$ where t is a point of the Shimura variety and $\psi_\rho(u,t)$ is a section of the line bundle indexed by ρ on the abelian variety over t. The similarity of two expressions is not a coincidence.

3. The modular definition of Fourier-Jacobi series.
There is a very complete discussion of Fourier-Jacobi series for Siegel modular forms in Chai and Faltings (1989), and a briefer discussion for automorphic forms on a Shimura variety of Hodge type in Brylinski (1983), §5.

4. A formal-algebraic definition of Fourier-Jacobi series.
Let (G,X) be a pair defining a Shimura variety, and let $W_\cdot^F$ be a Cayley filtration on G. In (VI.1.11) above, we derived from these data a triple $(P,W_\cdot,Y)$ defining a mixed Shimura variety. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let $\Gamma = G(\mathbb{Q}) \cap K$ and $\Gamma_P = P(\mathbb{Q}) \cap K$. Then $W_{-2}P(\mathbb{C})$ contains a canonical self-adjoint homogeneous cone C. Choose a $\bar{\Gamma}(F)$-admissible fan Δ in C. Then we can form the partial compactification $\mathrm{Sh}_K(G,X)_\Delta$ of $\mathrm{Sh}_K(G,X)$ along F. Assume that $\mathrm{Sh}_K(G,X)_\Delta$ is smooth, and that the boundary of $\mathrm{Sh}_K(G,X)$ in it is a divisor with normal crossings. We then write $\mathrm{Sh}_K(G,X)_{\hat{\Delta}}$ for the formal completion of $\mathrm{Sh}_K(G,X)$ along the boundary. We can also form the partial compactification $M_{K_P}(P,W_\cdot,Y)_\Delta$ of $M_{K_P}(P,W_\cdot,Y)$, and the formal completion $M_{K_P}(P,W_\cdot,Y)_{\hat{\Delta}}$ of $M_{K_P}(P,W_\cdot,Y)$ along its boundary in $M_{K_P}(P,W_\cdot,Y)_\Delta$.

CONJECTURE 4.1.. *There is a canonical isomorphism*

$$\mathrm{Sh}_K(G,X)_{\hat{\Delta}} \to G_\ell(\mathbb{Z})\backslash M_{K_P}(P,W_\cdot,Y)_{\hat{\Delta}}$$

The isomorphism should correspond to the isomorphism on the level of analytic spaces.

The statement should be regarded as giving a precise description of the structure of $\mathrm{Sh}(G, X)$ near the boundary component F. For Siegel modular varieties, it is proved in Chai and Faltings (1989), IV.

A $G_{\mathbb{C}}$-equivariant vector bundle $\mathcal{J}$ on $\check{X}$, defines automorphic vector bundles $\mathcal{V}(\mathcal{J})$ and $\mathcal{V}_M(\mathcal{J})$ on $\mathrm{Sh}(G, X)$ and $M(P, W., Y)$ respectively; extend the vector bundles to the partial compactifications; the isomorphism in (3.1) will give an isomorphism of the formal completions: $\mathcal{V}(\mathcal{J})_{\hat{\Delta}} \approx \mathcal{V}_M(\mathcal{J})_{\hat{\Delta}}$. A section f of $\mathcal{V}(\mathcal{J})$ will extend to a section of $\mathcal{V}(\mathcal{J})_{\Delta}$, and map to a section of $FJ^F(f)$ of $\mathcal{V}_M(\mathcal{J})_{\hat{\Delta}}$— this is the *Fourier-Jacobi series of f along F.*

5. Conjugates of Fourier-Jacobi series.
The map $f \mapsto FJ^F(f)$ should be compatible with the various maps $\phi^*_{\tau,x}$ (see V.5.1 and VI.4). The q-expansion principle should then allow us to deduce that an automorphic form is rational over a field L if and only if its Fourier-Jacobi series are.

Note that for noncompact Shimura varieties, this will give another description of the canonical model of minimal compactification: it is the Proj of the graded ring generated by automorphic forms whose Fourier-Jacobi series have coefficients in the reflex field. We mention that Baily and Karel have been attempting to give a totally different approach to some of the results in this article by directly constructing automorphic forms whose Fourier-Jacobi series are rational (in a suitable sense) over $E(G, X)$ and then showing that the Proj of the graded ring they define is the canonical model of the Shimura variety (see for example Baily (1985) and Karel (1986)).

6. Automorphic forms of half-integral weight.
Just as modular forms of half-integral weight for GL_2 correspond in a natural way to automorphic forms of integral weight on the mixed Shimura variety defined in (2.3) (see Eichler and Zagier 1985), so should all automorphic forms of half-integral weight on a Shimura variety correspond to automorphic forms of integral weight on a mixed Shimura variety.

Notes. There is an enormous literature on Fourier-Jacobi series. Apart from those referred to in the text, the following papers are most closely related to the main theme of this Chapter: Shimura

(1978b), (1978c); Garrett (1981), (1983); and Harris (1986). I understand that Richard Pink's Bonn thesis will examine the question of the formal-algebraic definition of Fourier-Jacobi series.

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Automorphic L-Functions: A Survey

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The purpose of this article is to report on the progress made on analytic properties of automorphic L-functions after Corvallis. The reader who is interested in the work done before that should consult [**2**], [**6**], [**15**], [**16**], [**21**], and [**26**]. For more details and references we refer the reader to the recent book of Gelbart and Shahidi [**21**]. We finally refer to [**17**] and [**39**] for two recent expository articles on the subject. I would like to thank Jean-Pierre Serre for several comments towards the precision of this article. We start with the following conjecture of Langlands.

1. The conjecture.

In this section our main reference is Borel's lectures in Corvallis [**6**].

1.1 Local Langlands L-Functions.

Let F be a non-archimedean local field. Denote by O its ring of integers and let P be its maximal ideal. We use q to denote the number of elements in the residue field O/P. If ψ is a non-trivial (additive) character of F, we shall say ψ is *unramified* if O is the largest ideal of F on which ψ is trivial.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. In this section we shall assume that $\mathbf{G}$ is *unramified*. This means that $\mathbf{G}$ is quasi-split to split over an unramified extension L of F. Let ${}^LG$ be the L-group of $\mathbf{G}$ (cf. [**6**] and [**36**]) and denote by ${}^LG^0$ its connected component. For our purposes we may assume ${}^LG = {}^LG^0 \rtimes \Gamma_{L/F}$, where $\Gamma_{L/F}$ is the Galois group of L over F. Let τ be the Frobenius conjugacy class of $\Gamma_{L/F}$. Since $\mathbf{G}$ is unramified we can talk of $\mathbf{G}(O)$ and take it as a hyperspecial maximal compact subgroup K of $G = \mathbf{G}(F)$. Let π be an irreducible admissible K-unramified representation of G. This simply means that there exists a vector in the space of π fixed by K. As it is explained in Sections 6 and 7 of [**6**], to every such π, Satake isomorphism attaches a unique ${}^LG^0$-semisimple conjugacy class $A \rtimes \tau$ in ${}^LG^0 \rtimes \tau$.

By a representation r of ${}^LG$, we shall mean a continuous homomorphism from ${}^LG$ into some $GL_N(\mathbb{C})$ whose restriction to ${}^LG^0$ is a complex analytic map. Let $\tilde{r}$ denote the contragredient of r.

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Fix a complex number s and set

$$L(s, \pi, r) = \det(I - r(A \rtimes \tau)q^{-s})^{-1}, \tag{1.1}$$

where $I = I_n$. This is the local Langlands L-function attached to π and r.

1.2. Langlands' conjecture on automorphic L-functions. In this section we let F be an $\mathbb{A}$-field, i.e. either a number field or a function field of one variable over a finite field. Denote by $\mathbb{A}_F$ its ring of adeles. We shall always fix a non-trivial character ψ of $F \setminus \mathbb{A}_F$.

Let $\mathbf{G}$ be a connected reductive algebraic group over F. Let $\pi = \underset{v}{\otimes}\pi_v$ be an automorphic form on $G = \mathbf{G}(\mathbb{A}_F)$. We refer to [**7**] for its precise definition.

Let ${}^LG$ and ${}^LG_v$ denote L-groups of $\mathbf{G}$ and $\mathbf{G} \underset{F}{\times} F_v$ ($\mathbf{G}$ as a group over F_v), respectively. Then there exists a natural homomorphism $\eta_v : {}^LG_v \to {}^LG$. Let r be a representation of ${}^LG$ as defined in 1.1. Then each $r_v = r \cdot \eta_v$ is one of ${}^LG_v$.

For almost all the places v of F, $\mathbf{G} \underset{F}{\times} F_v$ is unramified and π_v is unramified with respect to $\mathbf{G}(O_v)$. We always use S to denote a finite set of places of F, including all the archimedean ones, such that for every $v \notin S$, $\mathbf{G} \underset{F}{\times} F_v$, π_v, and ψ_v, $\psi = \underset{v}{\otimes}\psi_v$, are all unramified.

Given a set S as above and a representation r of ${}^LG$, let

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v), \tag{1.2}$$

where the factors on the right are defined as in 1.1. As explained in Theorem 13.2 of [**6**], given π and r, the Euler product (1.2) converges absolutely for $Re(s)$ sufficiently large and therefore defines a non-zero analytic function of s in that region. Langlands' conjecture on automorphic L-functions can then be stated as follows [**36**]:

CONJECTURE (LANGLANDS). *For every $v \in S$, it is possible to define a local L-function $L(s, \pi_v, r_v)$, inverse of a polynomial in q_v^{-s}, and a local root number $\varepsilon(s, \pi_v, r_v, \psi_v)$, a monomial in q_v^{-s}, in such a way that*

$$L(s, \pi, r) = \prod_{v} L(s, \pi_v, r_v) \tag{1.3}$$

extends to a meromorphic function of s on $\mathbb{C}$ with only a finite number of poles if F is number field, and a rational function of q^{-s}, if F is a function field whose field of constants has q elements, satisfying

$$L(s,\pi,r)=\varepsilon(s,\pi,r)L(1-s,\pi,\widetilde{r}), \tag{1.4}$$

where

$$\varepsilon(s,\pi,r)=\prod_v \varepsilon(s,\pi_v,r_v,\psi_v), \tag{1.5}$$

with $\varepsilon(s,\pi_v,r_v,\psi_v)=1$ if v is unramified, in particular if $v\notin S$.

In what follows we shall explain the progress made on the conjecture since Corvallis.

2. Rankin-Selberg L-functions.

These L-functions generalize those of Rankin [**55**] and Selberg [**57**]. They have been studied by Jacquet, Piatetski-Shapiro, and Shalika in a series of papers, but unfortunately their complete results have yet to appear.

2.1. The L-functions. Here $\mathbf{G}=GL(n)\times GL(m)$, where m and n are two positive integers. We may take ${}^LG=GL_n(\mathbb{C})\times GL_m(\mathbb{C})$. The representation r is equal to $r=\rho_n\otimes\rho_m$, where ρ_n and ρ_m are standard representations of $GL_n(\mathbb{C})$ and $GL_m(\mathbb{C})$, respectively. Let $\pi=\bigotimes_v\pi_v$ and $\pi'=\bigotimes_v\pi'_v$ be cusp forms on $GL_n(\mathbb{A}_F)$ and $GL_m(\mathbb{A}_F)$, respectively. The homomorphism η_v of 1.2 is the identity. If v is unramified, we set

$$L(s,\pi_v\times\pi'_v)=L(s,(\pi_v,\pi'_v),\rho_n\otimes\rho_m),$$

where the L-function on the right is as in 1.1. Then

$$L(s,\pi_v\times\pi'_v)=\prod_{\substack{1\le i\le n\\ 1\le j\le m}}(1-\alpha_{i,v}\alpha'_{j,v}q_v^{-s})^{-1},$$

where $A_v=\{\mathrm{diag}(\alpha_{1,v},\dots,\alpha_{n,v})\}$ and $A'_v=\{\mathrm{diag}(\alpha'_{1,v},\dots,\alpha'_{m,v})\}$ are the semisimple conjugacy classes attached to π_v and π'_v(cf. 1.1), respectively. With notation as in 1.2, we let

$$L_S(s,\pi\times\pi')=\prod_{v\notin S}L(s,\pi_v\times\pi'_v).$$

This is the partial Rankin-Selberg L-function attached to π, π', and S. For $m=n=2$, they were studied by Jacquet who generalized results of Rankin [**55**] and Selberg [**57**]. On the other hand, if $m=1$, they are the principal L-functions of Godement and Jacquet (cf. [**26**]).

2.2. The results. The results can be stated as follows:

a) The partial L-function $L_S(s, \pi \times \pi')$ converges absolutely for $\mathrm{Re}(s) > 1$ ([**29**]).

b) $L_S(s, \pi \times \pi')$ extends to a meromorphic function of s on $\mathbf{C}$ ([**34**]).

c) For $m \neq n$, $L_S(s, \pi \times \pi')$ is holomorphic on $\mathrm{Re}(s) \geq 1$ ([**29**,30]).

d) Assume $m = n$. Let

$$X = \{s \in \mathbf{C} \mid \mathrm{Re}(s) = 1, \alpha^{s-1} \otimes \pi \cong \widetilde{\pi'}\}.$$

Then $L_S(s, \pi \times \pi')$ has a pole at s_0 with $\mathrm{Re}(s_0) = 1$ if and only if $s_0 \in X$. This pole is simple [**30**]. Here $\alpha = |\det(\ )|$.

e) For $\mathrm{Re}(s) = 1$, $L_S(s, \pi \times \pi') \neq 0$ ([**59**], also Theorem 3.2.3 below).

f) If $v < \infty$, the L-function $L(s, \pi_v \times \pi'_v)$ and the root number $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$ are defined in [**28**].

g) If $v = \infty$, let $\varphi_v : W_v \to GL_n(\mathbf{C}) \times GL_m(\mathbf{C})$ be the homomorphism attached to $\pi_v \otimes \pi'_v$ by local class field theory [**6**,37], where W_v is the Weil group $W(\overline{F}_v/F_v)$. Denote by $L(s, r\cdot\varphi_v)$ and $\varepsilon(s, r\cdot\varphi_v, \psi_v)$, the Artin L-function and root number attached to $r \cdot \varphi_v$ [**69**], where $r = \rho_n \otimes \rho_m$. We then set

$$L(s, \pi_v \times \pi'_v) = L(s, r \cdot \varphi_v)$$

and

$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, r \cdot \varphi_v, \psi_v).$$

h) Let

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v)$$

and

$$\varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v),$$

where the factors are defined as in f) and g). Then

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi'}).$$

This is proved by combining the results in [**59**], [**60**], and [**62**].

i) The L-function $L(s, \pi \times \pi')$ is expected to be entire unless $m = n$ and $\pi \otimes \alpha^s \cong \widetilde{\pi'}$ for some s in which case poles are simple. More precisely $L(s, \pi \times \pi)$ is expected to have simple poles at $s = 0, 1$. A very recent preprint of Waldspurger [**71**] seems to have answered this question also positively and therefore the theory must now be complete (cf. [**41**]).

All the results are proved completely for number fields. Parts a, c, and d are also stated for function fields [**29**, 30]. Immediate extensions of all other parts to function fields are expected, but have never been stated anywhere.

2.3. Applications.

2.3.1. Classification of automorphic forms for $GL(n)$. Let $G = GL_r(\mathbb{A}_F)$. Fix a cusp form $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_u$ of $M = GL_{r_1}(\mathbb{A}_F) \times \cdots \times GL_{r_u}(\mathbb{A}_F)$, $r_1 + \cdots + r_u = r$, where M is considered as the standard Levi subgroup of the standard parabolic subgroup $P = MN$ of G. Let $\xi = \underset{v}{\otimes} \xi_v$ be the representation

$$\xi = \mathrm{Ind}(G, P, \sigma \otimes 1).$$

Similarly assume Q is another standard parabolic subgroup of G and τ a cusp form on its standard Levi subgroup. Set

$$\eta = \mathrm{Ind}(G, Q, \tau \otimes 1).$$

We choose a finite set S of places of F such that for $v \notin S$, σ_v and τ_v are both unramified. Then ξ_v and η_v have the same unramified components if and only if (σ, P) and (τ, Q) are conjugate , i.e. up to a permutation they are equivalent. This is proved in [**30**] and is a consequence of 2.2.a, 2.2.c, 2.2.d, and 2.2.e. When $M = G$ this is called the *Strong Multiplicity one Theorem*. A stronger version of this case is proved in [**42**].

2.3.2. Converse theorem. As explained in paragraph 14.6 of [**6**], it is expected that the analytic properties of these L-functions would lead to existence of automorphic forms on $GL_r(\mathbb{A}_F)$. But unfortunately the only published version of this is still [**27**].

2.3.3. Applications in Base change for $GL(n)$. Almost all the results in 2.2 are used by Arthur and Clozel in [**3**] to establish base change for forms on $GL(n)$.

3. Langlands' Euler products method. In a series of lectures in 1967, Langlands expressed constant terms of Eisenstein series on certain split algebraic groups as ratios of products of certain automorphic L-functions. From this he deduced the meromorphy of these

L-functions in a number of cases on the whole complex plane. This also gave him the most substantial evidence for his conjecture of Section 1.2. These lectures were later published as a book titled "Euler Products" [**34**]. Langlands' method was later pursued by Shahidi (cf. [**59**, 60, 63, 65], for example) who generalized and established further properties of these L-functions for the so called "generic" representations. The recent preprint of Waldspurger [**71**] must shed new lights on the whole theory, since in the case of $GL(N)$ it combines the results of this method with certain results of Jacquet, Piatetski-Shapiro, and Shalika to prove the holomorphy of Rankin-Selberg L-functions (cf. Section 2) for all $s \neq 0, 1$ (cf. §2.2.1); thus avoiding deeper analysis of the local integral representations of Jacquet, Piatetski-Shapiro, and Shalika for these L-functions.

3.1. The set up. Let $\mathbf{H}$ be a quasi-split connected reductive algebraic group over a *number field* F. Fix a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{H}$ and let $\mathbf{P} = \mathbf{MN}$ be a standard maximal parabolic ($\mathbf{U} \supset \mathbf{N}$) subgroup of $\mathbf{M}$. Let ${}^LM$ be the L-group of $\mathbf{M}$ and denote by ${}^L\mathfrak{n}$ the Lie algebra of the L-group ${}^LN$ of $\mathbf{N}$. If r denotes the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$, we write $r = \bigoplus_{i=1}^{m} r_i$ for its decomposition to irreducible components.

It is the group $\mathbf{M}$ and the representations r_i for which the conjecture can be addressed. To be in accordance with our general notation from now on we shall use $\mathbf{G}$ instead of $\mathbf{M}$. Let $\mathbb{A}_F$ denote the ring of adeles of F. For every group $\mathbf{L}$ over F we use L to denote $\mathbf{L}(\mathbb{A}_F)$. Fix a character $\chi = \underset{v}{\otimes}\chi_v$ of $\mathbf{U}(F) \setminus U$. We shall assume χ is *generic*. This simply means that the restriction of χ to every simple root group is non-trivial. Let $\mathbf{U}^0 = \mathbf{U} \cap \mathbf{G}$. We again use χ to denote $\chi | U^0$.

Let $\pi = \underset{v}{\otimes}\pi_v$ be a cuspidal representation of G. We shall say π is *globally χ-generic* if there exists a cusp form φ in the space of π such that

$$\int_{\mathbf{U}^0(F)\setminus U^0} \varphi(ug)\overline{\chi(u)}du \neq 0$$

for some $g \in G$.

Fix a non-trivial character $\psi = \underset{v}{\otimes}\psi_v$ of $F \setminus \mathbb{A}_F$. Then there is a natural generic character χ_0 of $\mathbf{U}(F) \setminus U$ defined by ψ. Changing the splitting on $\mathbf{G}$ we may assume that π is χ_0-generic. Otherwise said, we can find a cusp form in the L-packet of π, generic with respect to χ_0. Observe that $\chi = \chi_0 \cdot Ad(a)$, where $a \in \mathbf{A}_0(\overline{F})$ with $Ad(a)$ defined over F. Here $\mathbf{A}_0$ is the maximal split torus of π.

3.2. The results. The theory of local coefficients as developed in [**59**], [**60**], [**63**], and [**65**] leads to a number of general and deep results both in the theory of automorphic forms and representations of p-adic groups. We shall now state some of these results.

In what follows generic always means globally generic, and changing the splitting we may always assume that our representation is χ_0-generic. The following theorem is Theorem 3.5 of [**65**].

THEOREM 3.2.1. *([**59**, 60, 63, 65]) Assume π is χ_0-generic. Then each $L_S(s,\pi,r_i)$ converges absolutely for* Re$(s) > 2$ *and extends to a meromorphic function of s on* $\mathbf{C}$ *(Theorem 5.1 of [**63**] and Remark 12.4 of [**65**]). Moreover for each i, $1 \leq i \leq m$, and each $v \in S$, there exists a complex function $\gamma_i(s,\pi_v,\psi_v)$ (which is a rational function of $q_v^{-s}, v < \infty$), satisfying the following properties:*

a) If $v = \infty$ or π_v has a Iwahori fixed vector (in particular if π_v has a vector fixed by a special maximal compact subgroup) and $\varphi'_v : W'_{F_v} \to {}^LG$ is the homomorphism of the Deligne-Weil group attached to π_v, then

$$\gamma_i(s,\pi_v,\psi_v) = \varepsilon(s, r_{i,v}\cdot\varphi'_v,\psi_v)L(1-s,\widetilde{r}_{i,v}\cdot\varphi'_v)/L(s,r_{i,v}\cdot\varphi'_v),$$

*where $L(s,r_{i,v}\cdot\varphi'_v)$ and $\varepsilon(s,r_{i,v}\cdot\varphi'_v,\psi_v)$ are the Artin L-function and root number attached to $r_{i,v}\cdot\varphi'_v$ (cf. [**69**]).*

b) For each i, $1 \leq i \leq m$,

$$L_S(s,\pi,r_i) = \prod_{v\in S}\gamma_i(s,\pi_v,\psi_v)L_S(1-s,\pi,\widetilde{r}_i)$$

c) The factors γ_i are defined locally for every quasi-split local group, a Levi factor of it, and an irreducible admissible χ_0-generic representation σ of this Levi factor. They satisfy

$$\gamma_i(s,\sigma,\psi)\gamma_i(1-s,\widetilde{\sigma},\overline{\psi}) = 1,$$

where χ_0 is defined by means of ψ.

d) Together with an inductive property, the conditions a) and b) determine γ_i's uniquely.

The functional equation in part b) is a consequence of Theorem 4.1 of [**59**] and the inductive results of [**63**] and [**65**]. The fact that the local factors $\gamma_i(s,\pi_v,\psi_v)$ at the archimedean places are Artin factors is the main result (Theorem 3.1) of [**60**].

THEOREM 3.2.2. *([63], [65]) Assume π is χ_0-generic. Then for every $v \notin S$, every local L-function $L(s, \pi_v, r_{i,v})$ is holomorphic if* $\mathrm{Re}(s) \geq 1$, $1 \leq i \leq m$.

THEOREM 3.2.3. *(Theorem 5.1 of [59]) Assume π is χ_0-generic. Then*

$$\prod_{i=1}^{m} L_S(1, \pi, r_i) \neq 0.$$

Lists of all the possible $(\mathbf{H}, \mathbf{G}, r_i)$ are given in [34] and [63]. They include all the cases known by other methods. Examples will be given in the next several sections. We conclude this section with the following:

THEOREM 3.2.4. *(Theorem 6.1 of [63]) Assume $m = 1$ or 2 and moreover if $m = 2$, assume r_2 is one dimensional. Suppose π is χ_0-generic. Then for each $v \in S$, a local L-function $L(s, \pi_v, r_{1,v})$ can be defined in such a way that*

$$L(s, \pi, r_1) = \prod_v L(s, \pi_v, r_{1,v})$$

extends to a meromorphic function of $s \in \mathbb{C}$ with possibly only a finite number of poles, satisfying a functional equation. The factors at the archimedean places are Artin factors (cf. Theorem 3.2.1.a)

COROLLARY. *With assumptions as in Theorem 3.2.4, let S be a finite set of places of F, including all the ramified and archimedean ones, such that if $v \in s$ is ramified, then S contains all other places which lie over the same rational prime as v does. Then the partial L-function $L_S(s, \pi, r_1)$ extends to a meromorphic function of s with possibly only a finite number of poles on all of $\mathbb{C}$.*

3.3. Examples of Theorem 3.2.4. In all the following examples, besides Theorems 3.2.1–3.2.3, Theorem 3.2.4 applies and consequently the finiteness of poles on all of $\mathbb{C}$ also follows.

3.3.1. Rankin triple products. (Corollary 6.9 of [63]). Let $\mathbf{H} =$ Spin$(4,4)$ and take $\mathbf{G} = GL_2 \times SL_2 \times SL_2$. Fix a cusp form $\pi = \pi_1 \times \pi_2 \times \pi_3$ on $GL_2(\mathbb{A}_F) \times SL_2(\mathbb{A}_F) \times SL_2(\mathbb{A}_F)$. Assume $v \notin S$, $A(\pi_{1,v}) = \mathrm{diag}(\alpha_{1,v}, \alpha_{2,v})$, and $A(\pi_{2,v}) = \mathrm{diag}(\beta_{1,v}, \beta_{2,v})$, $A(\pi_{3,v}) = \mathrm{diag}(\gamma_{1,v}, \gamma_{2,v})$, both modulo the center of $GL_2(\mathbb{C})$. Then

$$L(s, \pi_v, r_1) = \prod_{i,j,k=1,2} (1 - \alpha_{i,v}\beta_{j,v}\gamma_{k,v} q_v^{-s})^{-1}. \qquad (3.3.1.1)$$

This is the first and only example of a triple Rankin product L-function of automorphic forms available at present. An integral representation for this L-function has been obtained by Garrett [**13**]. Using group representations this has also been treated by Piatetski-Shapiro and Rallis in [**49**]. We shall discuss these in Section 6 below.

3.3.2. Twisted triple products. (Cases ${}^3D_4 - 1$ and ${}^6D_4 - 1$ of [**63**]). Here **H** is the quasi-split orthogonal group of type D_4 defined by a separable extension E of degree 3 over F. We can take **G** such that there exists a surjection $\mathbf{G} \xrightarrow{\rho} \mathrm{Res}_{E/F} PGL_2 \longrightarrow 0$. The representation $r_1 \cdot {}^L\rho$ is an irreducible 8-dimensional representation of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rtimes \Gamma_{E/F}$ and $L(s, \pi_v, r_{1,v})$ generalizes the Asai's L-function (cf. [4,23]). If $v \notin S$ is inert, and $A(\pi_v) = (\mathrm{diag}(\alpha_v, \alpha_v^{-1}), I_2, I_2)$, then

$$\begin{aligned}(3.3.2.1)\quad & L(s, \pi_v, r_{1,v}) \\ & = (1 - \alpha_v q_v^{-s})^{-1}(1 - \alpha_v^{-1} q_v^{-s})^{-1}(1 - \alpha_v q_v^{-3s})^{-1}(1 - \alpha_v^{-1} q_v^{-3s})^{-1}.\end{aligned}$$

According as E/F is normal or not, this is the case ${}^3D_4 - 1$ or ${}^6D_4 - 1$ of [**63**]. Again we refer to [**13**] and [**49**] for an integral representation.

3.3.3. Second symmetric or exterior power L-functions for GL_n.
Using the cases (viii) and (iv) of [**34**], one can show that the results of the previous section all hold for $L_S(s, \pi, r)$, where π is a cusp form on $GL_n(\mathbb{A}_F)$ and r is either the symmetric or exterior square of the standard representation of $GL_n(\mathbb{C})$. They are also subject of a work in progress of Jacquet and Shalika, and Bump and Friedberg [**9**]. Finally when $n = 3$, we refer to [**46**] (cf. Section 7.1 for an application).

3.3.4. Exterior cube L-function for GL_6. Let **H** be the simply connected split group of type E_6. There is a parabolic subgroup whose Levi factor **G** is isomorphic to $(GL_1 \times SL_6)/\{\pm 1\}$. Let π_0 be a cusp form on $GL_6(\mathbb{A}_F)$ with central character ω. Then we use π to denote any irreducible component of $\omega^3 \otimes (\pi | SL_6(\mathbb{A}_F))$. It is a cusp form on G. If $\wedge^3 \rho_6$ denotes the exterior cube of the standard representation of $GL_6(\mathbb{C})$ (this is a 20-dimensional irreducible representation), then for $v \notin S$

$$L(s, \pi_v, r_1) = L(s, \pi_{0,v}, \wedge^3 \rho_6).$$

In fact if $A(\pi_{0,v}) = \mathrm{diag}(\alpha_{1,v}, \dots, \alpha_{6,v})$, then

$$L(s, \pi_{0,v}, \wedge^3 \rho_6) = \prod_{\substack{i \neq j,k \\ j \neq k \\ i,j,k=1}}^{6} (1 - \alpha_{i,v}\alpha_{j,v}\alpha_{k,v} q_v^{-s})^{-1}.$$

Now all the results of the previous section apply to $L_S(s, \pi_0, \wedge^3 \rho_6)$. This is example (x) of **[34]** (cf. Corollary 6.8 of **[63]** for PGL_6).

We should remark that in all these cases there is no restriction on the form since all the cusp forms on the group GL_n are globally generic **[66]** (with respect to any character).

3.4. Applications. The results of Section 3.2 can be used to compute Plancherel measures for quasi-split group [**59**, 65]. In particular, it leads to a proof of a conjecture of Langlands **[35]** on Plancherel measures **[62]**, **[64]**, **[65]**. Such results can be used to obtain deep results on non-supercuspidal tempered spectrum of many quasi-split groups (cf. **[64]** and **[65]**, for example).

4. Symmetric power L-functions for GL_2. Let $\pi = \bigotimes_v \pi_v$ be a cusp form on $GL_2(\mathbb{A}_F)$. Then for $v \notin S$, $A(\pi_v) = \mathrm{diag}(\alpha_v, \beta_v) \in GL_2(\mathbb{C})$. Given a positive integer m, let r_m denote the m-th symmetric power of the standard representation of $GL_2(\mathbb{C})$. This is an $(m+1)$-dimensional irreducible representation. Then, for $v \notin S$

$$L(s, \pi_v, r_m) = \prod_{0 \leq j \leq m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1}.$$

The L-functions $L_S(s, \pi, r_m)$ are quite important. They, basically, comprise all the automorphic L-functions for GL_2. Besides:

a) Assume that for every $m \in \mathbb{Z}^+$, $L_S(s, \pi, r_m)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Then in **[36]**, Langlands showed that for every $v \notin S$, $|\alpha_v| = |\beta_v| = 1$. This is the Ramanujan-Petersson's conjecture for π. One of the deepest and most difficult conjectures in number theory, whose proof in the case of holomorphic forms is due to Deligne **[10]**. For non-holomorphic forms the problem is still open (cf. 4.1.3 below), except for those forms which correspond to Galois representations (cf. the remark after Theorem 10.1 of **[72]**).

b) Assume a) and in addition that for every $m \in \mathbb{Z}^+$, the L-function $L_S(s, \pi, r_m)$ is non-zero and holomorphic for $\mathrm{Re}(s) = 1$. Then Sato-Tate's conjecture is valid **[58]**. It is a result of K. Murty **[44]** that if one knows the holomorphy for all m, then one has the non-vanishing for all m, and therefore the conjecture follows only from the holomorphy of all of these L-functions for $\mathrm{Re}(s) \geq 1$.

4.1. Results. Except for Murty's result mentioned above, there are no general results known about these L-functions. All that we know is for $m \leq 5$ which we shall now explain. We shall leave out the classical case $m = 1$ for which the conjecture is known following Hecke, Jacquet-Langlands, and Weil.

4.1.1. The case $m = 2$. This is the only non-classical case which we know the conjecture for $L(s, \pi, r_m)$ (cf. (1.3)). In fact it was proved by Shimura [**68**] that if π comes from a classical modular form, then $L(s, \pi, r_2)$ is entire unless there exists a non-trivial quadratic character η of $Q^* \setminus \mathbb{A}_Q^*$ such that $\pi \otimes \eta \cong \pi$, i.e. π is *monomial*. This was later extended to cusp forms on any $GL_2(\mathbb{A}_F)$ by Gelbart and Jacquet [**18**], where F is a $\mathbb{A}$-field.

4.1.2. Gelbart-Jacquet lift. Using the results of [**18**], it is now a simple application of the converse theorem for GL_3 (cf. [**27**]) that given a cusp form π on $GL_2(\mathbb{A}_F)$, there exists an automorphic representation Π on $GL_3(\mathbb{A}_F)$ such that

$$L(s, \Pi \otimes \omega) = L(s, \pi, r_2),$$

where the L-function on the left is the standard L-function of $\Pi \otimes \omega$ (cf. [**26**]) and ω is the central character of π. The representation Π is cuspidal unless $\pi \otimes \eta \cong \pi$ with η as in 4.1.1. The representation Π is what we call the *Gelbart-Jacquet lift* of π. We refer to [**11**] for a different approach using the trace formula.

4.1.3. Best estimates for Fourier coefficients. Now assume $\pi = \underset{v}{\otimes}\pi_v$ is a non-monomial (cf. 4.1.1) cusp form on $GL_2(\mathbb{A}_F)$. (For the monomial cusp forms the Ramanujan-Petersson's conjecture is automatically valid.) For $v \notin S$, let $A(\pi_v) = \mathrm{diag}(\alpha_v, \beta_v) \in GL_2(\mathbb{C})$. Then using Gelbart-Jacquet lift Π of π, it can be shown that

$$q_v^{-1/5} < |\alpha_v| < q_v^{1/5}$$

and

$$q_v^{-1/5} < |\beta_v| < q_v^{1/5},$$

(cf. 4.a above). When $F = \mathbb{Q}$ and in the form $p^{-1/5} \leq |\alpha_p| \leq p^{1/5}$, this was first proved by Serre in a letter to Deshouillers, but was never published. Published versions of the proof can be found in [**43**] and [**45**]. Unfortunately their proofs make use of certain unpublished results of Jacquet, Piatetski-Shapiro, and Shalika on Rankin-Selberg L-functions (cf. Section 2, here). It was precisely for this reason that Serre never published his proof.

In general (i.e. for $GL_2(\mathbb{A}_F)$ with F any number field) and with strict inequality, this is basically Corollary 5.5 of [**63**]. Its proof is complete and requires no unpublished results.

At the archimedean places, the best results are due to Gelbart-Jacquet [**18**] and Iwaniec [**25**]. We refer to [**63**] and [**40**] for estimates for other groups.

4.1.4. The case $m = 3$. This is one of the cases of Theorem 3.2.4 (observed by Langlands in [**34**]) which when mixed with the results of 4.1.2 and 2.2 leads to a functional equation with Artin L-functions at every place of F. Moreover, it can be shown that under a non-vanishing hypothesis for Jacquet-Langlands L-function $L(s, \pi)$ on an interval parallel to $[1/2, 1)$ (parallel to $(1/2, 1)$ if π is on $PGL_2(\mathbb{A}_F)$), $L(s, \pi, r_3)$ is entire [**64**]. The fact that $L(s, \pi, r_3) \neq 0$ for $\mathrm{Re}(s) \geq 1$ is basically proved in [**59**] (cf. Theorem 3.2.3).

4.1.5. The cases $m = 4$ and 5. Both L-functions extend to meromorphic functions of s on $\mathbb{C}$, each satisfying a functional equation (Theorem 3.2.1). Moreover, it is proved in [**59**], that for $\mathrm{Re}(s) = 1$, the L-function $L_S(s, \pi, r_4)$ is non-zero. When $m = 5$, it is proved in [**64**], that $L_S(s, \pi, r_5) \neq 0$ for $\mathrm{Re}(s) = 1$, except possibly for a simple zero at $s = 1$. As it is explained in [**64**], even this leads to non-trivial results in the direction of Sato-Tate's conjecture for holomorphic forms (cf. 4.b). This is due to Serre.

5. The work of Piatetski-Shapiro and Rallis. In around 1980, Waldspuger [**70**], using an ingenious method, described the Shimura correspondence [**67**] between automorphic forms on $\overline{SL_2}(\mathbb{A}_F)$, the two fold metaplectic covering of $SL_2(\mathbb{A}_F)$, and $PGL_2(\mathbb{A}_F)$ (a dual reductive pair; cf. [**24**]), by means of integration against the restriction of a theta function on a bigger group to $\overline{SL_2} \times PGL_2$. This idea was later generalized by Rallis [**54**], who, using a result of Kudla [**33**] and the Siegel-Weil formula expressed the norm of the corresponding lift F_f as an integral of an Eisenstein series on a bigger group against the product of f by itself (cf. [**21**], Section III.1.1 for more detail). These are the type of integrals which appear in the work of Piatetski-Shapiro and Rallis which we shall now explain.

5.1. The set up. Let $\mathbf{G}$ be a connected reductive algebraic group over F whose center $\mathbf{C}$ is anisotropic, i.e. $\mathbf{C}(F) \setminus \mathbf{C}(\mathbb{A}_F)$ is compact. Assume there exists another reductive group $\mathbf{H}$ over F in which $\mathbf{G}\times\mathbf{G}$ can be embedded. Let $\mathbf{G}^d$ be the image of $\mathbf{G}$ under the diagonal embedding of $\mathbf{G}$ in $\mathbf{H}$. Fix a parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$. Then $\mathbf{G} \times \mathbf{G}$ acts on $\mathbf{P} \setminus \mathbf{H} = X$. An orbit $X' \subset X$ of $\mathbf{G} \times \mathbf{G}$ is called *negligible* if the stabilizer $R' \subset \mathbf{G} \times \mathbf{G}$ of a point $x' \in X'$ contains the unipotent radical of a proper parabolic subgroup of $\mathbf{G} \times \mathbf{G}$. Let $x_0 \in X$ correspond to the coset $\mathbf{P}\cdot e$ and denote by X_0 its $\mathbf{G}\times\mathbf{G}$-orbit. This is called the main orbit. Then the stabilizer R_0 of x_0 in $\mathbf{G}\times\mathbf{G}$ is $\mathbf{P}\cap(\mathbf{G}\times\mathbf{G})$. We shall now assume that the following two conditions are satisfied:

a) $R_0 = \mathbf{G}^d$, and

b) every $X' \neq X_0$ is negligible.

If s is a complex number, there is a natural character ω_s of $\mathbf{P}(F)\setminus \mathbf{P}(\mathbb{A}_F)$ which is trivial on $\mathbf{G}^d(\mathbb{A}_F)$. Fix a function f in the space of $\operatorname{Ind}_{\mathbf{P}(\mathbb{A}_F)\uparrow\mathbf{H}(\mathbb{A}_F)} \omega_s$. We then let:

$$E(\omega_s, f, h) = \sum_{\gamma\in\mathbf{P}(F)\setminus\mathbf{H}(F)} f(\gamma h),$$

where $h \in \mathbf{H}(\mathbb{A}_F)$.

Let π be an irreducible cuspidal representation of $G = \mathbf{G}(\mathbb{A}_F)$. Choose a pair of cusp forms φ_1 and φ_2 in the spaces of π and its contragredient, respectively. Consider

$$(5.1.1)\quad Z(\omega_s,\varphi_1,\varphi_2,f) = \int_{\mathbf{G}(F)\times\mathbf{G}(F)\setminus G\times G} \varphi_1(g_1)\varphi_2(g_2)E(\omega_s, f, (g_1,g_2))dg_1dg_2.$$

It is easy to see that under assumptions a) and b) above,

$$(5.1.2)\qquad Z(\omega_s,\varphi_1,\varphi_2,f) = \int_G f(g,1) < \pi(g)\varphi_1,\varphi_2 > dg,$$

where $(g,1)$ is considered as an element of H by the embedding of $G\times G$ into H. Choosing φ_1 and φ_2 appropriately, we may assume (5.1.2) is Eulerian. Replacing E by the normalized E^*(cf. [**47**]) which has only a finite number of poles (this is accomplished if one multiplies E by a product of abelian L-functions which eliminates the infinitely many unwanted poles of E), we shall see that (5.1.1) provides us with an integral representation for certain automorphic L-functions. This would then lead to a proof of the finiteness of poles for these L-functions (since E^* has only a finite number of poles). We should remark that this may be considered as a generalization of the work of Godement and Jacquet on principal L-functions [**26**].

5.2. The results. [**47**, 48, 50] Choosing $\mathbf{H}$ appropriately, (5.1.1) will provide us with an integral representation for $L(s, \pi, r)$, where π is a cusp form on $\mathbb{A}_F$-points of either of the groups $\mathbf{G} = Sp_{2n}$, O_n, or U_n, and r is the standard representation of the corresponding L-group ${}^LG$. We should remark that the cusp forms no longer have to be generic. As mentioned above, this proves the finiteness of poles for each $L_S(s, \pi, r)$ on $\mathbb{C}$. The local factors at the ramified primes have not yet been all defined and therefore the conjecture of 1.2 has not yet been completely verified in these cases. We should remark that, using classical methods for holomorphic forms, some of these results have also been obtained by Andrianov [**1**], Gritsenko [**22**], as well as Böcherer and Schulze-Pillot (cf. [**5**]). We finally refer the reader to [**8**] for an integral representation for the L-function $L(s, \pi, r)$ where π is a globally generic cusp form on $GSp_6(\mathbb{A}_F)$, trivial on the center, and r is the irreducible eight dimensional representation of ${}^LGSp_6$ (which is isogenous to $\mathrm{Spin}(7, \mathbb{C})$). This L-function can also be found by the method of Chapter 3.

6. Rankin triple products. One of the striking developments in the theory of automorphic L-functions in the past few years has been the work of Paul Garrett [**13**] who has obtained an integral representation for the Rankin triple product L-functions (cf. §3.3.1 here). Even though many properties of these L-functions could already be concluded from the results of Chapter 3, this was the first time that an integral representation for these L-functions could be found, forty eight years after Rankin's work [**55**] on double L-functions. This was later on generalized in [**14**] to include the twisted cases as well. After his results were explained, it became clear that this is one of the cases that can be obtained from the Piatetski-Shapiro-Rallis' theory. This was done in [**49**], generalizing the work of Garrett to non-holomorphic forms. We shall now explain both works.

6.1. The work of Garrett [13]. Let φ_1, φ_2, and φ_3 be three holomorphic cusp forms on the upper half plane H. Denote by H_3 the Siegel upper half space of degree 3. Then $H \times H \times H$ can be embedded in H_3. Let $E(z, s)$ be the abelian Eisenstein series on H_3. Consider

$$\int_{H\times H\times H} \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)E((z_1, z_2, z_3), s)dz_1dz_2dz_3. \tag{6.1.1}$$

Then in **[13]**, Garrett shows that (6.1.1) is in fact Eulerian and if $E(z, s)$ is normalized (cf. Section 5.1) properly, the factors (for $v \notin S$) are equal to the Rankin triple product $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$, where π_1, π_2 and $\pi_3(\pi_i = \bigotimes_v \pi_{i,v})$ are automorphic representations attached to φ_1, φ_2, and φ_3, respectively. The L-function $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ is the L-function defined by the right hand side of (3.3.1.1). Even though not proved in **[13]**, this must at least lead to the finiteness of poles on $\mathbb{C}$ for $L_S(s, \pi_1 \times \pi_2 \times \pi_3)$ (cf. Section 5.1).

6.2. The work of Piatetski-Shapiro and Rallis [49]. Let K be a semi-simple abelian algebra of degree 3 over F. Then either

$$K = F \oplus F \oplus F, \tag{6.2.1}$$

$$K = E \oplus F, \qquad [E : F] = 2, \text{ or} \tag{6.2.2}$$

$$K = K \qquad [K : F] = 3. \tag{6.2.3}$$

Let $V = K \oplus K$ and define an alternating form A on V by

$$A[(x, y), (x', y')] = xy' - x'y,$$

where (x, y) and (x', y') are in V. Set $A' = tr_{K/F}A$. Then A' is a F-valued skew symmetric form on V and $GS_p(A') = GSp_6(F)$. The group $GL_2(K)$ acts on V. Let $GL_2(K)^0$ be the points of $GL_2(K)$ which under this action belong to $GS_p(A')$. Then, for example

$$GL_2(K)^0 = \{(g_1, g_2, g_3) | g_i \in GL_2(F), \det g_1 = \det g_2 = \det g_3\}$$

if we are in case (6.2.1), while

$$GL_2(K)^0 = \{g \in GL_2(K) | \det g \in F^*\}$$

in case (6.2.3). Next, let $\mathbf{P} = \mathbf{MN}$ be the parabolic subgroup of GSp_6 with $\mathbf{M} = GL_3 \times GL_1$. With notation as in Section 5.1, we choose f in the space of $\operatorname*{Ind}_{\mathbf{P}(\mathbb{A}_F)\uparrow GSp_6(\mathbb{A}_F)} \omega_s$ and let $E(\omega_s, f, x), x \in GSp_6(\mathbb{A}_F)$, be the corresponding Eisenstein series. Finally let Π be a cusp form on the adelized version of $GL_2(K)^0$ which we denote by $GL_2(K)^0(\mathbb{A}_F)$. Then $\Pi = \pi_1 \times \pi_2 \times \pi_3$ with each π_i a cusp form on $GL_2(\mathbb{A}_F)$ if we are in case (6.2.1), while Π is a cusp form on $GL_2(\mathbb{A}_K)$ in the other extreme. If $\mathbf{Z}'$ is the center of GSp_6 and φ is in the space of Π, we set

$$Z(s, \varphi, f) = \int_{\mathbf{Z}'(\mathbb{A}_F)GL_2(K)^0\backslash GL_2(K)^0(\mathbb{A}_F)} \varphi(x)E(\omega_s, \varphi, x)dx.$$

We are now in the situation of Section 5.1 and we must study the right orbits of $GL_2(K)^0$ in $\mathbf{P}(F) \setminus GSp_6(F)$. Conditions a) and b) of 5.1 are satisfied and $Z(s, \varphi, f)$ becomes Eulerian. Normalizing $E(\omega_s, f, x)$ appropriately then shows that for $v \notin S$ the local factors are equal to $L(s, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v})$ in case (6.2.1) (cf. §3.3.1) and are defined by

$$(1 - \alpha_v q_v^{-s})^{-1}(1 - \beta_v q_v^{-s})^{-1}(1 - \alpha_v \beta_v^2 q_v^{-3s})^{-1}(1 - \alpha_v^2 \beta_v q_v^{-3s})^{-1},$$

if we are in case (6.2.3) and v is inert; as in §3.3.2. We refer the reader to case (ii) in page 96 of [**21**] for the case (6.2.2) (cf. Corollary 6.9.b of [**63**]). We finally remark that case (6.2.3) extends Asai's result [4] from quadratic to cubic extensions. The results are formulated as the following theorem in [**49**]. Here we use $L_S(s, \Pi)$ to denote the product of the local factors discussed above.

THEOREM 6.2. [**49**]. *Under the assumption that F is totally real (and an assumption on the central character of Π), the partial L-function $L_S(s, \Pi)$ can be extended to all the finite ramified primes in such a way that the resulting L-function satisfies a functional equation. Moreover its possible poles are at $s = 0$, $\frac{1}{4}$, $\frac{3}{4}$, and 1.*

We remark that in view of Theorems 3.2.1 and 3.2.4 of Section 3 what is new and does not seem to follow from the method of Section 3 is the possible location of poles.

7. Rankin-Selberg type L-functions. Let $\mathbf{G}$ and $\mathbf{G}'$ be two connected reductive algebraic groups over a number field F. Then ${}^L G$ and ${}^L G'$ are naturally embedded in groups of type $GL_N(\mathbb{C}) \rtimes \Gamma_{\overline{F}/F}$. Let r and r' be these embeddings. These are what we call the standard representations of ${}^L G$ and ${}^L G'$. Fix two automorphic forms π

and π' on G and G'. Let (π, π') be the form on $G \times G'$. The L-function $L_S(s, (\pi, \pi'), r \otimes r')$ is usually called the *Rankin-Selberg L-function* for the pair (π, π'). As in Section 2, we shall denote it by $L_S(s, \pi \times \pi')$. When $\mathbf{G} = GL_n$ and $\mathbf{G}' = GL_m$, these L-functions were discussed in Section 2. This is the only case where the theory is now complete (cf. the recent work of Waldspurger [**71**]). In every case known at present, the second group is always a GL_n.

7.1. The work of Gelbart and Piatetski-Shapiro [20]. We start with the case $\mathbf{G} \times GL_n$, where $\mathbf{G} = SO_{2n+1}$. Then $\mathbf{G}$ has a subgroup $\mathbf{H}$ isomorphic to SO_{2n} and SO_{2n} has a Levi subgroup isomorphic to GL_n. Let π' be a cusp form on $GL_n(\mathbb{A}_F)$ and choose f in the space of π'. There is an Eisenstein series defined by f which we denote by $E(s, f, h)$, $h \in H$. Now let φ be a cusp form in the space of π and consider

$$Z(s, \varphi, f) = \int_{\mathbf{H}(F)\backslash H} \varphi(h) E(s, f, h)\, dh. \tag{7.1.1}$$

The normalizing factor for $E(s, f, h)$, (cf. 5.1), is now more delicate and is the L-function $L_S(2s, \pi', \wedge^2 \rho_n)$, where $\wedge^2 \rho_n$ is the exterior square of the standard representation ρ_n of $GL_n(\mathbb{C})$ (cf. §3.3.3).

Now, if π is also globally generic (π' always is [**66**]), it can be shown that $Z(s, \varphi, f)$ is Eulerian. Moreover if E is replaced by its normalization, the local factors at $v \notin S$, are $L(s, \pi_v \times \pi'_v)$.

Using the finiteness of poles for $L_S(2s, \pi', \wedge^2 \rho_n)$ (cf. §3.3.3), it is expected that (7.1.1) leads to a proof of the finiteness of poles for $L_S(s, \pi \times \pi')$.

Similar results are expected when $\mathbf{G} = SO_{2n}$ or Sp_{2n}. In the case $\mathbf{G} = GSp_4$ and $\mathbf{G}' = GL_2$ these results are also obtained by Piatetski-Shapiro and Soudry [**52**, 53]. Finally, we refer to [**19**], where $\mathbf{G} = U(2,1)$ defined by a quadratic extension E of F, and $\mathbf{G}' = \mathrm{Res}_{E/F}\, GL_1$ (cf. Sectin 8.1 below).

7.2. Examples from Euler products method. We refer to [**34**] and [**63**] for many examples of Rankin products including every case mentioned so far but:

7.3. The Case $G_2 \times GL_2$ [51]. This is a very new result just obtained by Piatetski-Shapiro, Rallis, and Schiffman. In fact, using their theory explained in Chapter 5, they have now been able to obtain an integral representation for $L_S(s, \pi \times \pi')$, where π is a globally generic cusp form on adelic points of a split exceptional group of type G_2, and π' is an automorphic form on $PGL_2(\mathbb{A}_F)$. Taking π' equal to the trivial representation, this also gives the L-function attached to the standard representation of G_2. This is very striking since neither of these L-functions can be obtained by any other method (also see 8.1 and 8.2 below).

8. Functoriality principle and L-functions. Going back to the general conjecture, let $\mathbf{G}$ be as in Section 1. A representation $r : {}^LG \to GL_N(\mathbb{C})$ is in fact a homomorphism from ${}^LG$ into ${}^LGL_N$ and therefore by Langlands' Functoriality Principle [**36**, 39], there must exist a map r_* from the space of automorphic forms on G into those on $GL_N(\mathbb{A}_F)$ such that

$$L(s, \pi, r) = L(s, r_*(\pi), \rho_n)$$

and

$$\varepsilon(s, \pi, r) = \varepsilon(s, r_*(\pi), \rho_n),$$

where the factors on the right are the standard L-function and root number for $r_*(\pi)$ which is an automorphic form on $GL_N(\mathbb{A}_F)$ [**26**]. Since Conjecture 1.2 is in fact proved for the standard L-functions for GL_N, the conjecture for $L(s, \pi, r)$ now follows. For briefness, we shall restrict ourselves to only two cases of functoriality (also see 4.1.2).

8.1. The unitary group in 3-variables. In [**56**], Rogawski has proved the existence of θ_*, where

$$\begin{aligned} \theta : {}^LU(2,1) = GL_3(\mathbb{C}) \rtimes \Gamma_{E/F} &\to (GL_3(\mathbb{C}) \times GL_3(\mathbb{C})) \rtimes \Gamma_{E/F} \\ &= {}^L(\mathrm{Res}_{E/F}\, GL_3), \end{aligned}$$

sends $g \rtimes \tau$ to $(g, g) \rtimes \tau$. Now let r be a representation of $(GL_3(\mathbb{C}) \times GL_3(\mathbb{C})) \rtimes \Gamma_{E/F}$, then

$$L_S(s, \pi, r \cdot \theta) = L_S(s, \theta_*(\pi), r). \tag{8.1.1}$$

Choosing r from the examples in [**63**] and using the theory of Section 3 must then lead to new L-functions for the unitary group $U(2,1)$ which can not be found by any other method.

8.2. Base change for GL_n. It is proved in [**3**] that if

$$\begin{aligned}\theta : {}^LGL_n \\ &= GL_n(\mathbb{C}) \times \Gamma_{E/F} \to (GL_n(\mathbb{C}) \times \cdots \times GL_n(\mathbb{C})) \rtimes \Gamma_{E/F} \\ &\qquad = {}^L(\mathrm{Res}_{E/F}\, GL_n)\end{aligned}$$

sends (g, τ) to $(g, \ldots, g) \rtimes \tau$, then θ_* exists (cf. [**12**] and [**38**] for $n = 3$ and 2, respectively). Here E/F is a cyclic extension. If r is a representation of ${}^L(\mathrm{Res}_{E/F}\, GL_n)$, then again (8.1.1) holds. It is intriging to see what new L-functions can be obtained, if one combines this with possibilities in [**63**].

9. Concluding remarks. It is clear that each of the methods discussed above has its advantages and limitations. While the method of Section 3 is powerful in establishing the functional equation with Artin factors at every place where the representation can be parametrized (Theorem 3.2.1), and even a proof of the finiteness of poles in many cases, it is the use of integral representations which has proved more useful in locating the poles. On the other hand when it comes to local analysis at the archimedean places, the method of integral representations has often been very cumbersome and unsuccessful. It may well turn out that, at least for those L-functions which appear in the constant terms of Eisenstein series (cf. Chapter 3), the most efficient way of obtaining complete results is in mixing the two methods. It is for this reason that the recent work of Waldspuger [**71**] on $GL(n) \times GL(m)$ (cf. §2.2.2) must be considered a breakthrough.

As experience has shown [**3**, 56], the use of analytic properties of those L-functions which appear in the constant terms of Eisenstein series, if not absolutely necessary, has greatly simplified any use of the trace formula in establishing the principle of functoriality. Whether this is the extent of which the analytic properties of automorphic L-functions can be used in establishing the principle of functoriality (cf. Section 8) in general remains to be seen.

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