Chapter 10

Algebraic schemes: geometry over an arbitrary field

In this chapter, we allow the base field to be arbitrary, and we allow the structure sheaves to contain nilpotent elements. Thus, we are moving away from geometry towards scheme theory.

We assume that the reader is familiar with the contents of the first 9 chapters, and we are more brief, since many arguments essentially repeat those in the first nine chapters.

Throughout, k is a field and k^{al} is an algebraic closure of k. Unadorned tensor products are over k. All k-algebras are finitely generated, and Alg_k denotes the category of such algebras. A reference n.mm is to the main notes *Algebraic Geometry*. CA= my *Commutative Algebra* notes. Hyperlinks may work if both pdf files are in the same folder.

a. Preliminaries

Sheaves

In the first nine chapters we considered only sheaves of functions, and so the restriction maps did, in fact, restrict a function to an open subset. Here we consider more general sheaves, and the restriction maps have to be included as part of the data.

10.1. A ***presheaf*** $\mathcal{F}$ on a topological space V is a map assigning to each open subset U of V a set $\mathcal{F}(U)$ and to each inclusion $U' \subset U$ a "restriction" map

$$a \mapsto a|U' : \mathcal{F}(U) \to \mathcal{F}(U');$$

when $U = U'$ the restriction map is required to be the identity map, and if

$$U'' \subset U' \subset U,$$

then the composite of the restriction maps

$$\mathcal{F}(U) \to \mathcal{F}(U') \to \mathcal{F}(U'')$$

is required to be the restriction map $\mathcal{F}(U) \to \mathcal{F}(U'')$. In other words, a presheaf is a contravariant functor to the category of sets from the category whose objects are the open subsets of V and whose morphisms are the inclusions. A ***homomorphism of presheaves*** $\alpha : \mathcal{F} \to \mathcal{F}'$ is a family of maps

$$\alpha(U) : \mathcal{F}(U) \to \mathcal{F}'(U)$$

commuting with the restriction maps, i.e., it is a morphism of functors. We sometimes write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$.

10.2. A presheaf $\mathcal{F}$ is a ***sheaf*** if it satisfies the ***sheaf condition***:

for any open covering $\{U_i\}$ of an open subset U of V and family of sections $a_i \in \mathcal{F}(U_i)$ agreeing on overlaps (that is, such that $a_i|U_i \cap U_j = a_j|U_i \cap U_j$ for all i, j), there is a unique element $a \in \mathcal{F}(U)$ such that $a_i = a|U_i$ for all i.

A ***homomorphism of sheaves*** on V is a homomorphism of presheaves. If the sets $\mathcal{F}(U)$ are abelian groups and the restriction maps are homomorphisms, then $\mathcal{F}$ is a ***sheaf of abelian groups***. Similarly one defines the notions of a ***sheaf of rings***, a ***sheaf of*** k***-algebras***, and a ***sheaf of modules*** over a sheaf of rings.

10.3. For $P \in V$, the ***stalk*** of a sheaf $\mathcal{F}$ (or presheaf) at P is

$$\mathcal{F}_P = \varinjlim \mathcal{F}(U) \quad \text{(limit over the open neighbourhoods of } P).$$

In other words, it is the set of equivalence classes of pairs (U, s) with U an open neighbourhood of P and $s \in \mathcal{F}(U)$; two pairs (U, s) and (U', s') are equivalent if $s|U'' = s'|U''$ on some open neighbourhood U'' of P contained in $U \cap U'$.

10.4. A ***ringed space*** is a pair $(V, \mathcal{O})$ consisting of topological space V together with a sheaf of rings. If the stalk $\mathcal{O}_P$ of $\mathcal{O}$ at all P is a local ring, then $(V, \mathcal{O})$ is called a ***locally ringed space***. A ***morphism*** $(V, \mathcal{O}) \to (V', \mathcal{O}')$ ***of ringed spaces*** is a pair (φ, ψ) comprising a continuous map $\varphi : V \to V'$ and a family of maps

$$\psi(U) : \mathcal{O}'(U) \to \mathcal{O}(\varphi^{-1}(U)), \quad U \text{ open in } V',$$

commuting with the restriction maps. Such a pair defines a homomorphism of rings $\psi_P : \mathcal{O}'_{\varphi(P)} \to \mathcal{O}_P$ for all $P \in V$. A ***morphism of locally ringed spaces*** is a morphism of ringed space such that ψ_P is a local homomorphism for all P.

10.5. Let V be a topological space. Recall that a collection $\mathcal{B}$ of open sets in V is a base for the topology if every open subset is a union of elements from $\mathcal{B}$. Regard $\mathcal{B}$ as a category with the inclusions as the only morphisms. A ***presheaf*** of sets on $\mathcal{B}$ is a functor $\mathcal{F}$ from $\mathcal{B}$ to sets. A ***sheaf*** of sets on $\mathcal{B}$ is a presheaf $\mathcal{F}$ of sets satisfying the sheaf condition: for any covering $\{U_i\}$ of a basic open subset U by basic open subsets U_i and family of sections $s_i \in \mathcal{F}(U_i)$ agreeing on overlaps, there is a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|U_i$ for all i. By s_i and s_j agreeing on the overlap $U_i \cap U_j$ we mean that $s_i|U' = s_j|U'$ for the sets U' in some covering of $U_i \cap U_j$ by basic open subsets. When $U_i \cap U_j$ is itself a basic open subset, this just means that $s_i|U_i \cap U_j = s_j|U_i \cap U_j$. Every sheaf on a base $\mathcal{B}$ of V extends uniquely to a sheaf on V. More precisely, the functor sending a sheaf on V to a sheaf on $\mathcal{B}$ is an equivalence of categories. See Stacks, 009H.

Extending scalars (extending the base field)

NILPOTENTS

Recall that a ring A is reduced if it has no nonzero nilpotents. A k-algebra A can be reduced without $A \otimes_k k^{\mathrm{al}}$ being reduced. Consider, for example,

$$A = k[X,Y]/(X^p + Y^p + a),$$

where $p = \mathrm{char}(k)$. If a is not a pth-power in k, then $X^p + Y^p + a$ is irreducible in $k[X,Y]$, and so A is an integral domain. As a becomes a pth power in k^{al}, say, $a = \alpha^p$,

$$X^p + Y^p + a = (X + Y + \alpha)^p,$$

in $k^{\mathrm{al}}[X,Y]$, and so $(x + y + \alpha)^p = 0$ in

$$A \otimes_k k^{\mathrm{al}} = k^{\mathrm{al}}[X,Y]/(X^p + Y^p + a) = k^{\mathrm{al}}[x,y].$$

In this subsection, we show that problems of this kind arise only because of inseparability. In particular, they do not occur if k is perfect.

Let p be the characteristic exponent of k (so p is 1 or a prime according as the characteristic of k is zero or nonzero). Let

$$k^{1/p} = \{\alpha \in k^{\mathrm{al}} \mid \alpha^p \in k\}.$$

It is a subfield of k^{al}, and $k^{1/p} = k$ if and only if k is perfect (for example, has characteristic zero). Let Ω be some (large) field containing k^{al}.

DEFINITION 10.6. Subfields K and K' of Ω containing k are said to be ***linearly disjoint*** over k if the map $K \otimes_k K' \to \Omega$ is injective.

Equivalent conditions:

- if $e_1, \ldots, e_m \in K$ are linearly independent over k and $e'_1, \ldots, e'_{m'} \in K'$ are linearly independent over k, then the elements $e_1e'_1, e_1e'_2, \ldots, e_me'_{m'}$ of Ω are linearly independent over k;

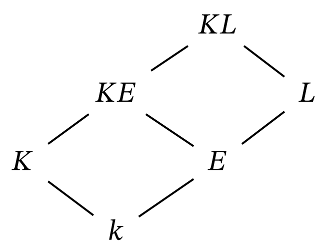
- if $e_1, \ldots, e_m \in K$ are linearly independent over k, then they are linearly independent over K'.

Similarly, we say that a k-algebra A in Ω is linearly disjoint from K over k if the map $A \otimes_k K \to \Omega$ is injective.

10.7. (a) Purely transcendental extensions of k are linearly disjoint over k from algebraic extensions of k.

(b) Separable algebraic extensions of k are linearly disjoint over k from purely inseparable algebraic extensions of k.

(c) Let $K \supset k$ and $L \supset E \supset k$ be subfields of Ω.



Then K is linearly disjoint from L over k if and only if K is linearly disjoint from E over k and KE is linearly disjoint from L over E,

DEFINITION 10.8. A ***separating transcendence basis*** for $K \supset k$ is a transcendence basis $\{x_1, \ldots, x_d\}$ such that K is separable over $k(x_1, \ldots, x_d)$.

The next proposition improves Theorem 9.27 of Milne 2022.

PROPOSITION 10.9. *Let K be a finitely generated field extension of k, and let Ω be an algebraically closed field containing K^{al}. The following statements are equivalent:*

(a) *K/k admits a separating transcendence basis;*

(b) *K is linearly disjoint from every purely inseparable extension of k in Ω;*

(c) *the fields K and $k^{1/p}$ are linearly disjoint over k.*

PROOF. (a)⇒(b). Let $\{x_1, \ldots, x_d\}$ be a separating transcendence basis for K over k, and let K' be a purely inseparable extension of k in Ω. Then $k(x_1, \ldots, x_d)$ is linearly disjoint from K' over k (by 10.7(a)), and $K'(x_1, \ldots, x_d)$ is linearly disjoint from K over $k(x_1, \ldots, x_d)$ (by 10.7(b)). Now apply 10.7(c).

(b)⇒(c). Trivial.

(c)⇒(a). Let $K = k(x_1, \ldots, x_n)$, and let d be the transcendence degree of K/k. After renumbering, we may suppose that $x_1, \ldots, x_d$ are algebraically independent (1.63(b)). We proceed by induction on n. If $n = d$ there is nothing to prove, and so we may suppose that $n \geq d + 1$. Then $f(x_1, \ldots, x_{d+1}) = 0$ for some nonzero irreducible polynomial $f(X_1, \ldots, X_{d+1})$ with coefficients in k. Not all $\partial f / \partial X_i$ are zero, for otherwise f would be a polynomial in $X_1^p, \ldots, X_{d+1}^p$, which contradicts the lemma below. After renumbering again, we may suppose that $\partial f / \partial X_{d+1} \neq 0$, and so $\{x_1, \ldots, x_d\}$ is a separating transcendence basis for $k(x_1, \ldots, x_{d+1})$ over k, which proves the proposition when $n = d + 1$. In the general case, $k(x_1, \ldots, x_{d+1}, x_{d+2})$ is algebraic over $k(x_1, \ldots, x_d)$ and x_{d+1} is separable over $k(x_1, \ldots, x_d)$, and so, by the primitive element theorem (Milne 2022, 5.1) there is an element y such that $k(x_1, \ldots, x_{d+2}) = k(x_1, \ldots, x_d, y)$. Thus K is generated by the $n - 1$ elements $x_1, \ldots x_d, y, x_{d+3}, \ldots, x_n$, and we apply induction. □

LEMMA 10.10. *Let $K = k(x_1, \dots, x_{d+1}) \subset \Omega$ with $x_1, \dots, x_d$ algebraically independent over F, and let $f \in k[X_1, \dots, X_{d+1}]$ be an irreducible polynomial such that $f(x_1, \dots, x_{d+1}) = 0$. If K is linearly disjoint from $k^{1/p}$, then $f \notin k[X_1^p, \dots, X_{d+1}^p]$.*

PROOF. Suppose otherwise, say, $f = g(X_1^p, \dots, X_{d+1}^p)$. Let $M_1, \dots, M_r$ be the distinct monomials in $X_1, \dots, X_{d+1}$ that actually occur in $g(X_1, \dots, X_{d+1})$, and let $m_i = M_i(x_1, \dots, x_{d+1})$. Then $m_1, \dots, m_r$ are linearly independent over k because they are distinct polynomials of degree less than that of f. However, $m_1^p, \dots, m_r^p$ are linearly dependent over k, because $g(x_1^p, \dots, x_{d+1}^p) = 0$. But

$$\sum a_i m_i^p = 0 \quad (a_i \in k) \Rightarrow \sum a_i^{1/p} m_i = 0 \quad (a_i^{1/p} \in k^{1/p})$$

contradicting the linear disjointness of K and $k^{1/p}$. □

DEFINITION 10.11. A finitely generated field extension $K \supset k$ is said to be ***regular*** if it satisfies the equivalent conditions of the proposition.

PROPOSITION 10.12. *Let A be a reduced k-algebra. The following statements are equivalent:*

(a) *$k^{1/p} \otimes_k A$ is reduced;*

(b) *$k^{\mathrm{al}} \otimes_k A$ is reduced;*

(c) *$K \otimes_k A$ is reduced for all fields $K \supset k$.*

When A is an integral domain, they are also equivalent to A and $k^{1/p}$ being linearly disjoint over k.

PROOF. The implications (c)⇒(b)⇒(a) are obvious, and so we only have to prove (a)⇒(c). After localizing A at a minimal prime, we may suppose that it is a field. Let $e_1, \dots, e_n$ be elements of A linearly independent over k. If they become linearly dependent over $k^{1/p}$, then $e_1^p, \dots, e_n^p$ are linearly dependent over k, say, $\sum a_i e_i^p = 0$, $a_i \in k$. Now $\sum a_i^{1/p} \otimes e_i$ is a nonzero element of $k^{1/p} \otimes_k A$, but

$$\left(\sum a_i^{1/p} \otimes e_i\right)^p = \sum a_i \otimes e_i^p = \sum 1 \otimes a_i e_i^p = 1 \otimes \sum a_i e_i^p = 0.$$

This shows that A and $k^{1/p}$ are linearly disjoint over k, and so A has a separating transcendence basis over k. From this it follows that $K \otimes_k A$ is reduced for all fields $K \supset k$. □

COROLLARY 10.13. *Let A be a k-algebra such that $k^{\mathrm{al}} \otimes_k A$ is reduced. Then $A \otimes_k B$ is reduced for all reduced k-algebras B (not necessarily finitely generated).*

PROOF. For any minimal prime ideal $\mathfrak{p}$ of B, the local ring $B_{\mathfrak{p}}$ is a field, and the map $A \otimes_k B \to \prod A \otimes_k B_{\mathfrak{p}}$ is injective. □

10.14. A ring A is said to be ***normal*** if $A_{\mathfrak{p}}$ is an integrally closed domain for all prime ideals $\mathfrak{p}$ in A. A k-algebra A is ***geometrically reduced*** (resp. ***normal***) if $k' \otimes_k A$ is reduced (resp. normal) for all extension fields k' of k. It suffices to check this condition for $k' = k^{1/p}$, where p is the characteristic exponent of k.[1]

[1] For "normal", this is Corollary 3.3 of Nagamachi, I., and Takamatsu, T., On behavior of conductors...over imperfect fields. J. Pure Appl. Algebra 228 (2024).

Idempotents

Even when a k-algebra A is an integral domain and $A \otimes_k k^{\mathrm{al}}$ is reduced, the latter need not be an integral domain. Suppose, for example, that A is a finite separable field extension of k. Then $A = k[X]/(f(X))$ for some monic irreducible separable polynomial $f(X)$, and so

$$A \otimes_k k^{\mathrm{al}} = k^{\mathrm{al}}[X]/(f(X)) = k^{\mathrm{al}}/(\prod(X - a_i)) \simeq \prod k^{\mathrm{al}}/(X - a_i)$$

(by Theorem 1.1). Thus if A contains a finite separable field extension of k, then $A \otimes_k k^{\mathrm{al}}$ cannot be an integral domain. The proposition below provides a converse.

Let A be an integral domain containing k. We say that k is ***algebraically closed in*** A if every element of A algebraic over k lies in k, i.e., an element a of A lies in k if $f(a) = 0$ for some nonzero $f \in k[X]$.

LEMMA 10.15. *Let k be algebraically closed in an extension field K, and let a be an element of K^{al} that is algebraic over k. Then K and $k[a]$ are linearly disjoint over k, and*

$$[K[a] : K] = [k[a] : k].$$

PROOF. Let $f(X)$ be the minimal polynomial of a over k. If h is a factor of f in $K[X]$, then the roots of h are among the roots of f, hence are algebraic over k, and so the coefficients of h are algebraic over k, hence lie in k. Thus $h \in k[X]$, and we deduce that f is irreducible in $K[X]$. Now the map

$$1 \otimes a \mapsto a \colon K \otimes_k k[a] \to K[a]$$

is an isomorphism because both K-algebras equal $K[X]/(f(X))$. □

PROPOSITION 10.16. *Let A be a k-algebra, and assume that A is an integral domain, and that $A \otimes_k k^{\mathrm{al}}$ is reduced. Then $A \otimes_k k^{\mathrm{al}}$ is an integral domain if and only if k is algebraically closed in A..*

PROOF. $\Leftarrow$: Let K be the field of fractions of A — it suffices to show that $K \otimes_k k^{\mathrm{al}}$ is an integral domain, and for this it suffices to show that K is linearly disjoint from L where L is any finite algebraic extension of k in K^{al} (because then $K \otimes_k L \simeq KL$, which is an integral domain). If L is separable over k, then it can be generated by a single element, and so this follows from the lemma. In the general case, we let E denote the largest subfield of L separable over k. From (10.7)(c), we see that it suffices to show that KE and L are linearly disjoint over E. From (10.12), we see that K and $k^{1/p}$ are linearly disjoint over k, and so K is a regular extension of k (see 10.9). It follows easily that KE is a regular extension of E, and KE is linearly disjoint from L by (10.7)(b).

$\Rightarrow$: If k is not algebraically closed in A, then $A \smallsetminus k$ contains an element a such that either $a^p \in k$ or a is separable over k. In the first case, $A \otimes_k k^{\mathrm{al}}$ is not reduced, and in the second it contains a nontrivial idempotent. □

COROLLARY 10.17. *Let A be a finitely generated k-algebra, and assume that A is an integral domain. Then*

(a) *$A \otimes_k k^{\mathrm{al}}$ has no nilpotents if and only if A and $k^{1/p}$ are linearly disjoint over k;*

(b) *$A \otimes_k k^{\mathrm{al}}$ has no idempotents if and only if k is separably closed in A.*

Thus, $A \otimes_k k^{\mathrm{al}}$ is an integral domain if and only if A and $k^{1/p}$ are linearly disjoint and k is separably closed in A.

b. Affine algebraic schemes

Let A be a (finitely generated) k-algebra.

10.18. Let V be the set of maximal ideals in A, and, for an ideal $\mathfrak{a}$ in A, let

$$V(\mathfrak{a}) = \{\mathfrak{m} \mid \mathfrak{m} \supset \mathfrak{a}\}.$$

Then

- $V(0) = V$, $V(A) = \emptyset$,
- $V(\mathfrak{ab}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for every pair of ideals $\mathfrak{a}, \mathfrak{b}$, and
- $V(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} \mathfrak{a}_i$ for every family of ideals $(\mathfrak{a}_i)_{i\in I}$.

For example, if $\mathfrak{m} \notin V(\mathfrak{a}) \cup V(\mathfrak{b})$, then there exist $f \in \mathfrak{a} \smallsetminus \mathfrak{m}$ and $g \in \mathfrak{b} \smallsetminus \mathfrak{m}$; but then $fg \notin \mathfrak{ab} \smallsetminus \mathfrak{m}$, and so $\mathfrak{m} \notin V(\mathfrak{ab})$ (cf. 2.10).

These statements show that the sets $V(\mathfrak{a})$ are the closed sets for a topology on V, called the ***Zariski topology***. We write $\mathrm{spm}(A)$ for V endowed with this topology.

For example, $\mathbb{A}^n \overset{\text{def}}{=} \mathrm{spm}(k[X_1, \dots, X_n])$ is ***affine*** n***-space*** over k. If k is algebraically closed, then the maximal ideals in A are exactly the ideals $(X_1 - a_1, \dots, X_n - a_n)$, and $\mathbb{A}^n$ can be identified with k^n endowed with its usual Zariski topology.

We now restate the Nullstellensatz and its immediate consequences for a nonalgebraically closed field k.

10.19 (NULLSTELLENSATZ). *Every proper ideal $\mathfrak{a}$ in $k[X_1, \dots, X_n]$ has a zero in $(k^{\mathrm{al}})^n$, i.e., there exists a point $(a_1, \dots, a_n) \in (k^{\mathrm{al}})^n$ such that $f(a_1, \dots, a_n) = 0$ for all $f \in \mathfrak{a}$.*

PROOF. We have to show that there exists a k-algebra homomorphism $k[X_1, \dots, X_n] \to k^{\mathrm{al}}$ containing $\mathfrak{a}$ in its kernel. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$. Then $k[X_1, \dots, X_n]/\mathfrak{m}$ is a field, which is finitely generated as a k-algebra. Therefore it is finite over k by Zariski's lemma (2.12), and so there exists a k-algebra homomorphism $k[X_1, \dots, X_n]/\mathfrak{m} \to k^{\mathrm{al}}$. The composite of this with the quotient map $k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]/\mathfrak{m}$ contains $\mathfrak{a}$ in its kernel. □

10.20 (STRONG NULLSTELLENSATZ). *For an ideal $\mathfrak{a}$ in $k[X_1, \dots, X_n]$, let $Z(\mathfrak{a})$ denote the set of zeros of $\mathfrak{a}$ in $(k^{\mathrm{al}})^n$. If a polynomial $h \in k[X_1, \dots, X_n]$ is zero on $Z(\mathfrak{a})$, then some power of h lies in $\mathfrak{a}$.*

PROOF. This can be deduced from 10.19 exactly as 2.16 is deduced from 2.11. □

COROLLARY 10.21. *The radical of an ideal $\mathfrak{a}$ in a k-algebra A is equal to the intersection of the maximal ideals containing it:*

$$\mathrm{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \mathfrak{m}.$$

In particular, if A is reduced, then the intersection of the maximal ideals in A is zero.

PROOF. The inclusion $\mathrm{rad}(\mathfrak{a}) \subset \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \mathfrak{m}$ holds in any ring (h lies in a maximal ideal if some power of h does).

Because of the correspondence between the ideals in a ring and in a quotient of the ring, it suffices to prove the reverse inclusion for $A = k[X_1, \dots, X_n]$.

Let h lie in all maximal ideals containing $\mathfrak{a}$, and let $(a_1, \ldots, a_n)$ be a zero of $\mathfrak{a}$ in $(k^{\text{al}})^n$. The image of the evaluation map

$$f \mapsto f(a_1, \ldots, a_n) \colon k[X_1, \ldots, X_n] \to k^{\text{al}}$$

is a subring of k^{al} which is algebraic over k, and hence is a field. Therefore, the kernel of the map is a maximal ideal, which contains $\mathfrak{a}$, and so also contains h. This shows that $h(a_1, \ldots, a_n) = 0$, and we conclude from the strong Nullstellensatz that $h \in \text{rad}(\mathfrak{a})$. □

10.22. For a subset S of $\text{spm}(A)$, let

$$I(S) = \bigcap \{\mathfrak{m} \mid \mathfrak{m} \in S\}.$$

Then

$$V(I(S)) = \text{Zariski-closure of } S,$$

and, for an ideal $\mathfrak{a}$ in A,

$$I(V(\mathfrak{a})) \stackrel{\text{def}}{=} \bigcap \{\mathfrak{m} \mid \mathfrak{m} \supset \mathfrak{a}\} \stackrel{10.21}{=} \text{rad}(\mathfrak{a}).$$

It follows that V and I are inverse bijections between the collections of radical ideals of A and closed subsets of $\text{spm}(A)$. Under this bijection, prime ideals correspond to irreducible sets, and maximal ideals correspond to points.

10.23. For $f \in A$, let $D(f) = \{\mathfrak{m} \mid f \notin \mathfrak{m}\}$. It is open in $\text{spm}(A)$ because its complement is the closed set $V((f))$. The sets of this form are called the ***basic open subsets*** of $\text{spm}(A)$. Let $V = V(\mathfrak{a})$ be a closed subset of $\text{spm}(A)$. According to the Hilbert basis theorem (2.8), A is noetherian, and so $\mathfrak{a} = (f_1, \ldots, f_m)$ for some $f_i \in A$, and

$$\text{spm}(A) \smallsetminus V = D(f_1) \cup \ldots \cup D(f_m).$$

This shows that every open subset of $\text{spm}(A)$ is a finite union of basic open subsets. In particular, the basic open subsets form a base for the Zariski topology on $\text{spm}(A)$.

10.24. Let $\alpha \colon A \to B$ be a homomorphism of k-algebras, and let $\mathfrak{m}$ be a maximal ideal in B. As B is finitely generated as a k-algebra, so also is $B/\mathfrak{m}$, which implies that it is a finite field extension of k (Zariski's lemma 2.12). Therefore the image of A in $B/\mathfrak{m}B$ is an integral domain of finite dimension over k, and hence is a field. This image is isomorphic to $A/\alpha^{-1}(\mathfrak{m})$, and so the ideal $\alpha^{-1}(\mathfrak{m})$ is maximal in A. Hence α defines a map

$$\alpha^* \colon \text{spm}(B) \to \text{spm}(A), \quad \mathfrak{m} \mapsto \alpha^{-1}(\mathfrak{m}),$$

which is continuous because $(\alpha^*)^{-1}(D(f)) = D(\alpha(f))$. In this way, spm becomes a functor from k-algebras to topological spaces.

10.25. Recall (1.10) that, for a multiplicative subset S of A, the ring of fractions having the elements of S as denominators is denoted by $S^{-1}A$. For example, if $S_f \stackrel{\text{def}}{=} \{1, f, f^2, \ldots\}$, then

$$A_f \stackrel{\text{def}}{=} S_f^{-1}A \simeq A[X]/(1 - fX).$$

Let $D = D(f)$ be a basic open subset of X. Then

$$S_D \stackrel{\text{def}}{=} A \smallsetminus \bigcup \{\mathfrak{m} \mid \mathfrak{m} \in D\}$$

is a multiplicative subset of A, and the map $S_f^{-1}A \to S_D^{-1}A$ defined by the inclusion $S_f \subset S_D$ is an isomorphism. If D' and D are both basic open subsets of X and $D' \subset D$, then $S_{D'} \supset S_D$, and so there is a canonical map

$$S_D^{-1}A \to S_{D'}^{-1}A. \tag{1}$$

10.26. There is a unique sheaf $\mathcal{O}_V$ of k-algebras on $V = \mathrm{spm}(A)$ such that

(a) for every basic open subset D of V,

$$\mathcal{O}_V(D) = S_D^{-1}A,$$

(b) for every pair $D' \subset D$ of basic open subsets of V, the restriction map

$$\mathcal{O}_V(D) \to \mathcal{O}_V(D')$$

is the map (1) for the pair.

To prove this, it suffices to show that the system satisfies the sheaf condition on the base (10.5). This can be shown by the same argument as in the second part of the proof of 3.11.

We write $\mathrm{Spm}(A)$ for $\mathrm{spm}(A)$ endowed with this sheaf of k-algebras. Note that, for every $f \in A$,

$$A_f \stackrel{\mathrm{def}}{=} S_f^{-1}A \simeq S_{D(f)}^{-1}(A) \stackrel{\mathrm{def}}{=} \mathcal{O}_V(D(f)).$$

10.27. By a ***k*-ringed space** we mean a topological space equipped with a sheaf of k-algebras. An ***affine algebraic scheme*** over k is a k-ringed space isomorphic to $\mathrm{Spm}(A)$ for some k-algebra A. A ***morphism*** (or ***regular map***) of affine algebraic schemes over k is a morphism of k-ringed spaces (it is automatically a morphism of *locally* ringed spaces).

10.28. The functor $A \rightsquigarrow \mathrm{Spm}(A)$ is a contravariant equivalence from the category of k-algebras to the category of affine algebraic schemes over k, with quasi-inverse $(V, \mathcal{O}_V) \rightsquigarrow \mathcal{O}_V(V)$. In particular

$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(\mathrm{Spm}(B), \mathrm{Spm}(A))$$

for all k-algebras A and B. (Same proof as for 3.24, 3.25.)

10.29. Let M be an A-module. There is a unique sheaf $\mathcal{M}$ of $\mathcal{O}_V$-modules on $V \stackrel{\mathrm{def}}{=} \mathrm{Spm}(A)$ with the following properties,

(a) for every basic open subset D of V, $\mathcal{M}(D) = S_D^{-1}M$, and

(b) for every pair $D' \subset D$ of basic open subsets, the restriction map $\mathcal{M}(D) \to \mathcal{M}(D')$ is the map $S_D^{-1}M \to S_{D'}^{-1}M$ defined by the inclusion $S_D \subset S_{D'}$.

A sheaf of $\mathcal{O}_V$-modules on V is said to be ***coherent*** if it is isomorphic to $\mathcal{M}$ for some finitely generated A-module M. The functor $M \rightsquigarrow \mathcal{M}$ is an equivalence from the category of finitely generated A-modules to the category of coherent $\mathcal{O}_V$-modules — it has quasi-inverse $\mathcal{M} \rightsquigarrow \mathcal{M}(V)$. Under this equivalence, finitely generated projective A-modules correspond to locally free $\mathcal{O}_V$-modules of finite rank (CA 12.6).

10.30. For fields $K \supset k$, the Zariski topology on K^n induces that on k^n. In order to prove this, we have to show (a) that every closed subset S of k^n is of the form $T \cap k^n$ for some closed subset T of K^n, and (b) that $T \cap k^n$ is closed for every closed subset of K^n.

(a) Let $S = V(f_1, \dots, f_m)$ with the $f_i \in k[X_1, \dots, X_n]$. Then

$$S = k^n \cap \{\text{zero set of } f_1, \dots, f_m \text{ in } K^n\}.$$

(b) Let $T = V(f_1, \dots, f_m)$ with the $f_i \in K[X_1, \dots, X_n]$. Choose a basis $(e_j)_{j\in J}$ for K as a k-vector space,[2] and write $f_i = \sum e_j f_{ij}$ (finite sum) with $f_{ij} \in k[X_1, \dots, X_n]$. Then

$$V(f_i) \cap k^n = \{\text{zero set of the family } (f_{ij})_{j\in J} \text{ in } k^n\}$$

for each i, and so $T \cap k^n$ is the zero set in k^n of the family (f_{ij}).

ASIDE 10.31. Let V be a Zariski-closed subset of k^n and $\bar{V}$ its closure in $(k^{\mathrm{al}})^n$. Do V and $\bar{V}$ have the same dimension (as noetherian topological spaces)? The answer is yes, if k of characteristic 0 and large in the following sense: every irreducible curve over k with a smooth k-point has infinitely many k-points. See mo479691.

c. Algebraic schemes

10.32. Let $(V, \mathcal{O}_V)$ be a k-ringed space. An open subset U of V is said to be ***affine*** if $(U, \mathcal{O}_V|U)$ is an affine algebraic scheme over k. An ***algebraic scheme over*** k is a k-ringed space $(V, \mathcal{O}_V)$ that admits a finite covering by open affines. A ***morphism of algebraic schemes*** (usually called a ***regular map***) over k is a morphism of k-ringed spaces. We often let V denote the algebraic scheme $(V, \mathcal{O}_V)$ and $|V|$ the underlying topological space of V. When the base field k is understood, we write "algebraic scheme" for "algebraic scheme over k".

The local ring at a point P of V is denoted by $\mathcal{O}_{V,P}$ or just $\mathcal{O}_P$, and the residue field at P is denoted by $\kappa(P)$. For example, if $V = \operatorname{Spm} A$ and $P = \mathfrak{m}$, then $\mathcal{O}_P = A_{\mathfrak{m}}$ and $\kappa(P) = A/\mathfrak{m}$.

10.33. An algebraic scheme V is said to be ***integral*** if it is reduced and irreducible. For example, $\operatorname{Spm}(A)$ is integral if and only if A is an integral domain. If V is integral, then $\mathcal{O}_V(U)$ is an integral domain for all open affine subsets U of V.

10.34. A regular map $\varphi\colon W \to V$ of algebraic schemes is said to be ***surjective*** (resp. ***injective, open, closed***, ***dominant***) if the map $|\varphi|\colon |W| \to |V|$ of the underlying topological spaces is surjective (resp. injective, open, closed, dominant, i.e., has dense image).[3] Note that the conditions depend only on the underlying topological spaces.

10.35. Let V be an algebraic scheme over k, and let A be a k-algebra. By definition, a morphism $\varphi\colon V \to \operatorname{Spm}(A)$ gives a homomorphism $\varphi^\natural\colon A \to \mathcal{O}_V(V)$ of k-algebras ($\mathcal{O}_V(V)$ is not necessarily finitely generated). In this way, we get an isomorphism (cf. 5.11)

$$\varphi \leftrightarrow \varphi^\natural\colon \operatorname{Mor}_k(V, \operatorname{Spm} A) \simeq \operatorname{Hom}_{k\text{-algebra}}(A, \mathcal{O}_V(V)). \tag{2}$$

10.36. Let V be an algebraic scheme over k. If V is affine, say, $V = \operatorname{Spm}(A)$, then the closed subsets of $|V|$ correspond to radical ideals in A, and hence satisfy the descending chain condition. In the general case, V is a finite union of open affines, and so its closed subsets satisfy the descending chain condition. In other words, $|V|$ is a noetherian topological space. It follows that $|V|$ can be written as a finite union of closed irreducible

[2] This may require the axiom of choice.

[3] These definitions are from EGA I, 2.3.3.

subsets, $|V| = W_1 \cup \cdots \cup W_r$; when we discard any W_i contained in another, the collection $\{W_1, \ldots, W_r\}$ is uniqely determined, and its elements are called the ***irreducible components*** of V (2.31).

A noetherian topological space has only finitely many connected components, each open and closed, and it is a disjoint union of them.

10.37. For an algebraic scheme V over k and k-algebra R, we let

$$V(R) = \mathrm{Hom}(\mathrm{Spm}(R), V).$$

For example, if $V = \mathrm{Spm}(A)$, then $V(R) = \mathrm{Hom}(A, R)$ (homomorphisms of k-algebras). The elements of $V(R)$ are called the R-***points*** of V (or the points of V ***with coordinates in*** R). To give a k-point of V is the same as giving a point P of $|V|$ such that $\kappa(P) = k$. We often identify $V(k)$ with the set of such P,

$$V(k) = \{P \in |V| \mid \kappa(P) = k\}.$$

For a ring R containing k, we define

$$V(R) = \varinjlim V(R_i),$$

where R_i runs over the (finitely generated) k-subalgebras of R. Again $V(R) = \mathrm{Hom}_k(A, R)$ if $V = \mathrm{Spm}(A)$. Then $R \rightsquigarrow V(R)$ is a functor from k-algebras (not necessarily finitely generated) to sets.

10.38. Let V be an algebraic scheme. An $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***coherent*** if, for every open affine subset U of V, the restriction of $\mathcal{M}$ to U is coherent (10.29). It suffices to check this condition for the sets in an open affine covering of V. Similarly, a sheaf $\mathcal{I}$ of ideals in $\mathcal{O}_V$ is ***coherent*** if its restriction to every open affine subset U is the subsheaf of $\mathcal{O}_V|U$ defined by an ideal in the ring $\mathcal{O}_V(U)$.

Subschemes

10.39 (OPEN SUBSCHEMES). Let V be an algebraic scheme over k. An ***open subscheme*** of V is a pair $(U, \mathcal{O}_V|U)$ with U open in V. It is again an algebraic scheme over k. To give an open subscheme of V is the same as giving an open subset of $|V|$.

10.40 (CLOSED SUBSCHEMES). Let $V = \mathrm{Spm}(A)$ be an affine algebraic scheme over k, and let $\mathfrak{a}$ be an ideal in A. Then $\mathrm{Spm}(A/\mathfrak{a})$ is an affine algebraic scheme with underlying topological space $V(\mathfrak{a})$.

Let V be an algebraic scheme over k, and let $\mathcal{I}$ be a coherent sheaf of ideals in $\mathcal{O}_V$. The support of the sheaf $\mathcal{O}_V/\mathcal{I}$ is a closed subset Z of V (13.5), and $(Z, (\mathcal{O}_V/\mathcal{I})|Z)$ is an algebraic scheme, called the ***closed subscheme*** of V defined by the sheaf of ideals $\mathcal{I}$. Note that if U is an open affine of V, then $Z \cap U$ is an open affine Z.

The closed subschemes of an algebraic scheme satisfy the descending chain condition. To see this, consider a chain of closed subschemes

$$Z \supset Z_1 \supset Z_2 \supset \cdots$$

of an algebraic scheme V. Because $|V|$ is noetherian (10.36), the chain $|Z| \supset |Z_1| \supset |Z_2| \supset \cdots$ becomes constant, and so we may suppose that $|Z| = |Z_1| = \cdots$. Write Z as a finite union of open affines, $Z = \bigcup U_i$. For each i, the chain $Z \cap U_i \supset Z_1 \cap U_i \supset \cdots$ of closed subschemes of U_i corresponds to an ascending chain of ideals in the noetherian ring $\mathcal{O}_Z(U_i)$, and therefore becomes constant.

10.41 (SUBSCHEMES). A ***subscheme*** of an algebraic scheme V is a closed subscheme of an open subscheme of V. Its underlying set is locally closed in V (i.e., open in its closure). Equivalently, it is the intersection of an open subset with a closed subset).

10.42. A regular map $\varphi\colon W \to V$ is said to be an ***immersion*** if it induces an isomorphism from W onto a subscheme Z of V. If Z is open (resp. closed), then φ is called an ***open*** (resp. ***closed***) ***immersion***.

Reduced schemes

10.43. For a ring A, the map

$$A \to \prod\nolimits_{\mathfrak{m}} A_{\mathfrak{m}} \qquad \text{(product over the maximal ideals of } A\text{)}$$

is injective. To see this, let a map to zero, and let $\mathfrak{a}$ be the annihilator of a. As a maps to zero in $A_{\mathfrak{m}}$, $\mathfrak{a}$ contains an element of $A \smallsetminus \mathfrak{m}$. Therefore the ideal $\mathfrak{a}$ is not contained in any maximal ideal of A, and so $\mathfrak{a} = A$. This implies that $a = 0$.

If A is reduced, then $S^{-1}A$ is reduced for any multiplicative subset S of A. It follows from the above remark that a ring A is reduced if and only if $A_{\mathfrak{m}}$ is reduced for all maximal ideals $\mathfrak{m}$ in A.

An algebraic scheme V is said to be ***reduced*** if $\mathcal{O}_{V,P}$ is reduced for all $P \in V$. For example, $\mathrm{Spm}(A)$ is reduced if and only if A is reduced. If V is reduced, then $\mathcal{O}_V(U)$ is reduced for all open affine subsets U of V.

10.44. Let V be an algebraic scheme over k. There is a unique reduced algebraic subscheme V_{red} of V with the same underlying topological space as V. For example, if $V = \mathrm{Spm}(A)$, then $V_{\mathrm{red}} = \mathrm{Spm}(A/\mathfrak{n})$ where $\mathfrak{n}$ is the nilradical of A.

Every regular map $W \to V$ from a reduced scheme W to V factors uniquely through the inclusion map $i\colon V_{\mathrm{red}} \to V$. In particular,

$$V_{\mathrm{red}}(R) \simeq V(R) \tag{3}$$

if R is a reduced k-algebra, for example, a field.

Every locally closed subset W of $|V|$ carries a unique structure of a reduced subscheme of V; we write W_{red} for W equipped this structure.

☡ Passage to the associated reduced scheme does not commute with extension of the base field. For example, an algebraic scheme V over k may be reduced without $V_{k^{\mathrm{al}}}$ being reduced (see p. 8 for the example $X^p + Y^p = a$).

Fibred products of algebraic schemes

10.45. Let $\varphi\colon V \to Z$ and $\psi\colon W \to Z$ be regular maps of algebraic schemes over k. Then the functor

$$R \rightsquigarrow V(R) \times_{Z(R)} W(R) \stackrel{\text{def}}{=} \{(x,y) \in V(R) \times W(R) \mid \varphi(x) = \psi(y)\}$$

is representable by an algebraic scheme $V \times_Z W$ over k, and $V \times_Z W$ is the fibred product of (φ, ψ) in the category of algebraic k-schemes, i.e., the diagram

$$\begin{array}{ccc} V \times_Z W & \longrightarrow & W \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ V & \xrightarrow{\ \varphi\ } & Z. \end{array}$$

is cartesian,

$$\mathrm{Hom}(T, V \times_Z W) \simeq \mathrm{Hom}(T, V) \times_{\mathrm{Hom}(T,Z)} \mathrm{Hom}(T, W).$$

For example, if $R \to A$ and $R \to B$ are homomorphisms of k-algebras, then $A \otimes_R B$ is a finitely generated k-algebra, and

$$\mathrm{Spm}(A) \times_{\mathrm{Spm}(R)} \mathrm{Spm}(B) = \mathrm{Spm}(A \otimes_R B).$$

When φ and ψ are the structure maps $V \to \mathrm{Spm}(k)$ and $W \to \mathrm{Spm}(k)$, the fibred product becomes the product, denoted $V \times W$, and

$$\mathrm{Hom}(T, V \times W) \simeq \mathrm{Hom}(T, V) \times \mathrm{Hom}(T, W).$$

The diagonal map $\Delta_V : V \to V \times V$ is the regular map whose composites with the projection maps equal the identity map of V. When V is affine, Δ_V is a closed immersion; in general, it is only an immersion (cf. 5.26).

Let $\varphi : W \to V$ be a regular map of algebraic schemes over k. The ***fibre*** $\varphi^{-1}(P)$ of φ over P is defined to be the fibred product,

$$\begin{array}{ccc} W & \longleftarrow & W \times_V P \stackrel{\mathrm{def}}{=} \varphi^{-1}(P) \\ \downarrow{\scriptstyle \varphi} & & \downarrow \\ V & \longleftarrow & P = \mathrm{Spm}(\kappa(P)). \end{array}$$

Thus, it is an algebraic scheme over the field $\kappa(x)$, which need not be reduced even if both V and W are reduced.

10.46. For a pair of regular maps $\varphi_1, \varphi_2 : V \to W$, the functor

$$R \rightsquigarrow \{x \in V(R) \mid \varphi_1(x) = \varphi_2(x)\}$$

is represented by a fibred product,

$$\begin{array}{ccc} \Delta_W \times_{W \times W} V & \hookrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle (\varphi_1, \varphi_2)} \\ \Delta_W & \hookrightarrow & W \times W. \end{array}$$

The subscheme $\Delta_W \times_{W \times W} V$ of V is called the ***equalizer*** $\mathrm{Eq}(\varphi_1, \varphi_2)$ of φ_1 and φ_2. Its underlying set is $\{x \in |V| \mid \varphi_1(x) = \varphi_2(x)\}$.

10.47. The intersection of two closed subschemes Z_1 and Z_2 of an algebraic scheme V is defined to be $Z_1 \times_V Z_2$ regarded as a closed subscheme of V with underlying set $|Z_1| \cap |Z_2|$. For example, if $V = \mathrm{Spm}(A)$, $Z_1 = \mathrm{Spm}(A/\mathfrak{a}_1)$, and $Z_1 = \mathrm{Spm}(A/\mathfrak{a}_2)$, then $Z_1 \cap Z_2 = \mathrm{Spm}(A/\mathfrak{a}_1 + \mathfrak{a}_2)$. This definition extends in an obvious way to finite, or even infinite, sets of closed subschemes. Because V has the descending chain condition on closed subschemes (10.40), every infinite intersection is equal to a finite intersection.

Separated schemes

10.48. An algebraic scheme V over k is said to be ***separated***[4] if, for any pair of regular maps $\varphi_1, \varphi_2\colon W \to V$, the subset of $|W|$ on which φ_1 and φ_2 agree is closed (so $\mathrm{Eq}(\varphi_1, \varphi_2)$ is a *closed* subscheme of W).

For example, affine algebraic schemes are separated (cf. 5.6).

10.49. The following conditions on an algebraic scheme V are equivalent:

(a) V is separated;

(b) the diagonal in $V \times V$ is closed (so Δ_V is a *closed* immersion);

(c) for every pair of open affines U, U' in V, $U \cap U'$ is an open affine subset of V, and the homomorphism

$$f \otimes g \mapsto f|_{U\cap U'} \cdot g|_{U\cap U'}\colon \mathcal{O}_V(U) \otimes \mathcal{O}_V(U) \to \mathcal{O}_V(U \cap U')$$

is surjective;

(d) the condition in (c) holds for the sets in some open covering of V.

See the proofs of 5.25 and 5.28.

Extension of the base field (extension of scalars)

10.50. Let K be a field containing k. There is a functor $V \rightsquigarrow V_K$ from algebraic schemes over k to algebraic schemes over K. For example, if $V = \mathrm{Spm}(A)$, then $V_K = \mathrm{Spm}(K \otimes A)$. If V is separated and $(U_i)_{i\in I}$ is a finite covering of V by open affines, then V_K is obtained by patching together the affine schemes U_{iK} using the open immersions of affine schemes $\left(U_i \cap U_j\right)_K \hookrightarrow U_{iK}$.

d. Algebraic varieties

10.51. An ***affine*** k-algebra[5] is a k-algebra A such that $k^{\mathrm{al}} \otimes A$ is reduced; in particular, A itself is reduced. If A is an affine k-algebra and B is a reduced ring containing k, then $A \otimes B$ is reduced (10.13); in particular $A \otimes K$ is reduced for all fields K containing k. The tensor product of two affine k-algebras is affine ($k^{\mathrm{al}} \otimes_k A$ is reduced if A is affine, and then $k^{\mathrm{al}} \otimes_k A \otimes_k B$ is reduced if B is also affine). When k is a perfect field, every reduced k-algebra is affine (10.12).

10.52. An algebraic scheme V is said to be ***geometrically reduced*** if $V_{k^{\mathrm{al}}}$ is reduced. For example, $\mathrm{Spm}(A)$ is geometrically reduced if and only if A is an affine k-algebra. If V is geometrically reduced, then V_K is reduced for all fields K containing k. If V is geometrically reduced and W is reduced (resp. geometrically reduced), then $V \times W$ is reduced (resp. geometrically reduced). If k is perfect, then every reduced algebraic scheme over k is geometrically reduced. These statements all follow from the affine case (10.51).

[4] The first edition of EGA I, required a scheme to be separated — otherwise it was called a prescheme. This was changed in the second edition and now, universally, a scheme is not required to be separated.

[5] Some authors define an affine k-algebra to be a reduced finitely generated k-algebra because these are the rings of functions on algebraic subsets. However, this class of rings is not closed under the formation of tensor products or extension of the base field.

10.53. An ***algebraic variety over*** k is an algebraic scheme over k that is both separated and geometrically reduced. Algebraic varieties remain algebraic varieties under extension of the base field, and products of algebraic varieties are again algebraic varieties.

☡ The fibred product of two algebraic varieties over an algebraic variety need not be an algebraic variety. Consider, for example,

$$\begin{array}{ccc} \mathbb{A}^1 & \longleftarrow & \mathbb{A}^1 \times_{\mathbb{A}^1} \{a\} = \mathrm{Spm}(k[X]/(X^p - a)) \\ {\scriptstyle x\mapsto x^p}\downarrow & & \downarrow \\ \mathbb{A}^1 & \longleftarrow & \{a\}. \end{array}$$

Fibred products computed in the category of algebraic varieties may differ from those computed in the category of algebraic schemes. Similar statements apply to intersections of subvarieties. For example, over a field of characteristic 2, the intersection of the diagonal in GL_2 with SL_2 is trivial in the category of algbraic varieties but is

$$\mu_2 \stackrel{\text{def}}{=} \mathrm{Spm}(k[X]/(X^2 - 1))$$

in the category of algebraic schemes.

e. The dimension of an algebraic scheme

10.54. Let A be a noetherian ring (not necessarily a k-algebra). Recall that the ***height*** of a prime ideal $\mathfrak{p}$ is the greatest length d of a chain of distinct prime ideals

$$\mathfrak{p} = \mathfrak{p}_d \supset \cdots \supset \mathfrak{p}_1 \supset \mathfrak{p}_0.$$

Let $\mathfrak{p}$ be minimal among the prime ideals containing an ideal $(a_1, \ldots, a_m)$; then

$$\mathrm{height}(\mathfrak{p}) \leq m.$$

Conversely, if $\mathrm{height}(\mathfrak{p}) = m$, then there exist $a_1, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is minimal among the prime ideals containing $(a_1, \ldots, a_m)$. (3.52, 3.53, or CA 21.6, 21.7).

The (Krull) dimension of A is $\sup\{\mathrm{height}(\mathfrak{p})\}$, where $\mathfrak{p}$ runs over the prime ideals of A (or just the maximal ideals — the two are obviously the same). Clearly, the dimension of a local ring with maximal ideal $\mathfrak{m}$ is the height of $\mathfrak{m}$, and for a general noetherian ring A,

$$\dim(A) = \sup(\dim(A_{\mathfrak{m}})).$$

Since all prime ideals of A contain the nilradical $\mathfrak{N}$ of A, we have

$$\dim(A) = \dim(A/\mathfrak{N}).$$

10.55. Let A be a finitely generated k-algebra, and assume that $A/\mathfrak{N}$ is an integral domain. According to the Noether normalization theorem (2.45), A contains a polynomial ring $k[x_1, \ldots, x_r]$ such that A is a finitely generated $k[x_1, \ldots, x_r]$-module. We call r the ***transcendence degree*** of A over k — it is equal to the transcendence degree of the field of fractions of $A/\mathfrak{N}$ over k. The length of every maximal chain of distinct prime ideals in A is $\mathrm{tr}\deg_k(A)$. In particular, every maximal ideal in A has height $\mathrm{tr}\deg_k(A)$, and so A has dimension $\mathrm{tr}\deg_k(A)$. The proofs of these facts in Section 2m and Section 3l do not require that k be algebraically closed.

10.56. Let V be an irreducible algebraic scheme over k. The ***dimension*** of V is the length of one (hence every) maximal chain of irreducible closed subschemes

$$V = V_0 \supset \cdots \supset V_d.$$

It is equal to the Krull dimension of $\mathcal{O}_{V,x}$, all $x \in |V|$, and to the Krull dimension of $\mathcal{O}_V(U)$, all open affines U in V. We have $\dim(V) = \dim(V_{\mathrm{red}})$, and if V is reduced, then $\dim(V)$ is equal to the transcendence degree of $k(V)$ over k.

An affine algebraic scheme $V = \operatorname{Spm}(A)$ is irreducible if and only if $A/\mathfrak{N}$ is an integral domain. In this case, the statements follow from (10.55). The general case follows easily.

The ***dimension*** of a reducible algebraic scheme over k is defined to be the maximum dimension of an irreducible component. When the irreducible components all have the same dimenions, the scheme is said to be ***equidimensional***.

10.57. Let V an irreducible algebraic variety. Let U be an open affine in V, and let $A = \mathcal{O}_V(U)$. Then A is an integral domain, and it satisfies the equivalent conditions of 10.12. In particular, A is linearly disjoint from $k^{1/p}$ over k.Therefore, its field of fractions $k(V)$ is linearly disjoint from $k^{1/p}$ over k, and so $k(V)$ is a regular extension of k (10.11). Thus $k(V)$ admits a separating transcendence basis over k. This means that V is birationally equivalent to a hypersurface $f(T_1, \ldots, T_{d+1})$, $d = \dim V$, such that $\partial f/\partial T_{d+1} \neq 0$ (cf. 3.37). It follows that the points x in V such that $\kappa(x)$ is separable over k form a dense subset of $|V|$. In particular, $V(k)$ is dense in $|V|$ if k is separably closed.

f. Tangent spaces and cones; regular and smooth points

10.58. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. The dimension of A is the height of $\mathfrak{m}$, and so (10.55),

$$\dim A \leq \text{minimum number of generators for } \mathfrak{m}.$$

When equality holds, A is said to be regular. Nakayama's lemma (1.3) shows that a set of elements of $\mathfrak{m}$ generates $\mathfrak{m}$ if and only if it spans the k-vector space $\mathfrak{m}/\mathfrak{m}^2$, where $k = A/\mathfrak{m}$. Therefore

$$\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$$

with equality if and only if A is regular. Every regular local ring is a unique factorization domain; in particular, it is an integrally closed domain. See Matsumura 1989, 20.3.

10.59. Let V be an algebraic scheme over k. A point $P \in |V|$ is ***regular*** if $\mathcal{O}_{V,P}$ is a regular local ring. The scheme V is ***regular*** if every point is regular. A connected regular algebraic scheme is integral, but not necessarily geometrically reduced.

10.60. Let $k[\varepsilon]$ be the ring of dual numbers (so $\varepsilon^2 = 0$), and let V be an algebraic scheme over k. From the k-algebra homomorphism $\varepsilon \mapsto 0\colon k[\varepsilon] \to k$, we get a map

$$V(k[\varepsilon]) \to V(k).$$

The fibre of this over a point $P \in V(k)$ is the ***tangent space*** $T_P(V)$ of V at P. Thus $T_P(V)$ is defined for all $P \in |V|$ with $\kappa(P) = k$. To give a tangent vector at P amounts to giving a local homomorphism $\alpha\colon \mathcal{O}_{V,P} \to k[\varepsilon]$ of k-algebras. Such a homomorphism can be written

$$\alpha(f) = f(P) + D_\alpha(f)\varepsilon, \quad f \in \mathcal{O}_P, \quad f(P),\ D_\alpha(f) \in k,$$

and D_α is a k-derivation $\mathcal{O}_P \to k$, which induces a k-linear map $\mathfrak{m}/\mathfrak{m}^2 \to k$. In this way, we get canonical isomorphisms

$$T_P(V) \simeq \mathrm{Der}_k(\mathcal{O}_P, k) \simeq \mathrm{Hom}_{k\text{-linear}}(\mathfrak{m}/\mathfrak{m}^2, k). \tag{4}$$

The formation of the tangent space commutes with extension of the base field:

$$T_P(V_{k'}) \simeq T_P(V)_{k'}.$$

10.61. Let V be an irreducible algebraic scheme over k, and let P be a point on V such that $\kappa(P) = k$. Then

$$\dim T_P(V) \geq \dim V$$

with equality if and only if P is regular (10.58).

10.62. Let V be an irreducible closed subscheme of $\mathbb{A}^n$, say,

$$V = \mathrm{Spm}\,A, \quad A = k[X_1, \dots, X_n]/\mathfrak{a}, \quad \mathfrak{a} = (F_1, \dots, F_r).$$

Consider the Jacobian matrix

$$J = \mathrm{Jac}(F_1, \dots, F_r) = \begin{pmatrix} \frac{\partial F_1}{\partial X_1}, & \dots, & \frac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial F_r}{\partial X_1}, & \dots, & \frac{\partial F_r}{\partial X_n} \end{pmatrix}.$$

Let $d = \dim V$. The ***singular locus*** V_{sing} of V is the closed subscheme of V defined by the $(n-d) \times (n-d)$ minors of this matrix.

For example, if V is the hypersurface defined by the polynomial $F(X_1, \dots, X_{d+1})$, then

$$\mathrm{Jac}(F) = \left(\frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_{d+1}}\right) \in M_{1,d+1}(A),$$

and the singular locus is the closed subscheme of V defined by the equations

$$\frac{\partial F}{\partial X_1} = 0, \dots, \frac{\partial F}{\partial X_{d+1}} = 0.$$

10.63. Let V be an affine algebraic variety over k. The choice of closed immersion of V into an affine space determines a closed subscheme V_{sing} of V which is independent of the embedding. For a general algebraic scheme V over k, the ***singular locus*** is defined to be the closed subscheme V_{sing} of V such that $V_{\mathrm{sing}} \cap U = U_{\mathrm{sing}}$ for every open affine U of V. In the next section, we shall see that V_{sing} is the complement of the open set where the sheaf $\Omega_{V/k}$ of differentials is locally free of rank $\dim(V)$.

From its definition, one sees that the formation of V_{sing} commutes with exension of the base field,

$$(V_{\mathrm{sing}})_{k'} = (V_{k'})_{\mathrm{sing}}.$$

Under the canonical bijection $V(k^{\mathrm{al}}) \simeq V_{k^{\mathrm{al}}}(k^{\mathrm{al}})|$, the elements of $V_{\mathrm{sing}}(k^{\mathrm{al}})$ correspond to the singular points of $V_{k^{\mathrm{al}}}(k^{\mathrm{al}})$ (those such that $\dim T_P(V) > \dim V$).

10.64. Let V be an algebraic scheme over k. A point P of V is ***singular*** or ***nonsingular*** according as P lies in the singular locus or not, and V is ***nonsingular*** (=***smooth***) or ***singular*** according as V_{sing} is empty or not. If P is such that $\varkappa(P) = k$, then P is nonsingular if and only if it is regular. A smooth variety is regular, and a regular variety is smooth if k is perfect. In general,

$$\begin{array}{ccc} V \text{ nonsingular} & \iff & V_{k^{\text{al}}} \text{ nonsingular} \\ \Downarrow & & \Downarrow \\ V \text{ regular} & \not\Rightarrow & V_{k^{\text{al}}} \text{ regular.} \end{array}$$

10.65. Let V be geometrically reduced and irreducible. Then V is birationally equivalent to a hypersurface $f(T_1,\ldots,T_{d+1}) = 0$ with $\partial f/\partial T_{d+1} \neq 0$ (see 10.57). It follows that the singular locus of V is a proper closed subscheme of V (10.62).

ASIDE 10.66. An algebraic scheme V over a field k is smooth if and only if, for all k-algebras R and ideals I in R such that $I^2 = 0$, the map $V(R) \to V(R/I)$ is surjective.

Tangent cones

Now that we are allowing nilpotents, we can give a more satisfactory definition of the tangent cone: the tangent cone of a k-point on an algebraic variety (or scheme) is an algebraic scheme.

DEFINITION 10.67. Let V be the curve in $\mathbb{A}^2$ defined by a polynomial $F(X,Y)$ without square factors. If $(0,0) \in V(k)$, we define the ***tangent cone at*** $(0,0)$ to be

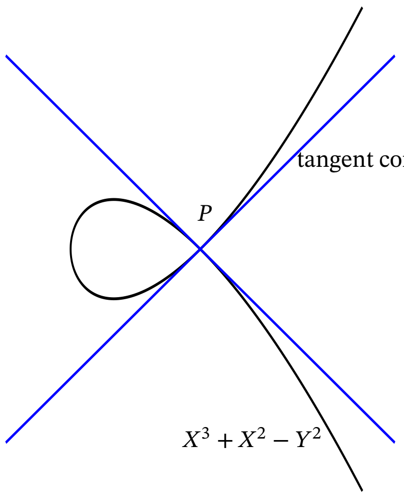
$$C_P(V) = \operatorname{Spm}(k[X,Y[/(F_*),$$

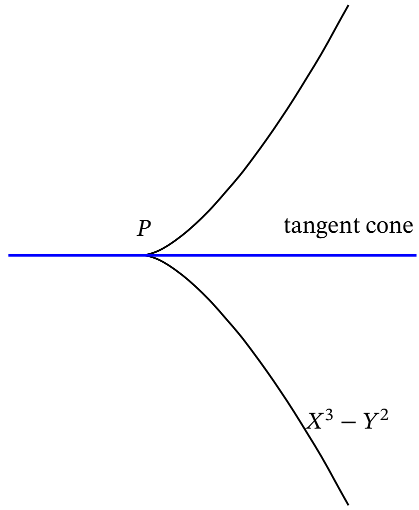
where F_* is the leading form of F. To obtain the tangent cone at any other k-point, translate to the origin, and then translate back.

EXAMPLE 10.68. See p. 86 for more examples.

Curve	Tangent Cone	
$X^3+X^2-Y^2$	X^2-Y^2	pair of line $Y=\pm X$
$X^3-X^2-Y^2$	X^2+Y^2	$\operatorname{Spm} k[X,Y]/(X^2+Y^2)$
X^3-Y^2	Y^2	$\operatorname{Spm} k[X,Y]/(Y^2)$

In each case, the curve is integral but its cone is reducible (first curve), becomes reducible after a field extension (second case), or is nonreduced (third case).





DEFINITION 10.69. Let V be closed algebraic subscheme of $\mathbb{A}^m$, and let $\mathfrak{a} = I(V)$. Assume that $P = (0, \dots, 0) \in V(k)$. Define $\mathfrak{a}_*$ to be the ideal generated by the leading forms F_* of the polynomials $F \in \mathfrak{a}$. We define the ***tangent cone*** to V at P to be

$$C_P(V) = \operatorname{Spm}(k[X_1, \dots, X_m]/\mathfrak{a}_*.$$

Let A be a local ring with maximal ideal $\mathfrak{n}$. The associated graded ring

$$\operatorname{gr}(A) = \bigoplus_{i \geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}.$$

Note that, if $A = B_\mathfrak{m}$ and $\mathfrak{n} = \mathfrak{m}A$, then $\operatorname{gr}(A) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ (because of 1.15).

PROPOSITION 10.70. *The homomorphism of k-algebras*

$$k[X_1, \dots, X_n]/\mathfrak{a}_* \to \operatorname{gr}(\mathcal{O}_P)$$

sending the class of X_i in $k[X_1, \dots, X_n]/\mathfrak{a}_*$ to the class of X_i in $\operatorname{gr}(\mathcal{O}_P)$ is an isomorphism.

PROOF. See 4.34.

DEFINITION 10.71. Let V be an algebraic scheme over k, and let $P \in V(k)$. The ***tangent cone*** to V at P is

$$C_P(V) = \operatorname{Spm}(\operatorname{gr}(\mathcal{O}_P)).$$

Because of Proposition 10.70, the three definitions are consistent.

10.72. The dimension of the tangent cone at P equals the dimension of V because the Krull dimension of a noetherian local ring is equal to that of its graded ring (Matsumura 1989, Theorem 13.9). Moreover, $\operatorname{gr}(\mathcal{O}_P)$ is a polynomial ring in $\dim V$ variables if and only if $\mathcal{O}_P$ is regular (ibid., Exercise 19.1). Let $P \in V(k)$. Then

$$P \text{ is nonsingular} \iff T_P(V) = k^{\dim V} \iff C_P(V) = \mathbb{A}^{\dim V}.$$

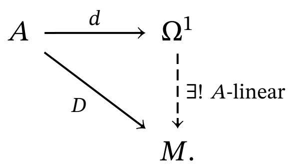
10.73. A regular map $\varphi\colon V \to W$ sending P to Q induces a homomorphism $\operatorname{gr}(\mathcal{O}_Q) \to \operatorname{gr}(\mathcal{O}_P)$, and hence a map $C_P(V) \to C_Q(V)$ of the tangent cones. We say that φ is ***étale*** at P if $\operatorname{gr}(\mathcal{O}_Q) \to \operatorname{gr}(\mathcal{O}_P)$ is an isomorphism. When P and Q are nonsingular points, this just says that the map $d\varphi\colon T_P(V) \to T_Q(W)$ on tangent spaces is an isomorphism.

g. Sheaves of differentials

Let A be a k-algebra, and let M be an A-module. Recall (from §5) that a k-derivation is a k-linear map $D: A \to M$ satisfying Leibniz's rule:

$$D(fg) = f \cdot Dg + g \cdot Df, \quad \text{all } f, g \in A.$$

DEFINITION 10.74. A pair $(\Omega^1_{A/k}, d)$ comprising an A-module $\Omega^1_{A/k}$ and a k-derivation $d: A \to \Omega^1_{A/k}$ is called the ***module of differential one-forms*** for A over k if it has the following universal property: for any k-derivation $D: A \to M$, there is a unique A-linear map $\alpha: \Omega^1_{A/k} \to M$ such that $D = \alpha \circ d$,



Thus,

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_{A\text{-linear}}(\Omega^1_{A/k}, M)).$$

It can be defined to be the free A-module with basis the symbols df, $f \in A$, modulo the relations

$$d(f+g) = df + dg, \quad d(fg) = f \cdot dg + g \cdot df, \quad dc = 0 \text{ if } c \in k.$$

EXAMPLE 10.75. Let $A = k[X_1, ..., X_n]$; then $\Omega^1_{A/k}$ is the free A-module with basis the symbols $dX_1, ..., dX_n$, and

$$df = \sum \frac{\partial f}{\partial X_i} dX_i.$$

EXAMPLE 10.76. Let $A = k[X_1, ..., X_n]/\mathfrak{a}$; then $\Omega^1_{A/k}$ is the free A-module with basis the symbols $dX_1, ..., dX_n$ modulo the relations:

$$df = 0 \text{ for all } f \in \mathfrak{a}.$$

EXAMPLE 10.77. A homomorphism $A \to A'$ of k-algebras gives rise to an isomorphism

$$A' \otimes_A \Omega^1_{A/k} \to \Omega^1_{A'/k}.$$

In particular, for any multiplicative subset S of A, we have canonical isomorphisms

$$S^{-1}\Omega^1_{A/k} \simeq S^{-1}A \otimes_A \Omega^1_{A/k} \simeq \Omega^1_{S^{-1}A/k}.$$

PROPOSITION 10.78. *Let V be an algebraic scheme over k. There is a unique sheaf of $\mathcal{O}_V$-modules $\Omega^1_{V/k}$ on V such that $\Omega^1_{V/k}(U) = \Omega^1_{\mathcal{O}_V(U)/k}$ for every open affine U of V.*

PROOF. When V is affine, say, $V = \mathrm{Spm}\, A$, then 10.77 shows that the coherent sheaf of $\mathcal{O}_V$-modules defined by the A-module $\Omega^1_{A/k}$ has the required properties. In the general case, the open affines form a base for the topology on V, and the statement follows from 10.5 (this is easier if V is separated). □

The sheaf $\Omega^1_{V/k}$ is called the ***sheaf of differential*** 1-***forms on*** V.

EXAMPLE 10.79. Let E be the affine curve

$$Y^2 = X^3 + aX + b,$$

and assume that $X^3 + aX + b$ has no repeated roots (so that E is nonsingular). Write x and y for the regular functions on E defined by X and Y. On the open set $D(y)$ where $y \neq 0$, let $\omega_1 = dx/y$, and on the open set $D(3x^2 + a)$, let $\omega_2 = 2dy/(3x^2 + a)$. Since $y^2 = x^3 + ax + b$,

$$2ydy = (3x^2 + a)dx.$$

and so ω_1 and ω_2 agree on $D(y) \cap D(3x^2 + a)$. Since $E = D(y) \cup D(3x^2 + a)$, we see that there is a differential ω on E that restricts to ω_1 on $D(y)$ and ω_2 on $D(3x^2 + a)$. It is an easy exercise in working with projective coordinates to show that ω extends to a differential one-form on the whole projective curve

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

In fact, $\Omega^1_{C/k}(C)$ is a one-dimensional vector space over k, with ω as basis. Note that

$$\omega = dx/y = dx/(x^3 + ax + b)^{1/2},$$

which cannot be integrated in terms of elementary functions. Integrals of the form $\int \omega$ arise when computing the arc length of an ellipse, and are called ***elliptic integrals***. The study of elliptic integrals was one of the starting points for the study of algebraic curves.

In general, if C is a complete nonsingular absolutely irreducible curve of genus g, then $\Omega^1_{C/k}(C)$ is a vector space of dimension g over k.

PROPOSITION 10.80. *Let V be an irreducible variety over k. There exists a nonempty open subvariety U of V such that $\Omega^1_{V/k}|U$ is free of rank* $\dim V$ *and* $\dim_{\kappa(v)}(\Omega^1_{V/k}(v)) > \dim V$ *if $v \notin |U|$.*

PROOF. We may suppose that k is algebraically closed. For $P \in V(k)$, we have

$$\mathrm{Hom}_{\mathcal{O}_P}(\Omega^1_{\mathcal{O}_P/k}.k) \simeq \mathrm{Der}_k(\mathcal{O}_p, k) \simeq T_P(V).$$

Thus,

$$\dim_k(\Omega^1_{V/k}(P)) = \dim_k T_P(V).$$

It follows from 13.6 (Chapter 13), that $\Omega^1_{V/k}$ is locally free of rank $\dim V$ over the open subset of V consisting of the nonsingular points, and at the remaining points it has fibre of dimension $> \dim V$. □

h. Algebraic schemes as functors

This section is a brief survey, which the reader can skip.

10.81. Recall that Alg_k is the category of *finitely generated* k-algebras. For a k-algebra A, let h^A denote the functor $R \rightsquigarrow \mathrm{Hom}(A, R)$ from k-algebras to sets. A functor $F\colon \mathsf{Alg}_k \to \mathsf{Set}$ is said to be ***representable*** if it is isomorphic to h^A for some k-algebra A. A pair (A, a), $a \in F(A)$, is said to ***represent*** F if the natural transformation

$$T_a\colon h^A \to F, \quad (T_a)_R(f) = F(f)(a),$$

is an isomorphism. This means that, for each $x \in F(R)$, there is a unique homomorphism $A \to R$ such that $F(A) \to F(R)$ sends a to x. The element a is said to be ***universal***. For example, (A, id_A) represents h^A. If (A, a) and (A', a') both represent F, then there is a unique isomorphism $A \to A'$ sending a to a'.

10.82 (YONEDA LEMMA). Let B be a k-algebra and let F be a functor $\mathsf{Alg}_k \to \mathsf{Set}$. An element $x \in F(B)$ defines a homomorphism

$$\mathrm{Hom}(B, R) \to F(R)$$

sending an f to the image of x under $F(f)$. This homomorphism is natural in R, and so we have a map of sets

$$F(B) \to \mathsf{Nat}(h^B, F).$$

The Yoneda lemma (q.v. Wikipedia) says that this is a bijection, natural in both B and F. For $F = h^A$, it becomes

$$\mathrm{Hom}(A, B) \simeq \mathsf{Nat}(h^B, h^A).$$

In other words, the contravariant functor $A \rightsquigarrow h^A$ is fully faithful. Its essential image consists of the representable functors.

10.83. Let h_V denote the functor $\mathrm{Hom}(-, V)$ from algebraic schemes over k to sets. The Yoneda lemma in this situation says that, for algebraic schemes V, W,

$$\mathrm{Hom}(V, W) \simeq \mathrm{Nat}(h_V, h_W).$$

Let h_V^{aff} denote the functor $R \rightsquigarrow V(R)\colon \mathsf{Alg}_k \to \mathsf{Set}$. Then $h_V^{\mathrm{aff}} = h_V \circ \mathrm{Spm}$, and can be regarded as the restriction of h_V to affine algebraic schemes.

Let V and W be algebraic schemes over k. Every natural transformation $h_V^{\mathrm{aff}} \to h_W^{\mathrm{aff}}$ extends uniquely to a natural transformation $h_V \to h_W$,

$$\mathrm{Nat}(h_V^{\mathrm{aff}}, h_W^{\mathrm{aff}}) \simeq \mathrm{Nat}(h_V, h_W),$$

and so

$$\mathrm{Hom}(V, W) \simeq \mathrm{Nat}(h_V^{\mathrm{aff}}, h_W^{\mathrm{aff}}).$$

In other words, the functor $V \rightsquigarrow h_V^{\mathrm{aff}}$ is fully faithful. We shall also refer to this statement as the *Yoneda lemma*. It allows us to identify an algebraic scheme over k with its "points-functor" $\mathsf{Alg}_k \to \mathsf{Set}$.

10.84. A morphism $\varphi\colon V \to W$ of functors is a ***monomorphism*** if $\varphi(R)$ is injective for all R. A morphism φ is an ***open immersion*** if it is open and a monomorphism (Demazure and Gabriel 1970, I, §1, 3.6, p. 10). Let $\varphi\colon V \to W$ be a regular map of algebraic schemes. If $\tilde{V} \to \tilde{W}$ is a monomorphism, then it is injective (ibid., 5.1, p. 24). If V is irreducible and $\tilde{V} \to \tilde{W}$ is a monomorphism, then there exists a dense open subset U of V such that $\varphi|U$ is an immersion.

ASIDE 10.85. Originally algebraic geometers considered algebraic varieties V over algebraically closed fields k. Here it sufficed to consider the set $V(k)$ of k-points. Later algebraic geometers considered algebraic varieties V over arbitrary fields k. Here $V(k)$ does not tell you much about V (it is often empty), and so people worked with $V(K)$ where K is some (large) algebraically closed field containing k. For algebraic schemes, even $V(K)$ is inadequate because it does not detect nilpotents. This suggests that we consider $V(R)$ for all k-algebras, i.e., we consider the functor $V: R \rightsquigarrow V(R)$ defined by V. This certainly determines V but leads to set-theoretic difficulties — putting a condition on $\tilde{V}$ involves quantifying over a proper class, and, in general, the natural transformations from one functor on k-algebras to a second functor form a proper class. These difficulties vanish when one restricts to k-algebras that are small in some sense. From this point-of-view, an algebraic scheme over k is determined by the functor it defines on *small* k-algebras, and it defines a functor on *all* k-algebras.

A criterion for a functor to arise from an algebraic scheme

By a functor in this subsection we mean a functor $\mathsf{Alg}_k \to \mathsf{Set}$. A subfunctor U of a functor V is ***open*** if, for all maps $\varphi: h^A \to V$, the subfunctor $\varphi^{-1}(U)$ of h^A is defined by an open subscheme of $\mathrm{Spm}(A)$. A family $(U_i)_{i\in I}$ of open subfunctors of V is an ***open covering*** of V if each U_i is open in V and $V = \bigcup U_i(K)$ for every field K. A functor V is ***local*** if, for all k-algebras R and all finite families $(f_i)_i$ of elements of A generating the ideal A, the sequence of sets

$$V(R) \to \prod\nolimits_i V(R_{f_i}) \rightrightarrows \prod\nolimits_{i,j} V(R_{f_i f_j})$$

is exact.

Let $\mathbb{A}^1$ denote the functor sending a k-algebra R to its underlying set. For a functor U, let $\mathcal{O}(U) = \mathrm{Hom}(U, \mathbb{A}^1)$ — it is a k-algebra. A functor U is ***affine*** if $\mathcal{O}(U)$ is finitely generated and the canonical map $U \to h^{\mathcal{O}(U)}$ is an isomorphism.

10.86. *A local functor admitting a finite covering by open affines is representable by an algebraic scheme (i.e., it is of the form $\tilde{V}$ for an algebraic scheme X).*

This is the *definition* of a scheme in Demazure and Gabriel 1970, I, §1, 3.11, p. 12.

i. Projective space; Grassmanians

10.87. The condition that k be algebraically closed in Section 6k (The functor defined by projective space) and Section 6m (Grassmann varieties) is unnecessary.

j. Dense points; dense subschemes

Because we allow nilpotents in the structure sheaf, a morphism $V \to W$ of algebraic schemes is not in general determined by its effect on $V(k)$, even when k is algebraically closed. We introduce some terminology to handle this.

DEFINITION 10.88. Let V be an algebraic scheme over k. We say that a subset S of $V(k)$ is ***schematically dense*** in V if the only closed subscheme Z of V such that $S \subset Z(k)$ is V itself.

For example, let $V = \mathrm{Spm}(A)$, and let S be a subset of $V(k)$. A closed subscheme $Z = \mathrm{Spm}(A/\mathfrak{a})$ of V is such that $S \subset Z(k)$ if and only if $\mathfrak{a} \subset \mathfrak{m}$ for all $\mathfrak{m} \in S$. Therefore, S is schematically dense in V if and only if $\bigcap\{\mathfrak{m} \mid \mathfrak{m} \in S\} = \{0\}$.

PROPOSITION 10.89. *Let V be an algebraic scheme over k and S a subset of $V(k) \subset |V|$. The following conditions are equivalent:*

(a) *S is schematically dense in V;*

(b) *V is reduced and S is dense in $|V|$;*

(c) *the family of homomorphisms*

$$f \mapsto f(s) : \mathcal{O}_V \to \kappa(s) = k, \quad s \in S,$$

is injective, i.e., $f = 0$ if $f(s) = 0$ for all $s \in S$.

PROOF. (a)⇒(b). Let $\bar{S}$ denote the closure of S in $|V|$. There is a unique reduced subscheme Z of V with underlying space $\bar{S}$ (10.44). As $S \subset |Z|$, the scheme $Z = V$, and so V is reduced with underlying space $\bar{S}$.

(b)⇒(c). Let U be an open affine subscheme of V, and let $A = \mathcal{O}_V(U)$. Let $f \in A$ be such that $f(s) = 0$ for all $s \in S \cap |U|$. Then $f(u) = 0$ for all $u \in |U|$ because $S \cap |U|$ is dense in $|U|$. This means that f lies in all maximal ideals of A, and so lies in the radical of A (10.21), which is zero because V is reduced.

(c)⇒(a). Let Z be a closed subscheme of V such that $S \subset Z(k)$. Because Z is closed in V, the homomorphism $\mathcal{O}_V \to \mathcal{O}_Z$ is surjective. Because $S \subset Z(k)$, the maps $f \mapsto f(s) : \mathcal{O}_V \to \kappa(s)$, $s \in S$, factor through $\mathcal{O}_Z$, and so $\mathcal{O}_V \to \mathcal{O}_Z$ is injective, hence an isomorphism, which implies that $Z = V$. □

PROPOSITION 10.90. *A schematically dense subset remains schematically dense under extension of the base field.*

PROOF. Let k' be a field containing k, and let $S \subset V(k)$ be schematically dense in V. We may suppose that V is affine, say, $V = \mathrm{Spm}(A)$. Let $s' : A \otimes k' \to k'$ be the map obtained from $s : A \to \kappa(s) = k$ by extension of scalars. The family s', $s \in S$, is injective because the family s, $s \in S$, is injective and k' is flat over k. □

COROLLARY 10.91. *If V admits a schematically dense subset $S \subset V(k)$, then it is geometrically reduced.*

PROOF. When regarded as a subset of $V(k^{\mathrm{al}})$, S is schematically dense in $V_{k^{\mathrm{al}}}$, which is therefore reduced (10.89). □

PROPOSITION 10.92. *Let $u, v : V \rightrightarrows W$ be morphisms from V to a separated algebraic scheme W over k. If S is schematically dense in V and $u(s) = v(s)$ for all $s \in S$, then $u = v$.*

PROOF. Because W is separated, the equalizer of the pair of maps is closed in V (10.48). As its underlying space contains S, it equals V. □

DEFINITION 10.93. Let V be an algebraic scheme over a field k, and let k' be a field containing k. We say that $V(k')$ is ***schematically dense*** in V if the only closed subscheme Z of V such that $Z(k') = V(k')$ is V itself.

PROPOSITION 10.94. *If $V(k')$ is schematically dense in V, then V is reduced. Conversely, if $V(k')$ is dense in $|V_{k'}|$ and V is geometrically reduced, then $V(k')$ is schematically dense in V.*

PROOF. Recall (10.44) that V_{red} is the (unique) reduced subscheme of V with underlying space $|V|$. Moreover $V_{\text{red}}(k') = V(k')$, and so $V_{\text{red}} = V$ if $V(k')$ is schematically dense in V.

Conversely, suppose that V is geometrically reduced and $V(k')$ is dense in $|V_{k'}|$. Let Z be a closed subscheme of V such that $Z(k') = V(k')$. Then $|Z_{k'}| = |V_{k'}|$ by the density condition. This implies that $Z_{k'} = V_{k'}$ because $V_{k'}$ is reduced, which in turn implies that $Z = V$ (see 10.156 below). □

COROLLARY 10.95. *If V is geometrically reduced and $k' \supset k$ is separably closed, then $V(k')$ is schematically dense in V.*

PROOF. By a standard result (10.57), $V(k')$ is dense in $|V_{k'}|$. □

COROLLARY 10.96. *Let Z and Z' be closed subvarieties of an algebraic scheme V over k. If $Z(k') = Z'(k')$ for some separably closed field k' containing k, then $Z = Z'$.*

PROOF. The closed subscheme $Z \cap Z'$ of Z has the property that $(Z \cap Z')(k') = Z(k')$, and so $Z \cap Z' = Z$. Similarly, $Z \cap Z' = Z'$. □

Thus, a closed subvariety Z of V is determined by the subset $Z(k^{\text{sep}})$ of $V(k^{\text{sep}})$. More explicitly, if $V = \operatorname{Spm}(A)$ and $Z = \operatorname{Spm}(A/\mathfrak{a})$, then $\mathfrak{a}$ is the set of $f \in A$ such that $f(P) = 0$ for all $P \in Z(k^{\text{sep}})$.

k. Finite schemes; connected components

Recall that an R-algebra A is said to be finite if it is finitely generated as an R-module.

Finite algebraic schemes

PROPOSITION 10.97. *The following conditions on a finitely generated k-algebra A are equivalent:*

(a) *A is artinian;*

(b) *A has Krull dimension zero;*

(c) *A is a finite k-algebra;*

(d) $\operatorname{spm}(A)$ *is discrete (in which case it is finite).*

PROOF. (a)⇔(b). Because finitely generated, A is noetherian, and hence artinian if and only if of dimension zero (CA 16.6).

(b)⇒(c). According to the Noether normalization theorem (2.45), there exist algebraically independent elements $x_1, \ldots, x_r$ in A such that A is finite over $k[x_1, \ldots, x_r]$. As $k[x_1, \ldots, x_r]$ has Krull dimension r (2.55) and $\dim k[x_1, \ldots, x_r] \leq \dim A$ (1.54), we see that (b) implies that $r = 0$ and that A is finite over k.

(c)⇒(a). Because A is finite-dimensional as a k-vector space, any descending chain of subspaces (a fortiori, ideals) terminates.

(d)⇒(b). Let $\mathfrak{m}$ be a maximal ideal in A. As $\{\mathfrak{m}\}$ is open in $\operatorname{spm}(A)$, it equals $\operatorname{spm}(A_f)$ for some $f \in A$. Every prime ideal in A_f is an intersection of maximal ideals (CA 13.11), and hence equals $\mathfrak{m}$. It follows that no prime ideal of A is properly contained in $\mathfrak{m}$. As this is true of all maximal ideals in A, its dimension is zero.

(a)⇒(d). Because A is artinian, it has only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$, and some product, say, $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r}$, equals 0 (CA §16). According Theorem 1.1, $A \simeq A/\mathfrak{m}_1^{n_1} \times \cdots \times A/\mathfrak{m}_r^{n_r}$ and so $\mathrm{spm}(A) = \bigsqcup \mathrm{spm}(A/\mathfrak{m}_i^{n_i}) = \bigsqcup \{\mathfrak{m}_i\}$ (disjoint union of open one-element sets). □

PROPOSITION 10.98. *The following conditions on an algebraic scheme V over k are equivalent:*

(a) *V is affine and $\mathcal{O}_V(V)$ is a finite k-algebra;*

(b) *V has dimension zero;*

(c) *$|V|$ is discrete (in which case it is finite).*

PROOF. The implications (a)⇒(b)⇒(c) follow immediately from 10.97. It remains to prove that (c)⇒(a). If $|V|$ is discrete, then (by 10.97) every open affine subscheme is a finite disjoint union $U = \bigsqcup \mathrm{Spm}(A_i)$ with A_i a finite local k-algebra. Therefore, the same is true of V, say, $V = \bigsqcup \mathrm{Spm}(A_j) = \mathrm{Spm}(\prod A_j)$, and $\prod A_j$ is a finite k-algebra. □

An algebraic scheme over k is ***finite*** if it satisfies the equivalent conditions of 10.98.

Finite algebraic varieties (étale schemes)

We summarize part of Chapter 8 of Milne 2022 (=FT).

DEFINITION 10.99. A k-algebra A is ***diagonalizable*** if it is isomorphic to the product algebra k^n for some $n \in \mathbb{N}$, and it is ***étale*** if $k' \otimes A$ is diagonalizable for some field k' containing k.

In particular, an étale k-algebra is finite.

10.100 (FT 8.6, 8.7). The following conditions on a finite k-algebra A are equivalent:

(a) A is étale;

(b) A is a product of separable field extensions of k;

(c) $k' \otimes A$ is reduced for all fields k' containing k (i.e., A is an affine k-algebra);

(d) $k^{\mathrm{sep}} \otimes A$ is diagonalizable.

10.101 (FT 8.8). A k-algebra $k[T]/(f(X))$ is étale if and only if f is separable, i.e., has no multiple roots. Every étale k-algebra is a finite product of such algebras.

10.102 (FT 8.9, 8.10, 8.11). Finite products, tensor products, and quotients of diagonalizable (resp. étale) k-algebras are diagonalizable (resp. étale). The composite of any finite set of étale subalgebras of a k-algebra is étale. If A is étale over k, then $k' \otimes A$ is étale over k' for every field k' containing k.

DEFINITION 10.103. An algebraic scheme over k is ***étale*** if it is affine and $\mathcal{O}_V(V)$ is an étale k-algebra.

Almost by definition, a finite k-algebra is étale if and only if the ring $k^{\mathrm{al}} \otimes A$ reduced. It follows from (10.98) that the étale algebraic schemes over k are exactly the algebraic varieties of dimension zero.

10.104 (FT 8.23). Let k^{sep} be a separable closure of k, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. By a ***Γ-set*** we meet a set S equipped with an action of Γ. A Γ-set S is ***discrete*** if the action $\Gamma \times S \to S$ is continuous relative to the Krull topology on Γ and the discrete topology on S. If V is a zero-dimensional variety over k, then $V(k^{\mathrm{sep}})$ is a finite discrete Γ-set, and the functor

$$V \rightsquigarrow V(k^{\mathrm{sep}}) \tag{5}$$

is an equivalence from the category of zero-dimensional algebraic varieties over k to the category of finite discrete Γ-sets.

The algebraic variety of connected components of an algebraic scheme

Let f be a nontrivial idempotent in a ring A, i.e., $f^2 = f$ and $f \neq 0, 1$. As idempotents in integral domains are trivial, each prime ideal in A contains exactly one of f or $1-f$. Therefore $\mathrm{spm}(A)$ is a disjoint union of the closed-open subsets $D(f)$ and $D(1-f)$. More generally, let V be an algebraic scheme over k. Then $\mathcal{O}(V)$ is a k-algebra (not necessarily finitely generated), and a nontrivial idempotent in $\mathcal{O}(V)$ decomposes V into a disjoint union of two nonempty closed-open subsets.

PROPOSITION 10.105. *Let V be an algebraic scheme over k. There exists a largest étale k-subalgebra $\pi(V)$ in $\mathcal{O}(V)$.*

PROOF. Let A be an étale subalgebra of $\mathcal{O}(V)$. Then $A \otimes k^{\mathrm{sep}} \simeq (k^{\mathrm{sep}})^n$ for some n, and so

$$1 = f_1 + \cdots + f_n$$

with the f_i orthogonal idempotents in $\mathcal{O}(V_{k^{\mathrm{sep}}})$. The f_i decompose $|V_{k^{\mathrm{sep}}}|$ into a disjoint union of n closed-open subsets, and so n is not more than the number of connected components of $|V_{k^{\mathrm{sep}}}|$. Thus the number $[A : k] = [A \otimes k^{\mathrm{sep}} : k^{\mathrm{sep}}]$ is bounded. It follows that the composite of all étale k-subalgebras of $\mathcal{O}(V)$ is an étale k-subalgebra which contains all others. □

Define

$$\pi_0(V) = \mathrm{Spm}(\pi(V)).$$

Under the canonical isomorphism (see 10.35)

$$\mathrm{Hom}_{k\text{-algebra}}(\pi(V), \mathcal{O}(V)) \simeq \mathrm{Hom}_{k\text{-scheme}}(V, \mathrm{Spm}(\pi(V))),$$

the inclusion $\pi(V) \hookrightarrow \mathcal{O}(V)$ corresponds to a morphism $\varphi\colon V \to \pi_0(V)$, which is universal among morphisms from V to an étale k-scheme.

PROPOSITION 10.106. *Let V be an algebraic scheme over k.*

(a) *For all fields k' containing k,*

$$\pi_0(V_{k'}) \simeq \pi_0(V)_{k'}.$$

(b) *Let W be a second algebraic scheme over k. Then*

$$\pi_0(V \times W) \simeq \pi_0(V) \times \pi_0(W).$$

PROOF. (a) Let $\pi = \pi(\mathcal{O}(V))$ and $\pi' = \pi(\mathcal{O}(V_{k'}))$. Then $\pi \otimes k' \subset \pi'$, and it remains to prove equality.

Suppose first that $k' = k^{\text{sep}}$, and let $\Gamma = \text{Gal}(k^{\text{sep}}/k)$. By uniqueness, π' is stable under Γ, and by Galois theory (FT, 7.13), π'^{Γ} is étale over k and $\pi'^{\Gamma} \otimes k' \simeq \pi'$. On the other hand $\pi \subset \pi'^{\Gamma}$, and so $\pi = \pi'^{\Gamma}$ by maximality. Hence $\pi \otimes k' \simeq \pi'$.

Now suppose that $k = k^{\text{sep}}$ and $k' = k^{\text{al}}$. If $k^{\text{al}} \neq k$, then k has characteristic $p \neq 0$. Let $e_1, \ldots, e_m$ be a basis for π' as a k^{al}-vector space consisting of idempotents, and let $e_j = \sum a_i \otimes c_i$ with $a_i \in \mathcal{O}(V)$ and $c_i \in k^{\text{al}}$. For some r, all $c_i^{p^r} \in k$. As e_j is an idempotent, $e_j = e_j^{p^r} = \sum a_i^{p^r} \otimes c_i^{p^r} \in \mathcal{O}(V)$. Hence $\pi \otimes k^{\text{al}} \simeq \pi'$.

Next suppose that k and k' are algebraically closed. We have to show that V is connected if and only if $V_{k'}$ is connected. If $\pi' = k'$, then $\pi = k$ because $\pi \otimes k' \subset \pi'$. Conversely, if V is connected, then $V_{k'}$ is connected because $|V|$ is dense in $|V_{k'}|$.

In the general case, let $k^{\text{al}} \subset k'^{\text{al}}$ be algebraic closures of k and k'. If $\pi \otimes k' \neq \pi'$ then $\pi \otimes k' \otimes_{k'} k'^{\text{al}} \neq \pi' \otimes_{k'} k'^{\text{al}}$, and so $(\pi \otimes k^{\text{al}}) \otimes_{k^{\text{al}}} k'^{\text{al}} \neq \pi' \otimes_{k'} k'^{\text{al}}$. But this contradicts the previous statements.

(b) After (a), we may suppose that $k = k^{\text{al}}$, and then we have to show that $V \times W$ is connected if V and W are. But $V \times W$ is a union of the connected subvarieties $v \times W$ and $V \times w$ with $v \in |V|$ and $w \in |W|$, and so this is obvious. □

If $\pi(V)$ is a field, then $\mathcal{O}(V)$ has no nontrivial idempotents, and so V is connected. If k is algebraically closed in[6] $\mathcal{O}(V)$, then it is algebraically closed in $\pi(V)$, and so $\pi(V) = k$; in this case, $\pi(V_{k^{\text{al}}}) = k^{\text{al}}$ and $V_{k^{\text{al}}}$ is connected.

PROPOSITION 10.107. *Let V be an algebraic scheme over k.*

(a) *The fibres of the map $\varphi \colon V \to \pi_0(V)$ are the connected components of V.*

(b) *For all $v \in |\pi_0(V)|$, the fibre $\varphi^{-1}(v)$ is a geometrically connected scheme over $\kappa(v)$.*

PROOF. Let $v \in |\pi_0(V)|$. For the fibre $V_v = \varphi^{-1}(v)$, we have $\pi(V_v) = \kappa(v)$. Therefore the statements follow from the above discussion. □

COROLLARY 10.108. *Let V be a connected algebraic scheme over k such that $V(k) \neq \emptyset$. Then V is geometrically connected, and $V \times W$ is connected for any connected algebraic scheme W over k.*

PROOF. By definition, $A = \pi(V)$ is a finite product of separable field extensions of k. If A had more than one factor, $\mathcal{O}(V)$ would contain a nontrivial idempotent, and V would not be connected. Therefore, A is a field containing k. Because $V(k)$ is nonempty, there is a k-homomorphism $A \to k$, and so $A = k$. Now $V_{k^{\text{al}}}$ is connected (see the above discussion). Moreover,

$$\pi_0(V \times W) \simeq \pi_0(V) \times \pi_0(W) \simeq \pi_0(W),$$

and so $V \times W$ is connected. □

REMARK 10.109. Let V be an algebraic scheme over k.

(a) The connected components of $V_{k^{\text{sep}}}$ form a finite set on which $\text{Gal}(k^{\text{sep}}/k)$ acts continuously, and $\pi_0(V)$ is the étale scheme over k corresponding to this set under the equivalence $Z \rightsquigarrow Z(k^{\text{sep}})$ in (5).

[6] This means that an element a of $\mathcal{O}(V)$ lies in k if $f(a) = 0$ for some nonzero $f(X) \in k[X]$.

(b) For $v \in \pi_0(V)$, $\varphi^{-1}(v) \to \mathrm{Spm}(\kappa(v))$ is flat because $\kappa(v)$ is a field. Therefore, the morphism $\varphi\colon V \to \pi_0(V)$ is faithfully flat.

(c) The formation of $\varphi\colon V \to \pi_0(V)$ commutes with extension of the base field. This is what the proof of 10.106 shows.

SUMMARY 10.110. Let V be an algebraic scheme over k. Among the regular maps from V to a zero-dimensional algebraic variety there is one $V \to \pi_0(V)$ that is universal. The fibres of the map $V \to \pi_0(V)$ are the connected components of V. The map $V \to \pi_0(V)$ commutes with extension of the base field, and $\pi_0(V \times W) \simeq \pi_0(V) \times \pi_0(W)$. The variety $\pi_0(V)$ is called the ***variety of connected components*** of V.

l. Properties of morphisms

In this section, we review the definitions of the different types of morphisms and their properties. Most proofs in the first nine chapters generalize without difficulty to the new situation.

Separated maps

10.111. For a regular map $V \to S$ of algebraic schemes over k, we define $\Delta_{V/S}$ to be the equalizer of the projection maps $\Delta_V \rightrightarrows S$. It is a subscheme of $V \times_S V$. The map $V \to S$ is said to be ***separated*** if $\Delta_{V/S}$ is closed. For example, if V is an algebraic scheme over k, then $\Delta_{V/\mathrm{Spm}(k)} = \Delta_V$, and so the structure map $V \to \mathrm{Spm}(k)$ is separated if and only if V is separated.

10.112. A regular map $\varphi\colon V \to S$ is separated if there exists an open covering $S = \bigcup S_i$ of S such that $\varphi^{-1}(S_i) \xrightarrow{\varphi} S_i$ is separated for all i.

10.113. A regular map $\varphi\colon V \to S$ is separated if V and S are separated. (As V is separated, the diagonal Δ_V in $V \times V$ is closed; as S is separated, the equalizer of the projections $\Delta_V \rightrightarrows S$ is closed.)

Affine maps

10.114. A regular map $\varphi\colon V \to S$ is said to be ***affine*** if, for all open affines U in S, $\varphi^{-1}(U)$ is an open affine in V. It suffices to check the condition for the U in an open affine cover of V.

10.115. Every affine map is separated. (Affine algebraic schemes over k are separated, and so regular maps of affine algebraic schemes are separated (10.113). Therefore, this follows from (10.112).)

Flat maps

A flat morphism is the algebraic analogue of a map whose fibres form a continuously varying family. For example, a surjective morphism of smooth varieties is flat if and only if all fibres have the same dimension (10.130). A finite morphism to a reduced algebraic scheme is flat if and only if, over every connected component, all fibres have the same number of points (counting multiplicities) (10.123). A flat morphism of finite type of algebraic schemes is open, and surjective flat morphisms are epimorphisms in a very strong sense (10.75).

10.116. Recall from Section 9c: A homomorphism $A \to B$ of rings is ***flat*** if the functor $M \rightsquigarrow B \otimes_A M$ of A-modules is exact. It is ***faithfully flat*** if, in addition,

$$B \otimes_A M = 0 \Rightarrow M = 0.$$

(a) If $f: A \to B$ is flat, then so also is $S^{-1}f: S^{-1}A \to S^{-1}B$ for any multiplicative subsets S of A.

(b) A homomorphism $f: A \to B$ is flat if and only if $A_{f^{-1}(\mathfrak{n})} \to B_{\mathfrak{n}}$ is flat for all maximal ideals $\mathfrak{n}$ in B.

(c) Let $A \to A'$ be a homomorphism of rings. If $A \to B$ is flat (resp. faithfully flat), then $A' \to A' \otimes B$ is flat (resp. faithfully flat).

(d) Faithfully flat homomorphisms are injective.

10.117. A regular map $\varphi: W \to V$ of algebraic schemes over k is said to be ***flat*** if, for all $w \in |W|$, the map $\mathcal{O}_{V,\varphi w} \to \mathcal{O}_{W,w}$ is flat. A flat map φ is ***faithfully flat*** if it is flat and $|\varphi|$ is surjective.

For example, the map $\mathrm{Spm}(B) \to \mathrm{Spm}(A)$ defined by a homomorphism of k-algebras $A \to B$ is flat (resp. faithfully flat) if and only if $A \to B$ is flat (resp. faithfully flat).

10.118. A flat map $\varphi: W \to V$ of algebraic schemes is open, and hence universally open.

10.119 (Generic flatness). Let $\varphi: W \to V$ be a dominant map of algebraic schemes. If V is integral, there exists a dense open subset U of V such that $\varphi^{-1}(U) \xrightarrow{\varphi} U$ is faithfully flat.

After passing to suitable open affine subschemes, we may suppose φ is defined by a homomorphism $A \to B$ of finitely generated k-algebras with A an integral domain. According to CA, 11.21, there are nonzero elements $a \in A$ and $b \in B$ such that $\mathrm{Spm}\, B_b \to \mathrm{Spm}\, A_a$ is faithfully flat.

10.120. Let $\varphi: W \to V$ be a regular map of algebraic schemes. If $\mathrm{pr}_1: W \times_V W \to W$ is faithfully flat, then so also is φ.

Finite maps and quasi-finite maps

10.121. A regular map $\varphi: W \to V$ of algebraic schemes over k is ***finite*** if, for all open affine $U \subset V$, $\varphi^{-1}(U)$ is affine and $\mathcal{O}_W(\varphi^{-1}(U))$ is a finite $\mathcal{O}_V(U)$-algebra. It suffices to check the condition for U in an open affine cover of V.

For example, the map $\mathrm{Spm}(B) \to \mathrm{Spm}(A)$ defined by a homomorphism of k-algebras $A \to B$ is finite if and only if $A \to B$ is finite.

10.122. A regular map $\varphi: W \to V$ of algebraic schemes over k is ***quasi-finite*** if, for all $v \in V$, the fibre $\varphi^{-1}(v)$ is a finite scheme over $k(v)$. Finite maps are quasi-finite.

For $v \in V$, we let

$$\deg_v(\varphi) = \dim_k(\mathcal{O}_{\varphi^{-1}(v)}(\varphi^{-1}(v)).$$

For example, if φ is the map of affine algebraic schemes defined by a homomorphism of k-algebras $A \to B$, then then

$$\deg_v(\varphi) = \dim_k B \otimes_A (A/\mathfrak{m}_v).$$

10.123. A regular map $\varphi\colon W \to V$ of algebraic schemes with V integral is flat if and only if $\deg_v(\varphi)$ is independent of $v \in V$.

10.124. Let $\varphi\colon W \to V$ be a finite map of integral schemes. The ***degree*** of φ is the degree of $k(W)$ over $k(V)$, and the ***separable degree*** of φ is the degree of the greatest separable subextension of $k(W)$ over $k(V)$.

(a) For all $v \in V$,

$$\deg_v(\varphi) \leq \deg(\varphi),$$

and the points v for which equality holds form a dense open subset of V.

(b) Assume that k is algebraically closed. For all $v \in V$,

$$\#\left|\varphi^{-1}(v)\right| \leq \text{sep}\ \deg(\varphi),$$

and the points v for which equality holds form a dense open subset of V.

10.125 (ZARISKI'S MAIN THEOREM). Every separated map $\varphi\colon W \to V$ factors into the composite

$$W \xrightarrow{\iota} W' \xrightarrow{\varphi'} V$$

of an open immersion ι and a finite map φ'.

10.126. Let $\varphi\colon W \to V$ be a quasi-finite map of integral algebraic schemes. If φ is birational (i.e., of degree 1) and V is normal, then φ is an open immersion.

The fibres of regular maps

Let $\varphi\colon W \to V$ be a dominant map of integral schemes.

10.127. There exists a dense open subset U of W such that $\varphi(U)$ is open, $U = \varphi^{-1}(\varphi(U))$, and $U \xrightarrow{\varphi} \varphi(U)$ is flat.

10.128. Let $P \in \varphi(V)$. Then

$$\dim(\varphi^{-1}(P)) \geq \dim(W) - \dim(V).$$

Equality holds for $P \in \varphi(U)$, where U is as in 10.127.

In particular, equality holds for all $P \in V$ if φ is faithfully flat.

10.129. Let S be an irreducible closed subset of V, and let T be an irreducible component of $\varphi^{-1}(S)$ such that $\varphi(T)$ is dense in S. Then

$$\dim(T) \geq \dim(S) + \dim(W) - \dim(V).$$

With U as in 10.127, if S intersects $\varphi(U)$ and T intersects U, then equality holds.

10.130. A surjective morphism of smooth algebraic k-schemes is flat (hence faithfully flat) if its fibres all have the same dimension.

Étale maps

DEFINITION 10.131. A regular map $\varphi : W \to V$ of algebraic schemes over k is ***étale*** if (a) it is flat and (b) for every $v \in |V|$, the fibre $\varphi^{-1}(v)$ over v is an étale scheme (=algebraic variety of dimension zero).

10.132. A regular map $\varphi : W \to V$ of algebraic schemes over an algebraically closed field k is étale if and only the map $C_Q(W) \to C_{\varphi Q}(V)$ is an isomorphism for all $Q \in |W|$.

Thus, for algebraic varieties over an algebraically closed field, Definition 10.131 agrees with those in Chapter V. There is much to be said about étale morphisms, but, for the moment, we refer the reader to I.2 of my notes, *Lectures on Étale Cohomology.*

Smooth maps

DEFINITION 10.133. A regular map $\varphi : W \to V$ of algebraic schemes over k ***smooth*** if (a) it is flat and (b) for every $v \in |V|$, $\varphi^{-1}(v)$ is a smooth scheme (=nonsingular variety).

10.134. A regular map $\varphi : W \to V$ is smooth if and only if it is locally of the form

$$U \xrightarrow{\text{étale}} \mathbb{A}^n_V \to V,$$

i.e., every $Q \in |W|$ has an open neighbourhood U such that $\varphi|U$ admits such a factorization.

10.135. Let A be a finitely generated k-algebra. The map

$$\operatorname{Spm}(A[X_1, \dots, X_n]/(P_1, \dots, P_r)) \to \operatorname{Spm} A$$

defined by the k-algebra homomorphism $A \to A[X_1, \dots, X_n]/(P_1, \dots, P_r)$ is smooth if and only if the matrix

$$\left(\frac{\partial P_i}{\partial X_j}(a_1, \dots a_n)\right)$$

has rank r for all $(a_1, \dots, a_n) \in A^n$. A regular map $\varphi : W \to V$ is smooth if and only if it is locally of this form, i.e., for all $w \in W$, there are open affine neighbourhoods U' of w and U of $\varphi(w)$ such that $\varphi(U') \subset U$ and the restriction of φ to $U' \to U$ is of the above form.

10.136. A regular map $\varphi : W \to V$ of algebraic schemes over an algebraically closed field k is smooth if and only if the map $C_Q(W) \to C_{\varphi Q}(V)$ is surjective for all $Q \in |W|$. In particular, if W and V are smooth varieties, then φ is smooth if and only if it induces surjective maps on the tangent spaces.

Separable maps

10.137. A dominant map $\varphi : W \to V$ of integral algebraic schemes is ***separable*** if $k(W)$ is a separably generated field extension of $k(V)$.

10.138. Let $\varphi : W \to V$ be a dominant map of integral algebraic schemes.

(a) If there exists a nonsingular point $Q \in W$ such that $\varphi(Q)$ is nonsingular and $(d\varphi)_Q$ is surjective, then φ is separable.

(a) If φ is separable, then the set of points $Q \in W$ satisfying the condition in (a) is a dense open subset of W.

10.139. The pull-back of a separable map of irreducible algebraic varieties is separable.

10.140. Let Z_1 and Z_2 be closed subschemes of an algebraic scheme V. Then $Z_1 \cap Z_2 \stackrel{\text{def}}{=} Z_1 \times_V Z_2$ is a closed algebraic subscheme of V. If V, Z_1, and Z_2 are all algebraic varieties, then $Z_1 \cap Z_2$ is an algebraic variety provided $T_P(Z_1)$ and $T_P(Z_2)$ cross transversally (in $T_P(V)$) for all P in an open subset of V.

m. Complete schemes; proper morphisms

Complete algebraic schemes

10.141. An algebraic scheme V is said to be ***complete*** if it is separated and if, for all algebraic schemes T, the projection map $q\colon V \times T \to T$ is closed. (It suffices to check this with $T = \mathbb{A}^n$.)

10.142. (a) Closed subschemes of complete schemes are complete.

(b) An algebraic scheme is complete if and only if its irreducible components are complete.

(c) Products of complete schemes are complete.

(d) Let $\varphi\colon V \to S$ be a regular map of algebraic varieties. If V is complete, then $\varphi(V)$ is a complete closed subvariety of S. In particular,

  i) if $\varphi\colon V \to S$ is dominant and V is complete, then φ is surjective and S is complete;

  ii) complete subvarieties of algebraic varieties are closed.

(e) A regular map $V \to \mathbb{P}^1$ from a complete connected algebraic variety V is either constant or surjective.

(f) The only regular functions on a complete connected algebraic variety are the constant functions.

(g) The image of a regular map from a complete connected algebraic scheme to an affine algebraic scheme is a point. The only complete affine algebraic schemes are the finite schemes.

10.143. Projective algebraic schemes are complete.

10.144. Every quasi-finite map $W \to V$ with W complete is finite.

Proper morphisms

10.145. A regular map $\varphi\colon V \to S$ of algebraic schemes is ***proper*** if it is separated and universally closed (i.e., for all regular maps $T \to S$, the projection map $q\colon V \times_S T \to T$ is closed).

10.146. A finite map is proper.

10.147. An algebraic scheme V is complete if and only if the map $V \to \mathrm{Spm}(k)$ is proper. The base change of a proper map is proper. In particular, if $\pi\colon V \to S$ is proper, then $\pi^{-1}(P)$ is a complete subscheme of V for all $P \in S$.

10.148. If $V \to S$ is proper and S is complete, then V is complete.

10.149. The inverse image of a complete algebraic scheme under a proper map is complete.

10.150. Let $\varphi\colon V \to S$ be a proper map. The image φZ of any complete algebraic subscheme Z of V is a complete algebraic subscheme of S.

10.151. Let $A = \bigoplus_{d\geq 0} A_d$ be a graded ring such that

(a) as an A_0-algebra, A is generated by A_1, and

(b) for every $d \geq 0$, A_d is finitely generated as an A_0-module.

A map $\pi\colon \mathrm{Proj}(A) \to \mathrm{Spm}(A_0)$ is defined (to be added).

10.152. The map $\pi\colon \mathrm{proj}(A) \to \mathrm{spm}(A_0)$ is closed.

n. Restriction of the base field

Also called Weil restriction of scalars.

Let A be a finite k-algebra. A functor F from A-algebras to sets defines a functor

$$(F)_{A/k}\colon \mathsf{Alg}_k \to \mathsf{Set}, \quad R \rightsquigarrow F(A\otimes R).$$

If F is representable, is $(F)_{A/k}$ also representable? We prove that it is in two cases.

PROPOSITION 10.153. *If $F\colon \mathsf{Alg}_A \to \mathsf{Set}$ is represented by a finitely generated A-algebra, then $(F)_{A/k}$ is represented by a finitely generated k-algebra.*

PROOF. Let

$$A = ke_1 \oplus \cdots \oplus ke_d, \quad e_i \in A.$$

Consider first the case that $F = \mathbb{A}^n$, so that $F(R) = R^n$ for all A-algebras R. For a k-algebra R,

$$R' \stackrel{\text{def}}{=} A\otimes R \simeq Re_1 \oplus \cdots \oplus Re_d,$$

and so there is a bijection

$$(a_i)_{1\leq i\leq n} \mapsto (b_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq d}}\colon R'^n \to R^{nd}$$

which sends (a_i) to the family (b_{ij}) defined by the equations

$$a_i = \sum\nolimits_{j=1}^{d} b_{ij}e_j, \quad i = 1,\ldots,n. \tag{6}$$

The bijection is natural in R, and shows that $(F)_{A/k} \approx \mathbb{A}^{nd}$ (the isomorphism depends only on the choice of the basis $e_1,\ldots,e_d$).

If F is represented by a finitely generated A-algebra, then F is a closed subfunctor of $\mathbb{A}^n$ for some n. Therefore $(F)_{A/k}$ is a closed subfunctor of $(\mathbb{A}^n)_{A/k} \approx \mathbb{A}^{dn}$, and so $(F)_{A/k}$ is represented by a quotient of $k[T_1,\ldots,T_{dn}]$.

Alternatively, suppose that F is the subfunctor of $\mathbb{A}^n$ defined by a polynomial $f(X_1,\dots,X_n)$ in $A[X_1,\dots,X_n]$. On substituting

$$X_i = \sum_{j=1}^{d} W_{ij} e_j$$

into f, we obtain a polynomial $g(Y_{11}, Y_{12}, \dots, Y_{nd})$ with the property that

$$f(a_1,\dots,a_n) = 0 \iff g(b_{11}, b_{12}, \dots, b_{nd}) = 0$$

when the a and b are related by (6). The polynomial g has coefficients in A, but we can write it (uniquely) as a sum

$$g = g_1 e_1 + \cdots + g_d e_d, \quad g_i \in k[Y_{11}, Y_{12}, \dots, Y_{nd}].$$

Clearly,

$$g(b_{11}, b_{12}, \dots, b_{nd}) = 0 \iff g_i(b_{11}, b_{12}, \dots, b_{nd}) = 0 \text{ for } i = 1, \dots, d,$$

and so $(F)_{A/k}$ is isomorphic to the subfunctor of $\mathbb{A}^{nd}$ defined by the polynomials $g_1, \dots, g_d$. This argument extends in an obvious way to the case that F is the subfunctor of $\mathbb{A}^n$ defined by a finite set of polynomials. □

PROPOSITION 10.154. *Let V be an algebraic scheme over A such that every finite subset of $|V|$ is contained in an open affine subscheme (e.g., V quasi-projective). Then $(V)_{A/k}$ is an algebraic scheme over k.*

PROOF. We use two obvious facts: (a) if U is an open subfunctor of F, then $(U)_{A/k}$ is an open subfunctor of $(F)_{A/k}$; (b) if F is local (see 10.86), then $(F)_{A/k}$ is local. Let U be an open affine subscheme of V. Then $(U)_{A/k}$ is an open subfunctor of $(V)_{A/k}$ and it is an affine scheme over k by (10.153. It remains to show that a finite number of the functors $(U)_{A/k}$ cover $(V)_{A/k}$ (10.86).

Let $d = [A : k]$, and let $|V|^d$ be the topological product of d copies $|V|$. By assumption, the sets U^d with U open affine in $|V|$ cover $|V|^d$. As $|V|^d$ is quasi-compact, a finite collection $U_1, \dots, U_n$ cover $|V|^d$.

Let U be the union of the subfunctors $(U_i)_{A/k}$ of $(V)_{A/k}$. It is an open subfunctor of $(V)_{A/k}$, and so if $U \neq (V)_{A/k}$, then $U(K) \neq (V)_{A/k}(K)$ for some field K containing k. A point $Q \in (V)_{A/K}(K)$ is an A-morphism $\operatorname{Spm}(A \otimes K) \to V$. The image of $|Q|$ is contained in a subset of $|V|$ with at most d elements, and so Q factors through some U_i. Therefore $(V)_{A/k} = \bigcup (U_i)_{A/k}$. □

o. Galois descent

10.155. Let $\Omega \supset k$ be an extension of fields, and let $\Gamma = \operatorname{Aut}(\Omega/k)$. Assume that $\Omega^\Gamma = k$. This is true, for example, if Ω is a Galois extension of k. Then the functor $V \rightsquigarrow \Omega \otimes_k V$ from vector spaces over k to vector spaces over Ω equipped with a continuous action of Γ is an equivalence of categories.

10.156. Let V be an algebraic scheme over a field k, and let $V' = V_{k'}$ for some field k' containing k. Let W' be a closed subscheme of V'. There exists at most one closed subscheme W of V such that $W_{k'} = W'$ (as a subscheme of V').

Let $\Gamma = \mathrm{Aut}(k'/k)$, and assume that $k'^{\Gamma} = k$. Then W' arises from an algebraic subscheme of V if and only if it is stable under the action of Γ on V'. When V and W' are affine, say, $V = \mathrm{Spm}(A)$ and $W' = \mathrm{Spm}(A_{k'}/\mathfrak{a})$, to say that W' is stable under the action of Γ means that $\mathfrak{a}$ is stable under the action of Γ on $A_{k'} \stackrel{\text{def}}{=} A \otimes k'$. More generally, it means that the ideal defining W' in $\mathcal{O}_{V'}$ is stable under the action of Γ on $\mathcal{O}_{V'}$.

Let $k' = k^{\mathrm{sep}}$. An algebraic subvariety W' of V' is stable under the action of Γ on V' if and only if the set $W'(k')$ is stable under the action of Γ on $V(k')$.

10.157. Let V and W be algebraic schemes over k with W separated, and let $V' = V_{k'}$ and $W' = W_{k'}$ for some field k' containing k. Let $\varphi' : V' \to W'$ be a regular map. Because W' is separated, the graph $\Gamma_{\varphi'}$ of φ' is closed in $V \times W$, and so we can apply (10.156) to it. We deduce:

- There exists at most one regular map $\varphi : V \to W$ such that $\varphi' = \varphi_{k'}$.
- Let $\Gamma = \mathrm{Aut}(k'/k)$, and assume that $k'^{\Gamma} = k$. Then $\varphi' : V' \to W'$ arises from a regular map over k if and only if its graph is stable under the action of Γ on $V' \times W'$.
- Let $k' = k^{\mathrm{sep}}$, and assume that V and W are algebraic varieties. Then φ' arises from a regular map over k if and only if the map

  $$\varphi'(k') : V(k') \to W(k')$$

  commutes with the actions of Γ on $V(k')$ and $W(k')$.

See my article, Descent for algebraic schemes, arXiv:2406.05550.

p. Schemes in general

In this chapter, we have studied schemes of finite type over a base field. In the 1950s (if not earlier), it became clear that one needed to consider schemes over more general base rings. For example, the study of algebraic schemes over $\mathbb{Q}$ and their reductions to algebraic schemes over the finite fields $\mathbb{F}_p$ amounts to the study of algebraic schemes over $\mathbb{Z}$. This suggested attaching a geometric object to *every* commutative ring, not just those finitely generated over a field. Unfortunately, the map $A \mapsto \mathrm{spm}\, A$ is not functorial in this wider context: if $\varphi : A \to B$ is a homomorphism of rings, then the inverse image $\varphi^{-1}(\mathfrak{m})$ of a maximal ideal $\mathfrak{m}$ of B need not be maximal — consider for example the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Thus one is forced to replace $\mathrm{spm}(A)$ with $\mathrm{spec}(A)$, the set of all prime ideals in A. One then attaches a locally ringed space $\mathrm{Spec}(A)$ to A, and defines an ***affine scheme*** to be any locally ringed space isomorphic to $\mathrm{Spec}(A)$ for some ring A. A ***scheme*** is a locally ringed space that admits an open covering by affine schemes. A scheme over a ring A is a scheme V equipped with a morphism $V \to \mathrm{Spec}(A)$. This point of view was developed with great vigour by Grothendieck and his associates in the 1960s (see the seminars SGA and the treatise EGA).

There is a natural functor $V \rightsquigarrow V^*$ from the category of algebraic schemes over a field k in our sense to the category of schemes of finite-type over k in the sense of EGA, which gives an equivalence of categories. Under the equivalence, algebraic varieties over k correspond to the geometrically-reduced separated schemes of finite-type over k. To construct the underlying set of V^* from that of V, one only has to add one point p_Z for each irreducible closed subset Z of V of dimension > 0. In other words, $|V^*|$ is the set of irreducible closed subsets of $|V|$. For an open subset U of V, let U^* denote the subset of

V^* containing the points of U together with the points p_Z such that $U \cap Z$ is nonempty. Then $U \mapsto U^*$ is a bijection from the set of open subsets of V onto the set of open subsets of V^* with inverse $U^* \mapsto V \cap U^*$. The closure of $\{p_Z\}$ is Z, and so the map $V \to V^*$ identifies V with the set of closed points of V^*. Define $\Gamma(U^*, \mathcal{O}_{V^*}) = \Gamma(U, \mathcal{O}_V)$ for each open subset U of V. Note that the topologies (families of open subsets) and sheaves of V and V^* are the same — only the underlying sets differ. For a closed irreducible subset Z of V, the local ring $\mathcal{O}_{V^*,p_Z} = \varinjlim_{U \cap Z \neq \emptyset} \Gamma(U, \mathcal{O}_U)$. The inverse functor is even easier to describe: simply omit the nonclosed points from the base space.[7]

Every aspiring algebraic and (especially) arithmetic geometer needs to learn the basic theory of schemes, and for this I recommend reading Chapters II and III of Hartshorne 1997.

Comparison with algebraic schemes in the sense of EGA

10.158. In the language of EGA, we are ignoring the nonclosed points in our algebraic schemes. In other words, we are working with ultraschemes rather than schemes (EGA I, Appendice). For schemes of finite type over a field k (i.e., algebraic over k), we provide a short dictionary. Note that, in a finitely generated k-algebra, every prime ideal is an intersection of maximal ideals (Nullstellensatz 10.21), so the maximal ideals determine the prime ideals.

(a) Let V be an algebraic scheme over k in the sense of EGA, and let V_0 be the set of closed points in V with the induced topology. The map $S \mapsto S \cap V_0$ is an isomorphism from the lattice of closed (resp. open, constructible) subsets of V to the lattice of similar subsets of V_0. In particular, V is connected if and only if V_0 is connected. To recover V from V_0, add a point z for each irreducible closed subset Z of V_0 not already a point; the point z lies in an open subset U if and only if $U \cap Z$ is nonempty. Thus the ringed spaces $(V, \mathcal{O}_V)$ and $(V_0, \mathcal{O}_V|V_0)$ have the same lattice of open subsets and the same k-algebra for each open subset; they differ only in the underlying sets. See EGA IV, §10.

(b) Let V be an algebraic scheme over k in the sense of EGA. Then V is normal (resp. regular) in the sense of EGA if and only if $\mathcal{O}_{V,v}$ is normal (resp. regular) for all closed points v of V. Moreover, V is smooth over k, i.e., the morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(k)$ is smooth, if and only if $V_{k^{\mathrm{a}}}$ is regular, which again is a condition on the closed points.

(c) Morphisms of algebraic schemes over k map closed points to closed points. The functor $(V, \mathcal{O}_V) \rightsquigarrow (V_0, \mathcal{O}_V|V_0)$ is an equivalence from the category of algebraic schemes over k to the category of ultraschemes over k.

(d) Let $\varphi\colon V \to W$ be a morphism of algebraic schemes over k in the sense of EGA. Then

- φ is surjective if and only if it is surjective on closed points (use (a) and that φ maps constructible sets to constructible sets);

[7] Some authors call a geometrically reduced scheme of finite-type over a field a variety. Despite their similarity, it is important to distinguish such schemes from varieties (in the sense of these notes). For example, if W and W' are subvarieties of a variety, their intersection in the sense of schemes need not be reduced, and so may differ from their intersection in the sense of varieties. For example, if $W = V(\mathfrak{a}) \subset \mathbb{A}^n$ and $W' = V(\mathfrak{a}') \subset \mathbb{A}^{n'}$ with $\mathfrak{a}$ and $\mathfrak{a}'$ radical, then the intersection W and W' in the sense of schemes is $\mathrm{Spec}\, k[X_1, \ldots, X_{n+n'}]/(\mathfrak{a}, \mathfrak{a}')$ while their intersection in the sense of varieties is $\mathrm{Spec}\, k[X_1, \ldots, X_{n+n'}]/\mathrm{rad}(\mathfrak{a}, \mathfrak{a}')$.

- φ is quasi-finite if and only if $\varphi^{-1}(w)$ is finite for all closed points w of W;
- φ is flat if and only if $\mathcal{O}_{W,\varphi(v)} \to \mathcal{O}_{V,v}$ is flat for all closed points v of V;
- φ is smooth if and only if it is flat and its closed fibres are smooth.

See Demazure and Gabriel 1970, p. 95-96, 6.5–6.10.

Bibliography

DEMAZURE, M. AND GABRIEL, P. 1970. Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam.

MATSUMURA, H. 1989. Commutative ring theory, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, first paperback edition. Translated from the Japanese by M. Reid.

MILNE, J. S. 2022. Fields and Galois Theory. Kea Books, Ann Arbor, MI.

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