Chapter 11

Surfaces

The theory of surfaces is one of the most beautiful parts of algebraic geometry. It is more complex than the theory of curves, but is more complete than the theory of higher dimensional varieties and serves as a model for it. In this chapter, we prove the Riemann-Roch theorem for a surface, and deduce the Hodge index theorem. From this, the Riemann hypothesis for curves over finite fields follows easily — this remains the most illuminating proof.

We fix an algebraically closed field k. Algebraic varieties over k are assumed to be irreducible. Points on varieties are closed. A surface is an algebraically variety of dimension 2.

a. Divisors and their intersections

Let V be a smooth surface over k. Recall that smoothness means that the local ring $\mathcal{O}_P$ at a point P of V is regular. In particular, it is factorial, and so we have a good theory of Weil divisors (cf. Chapter 12). If x, y generate the maximal ideal in $\mathcal{O}_P$, the the completion of $\mathcal{O}_P$ is the power series ring $k[[x,y]]$.

Definitions

If V is affine, say, $V = \operatorname{Spec} A$, then the prime ideals $\mathfrak{p}$ of A are of the following types according as their height is 0, 1, or 2:

$$\begin{aligned} \mathfrak{p} = 0 &\iff \operatorname{tr.deg}_k A/\mathfrak{p} = 2 &&\iff V(\mathfrak{p}) = V \\ \mathfrak{p} \text{ minimal nonzero} &\iff \operatorname{tr.deg}_k A/\mathfrak{p} = 1 &&\iff V(\mathfrak{p}) = \text{a curve on } V \\ \mathfrak{p} \text{ maximal} &\iff \operatorname{tr.deg}_k A/\mathfrak{p} = 0 \text{ (so } A/\mathfrak{p} = k) &&\iff V(\mathfrak{p}) = \text{a point.} \end{aligned}$$

By a ***curve*** on V, we mean an irreducible closed subvariety of V of dimension one (hence also codimension one). Equivalently, it is an irreducible closed subset of $|V|$ of dimension 1 equipped with its canonical structure as a reduced scheme. By a ***divisor*** on V, we mean a finite formal sum

$$D = \sum n_i Z_i, \quad n_i \in \mathbb{Z}, \quad Z_i \text{ a curve on } V.$$

This is Chapter 11 of Algebraic Geometry by J.S. Milne. It is based on lectures of Arthur Mattuck from 1967 and a lecture of the author from 1968. Version November 4, 2024.

We say that D is ***positive*** (or ***effective***), denoted $D \geq 0$, if all $n_i \geq 0$.

Let Z be a curve on V. If U is an open affine of V that intersects Z, then Z corresponds to a prime ideal $\mathfrak{p}$ in $\mathcal{O}_V(U)$ of height 1, and so $\mathcal{O}_Z \overset{\text{def}}{=} \mathcal{O}_V(U)_\mathfrak{p}$ is a normal noetherian local ring of dimension 1. It is therefore a discrete valuation ring (*Commutative Algebra*, 20.2), and we let

$$\mathrm{ord}_Z \colon k(V)^\times \twoheadrightarrow \mathbb{Z}$$

denote the corresponding normalized valuation on $k(V)$. The ***divisor*** of f is

$$(f) = \sum \mathrm{ord}_Z(f) \cdot Z$$

(sum over the finitely many curves Z on V such that $\mathrm{ord}_Z(f) \neq 0$). Write (f) as the difference of two positive divisors

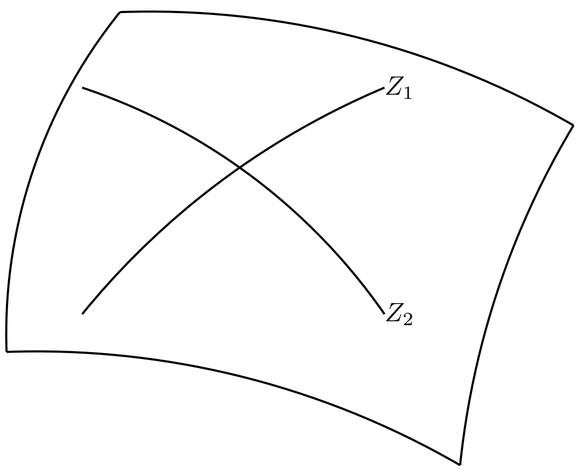
$$(f) = (f)_0 - (f)_\infty,$$

— $(f)_0$ is the divisor of zeros of f and $(f)_\infty$ the divisor of poles. A divisor of the form (f) is said to be ***principal***. Two divisors D_1 and D_2 are said to be ***linearly equivalent***, denoted $D_1 \sim D_2$, if they differ by a principal divisor,

$$D_1 - D_2 = (f).$$

Intersections

We first consider the problem of defining the intersection of two curves Z_1 and Z_2 on a smooth surface V (note that Z_1 and Z_2 may be singular).



PROPOSITION 11.1. *If Z_1 and Z_2 are distinct curves on V, then $Z_1 \cap Z_2$ is a finite set of points.*

PROOF. It suffices to prove this when V is affine, say $V = \mathrm{Spec}(A)$. Then Z_1 and Z_2 correspond to prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in A, and

$$Z_1 \cap Z_2 = V(\mathfrak{p}_1, \mathfrak{p}_2).$$

Let

$$(\mathfrak{p}_1, \mathfrak{p}_2) = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

be a minimal primary decomposition of $(\mathfrak{p}_1, \mathfrak{p}_2)$, and let $\mathfrak{p}'_i = \mathrm{rad}(\mathfrak{q}_i)$ (*Commutative Algebra*, 19.7 et seq.). If $\mathfrak{p}'_i$ were a minimal nonzero prime ideal in A, then it would have to equal both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ because it contains them, but $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and so this is impossible. Therefore $\mathfrak{p}'_i$ is maximal. Now

$$Z_1 \cap Z_2 = V\left(\bigcap\nolimits_{i=1}^{n} \mathfrak{q}_i\right) = \bigcup\nolimits_{i=1}^{n} V(\mathfrak{p}'_i),$$

which is a finite set of points. □

Let P be a point on V. Let $\mathcal{O}_P$ be the local ring at P and let $\mathfrak{m}_P$ be its maximal ideal. A curve Z on V defines an ideal $\mathfrak{p}$ in $\mathcal{O}_P$ (for example, if $V = \operatorname{Spec} A$ and $Z = V(\mathfrak{p}')$, then $\mathfrak{p} = \mathfrak{p}'\mathcal{O}_P$). Because $\mathcal{O}_P$ is factorial, and $\mathfrak{p}$ has height 1, it is principal, say $\mathfrak{p} = (f)$ (1.25). Now

$$Z = (f) + \text{components not passing through } P.$$

We call $f = 0$ a ***local equation*** for Z near P. Note that f is a unit in $\mathcal{O}_P$ if and only if $P \notin Z$.

Let Z_1 and Z_2 be curves on V with local equations $f = 0$ and $g = 0$ near P. Then

$$(\mathfrak{p}_1, \mathfrak{p}_2)\mathcal{O}_P = (f, g)\mathcal{O}_P.$$

If $P \notin Z_1 \cap Z_2$, then $(\mathfrak{p}_1, \mathfrak{p}_2) = \mathcal{O}_P$. On the other hand, if $P \in Z_1 \cap Z_2$, then $(f,g)\mathcal{O}_P$ is primary for $\mathfrak{m}_P$, and so $\mathrm{rad}(f,g) = \mathfrak{m}_P$. Because $\mathcal{O}_P$ is noetherian, this implies that (f,g) contains some power $\mathfrak{m}_P^{r+1}$ of $\mathfrak{m}_P$. Therefore

$$\dim_k(\mathcal{O}_P/(f,g)) \leq \dim_k(\mathcal{O}_P/\mathfrak{m}_P^{r+1}) = \dim_k k[[x,y]]/(x,y)^{r+1} < \infty.$$

DEFINITION 11.2. Let Z_1 and Z_2 be distinct curves on a smooth surface V, and let P be a point on V. We set

$$(Z_1 \cdot Z_2)_P = \dim_k \mathcal{O}_P/(f,g),$$

where f and g are local equations for Z_1 and Z_2 near P.

ASIDE 11.3. Let X_1, X_2 be local parameters at P.[1] Then every $f \in \mathcal{O}_P$ can be written uniquely in the form

$$f = (\text{polynomial of degree } \leq r \text{ in } X_1, X_2 \text{ with coefficients in } k) + f_0, \quad f_0 \in \mathfrak{m}^{r+1}$$

(because $\mathrm{gr}(A) \simeq k[X_1, X_2]$) and so the calculation of $(Z_1 \cdot Z_2)_P$ comes down to a calculation in a polynomial ring.

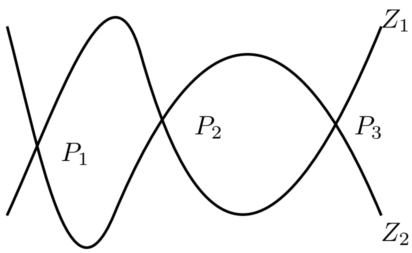
Note that

$$(Z_1 \cdot Z_2)_P = 0 \iff P \notin Z_1 \cap Z_2.$$

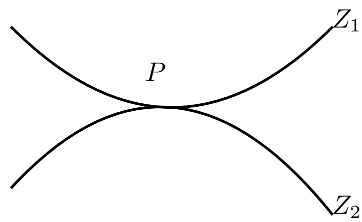
The ***support*** of a divisor $D = \sum n_i Z_i$ is the union of the set of curves Z_i with $n_i \neq 0$. We say that $D_1 \cap D_2$ is ***defined*** (or that D_1 and D_2 are in ***general position***) if $\mathrm{supp}(D_1)$ and $\mathrm{supp}(D_2)$ have no common curves. We extend the above definition to all divisors $D_1 = \sum m_i Z_i$ and $D_2 = \sum n_j Z_j$ such that $D_1 \cap D_2$ is defined by setting

$$(D_1 \cdot D_2)_P = \sum m_i n_j (Z_i \cdot Z_j)_P.$$

[1] Let P be a nonsingular point on a variety V. By a system of local parameters at P we mean a family $\{f_1, \ldots, f_d\}$ of germs of regular functions at P generating the maximal ideal in $\mathcal{O}_P$. Equivalent condition: $(df_1)_P, \ldots, (df_d)_P$ is a basis for the dual space to $T_P(V)$. Such a system defines an étale map $U \to \mathbb{A}^d$ on an open neighbourhood of P. We also say that $f_1, \ldots, f_n$ are local uniformizing parameters, or just local parameters, at P. See Section 5o.



$Z_1 \cdot Z_2 = P_1 + P_2 + P_3$



$Z_1 \cdot Z_2 = 2P$

Let $D = \sum n_i Z_i$ be a positive divisor. Let $f_i = 0$ be a local equation for Z_i near P, and let $f = \prod_i f_i^{n_i}$. Then

$$D = (f) + \text{components not passing through } P,$$

and we call $f = 0$ a ***local equation*** for D near P.

PROPOSITION 11.4. *Let $D_1, D_2 \geq 0$ be divisors on V such that $D_1 \cap D_2$ is defined. Then*

$$(D_1 \cdot D_2)_P = \dim_k(\mathcal{O}_P/(f, g)),$$

where f and g are local equations for D_1 and D_2 near P.

PROOF. From the above description of a local equation for D_1 (and induction), one sees that it suffices to prove that

$$\dim_k(\mathcal{O}_P/(f_1 f_2, g)) = \dim_k(\mathcal{O}_P/(f_1, g)) + \dim_k(\mathcal{O}/(f_2, g)) \tag{1}$$

for all nonzero f_1, f_2, $g \in \mathcal{O}_P$ with $f_1 f_2$ and g relatively prime. Consider the quotient map

$$f \mapsto \bar{f} \colon \mathcal{O}_P \to \bar{\mathcal{O}} \stackrel{\text{def}}{=} \mathcal{O}_P/(g).$$

This gives an isomorphism

$$\mathcal{O}_P/(f, g) \simeq \bar{\mathcal{O}}/(\bar{f}),$$

and so equation (1) can be rewritten as

$$\dim_k(\bar{\mathcal{O}}/(\bar{f}_1 \bar{f}_2)) = \dim_k(\bar{\mathcal{O}}/(\bar{f}_1)) + \dim_k(\bar{\mathcal{O}}/(\bar{f}_2)).$$

Consider the map

$$\bar{\alpha} \mapsto \bar{f}_1 \bar{\alpha} \colon \bar{\mathcal{O}} \to \bar{f}_1 \bar{\mathcal{O}} \quad \text{(i.e., } \mathcal{O}_P/(g) \to (f_1, g)/(g))$$

If $\bar{\alpha}$ maps to zero, then

$$f_1 \alpha = \beta g \text{ for some } \beta \in \mathcal{O}_P,$$

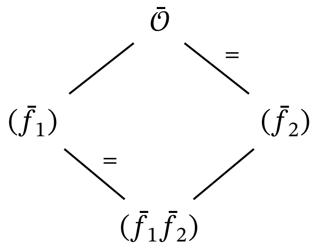
but $\mathcal{O}_P$ is factorial and f_1 and g are relatively prime, and so this implies that $g | \alpha$, i.e., $\bar{\alpha} = 0$. Therefore the map is an isomorphism. It maps $(\bar{f}_2)$ isomorphically onto $(\bar{f}_1 \bar{f}_2)$, and so

$$\dim_k(\bar{\mathcal{O}}/(\bar{f}_2)) = \dim_k((\bar{f}_1)/(\bar{f}_1 \bar{f}_2)).$$

Now

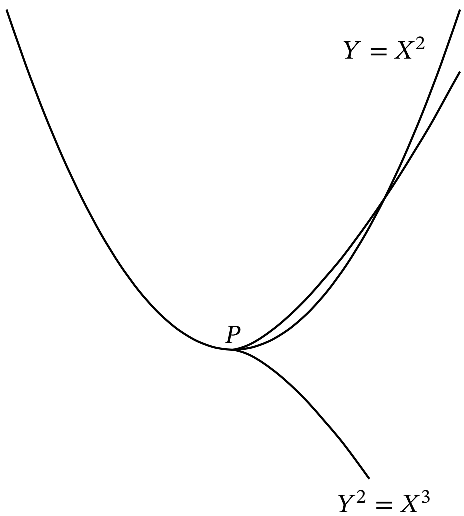
$$\dim_k(\bar{\mathcal{O}}/(\bar{f}_1 \bar{f}_2)) = \dim_k(\bar{\mathcal{O}}/(\bar{f}_1)) + \dim_k((\bar{f}_1)/(\bar{f}_1 \bar{f}_2)) = \dim_k(\bar{\mathcal{O}}/(\bar{f}_1)) + \dim_k(\bar{\mathcal{O}}/(\bar{f}_2))$$

as required,



□

EXAMPLE 11.5. Consider the curves $Y^2 = X^3$ and $Y = X^2$ in $\mathbb{A}^2$ at the origin $P = (0,0)$.



Here $f = Y^2 - X^3$ and $g = Y - X^2$. What is $\dim_k(\mathcal{O}_P/(f,g))$? The ideal (f,g) contains $Y^2 - X^3$, $Y - X^2$, and $X^3 - X^4$. In the quotient ring $\mathcal{O}_P/(f,g)$, $y = x^2$, and so we can forget powers of y; we have $x^3(x-1) = 0$, and so $x^3 = 0$ because $x - 1$ is a unit in $\mathcal{O}_P$. Therefore, $1, x, x^2$ is a basis for $\mathcal{O}_P/(f,g)$, and so $(Z_1 \cdot Z_2)_P = 3$.

In the alternative (old Italian) approach, one moves one curve slightly, say, to $Y + \varepsilon = X^2$. In the quotient ring, we then have the relation

$$(x^2 - \varepsilon)^2 - x^3 = 0, \text{ i.e.,}$$
$$x^4 - x^3 - 2x^2\varepsilon + \varepsilon^2 = 0,$$

which has one root near 1 and 3 near zero.

Brief review of cohomology (see Chapter 13)

Let

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0 \tag{2}$$

be an exact sequence of coherent sheaves of $\mathcal{O}_V$-modules on an algebraic variety V. When we tensor this with a locally free sheaf $\mathcal{N}$ of finite rank r, then

$$0 \to \mathcal{M}' \otimes \mathcal{N} \to \mathcal{M} \otimes \mathcal{N} \to \mathcal{M}'' \otimes \mathcal{N} \to 0$$

is exact because locally it is just a direct sum of r copies of the original sequence. In general, when $\mathcal{N}$ is only coherent, only the sequence

$$\mathcal{M}' \otimes \mathcal{N} \to \mathcal{M} \otimes \mathcal{N} \to \mathcal{M}'' \otimes \mathcal{N} \to 0$$

is exact. However, this sequence extends to an exact sequence

$$\cdots \to \mathcal{T}or^1(\mathcal{M},\mathcal{N}) \to \mathcal{T}or^1(\mathcal{M}'',\mathcal{N}) \to \mathcal{M}' \otimes \mathcal{N} \to \mathcal{M} \otimes \mathcal{N} \to \mathcal{M}'' \otimes \mathcal{N} \to 0$$

with well-defined sheaves $\mathcal{T}or^i$.

For a coherent sheaf $\mathcal{M}$ of $\mathcal{O}_V$-modules, we define $H^i(V,\mathcal{M})$ to be the Čech cohomology groups of $\mathcal{M}$ relative to a finite covering $V = \bigcup_i U_i$ of V by open affines U_i. Thus, $H^i(V,\mathcal{M})$ is the ith cohomology group of a complex

$$\prod\nolimits_i \mathcal{M}(U_i) \to \prod\nolimits_{i,j} \mathcal{M}(U_i \cap U_j) \to \prod\nolimits_{i,j,l} \mathcal{M}(U_i \cap U_j \cap U_l) \to \cdots .$$

Up to a canonical isomorphism, the groups are independent of the covering. As $\mathcal{M}$ is a sheaf, $H^0(V,\mathcal{M}) = \mathcal{M}(U)$. An exact sequence (2) gives rise to an exact cohomology sequence (of k-vector spaces)

$$0 \to H^0(V,\mathcal{M}') \to H^0(V,\mathcal{M}) \to H^0(V,\mathcal{M}'') \to H^1(V,\mathcal{M}') \to H^1(V,\mathcal{M}) \to \cdots .$$

If V has dimension n, then $H^i(V,\mathcal{M}) = 0$ for $i > n$, and if V is complete, then the k-vector spaces $H^i(V,\mathcal{M})$ are finite-dimensional.

Let V be a normal closed subvariety of dimension ≥ 2 in some projective space, and let $\mathcal{L}$ be a locally free sheaf on V of finite rank. Then “Theorem B”:

$$H^1(V,\mathcal{L}(-n)) = 0 \quad \text{for all sufficiently large } n.$$

Serre FAC,[2] §76). This statement is related to the “Enriques-Severi lemma” proved by Zariski (1952)[3] and used by him to prove the Riemann-Roch theorem in nonzero characteristic.

Intersections and cohomology

Given a divisor D, we define the sheaf $\mathcal{O}(D)$ by

$$\Gamma(U,\mathcal{O}(D)) = \{f \in k(V)^\times \mid (f) + D \geq 0\} \cup \{0\}.$$

When D is positive, we define the sheaf $\mathcal{O}_D$ by the exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

Then $\mathrm{supp}(\mathcal{O}_D) = \mathrm{supp}(D)$. For example, if D is a curve Z, then $\mathcal{O}_Z$ is the structure sheaf on Z extended by zero to V, i.e., $\mathcal{O}_Z = j_*(\text{structure sheaf on } Z)$, where j is the inclusion $Z \hookrightarrow V$.

PROPOSITION 11.6. *Let D_1 and D_2 be divisors on V:*

[2] Serre, Jean-Pierre. Faisceaux algébriques cohérents. Ann. of Math. (2) 61, (1955), 197–278. Translation available.

[3] Zariski, Oscar, Complete linear systems on normal varieties and a generalization of a lemma of Enriques-Severi. Ann. of Math. (2) 55, (1952). 552–592.

(a) $\mathrm{supp}(\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}) = \mathrm{supp}(D_1) \cup \mathrm{supp}(D_2)$;
(b) $\dim_k(H^0(\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2})) = (D_1 \cdot D_2)$;
(c) $\mathcal{T}\mathrm{or}^1_{\mathcal{O}_V}(\mathcal{O}_{D_1}, \mathcal{O}_{D_2}) = 0$.

PROOF. These are all local statements. For (a) and (b), it suffices to show that

$$\mathcal{O}_P/(f,g) \simeq \mathcal{O}_P/(f) \otimes \mathcal{O}_P/(g),$$

and for (c) it suffices to show that

$$\mathcal{T}or^1_{\mathcal{O}_P}(\mathcal{O}_P/(f), \mathcal{O}_P/(g)) = 0.$$

On tensoring

$$0 \to (g) \to \mathcal{O} \to \mathcal{O}/(g) \to 0$$

with (f), we obtain an isomorphism

$$(f) \otimes \mathcal{O}/(g) \simeq (f)/(fg).$$

On tensoring

$$0 \to (f) \to \mathcal{O} \to \mathcal{O}_P/(f) \to 0$$

with $\mathcal{O}/(g)$, we obtain an exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{T}or^1_{\mathcal{O}}(\mathcal{O}/(f), \mathcal{O}/(g)) & \longrightarrow & (f) \otimes \mathcal{O}/(g) & \longrightarrow & \mathcal{O}/(g) & \longrightarrow & \mathcal{O}/(f) \otimes \mathcal{O}/(g) & \longrightarrow 0 \\ & & & \| & & & & & \\ & & & (f)/(fg) & & & & & \end{array}$$

— at left $\mathcal{T}or^1_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/(g)) = 0$ because $\mathcal{O}$ is (locally) free. But the map $(f)/(fg) \to \mathcal{O}/(g)$ is injective because $\mathcal{O}_P$ is factorial, and so

$$\mathcal{T}or^1_{\mathcal{O}}(\mathcal{O}/(f), \mathcal{O}/(g)) = 0$$

and

$$\mathcal{O}_P/(f,g) \simeq \mathcal{O}_P/(f) \otimes \mathcal{O}_P/(g).$$

□

For a divisor D, we let

$$\chi(D) = \chi(\mathcal{O}(D)) \stackrel{\text{def}}{=} \dim_k H^0(V, \mathcal{O}(D)) - \dim_k H^1(V, \mathcal{O}(D)) + \dim_k H^2(V, \mathcal{O}(D)).$$

PROPOSITION 11.7. *Let V be a smooth complete surface over k, and let $D_1, D_2 \geq 0$ be divisors on V such that $D_1 \cap D_2$ is defined. Then*

$$(D_1 \cdot D_2) = \chi(\mathcal{O}) - \chi(-D_1) - \chi(-D_2) + \chi(-D_1 - D_2).$$

PROOF. Tensoring

$$0 \to \mathcal{O}(-D_1) \to \mathcal{O} \to \mathcal{O}_{D_1} \to 0$$

with $\mathcal{O}_{D_2}$ gives an exact sequence

$$0 \to \mathcal{O}(-D_1) \otimes \mathcal{O}_{D_2} \to \mathcal{O}_{D_2} \to \mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2} \to 0.$$

Therefore,

$$\begin{aligned}(D_1 \cdot D_2) &= \dim_k H^0(\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}) &&\text{(by 11.6)}\\ &= \chi(\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}) &&(\operatorname{supp}(\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}) \text{ has dimension } 0)\\ &= \chi(\mathcal{O}_{D_2}) - \chi(\mathcal{O}(-D_1) \otimes \mathcal{O}_{D_2}) &&\text{(see the above sequence)}\\ &= -\chi(-D_2) + \chi(\mathcal{O}) - \chi(-D_1) + \chi(-D_1 - D_2).\end{aligned}$$

For the last equality, we used the exact sequences

$$0 \to \mathcal{O}(-D_2) \to \mathcal{O} \to \mathcal{O}_{D_2} \to 0$$
$$0 \to \mathcal{O}(-D_1) \otimes \mathcal{O}(-D_2) \to \mathcal{O}(-D_1) \to \mathcal{O}(-D_1) \otimes \mathcal{O}_{D_2} \to 0$$

(the first is the definition of $\mathcal{O}_{D_2}$ and the second is obtained from it by tensoring with $\mathcal{O}(-D_1)$; note that $\mathcal{O}(-D_1) \otimes \mathcal{O}(-D_2) \simeq \mathcal{O}(-D_1 - D_2)$). □

COROLLARY 11.8. *Let D_1 and D_2 be divisors on V (not necessarily positive). If D_1 is principal and $D_1 \cap D_2$ is defined, then $(D_1 \cdot D_2) = 0$.*

PROOF. By linearity, we may suppose that $D_2 \geq 0$. Let $D_1 = E_1 - E_2$, $E_i \geq 0$. Then

$$(D_1 \cdot D_2) = (E_1 \cdot D_2) - (E_2 \cdot D_2) = 0$$

because $E_1 \sim E_2$ implies $\mathcal{O}(E_1) \approx \mathcal{O}(E_2)$.

(More directly, if D_2 is a smooth curve C on V and $D = (f)$, then $C \cdot (f)$ is the divisor of $f|C$ on C, and

$$(C \cdot D) = \deg(C \cdot D) = \deg(f|C) = 0.)$$ □

DEFINITION 11.9. Let V be a smooth complete surface, and let D_1 and D_2 be divisors on V such that $D_1 \cap D_2$ is defined. We set

$$D_1 \cdot D_2 = \sum (D_1 \cdot D_2)_P P$$
$$(D_1 \cdot D_2) = \sum (D_1 \cdot D_2)_P.$$

ASIDE 11.10. Every smooth complete surface is projective (Zariski 1958[4]), and so we can use “smooth projective” and “smooth complete” interchangeably. A singular surface need not be projective.

LEMMA 11.11. *Let C be a curve on a smooth surface V. For any divisor $D \geq 0$ such that $C \cap D$ is defined,*

$$\mathcal{O}_C(D \cdot C) = \mathcal{O}_C \otimes_{\mathcal{O}_V} \mathcal{O}(D) \quad (= \textit{restriction of } \mathcal{O}(D) \textit{ to } C).$$

PROOF. On tensoring

$$0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0$$

with $\mathcal{O}(D)$, we get an exact sequence

$$0 \to \mathcal{O}(-C) \otimes \mathcal{O}(D) \to \mathcal{O}(D) \to \mathcal{O}_C \otimes \mathcal{O}(D) \to 0.$$

[4]Zariski, Oscar. Introduction to the problem of minimal models in the theory of algebraic surfaces. Publications of the Mathematical Society of Japan, no. 4 The Mathematical Society of Japan, Tokyo 1958.

Let $f = 0$ (resp. $g = 0$) be a local equation for D (resp. C) near P. Then, near P, these sequences become

$$0 \to (g) \to \mathcal{O} \to \bar{\mathcal{O}} \to 0$$

and

$$(f) \otimes (g) \to (f) \to \bar{\mathcal{O}} \otimes (f) \to 0.$$

Therefore

$$\bar{\mathcal{O}} \otimes (f) \simeq (f)/(fg).$$

But there is an $\mathcal{O}_D$-isomorphism

$$(f)/(fg) \xrightarrow{\simeq} (\bar{f})$$

(see the proof of (11.4)). □

DEFINITION 11.12. Curves Z_1 and Z_2 on V are said to intersect ***transversally*** at P if $(Z_1 \cdot Z_2)_P = 1$.

In other words, Z_1 and Z_2 intersect transversally at P if their local equations near P form a system of local parameters at P.

We wish to define $D_1 \cdot D_2$ for arbitrary D_i, e.g., for $D_1 = D_2$.

LEMMA 11.13 (MOVING LEMMA). *Given divisors D_1, D_2, there exists $D_1' \sim D_1$ such that $D_1' \cap D_2$ is defined.*

PROOF. The surface V is normal (because smooth), and so each curve Z on V defines a discrete valuation ord_Z of $k(V)$. The weak approximation theorem for valuations[5] says that, for any finite set $\{Z_1, \dots, Z_m\}$ of distinct curves on V and integers $n_1, \dots, n_m$, there exists an $f \in k(V)$ such that $\mathrm{ord}_{Z_i}(f) = n_i$ for all i.

Write $D_1 = \sum \mathrm{ord}_Z(D)Z$, and choose f so that

$$\mathrm{ord}_Z(f) = \begin{cases} \mathrm{ord}_Z(D_1) & \text{all } Z \subset \mathrm{supp}(D_1) \\ 0 & \text{all } Z \subset \mathrm{supp}(D_2) \text{ but not in } \mathrm{supp}(D_1). \end{cases}$$

Let $D_1' = D_1 - (f)$, and consider a curve $Z \in \mathrm{supp}(D_2)$. Then

$$\begin{cases} Z \subset \mathrm{supp}(D_1) \Rightarrow \mathrm{ord}_Z(D_1') = \mathrm{ord}_Z(D_1) - \mathrm{ord}_Z(f) = 0 \\ Z \notin \mathrm{supp}(D_1) \Rightarrow \mathrm{ord}_Z(D_1') = \mathrm{ord}_Z(f) = 0, \end{cases}$$

and so $D_1' \cap D_2$ is defined. □

Now we can define

$$(D_1 \cdot D_2) = (D_1' \cdot D_2)$$

with D_1' as in the lemma. This makes sense, because if $D_1'' \sim D_1$ is such that $D_1'' \cap D_2$ is also defined, then

$$(D_1' \cdot D_2) = (D_1'' \cdot D_2) \text{ because } D_1' \sim D_1''.$$

In particular, $(D \cdot D)$ is defined — we denote it by (D^2).

[5] My notes *Algebraic Number Theory*, Theorem 7.20. Alternatively, let $A = \mathcal{O}_{Z_1} \cap \cdots \cap \mathcal{O}_{Z_m}$, and let $\mathfrak{p}_i = \mathfrak{m}_{Z_i} \cap A$. Then A is a Dedekind domain with $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ as its nonzero prime ideals. For each i, choose $f_i \in \mathfrak{p}_i^{n_i} \setminus \mathfrak{p}_i^{n_i+1}$ and apply the Chinese remainder theorem to get f.

SUMMARY 11.14. We have a symmetric bi-additive pairing

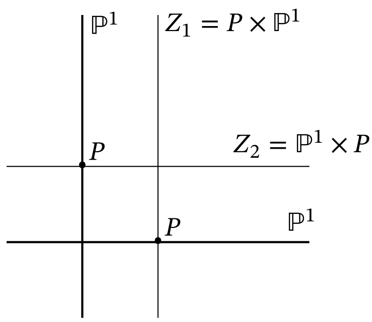
$$D_1, D_2 \mapsto (D_1 \cdot D_2)\colon \operatorname{Div}(V) \times \operatorname{Div}(V) \to \mathbb{Z}.$$

When V is complete, the Euler formula holds:

$$(D_1 \cdot D_2) = \chi(\mathcal{O}) - \chi(-D_1) - \chi(-D_2) + \chi(-D_1 - D_2).$$

(We proved this for positive divisors, but it extends by linearity.)

EXAMPLE 11.15. Let $V = \mathbb{P}^1 \times \mathbb{P}^1$, $Z_1 = P \times \mathbb{P}^1$, $Z_2 = \mathbb{P}^1 \times P$, $\Delta =$ diagonal.



Then

$$(Z_1 \cdot Z_2) = 1, \quad (Z_1^2) = 0 = (Z_2^2).$$

The diagonal Δ is the zero-set of $X - X'$ on V, which has poles, where X and X' have poles, namely, on $P_\infty \times \mathbb{P}^1$ and $\mathbb{P}^1 \times P_\infty$. Therefore $\Delta \sim Z_1 + Z_2$, and so

$$(\Delta \cdot \Delta) = (\Delta \cdot Z_1) + (\Delta \cdot Z_2) = 1 + 1 = 2.$$

In general, if C is a curve of genus g, then on $C \times C$,

$$(\Delta \cdot \Delta) = 2 - 2g$$

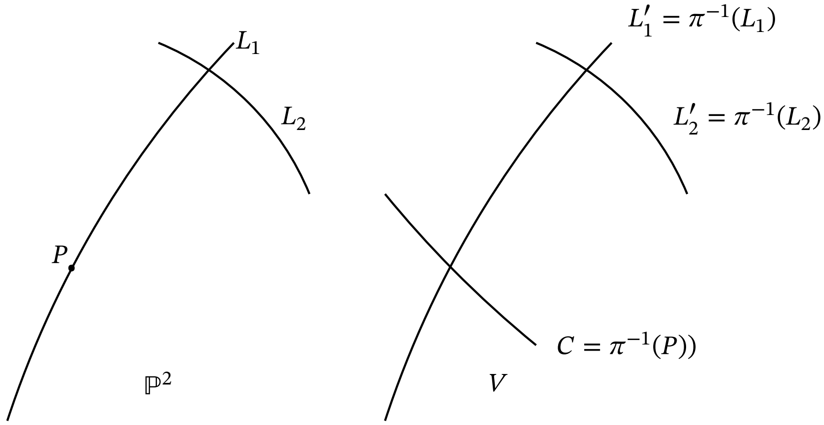
(see later 11.32).

EXAMPLE 11.16. Let V be the projective plane with a point P "blown up" to a line, i.e., we have a monoidal transformation

$$V \xrightarrow{\pi} \mathbb{P}^2.$$

Then V is smooth, and π is an isomorphism outside $\pi^{-1}(P)$, which is a curve C on V.

We claim that $(C \cdot C) = -1$.



The curve C parametrizes the directions through P (the lines through the origin in $T_P(V)$). The line L_1' intersects C at the point representing the direction of L_1, and $L_2' \cap C = \emptyset$. Because $L_1 \sim L_2$ on $\mathbb{P}^2$, we have $C + L_1' \sim L_2'$ for their inverse images on V. Therefore

$$(C \cdot C) = (C \cdot (L_2' - L_1')) = 0 - 1 = -1.$$

ASIDE 11.17. When C and D move in an algebraic family, the intersection number $(C \cdot D)$ is preserved. Hence, if $(C^2) < 0$, then C cannot move in a family of effective divisors (i.e., C is "rigid" inside V); otherwise, if C moved to distinct curves $C'$and C'', then

$$(C \cdot C) = (C' \cdot C'') \geq 0.$$

ASIDE 11.18. Let V be a smooth complete surface. A divisor D on V is ***numerically equivalent*** to zero, $D \approx_n 0$ if $(D \cdot C) = 0$ for all curves C. Such divisor classes form a subgroup $\mathcal{N} = \mathrm{Pic}^{\tau}(V)$ of $\mathrm{Pic}(V)$, and the quotient $\mathrm{Pic}(V)/\mathcal{N}$ is the ***Néron-Severi group*** of V. It is a free abelian group of finite rank ρ, called the ***Picard number*** of V. There is a filtration of $\mathrm{Pic}(V)$ with the quotients at right:

$$\begin{array}{cl} \mathrm{Pic}(V) & \\ \cup & \text{Néron-Severi group} \\ \mathcal{N} & \\ \cup & \text{finite group} \\ \mathrm{Pic}^0(V) & \\ \cup & \text{abelian variety} \\ 0 & \end{array}$$

Over $\mathbb{C}$,

$$\rho = \dim_{\mathbb{Q}} H^{1,1}(V,\mathbb{Q}) \stackrel{\text{def}}{=} H^{1,1}(V,\mathbb{C}) \cap H^2(V,\mathbb{Q}).$$

ASIDE 11.19. The first rigorous general definition of the intersection numbers of algebraic cycles on smooth abstract varieties over arbitrary fields was given by Weil in his Foundations.[6] For a "static" approach, see Serre's Multiplicity notes.[7] For an introduction to modern intersection theory, see Fulton 1984.[8]

[6]Weil, André, Foundations of Algebraic Geometry. American Mathematical Society Colloquium Publications, vol. 29. American Mathematical Society, New York, 1946.

[7]Serre, Jean–Pierre, Algèbre Locale Multiplicités, LNM 11, 1965.

[8]Fulton, W., Introduction to Intersection Theory in Algebraic Geometry, AMS, 1984.

b. Differentials

Following Zariski (Harvard notes, 1957–58),[9] we give a down-to-earth definition of differentials, which is especially suitable for computation. Later we shall show that, in the case of interest to us, namely, smooth varieties, they agree with the Kähler differentials.

Let V be a variety (possibly incomplete, singular) of dimension n over an algebraically closed field k. We consider derivations of $k(V)/k$, i.e. maps $D\colon k(V) \to k(V)$ satisfying

$$D(x+y) = Dx + Dy, \quad D(xy) = xDy + yDx, \quad Dc = 0, \quad x, y \in k(V),\ c \in k.$$

The derivations of $k(V)/k$ become a $k(V)$-vector space of dimension n with the definition

$$(\alpha D)(x) = \alpha(Dx), \quad \alpha, x \in k(V).$$

For example, if $V = \mathbb{P}^n$, then $k(V) = k(X_1, \dots, X_n)$, and $\left\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right\}$ is a basis for the space of derivations.

THEOREM 11.20. *Let $\{X_1, \dots, X_n\}$ be a transcendence basis for $k(V)/k$. Then $k(V)/k$ is separable if and only if every derivation of $k(X_1, \dots, X_n)/k$ extends uniquely to $k(V)$.*

PROOF. Let D be a derivation of $k(X_1, \dots, X_n)/k$, and let $y \in k(V)$. Then

$$f(y, X_1, \dots, X_n) = 0$$

for some polynomial f, and

$$\frac{\partial f}{\partial Y} Dy + \sum \frac{\partial f}{\partial X_i} DX_i = 0.$$

If $\frac{\partial f}{\partial Y} \neq 0$, then this equation defines Dy uniquely. □

Recall that a transcendence basis $\{X_1, \dots, X_n\}$ for $k(V)/k$ is ***separating*** if $k(V)$ is separable over $k(X_1, \dots, X_n)$. Any transcendence basis $\{X_1, \dots, X_n\}$ such that the derivations $\partial/\partial X_i$ extend to $k(V)$ is separating (and then the $\partial/\partial X_i$ form a basis for the space of derivations). Such transcendence bases exists (*Field Theory*, 9.27).

The ***differential* 1*-forms*** of $k(V)/k$ are the elements of the $k(V)$-dual of the space of derivations. Hence, if $X_1, \dots, X_n$ form a separating transcendence basis of F/k, then $dX_1, \dots, dX_n$ (where $dX_i(D) = D(X_i)$) form a basis for the differential 1-forms, and so every differential 1-form has a unique expression $\sum \alpha_i dX_i$, $\alpha_i \in k(V)$. If $X_1', \dots, X_n'$ is second such basis, then

$$\sum \alpha_i dX_i = \sum_{i,j} \alpha_i \frac{\partial X_i}{\partial X_j'} dX_j'$$

We let $\mathcal{D}_V$ denote the $k(V)$-space 1-forms. Then $\bigwedge^p \mathcal{D}_V$ is the space of p-forms. Relative to a separating transcendence basis $\{X_1, \dots, X_n\}$, such a form can be written uniquely as

$$\omega = \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \cdots i_p} dX_{i_1} \cdots dX_{i_p}, \quad \alpha_{i_1 \cdots i_p} \in k(V).$$

[9]Zariski, Oscar. An Introduction to the theory of algebraic surfaces (notes of a course at Harvard, 1957-58). Lecture Notes in Mathematics, No. 83 Springer, 1969.

We are especially interested in the space of n-forms. This is a one-dimensional space over $k(V)$, with basis $dX_1dX_2\cdots dX_n$ for any separating transcendence basis $\{X_1,\dots,X_n\}$. Note that if $\{X_1',\dots,X_n'\}$ is a second such transcendence basis

$$dX_1\cdots dX_n=\frac{\partial(X_1,\dots,X_n)}{\partial(X_1',\dots,X_n')}dX_1'\cdots dX_n'.$$

LEMMA 11.21. *Let P be a smooth point of V, and let $X_1,\dots,X_n$ be a system of local parameters at P. Then*

(a) *$X_1,\dots,X_n$ is a separating transcendence basis for F/k;*

(b) $\frac{\partial}{\partial X_i}\mathcal{O}_P\subset\mathcal{O}_P,\quad \frac{\partial}{\partial X_i}\mathfrak{m}^{r+1}\subset\mathfrak{m}^r.$

PROOF. We shall use that $f\in\mathcal{O}_P$ can be written

$$f=(\text{polynomial in the }X_i\text{with coefficients in }k)+f_0,\quad f_0\in\mathfrak{m}^{r+1}$$

(algebraic Taylor's formula).

(a) It suffices to show that a k-derivation D of $k(V)/k$ is zero if it is zero on $X_1,\dots,X_n$ (for then $dX_1,\dots,dX_n$ is a basis for the space of 1-forms; hence $\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_1}$ is a basis for the k-derivations of $k(V)/k$, and so $X_1,\dots X_n$ form a separating transcendence basis for $k(V)/k$).

I claim that

$$D'\mathcal{O}_P\subset\mathcal{O}_P.$$

for some multiple $D'=gD$ of D with $g\in\mathcal{O}_P$. To see this, note that

$$D(f/h)=\frac{hDf-fDh}{h^2},\tag{3}$$

and so, if $\mathcal{O}_P$ is a localization of $k[f_1,\dots,f_m]$, then it suffices to take g to be the product of the denominators of the Df_i. Now

$$D'\mathfrak{m}^{r+1}\subset\mathfrak{m}^r$$

because

$$D'(f_1\cdots f_{r+1})=D'(f_1)f_2\cdots f_{r+1}+\cdots+f_1\cdots f_rD'(f_{r+1}).\tag{4}$$

Let $f\in\mathcal{O}_P$. We are given that $D'X_i=0$ all i, and so

$$D'f=D'f_0\in\mathfrak{m}^r.$$

As this is true for all r, Krull's intersection theorem (1.8) shows that $D'f=0$. Now $D=g^{-1}D'$ is zero on $\mathcal{O}_P$, and this implies that it is zero on $k(V)$ by (3).

(b) Let $D=\frac{\partial}{\partial X_i}$. Choose a g as in (a), so that $(gD)\mathcal{O}_P\subset\mathcal{O}_P$. Let $f\in\mathcal{O}_P$; then

$$(gD)X_j\in(g)\quad(\text{all }j)\ \Rightarrow\ (gD)f\in(g)+\mathfrak{m}^r\quad(\text{all }r)\ \Rightarrow\ Df\in\mathcal{O}_P.$$

Now (4) shows that $\frac{\partial}{\partial X_i}\mathfrak{m}^{r+1}\subset\mathfrak{m}^r$. □

We now assume that V is smooth.

A differential 1-form ω is ***holomorphic*** at P if, when expressed in terms of a system of local parameters X_i at P,

$$\omega = \sum \alpha_i dX_i,$$

the $\alpha_i \in \mathcal{O}_P$. This definition is independent of the choice of the X_i because, for another choice X'_j of a system of local parameters, $\frac{\partial X_i}{\partial X'_i}$ and its inverse lie in $\mathcal{O}_P$, by the lemma. Similarly, a differential n-form $\omega = \alpha dX_1 \cdots dX_n$ is said to be ***holomorphic*** at P if $\alpha \in \mathcal{O}_P$. Again this is independent of the choice of the X_i because the Jacobian $\frac{\partial(X_1,\ldots,X_n)}{\partial(X'_1,\ldots,X'_n)}$ and its inverse lie in $\mathcal{O}_P$.

The sheaf Ω_V^p of holomorphic p-forms on V is defined by setting

$$\Gamma(U, \Omega_V^p) = \{p\text{-forms } \omega \mid \omega \text{ holomorphic at all points of } U\}$$

for all open U in V. For example

$$\begin{aligned}
\Omega^1 &= \text{“cotangent sheaf”} \\
\Omega^n &= \text{“canonical sheaf”, is invertible} \\
&= \mathcal{O}_V(K) \text{ where } K \text{ is a canonical divisor.}
\end{aligned}$$

PROPOSITION 11.22. *The sheaf Ω_V^p is locally free of rank*

$$\mathrm{rank}(\Omega^p) = \binom{n}{p};$$

in particular, it is coherent.

We need the following basic fact (5.52):

11.23. *Let $f_1, \ldots, f_n$ be a system of local parameters at a smooth point P. Then there is an open neighbourhood U of P such that $f_1, \ldots, f_n$ are represented by pairs $(\tilde{f}_1, U), \ldots, (\tilde{f}_n, U)$ and the map $(\tilde{f}_1, \ldots, \tilde{f}_n)\colon U \to \mathbb{A}^n$ is étale.*

In particular, $\tilde{f}_1 - \tilde{f}_1(Q), \ldots, \tilde{f}_n - \tilde{f}_n(Q)$ is a system of local parameters at Q for all points $Q \in U$. It follows that the family $d\tilde{f}_1, \ldots, d\tilde{f}_n$ is a basis for $\Omega_V^1|U$, which is therefore free of rank n. Similarly,

$$d\tilde{f}_{i_1} \cdots d\tilde{f}_{i_p}, \quad i_1 < \cdots < i_p,$$

form a basis for $\Omega_V^p|U$.

Now assume that V is complete. We define

$$h^{p,q} = \dim_k H^q(V, \Omega^p).$$

Thus, $h^{p,0} \overset{\text{def}}{=} \dim_k H^0(V, \Omega^P)$ is number of independent holomorphic p-forms on V. The integer

$$h^{n,0} = \dim_k H^0(V, \Omega^n)$$

is called the ***geometric genus*** of V and is denoted p_g.

In characteristic 0, the theory of harmonic integrals shows that

$$h^{p,q} = h^{q,p}$$
$$h^{p,q} = h^{n-p,n-q}$$

and $h^{1,0}$ is the irregularity of V.

In characteristic $\neq 0$, then Mumford[10], showed that there exists a surface with

$$h^{1,0} \neq h^{0,1}.$$

(See also Igusa 1955). Serre duality implies that $h^{0,q} = h^{n,n-q}$ in all characteristics.

The canonical class

The sheaf Ω_V^n can also be described as $\mathcal{O}_V(K)$, where K is the divisor of an n-form ω, as we now explain.

Let ω be an n-form on V (smooth, not necessarily complete), and let P be a (closed) point on V. We want to define the divisor (ω) of ω. Let Z be an irreducible closed subvariety of V of codimension 1, and suppose that

$$\omega = \alpha dX_1 \dots dX_n,$$

where the X_i are a system of local parameters at some nonsingular point in Z. Set

$$\mathrm{ord}_Z(\omega) = \mathrm{ord}_Z(\alpha).$$

We first check that this is independent of the choice of $X_1, \dots, X_n$.We have

$$\frac{\partial}{\partial X_i}\mathcal{O}_Z \subset \mathcal{O}_Z$$

(write $\beta \in \mathcal{O}_Z$ as $\beta = f/g$, $f, g \in \mathcal{O}_P$, $g|Z \neq 0$, and note that $\frac{\partial f}{\partial X_i}, \frac{\partial g}{\partial X_i} \in \mathcal{O}_P$). Therefore,

$$\begin{aligned}\omega &= \alpha dX_1 \cdots dX_n \\ &= \alpha J dX_1' \cdots dX_n'\end{aligned}$$

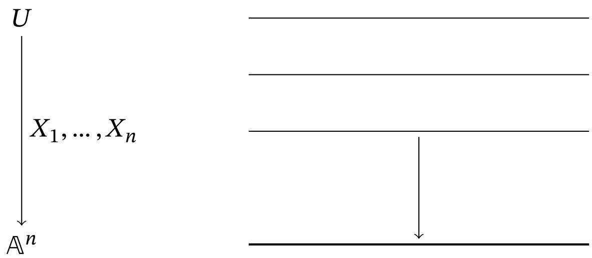
with $J = \frac{\partial(X_1,\dots,X_n)}{\partial(X_1',\dots,X_n')}$ a unit in $\mathcal{O}_Z$, and so $\mathrm{ord}_Z(\alpha) = \mathrm{ord}_Z(\alpha J)$.

Secondly, one checks that it is independent of the choice of P.

Thirdly, one check that the sum $(\omega) = \sum_Z \mathrm{ord}_Z(\omega)Z$ is finite, i.e., $\mathrm{ord}_Z(\omega) \neq 0$ for only finitely many Z. For this, we use the following fact:

> If $X_1, \dots, X_n$ is a separating transcendence basis for $k(V)/k$, then there exists an open subset U of V such that (a) $X_1, \dots, X_n$ are all holomorphic on U, and (b) for all points P in U, the functions $X_i - X_i(P)$ form a system of local parameters at P.

[10]Amer. J. Math. 83 (1961), 339–342; 84 (1962), 642–648.



Here $X_1, \ldots, X_n$ are a system of local parameters at all unramified points (the X_i define a regular map to $\mathbb{A}^n$ on an open subset U of V, which is étale over an open subset of U because it is at the generic point; cf. 11.23). On U, we can write $\omega = \alpha dX_1 \cdots dX_n$, where $dX_i = d(X_i - X_i(P))$. The only Z that enter into (ω) are (a) the Z in (α) (finite number), and (b) the Z that occur in $V \smallsetminus U$ (finite number).

Let $K = (\omega)$. Replacing ω with $f\omega$, $f \in k(V)^\times$, replaces K with a linearly equivalent divisor. The class of K is said to be ***canonical***.

Fix a ω_0. Then any other n-form ω on an open subset U of V can be written in the form $\omega = f\omega_0$, and the map $\omega \mapsto f$ defines an isomorphism

$$\Omega^n \to \mathcal{O}(K).$$

EXAMPLE 11.24. Consider $V = \mathbb{P}^n$. Let

$T_0, \ldots, T_n$ be homogeneous coordinates

$X_i = T_i/T_0$

$Y_1 = T_0/T_1$ and $Y_i = T_i/T_1, \quad i \geq 2.$

Then $X_1 = 1/Y_1, X_i = Y_i/Y_1, i \geq 2$. Thus

$$\begin{aligned} dX_1 \cdots dX_n \leftrightarrow & \left(-\frac{dY_1}{Y_1^2}\right) \cdot \left(\frac{Y_1 dY_2 - Y_2 dY_1}{Y_1^2}\right) \cdots \\ &= \frac{-dY_1 dY_2 \cdots dY_n \cdot Y_1^{n-1}}{Y_1^{2n}} \qquad dY_1 dY_1 \cdots = 0 \\ &= \frac{1}{Y_1^{n+1}} dY_1 \cdots dY_n. \end{aligned}$$

Note that $Y_1 = 0$ is the hyperplane at infinity, H_∞, and so

$$(dX_1 \cdots dX_n) = -(n+1)H_\infty.$$

In fact,

$$\begin{aligned} K_{\mathbb{P}^n} &= -(n+1)H, \quad H \text{ any hyperplane.} \\ p_g &= \dim H^0(V, \Omega^p) = 0 \text{ (because } K < 0). \end{aligned}$$

EXAMPLE 11.25. Let V be the product of two complete smooth curves,

$$V = C_1 \times C_2.$$

I claim that

$$K_{C_1\times C_2} = K_{C_1}\times C_2 + C_1 \times K_{C_2}.$$

Let ω_1 (resp. ω_2) be a 1-form on C_1 (resp. C_2); then $\omega_1\wedge\omega_2$ is a 2-form on $C_1\times C_2$, and

$$(\omega_1\wedge\omega_2) = (\omega_1)\times C_2 + C_1\times(\omega_2),$$

and so

$$\Omega^2_{C_1\times C_2} = \Omega^1_{C_1}\otimes\Omega^1_{C_2}.$$

By the Künneth formula,[11]

$$H^0(\Omega^2_{C_1\times C_2}) = H^0(\Omega^1_{C_1})\otimes H^0(\Omega^1_{C_2}),$$

and so

$$p_g(C_1\times C_2) = p_g(C_1)\times p_g(C_2).$$

(In fact, we didn't use that the C_i have dimension 1).

The residue map and the adjunction formula

Now let V be a smooth variety over k, and let Z be a smooth closed subvariety of codimension 1. We wish to relate K_V to K_Z, by showing that there is an exact sequence

$$0\longrightarrow\Omega^n_V\longrightarrow\Omega^n_V(Z)\xrightarrow{\text{residue}}\Omega^{n-1}_Z\longrightarrow 0. \tag{5}$$

Here $\Omega^n_V(Z)=\Omega^n_V\otimes\mathcal{O}(Z)$ ="n-forms with at worst a simple pole on Z".

First we must define the (Poincaré) residue map. Let P be a point in Z, and let $z=0$ be a local equation for Z near P. Write $\mathcal{O}_{V,P}\to\mathcal{O}_{Z,P}=\mathcal{O}_{V,P}/(z)$ as $f\mapsto\bar f:\ \mathcal{O}_P\to\bar{\mathcal{O}}_P$. Let $X_1,\dots,X_{n-1}\in\mathcal{O}_P$ be such that $\bar X_1,\dots,\bar X_{n-1}$ are a system of local parameters at P on Z; then $X_1,\dots,X_{n-1},z$ are a system of local parameters for P on V. Define the residue of the n-form $\omega\geq -Z$ at P as follows: by assumption, $z\omega$ is a holomorphic n-form at P, say, $z\omega = f\,dX_1\cdots dX_{n-1}dZ$ with $f\in\mathcal{O}_P$; then

$$\mathrm{res}_P(\omega)=\bar f\,d\bar X_1\cdots d\bar X_{n-1}.$$

LEMMA 11.26. *This definition is independent of the choice of X_i and z.*

PROOF. Suppose that $z',X'_1,\dots,X'_n$ are used instead. Note that $\left(\overline{\frac{\partial}{\partial X'_i}}\right)$ is a well-defined derivation on $\bar{\mathcal{O}}$. Moreover, $\left(\overline{\frac{\partial y}{\partial X'_i}}\right)=\frac{\partial\bar y}{\partial\bar x_i}$ for $y=X_1,\dots,X_{n-1}$, and therefore equality. holds for all $y\in\mathcal{O}$. Note also that $z=\varepsilon z'$ with ε a unit in $\mathcal{O}_P$ because $(z)=(z')$ as ideals in $\mathcal{O}_P$ (by definition). We have

$$\begin{cases} z\omega = f\,dX_1\cdots dX_{n-1}dz\\ z'\omega = f'\,dX'_1\cdots dX'_{n-1}dz'.\end{cases}$$

[11] Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on algebraic varieties V and W. Define a coherent sheaf on $V\times W$ by setting $\mathcal{F}\boxtimes\mathcal{G}=\mathcal{F}\otimes_{\mathcal{O}_V}\mathcal{O}_{V\times W}\otimes_{\mathcal{O}_W}\mathcal{G}$. Then

$$H^*(V\times W,\mathcal{F}\boxtimes\mathcal{G})\simeq H^*(V,\mathcal{F})\otimes H^*(W,\mathcal{G}).$$

See Serre, GACC, 1959, VII, n°17.

Therefore,

$$\begin{cases} z\omega = fJdX_1' \cdots dX_{n-1}'dz' \\ z'\omega = (f/\varepsilon)JdX_1' \cdots dX_{n-1}'dz' \end{cases}$$

and so

$$\begin{aligned} f' &= (f/\varepsilon)J \\ \bar{f}' &= (\bar{f}/\bar{\varepsilon})\bar{J}. \end{aligned}$$

We calculate $\bar{J}$. First

$$J = \begin{pmatrix} \frac{\partial X_1}{\partial X_1'} & \frac{\partial X_1}{\partial X_2'} & \cdots & * \\ \vdots & & & \vdots \\ \vdots & & & * \\ \frac{\partial Z}{\partial X_1'} & \cdots & \cdots & \frac{\partial z}{\partial z'} \end{pmatrix}$$

Now

$$\frac{\partial z}{\partial z'} = \varepsilon + z'\frac{\partial \varepsilon}{\partial z'}, \quad (\frac{\partial \varepsilon}{\partial z'} \in \mathcal{O}_p),$$

therefore

$$\begin{aligned} &\overline{\left(\frac{\partial z}{\partial z'}\right)} = \bar{\varepsilon}, \quad \bar{z}' = 0 \\ &\overline{\left(\frac{\partial z}{\partial x_j'}\right)} = \frac{\partial \bar{z}}{\partial \bar{x}_j} = 0 \quad (\text{as } \bar{z} = 0). \end{aligned}$$

We conclude that

$$\begin{aligned} \bar{J} &= \begin{pmatrix} \frac{\partial \bar{X}_1}{\partial \bar{X}_1'} & \cdots & \frac{\partial \bar{X}_1}{\partial \bar{X}_{n-1}'} & * \\ \vdots & & \vdots & \vdots \\ \frac{\partial \bar{X}_{n-1}}{\partial \bar{X}_1'} & \cdots & \frac{\partial \bar{X}_{n-1}}{\partial \bar{X}_{n-1}'} & * \\ 0 & 0 & 0 & \varepsilon \end{pmatrix} \\ &= \frac{\partial(\bar{X}_1 \cdots \bar{X}_{n-1})}{\partial(\bar{X}_1' \cdots \bar{X}_{n-1}')}\bar{\varepsilon}. \end{aligned}$$

Therefore,

$$\bar{f}' = \bar{f}\frac{\partial(\bar{X}_1 \cdots \bar{X}_{n-1})}{\partial(\bar{X}_1' \cdots \bar{X}_{n-1}')}$$

and so

$$\bar{f}d\bar{X}_1 \cdots d\bar{X}_n = \bar{f}'d\bar{X}_1' \cdots d\bar{X}_n'$$

as required.

At this point, it is easy to check the exactness of the sequence. □

THEOREM 11.27 (ADJUNCTION FORMULA). *Let Z be a smooth curve on a smooth surface V. Then*

$$K_Z = (K_V + Z) \cdot Z.$$

PROOF. We write the residue sequence in the form

$$0 \to \mathcal{O}_V(K_V) \to \mathcal{O}_V(K_V + Z) \to \mathcal{O}_Z(K_Z) \to 0$$

Tensoring the following exact sequence (which is the definition of $\mathcal{O}_Z$)

$$0 \to \mathcal{O}(-Z) \to \mathcal{O} \to \mathcal{O}_Z \to 0$$

with $\mathcal{O}(K_V + Z)$ gives an exact sequence

$$0 \to \mathcal{O}(K_V) \to \mathcal{O}(K_V + Z) \to \mathcal{O}_Z(K_V + Z \cdot Z) \to 0$$

(recall (11.11) that $\mathcal{O}_Z \otimes \mathcal{O}(D) = \mathcal{O}_Z(D \cdot Z)$). On comparing the two sequences, we find that

$$\mathcal{O}_Z(K_V + Z \cdot Z) \simeq \mathcal{O}_Z(K_Z)$$

and hence that

$$K_V + Z \cdot Z \sim K_Z.$$

□

ASIDE 11.28. The theorem holds for any smooth closed subvariety Z of dimension $n-1$ in a smooth variety V of dimension n. We have to define $D \cdot Z$ for D a divisor on V. For this, write $D = (f)$ locally near P, and define $D \cdot Z$ locally to be $(f|Z)$. Now the equality $\mathcal{O}_Z \otimes \mathcal{O}(D) = \mathcal{O}_Z(D \cdot Z)$ holds in general (with the same proof).

EXAMPLE 11.29. Let C be a smooth curve on a smooth complete surface V. From the theory of curves,

$$\deg(K_C) = 2g(C) - 2.$$

From the adjunction formula

$$\deg(K_C) = ((K_V + C) \cdot C).$$

Therefore

$$g(C) = \frac{1}{2}(K_V + C) \cdot C + 1.$$

For a curve C (possibly singular) on a surface V, define

$$p_a(C) = \frac{1}{2}(K_V + C \cdot C) + 1.$$

This is the "virtual arithmetic genus of C on V". Note that it is an invariant of C ***on*** V.

THEOREM 11.30. *$p_a(C) \geq g(C)$, and equality holds if and only if C is smooth.*

PROOF. Zariski, Harvard notes, 11.3. □

EXAMPLE 11.31. Let C be a smooth curve of degree n in $\mathbb{P}^2$. Then $K_V = -3H$ by 11.24 and $C \sim nH$, where H is a hyperplane in $\mathbb{P}^2$. From the adjunction formula,

$$\begin{aligned} K_C &= (-3H + C) \cdot C \\ &= (n-3)H \cdot C \\ &= \text{hypersurface section of } C \text{ of degree } n-3 \end{aligned}$$

and so

$$\deg(K_C) = (n-3)n.$$

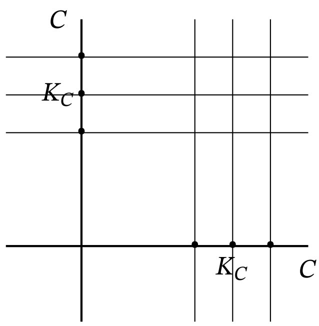
From the formula $\deg(K_C) = 2g(C) - 2$, we deduce that

$$g(C) = \frac{1}{2}(n-3)n + 1 = \frac{(n-1)(n-2)}{2}.$$

For example, a smooth cubic curve C in $\mathbb{P}^2$ has genus 1 and canonical class $K_C = 0$.

EXAMPLE 11.32. Let $V = C \times C$ with C a smooth curve of genus g. Then (see 11.25),

$$K_V = K_C \times C + C \times K_C$$



The diagonal Δ is a smooth curve on $C \times C$, isomorphic to C, and so $K_\Delta = K_C$ when we identify the two curves. We have

$$\begin{aligned} K_\Delta &= (K_C \times C + C \times K_C + \Delta) \cdot \Delta \quad \text{(adjunction formula)} \\ &= K_\Delta + K_\Delta + \Delta \cdot \Delta \quad (\text{because } P \times C \cdot \Delta = (P,P)\text{``=''}P). \end{aligned}$$

Therefore

$$\Delta \cdot \Delta = -K_\Delta.$$

(Weil 1945[12] uses this as the definition of $K_\Delta = K_C$). On taking degrees, we find that

$$(\Delta \cdot \Delta) = 2 - 2g.$$

Topologically (over $\mathbb{C}$),

$$\begin{aligned} (\Delta \cdot \Delta) &= \text{Euler characteristic } b_0 - b_1 + b_2 \text{ of } C \\ &= 1 - 2g + 1. \end{aligned}$$

EXAMPLE 11.33. Consider $\mathbb{P}^2$ with a point P "blown up" to a line. Thus, we have a surface V and a regular birational map

$$\pi : V \longrightarrow \mathbb{P}^2$$

such that the restriction of π to $V \smallsetminus C \to \mathbb{P}^2 \smallsetminus P$ is an isomorphism. Here $C \approx \mathbb{P}^1$ is the curve $\pi^{-1}(P)$, which is the set of "directions" through P. Let ω be a 2-form on $\mathbb{P}^2$. Its

[12] Weil, André, Sur les courbes algébriques et les variétés qui s'en déduisent. Publications de l'Institut de Mathématiques de l'Université de Strasbourg, 7 (1945).

divisor $(\omega) = -3H$, and $\pi^*\omega$ is a 2-form on V. Choose a hyperplane (i.e., a line) H in $\mathbb{P}^2$ not containing P, and let $H' = \pi^{-1}H$. Then

$$K_V = (\pi^*\omega) = -3H' + nC, \text{ some } n.$$

Take degrees in the adjunction formula, we find that

$$\begin{aligned}(K_C) &= ((K_V + C) \cdot C) \\ -2 &= \left(\left(-3H' + (n+1)C\right) \cdot C\right) \\ &= (n+1)(C^2) \quad \text{(note that } H' \cdot C = 0 \text{ because } H \not\ni P).\end{aligned}$$

But $C \cdot C = -1$ (see 11.16), and so

$$-2 = -(n+1)$$

and $n = 1$. We have shown that

$$K_V = -3H' + C.$$

(More generally, when blowing up a point on a smooth surface,

$$K_V = \pi^*(\text{canonical class}) + C.)$$

EXAMPLE 11.34. Let V be a smooth surface of degree n in $\mathbb{P}^3$. Then

$$\begin{aligned}K_{\mathbb{P}^3} &= -4H \\ K_V &= (-4H + V) \cdot V,\end{aligned}$$

where H is a hyperplane in $\mathbb{P}^3$. We have $V \sim nH$, and so

$$K_V = \text{hypersurface section of degree } n-4.$$

$n = 4$, quartic surface, $K = 0$ ($K3$ surfaces, Kummer, Kähler, Kodaira).
$n = 3$, cubic.
$n = 2$, quadric surface

$$\begin{aligned}K &= -2 \text{ (hyperplane section)} \\ V &= \mathbb{P}^1 \times \mathbb{P}^1 \text{ if } V \text{ smooth} \\ K_V &= K \times \mathbb{P}^1 + \mathbb{P}^1 \times K.\end{aligned}$$

Comparison with Kähler differentials

Let A be a k-algebra, and let M be an A-module. Recall (from §5) that a k-derivation is a k-linear map $D\colon A \to M$ satisfying Leibniz's rule:

$$D(fg) = f \circ Dg + g \circ Df, \quad \text{all } f, g \in A.$$

A pair $(\Omega_{A/k}, d)$ comprising an A-module $\Omega_{A/k}$ and a k-derivation $d\colon A \to \Omega_{A/k}$ is the ***(Kähler) module of differential one-forms*** for A over k if it has the following universal property: for any k-derivation $D\colon A \to M$, there is a unique A-linear map $\alpha\colon \Omega_{A/k} \to M$ such that $D = \alpha \circ d$,

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega_{A/k} \\ & \searrow^{D} & \downarrow{\scriptstyle \alpha} \\ & & M. \end{array}$$

Thus,

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_{A\text{-linear}}(\Omega_{A/k}, M).$$

For any multiplicative subset S of A,

$$S^{-1}\Omega_{A/k} \simeq \Omega_{S^{-1}A/k}. \tag{6}$$

Now let V be an irreducible algebraic variety over k. The sheaf of Kähler differentials on V is the coherent $\mathcal{O}_V$-module $\Omega_{V/k}$ such that

$$\Gamma(U, \Omega_{V/k}) = \Omega_{\mathcal{O}(U)/k}$$

for all open affines U. From (6), we deduce that

$$\Gamma(U, \Omega_{V/k}) = \{\omega \in \Omega_{k(V)/k} \mid \omega \in \Omega_{\mathcal{O}_p/k} \text{ for all } P \in U\}.$$

Now assume that V is nonsingular. Then

$$\Omega_{k(V)/k} \simeq \Omega^1_{k(V)/k}$$

since both are (by definition) the $k(V)$-linear dual of $\mathrm{Der}_k(k(V), k(V))$. Similarly,

$$(\Omega_{V/k})_P \overset{(6)}{\simeq} \Omega_{\mathcal{O}_P/k} \simeq \Omega^1_{\mathcal{O}_P/k} \overset{\text{def}}{=} (\Omega^1_{V/k})_P.$$

Therefore $\Omega_{V/k}$ and $\Omega^1_{V/k}$ are equal as subsheaves of the constant sheaf $\Omega^1_{k(V)/k}$ (and the arguments in the text show that both are locally free of rank $\dim V$).

c. The Riemann-Roch theorem

Let V be a smooth projective surface over k, and let D be a divisor on V. Define

$$\begin{aligned} h^i(D) &= \dim_k H^i(V, \mathcal{O}(D)) \\ \chi_V(D) &= \sum (-1)^i h^i(D) \\ \chi_V(\mathcal{O}_V) &= \chi_V = \text{“Euler characteristic” of } V \\ &= p_a(V) + 1, \quad p_a(V) = \text{arithmetic genus of } V. \end{aligned}$$

For example, for a surface,

$$\begin{aligned} \chi(D) &= h^0(D) - h^1(D) + h^2(D) \\ p_a(V) &= -h^1(\mathcal{O}_V) + h^2(\mathcal{O}_V). \end{aligned}$$

The weak Riemann-Roch theorem

Recall that, for a curve C, the weak Riemann-Roch theorem (i.e., Riemann's theorem) says that

$$\chi(D) = 1 - g + \deg(D),$$

where $1 - g = \chi_C$. This holds for $D = 0$ by definition, and it can be proved for a general D by noting that adding a point P to a divisor D adds 1 to both sides. Indeed, on tensoring

$$0 \to \mathcal{O}(-P) \to \mathcal{O} \to \mathcal{O}_P \to 0$$

with $\mathcal{O}(D+P)$, we obtain an exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to \mathcal{O}(D+P) \otimes \mathcal{O}_P \to 0;$$

but $\mathcal{O}(D+P) \otimes \mathcal{O}_P = \mathcal{O}_P$ (i.e., it is the sheaf supported on the point P having fibre k), and so

$$\chi(D+P) - \chi(D) = \chi(\mathcal{O}_P) = 1.$$

The proof of the similar result for surfaces is more complicated because curves on surfaces are more complicated than points in curves.

THEOREM 11.35 (WEAK RIEMANN-ROCH FOR SURFACES). *Let D be a divisor on a smooth projective surface. Then*

$$\chi(D) = \chi_V + \frac{1}{2}(D \cdot D - K).$$

PROOF. If $D = 0$, then the statement is true by definition. Thus, to prove the theorem it suffices to show that the statement is true for $D + C$ if and only if it is true for D, where C is any curve on D. We prove this first for a smooth curve C.

On tensoring

$$0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0$$

with $\mathcal{O}(D+C)$, we obtain an exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+C) \to \mathcal{O}_C(D+C \cdot C) \to 0$$

(we used (11.11) to replace $\mathcal{O}(D+C) \otimes \mathcal{O}_C$ with $\mathcal{O}_C(D+C \cdot C)$). Hence

$$\begin{aligned}
\chi(D+C) - \chi(D) &= \chi_C(D+C \cdot C) \\
&= \chi_C + (D+C \cdot C) \quad \text{by R-R for a curve} \\
&= 1 - g(C) + (D+C \cdot C) \\
&= -\frac{1}{2}(K+C \cdot C) + (D+C \cdot C).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{2}(D+C \cdot D+C-K) - \frac{1}{2}(D \cdot D-K) &= \frac{1}{2}(D+C \cdot C+C \cdot D-K) \\
&= \frac{1}{2}(D+C \cdot C+D+C \cdot C-C \cdot C+K) \\
&= \chi(D+C) - \chi(D)
\end{aligned}$$

This completes the proof of the theorem when D is a sum of smooth curves.

To complete the proof of the theorem, we need to use a weak form of Bertini's theorem (proved later 11.45).

Let V be a smooth projective variety, and let D be a divisor on V. Then there exists a hypersurface section C of V such that $D+C$ is linearly equivalent to a sum $\sum E_i$ with the E_i smooth and irreducible and such that $E_i \cap E_j = \emptyset$ for $i \neq j$.

We can now prove the weak Riemann-Roch theorem for an arbitrary divisor D. According to the Bertini theorem,

$$D \sim \sum E_i - C = \sum E_i - \sum C_j$$

with the E_i and C_j all smooth. This completes the proof. □

REMARK 11.36. We use this to compute the Hilbert polynomial of a surface:

$$\begin{aligned}
\chi(\mathcal{O}(n)) &\overset{\text{def}}{=} \chi(C_n) \quad (C_n = nH, H \text{ a hyperplane section}) \\
&= \chi_V + \frac{1}{2}(C_n \cdot C_n - K) \quad \text{(weak R-R)} \\
&= \frac{(H \cdot H)}{2} n^2 - \frac{1}{2} K \cdot H + \chi_V.
\end{aligned}$$

In general, for a smooth projective variety V of dimension d and degree δ,

$$\chi(\mathcal{O}(n)) = \frac{\delta}{d!} n^d + \text{terms of lower degree in } n.$$

Recall that the degree of V is the intersection number of V with a linear subvariety of $\mathbb{P}^m$ of codimension 2. This agrees with our calculation. For n sufficiently positive, $h^i(C_n) = 0$ for $i \geq 1$, and so

$$\chi(\mathcal{O}(n)) = h^0(C_n) = \dim_k(k_{\text{hom}}[V]_n) \text{ for } n \geq 0,$$

where $k_{\text{hom}}[V]$ is the homogeneous coordinate ring of V and $k_{\text{hom}}[V]_n$ is the part of degreee n.

Serre duality

THEOREM 11.37 (SERRE DUALITY). *Let V be a smooth projective variety of dimension n over k, let $\mathcal{E}$ be a locally free sheaf on V, and let Ω^n be the sheaf of holomorphic n-forms on V. Then the cup-product pairing*

$$H^p(V, \mathcal{E}) \times H^{n-p}(V, \mathcal{E}^\vee \otimes \Omega^n) \to H^n(V, \Omega^n) \simeq k$$

is a perfect pairing (i.e., it identifies each of H^p and H^{n-p} with the dual of the other).

Therefore,

$$\dim_k H^p(V, \mathcal{E}) = \dim_k H^{n-p}(V, \mathcal{E}^\vee \otimes \Omega^n).$$

When $\mathcal{E} = \mathcal{O}(D)$, this becomes

$$h^p(D) = h^{n-p}(K - D).$$

In particular, for V a surface,

$$\begin{aligned}
h^2(D) &= h^0(K - D) \\
h^1(D) &= h^1(K - D).
\end{aligned}$$

Hence, Riemann-Roch for a surface becomes

$$h^0(D) - h^1(D) + h^0(K - D) = \chi_V + \frac{1}{2}(D \cdot D - K).$$

The general theorem is proved in Grothendieck's Séminaire Bourbaki 149. See also Grothendieck's talk at the 1958 ICM (Edinburgh), and the last section of Serre's FAC. We will prove only that $h^n(D) = h^0(K - D)$.[13] This will suffice to complete the proof of the Riemann-Roch theorem for a surface.

LEMMA 11.38. *Let V be a complete smooth surface embedded in some projective space $\mathbb{P}^N$, and let C_n be a section of V by a hypersurface of degree n (i.e., a section of $\mathcal{O}(n)$). Let D be a divisor on V. Then*

$$D > 0 \Rightarrow D \cdot C_n > 0 \text{ on } C_n,$$

and so $(D \cdot C_n) > 0$. For example, $(C_n^2) > 0$.

PROOF. Choose C_n so that its intersection with all the components of D are defined. Then

$$\operatorname{supp}(D \cdot C_n) = \operatorname{supp}(D) \cap \operatorname{supp}(C_n) \neq \emptyset.$$

Indeed, $C_n = H_n \cap V$ with H_n a hypersurface of degree n in $\mathbb{P}^N$, and

$$\operatorname{supp}(D) \cap \operatorname{supp}(C_n) = \operatorname{supp}(D) \cap H_n$$

is nonempty because otherwise each component of D would be contained an affine variety $\mathbb{P}^N \smallsetminus H_n$. □

LEMMA 11.39. *Let V be a smooth projective surface over k.*

(a) $H^2(V, \Omega_V^2) \simeq k$;

(b) $H^2(V, \Omega_V^2(D)) = 0$ *if* $D > 0$.

PROOF. (a) From the residue sequence

$$0 \to \Omega_V^2 \to \Omega_V^2(C_n) \to \Omega_{C_n}^1 \to 0,$$

we get an exact cohomology sequence

$$H^1(V, \Omega_V^2(n)) \to H^1(C, \Omega_{C_n}^1) \xrightarrow{\alpha} H^2(V, \Omega_V^2) \to H^2(V, \Omega_V^2(n)) \to 0.$$

But by Theorem B in cohomology (p. 6), the two end terms are zero for large n, and so α is an isomorphism for large n. But

$$H^1(C, \Omega_{C_n}^1) \simeq k$$

by the theory of curves.

(b) On tensoring the residue sequence with $\mathcal{O}(D)$, we get an exact sequence

$$0 \to \Omega_V^2(D) \to \Omega_V^2(D + C_n) \to \Omega_{C_n}^1 \otimes \mathcal{O}(D) \to 0.$$

But $\Omega_{C_n}^1 = \mathcal{O}(K_{C_n})$, and so $\Omega_{C_n}^1 \otimes \mathcal{O}(D) = \mathcal{O}_{C_n}(K_{C_n} + D \cdot C_n)$. Now the cohomology sequence of this sequence and Theorem B show that

$$\begin{aligned} H^2(V, \Omega_V^2(D)) &\simeq H^1(C_n, \mathcal{O}_{C_n}(K_{C_n} + D \cdot C_n)) \\ &\simeq H^0(C_n, -D \cdot C_n)^\vee \quad \text{(Serre duality on a curve)} \\ &= 0 \qquad \text{as } -D \cdot C_n < 0. \end{aligned}$$

□

[13] Following Zariski, Bulletin of the AMS, 1958 — see the end of the report.

Let V be a smooth projective surface.

DEFINITION 11.40. We let $J(D)$ denote the dual k-vector space to $H^2(\mathcal{O}(D))$.

If $D' \geq D$, then there is an exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D') \to \mathcal{L} \to 0,$$

and the sheaf $\mathcal{L}$ has support in $\mathrm{supp}(D' \setminus D)$ (let P be a point of V; if there exists an f such that $(f) \geq -D'$ but $(f) \ngeq -D$ at P, then clearly $P \in \mathrm{supp}(D' - D)$). Thus, $\mathrm{supp}(\mathcal{L})$ is of dimension ≤ 1, and so $H^2(V, \mathcal{L}) = 0$, and we have an exact sequence

$$H^2(\mathcal{O}(D)) \to H^2(\mathcal{O}(D')) \to 0.$$

Dually, this becomes

$$J(D) \leftarrow J(D') \leftarrow 0.$$

In other words,

$$D' \geq D \implies J(D') \subset J(D).$$

DEFINITION 11.41. The "Weil differential 2-forms" are defined to be

$$J = \varinjlim J(D) = \bigcup_D J(D)$$

(limit as D becomes more negative).

Let $f \in k(V)$ and $\sigma \in H^2(D)$, then

$$f(\sigma) \in H^2(D - (f)) \quad \text{(roughly, } f\sigma \geq -D + (f)\text{)}.$$

Let $\lambda \in J(D)$ and $f \in k(V)$; then $f\lambda \in J(D + (f))$ is the element

$$H^2(D + (f)) \xrightarrow{f} H^2(D) \xrightarrow{\lambda} k(V),$$

i.e.,

$$(f\lambda)(\sigma) = \lambda(f(\sigma)) \quad \text{for all } \sigma \in H^2(D + (f)).$$

In this way, J becomes a $k(V)$-vector space.

THEOREM 11.42. *J is a one-dimensional $k(V)$-vector space.*

PROOF. Clearly, $\dim_{k(V)} J \geq 1$, because $J(K)$ is dual to $H^2(V, \Omega^2) = k \neq 0$. It remains to show that $\dim_{k(V)} J \leq 1$. Suppose not, and let λ_1 and λ_2 be two $k(V)$-independent elements of J, which we may suppose lie in $J(D)$ for some $D < 0$. The map

$$\Phi\colon H^0(V, \mathcal{O}(n)) \oplus H^0(V, \mathcal{O}(n)) \to J, \quad f, g \mapsto f\lambda_1 + g\lambda_2$$

is injective, because λ_1, λ_2 are independent over $k(V)$, and it takes values in $J(D - C_n)$, because

$$f \geq -C_n,\ \lambda \in J(D) \implies f\lambda \in J(D + (f)) \subset J(D - C_n)).$$

We obtain a contradiction by estimating the dimensions of the k-vector spaces for $n \gg 0$.

For the left hand side,

$$\dim H^0(V, \mathcal{O}(n)) \sim \frac{(C_1^2)}{2} n^2 + \text{lower powers.}$$

— see (11.36), or use the weak Riemann-Roch theorem,

$$h^0(C_n) \sim \frac{1}{2}(C_n^2 - K) \sim \frac{1}{2}(n^2 C_1^2)$$

Therefore

$$\dim_k (H^0(V, \mathcal{O}(n)) \oplus H^0(V, \mathcal{O}(n))) \sim n^2 (C_1^2).$$

On the other hand,

$$\dim_k (J(D - C_n)) = \dim H^2(\mathcal{O}(D - C_n)).$$

From the cohomology sequence of

$$0 \to \mathcal{O}(D - C_n) \to \mathcal{O}(D) \to \mathcal{O}_{C_n}(D \cdot C_n) \to 0$$

we find that

$$h^2(D - C_n) \sim h^1_{C_n}(D \cdot C_n),$$

and

$$h^1_{C_n}(D \cdot C_n) = 1 - g(C_n) + \deg(D \cdot C_n)$$

by the Riemann-Roch theorem for curves (note that $h^0_{C_n}(D \cdot C_n) = 0$ as $D < 0$). Now

$$\deg(D \cdot C_n) = n \deg(D \cdot C_1)$$

and so we can ignore it. On the other hand,

$$\begin{aligned} g(C_n) &= \frac{1}{2}(C_n \cdot C_n + K) + 1 \quad \text{(adjunction formula)} \\ &\sim \frac{1}{2} n^2 (C_1^2) \quad \text{(we may suppose } C_n \text{ smooth by Bertini).} \end{aligned}$$

Hence

$$\dim_k (J(D - C_n)) \sim \frac{n^2}{2}(C_1^2),$$

which is a contradiction (because $(C_1^2) > 0$). □

Now consider the pairing

$$\begin{array}{cccc} H^0(\Omega^2(-D)) & \times & H^2(\mathcal{O}(D)) & \to & H^2(D) \simeq k. \\ \omega > D & & \begin{array}{c} \sigma = \{f_{ijk}\} \\ (f_{ijk}) > -D \end{array} & \mapsto & \begin{array}{c} \omega\sigma = \{\omega f_{ijk}\} \\ \omega f_{ijk} \text{ holomorphic} \end{array} \end{array}$$

The pairing gives a commutative diagram (all $D' > D$):

$$\begin{array}{ccccc} H^0(\Omega^2(-D)) & \xrightarrow{\varphi_D} & J(D) & & H^2(\mathcal{O}(D)) \\ \uparrow & & \uparrow & \text{dual to} & \downarrow \\ H^0(\Omega^2(-D')) & \xrightarrow{\varphi_{D'}} & J(D') & & H^2(\mathcal{O}(D')). \end{array}$$

On passing to the direct limit, we get a map

$$\varphi\colon \Omega^2_{k(V)/k} \to J,$$

where $\Omega^2_{k(V)/k}$ is the space of 2-forms of $k(V)/k$. This is a nonzero $k(V)$-linear map, and both spaces are one-dimensional $k(V)$-vector spaces, and so φ is an isomorphism.

It follows that each map φ_D is injective, but we shall in fact show that φ_D is also surjective. For this it suffices to show:

$$\{\omega \text{ a 2-form, } \omega \geq D \text{ and } \varphi(\omega) \in J(D'), \quad D' > D\} \Rightarrow \{\omega \geq D'\}.$$

It suffices prove this with $D' = D + E$ with E irreducible, because, if $D' = D + E_1 + \cdots + E_r$, then

$$\begin{aligned} &\omega \geq D, \quad \varphi(\omega) \in J(D') \subset J(D + E_1) \\ \Rightarrow\ &\omega \geq D + E_1, \quad \varphi(\omega) \in J(D') \subset J(D + E_1 + E_2) \\ \Rightarrow\ &\text{etc.} \end{aligned}$$

Note that $\varphi(\omega) \in J(D') \iff \varphi(\omega)$ vanishes on $\mathrm{Ker}(\psi)$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(D) & \longrightarrow & \mathcal{O}(D+E) & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^2 & \longrightarrow & \Omega^2(E) & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \end{array}$$

in which the vertical arrows at left are multiplication by ω, and the dashed arrow on the cokernels is induced by the other two. If E is smooth, then $\mathcal{M}_1 = \mathcal{O}_E(D + E \cdot E)$ and $\mathcal{M}_2 = \Omega^1_E$. In any event, $\mathrm{supp}(\mathcal{M}_1) \subset E$. Consider

$$\begin{array}{ccccccc} H^1(E, \mathcal{M}_1) & \xrightarrow{\rho} & H^2(\mathcal{O}(D)) & \xrightarrow{\psi} & H^2(\mathcal{O}(D+E)) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi(\omega)} & & \downarrow & & \\ H^1(E, \mathcal{M}_2) & \xrightarrow{a} & H^2(\Omega^2) & \longrightarrow & H^2(\Omega^2(E)) & \longrightarrow & 0 \\ & & \| & & \| & & \\ & & k & & 0 & & \end{array}$$

The zeros at right are because E is one-dimensional, and $H^2(\Omega^2(E)) = 0$ because $E > 0$ (see an earlier lemma). Now $\varphi(\omega)$ vanishes on $\mathrm{Ker}(\psi) = \mathrm{Im}(\rho)$, and so $\varphi(\alpha) \circ \rho = 0$; therefore $a \circ \alpha = 0$, and so α is not surjective.

LEMMA 11.43. *If $\omega \ngeq D + E$, then α must be surjective.*

PROOF. Let $(\omega) = D + D''$, $D'' > 0$, $\mathrm{supp}(E) \not\subset \mathrm{supp}(D'')$.

In order to show that $\alpha\colon H^1(E, \mathcal{M}_1) \to H^1(E, \mathcal{M}_2)$ is surjective, it suffices to show that $\mathcal{M}_1 \to \mathcal{M}_2$ is surjective except at a finite number of points. To see this, consider,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & \mathcal{M}_1 & & \longrightarrow & & \mathcal{M}_2 & \longrightarrow & K_2 & \longrightarrow & 0. \\ & & & & & \searrow & & \nearrow & & & & & \\ & & & & & & C & & & & & & \\ & & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & & 0 & & & & \end{array}$$

If $\mathcal{M}_1 \to \mathcal{M}_2$ is surjective for almost all points, then $\operatorname{supp}(K_2)$ has dimensional zero, and the cohomology sequences of the two short exact sequences give surjections

$$H^1(C) \twoheadrightarrow H^1(\mathcal{M}_2)$$
$$H^1(\mathcal{M}_1) \twoheadrightarrow H^1(C).$$

It remains to show that $\mathcal{M}_1 \to \mathcal{M}_2$ is almost surjective. It suffices to show that

$$\mathcal{O}(D+E)_P \to \Omega(E)_P$$

is surjective for almost all $P \in E$. But

$$\mathcal{O}(D)_P \to \Omega_P^2$$

is surjective whenever $(\omega)_P = D_P$, i.e., whenever $P \notin D$. The same is true after tensoring with $\mathcal{O}(E)$: the map $\mathcal{O}(D+E)_P \to \Omega^2(E)_P$ is surjective whenever $P \notin D''$, and $D'' \cap E$ has only finitely many points. This completes the proof. □

This type of argument works only to show that $H^0(\mathcal{O}(D))$ and $H^n(\mathcal{O}(K-D))$ are dual.

The full Riemann-Roch theorem

THEOREM 11.44 (RIEMANN-ROCH). *Let D be a divisor on a smooth projective surface. Then*

$$h^0(D) - h^1(D) + h^0(K-D) = \chi_V + \frac{1}{2}(D \cdot D - K).$$

PROOF. Combine the weak version of the theorem (11.35) with the equality $h^2(D) = h^0(K-D)$ proved in the last subsection. □

Compare this with the original version (of the early Italian geometers):

$$h^0(D) + h^0(K-D) \geq \chi_V + \frac{1}{2}(D \cdot D - K)$$

The difference of the two sides was denoted $i(D)$, and called the "superabundance" of D. The early Italian geometers could prove that $i(D + C_n) = 0$ for $n \gg 0$.

When $D = 0$, $\mathcal{O}(D) = \mathcal{O}_V$, and

$$1 - h^1(\mathcal{O}) + h^0(K) = \chi_V$$
$$h^0(K) = \dim_k H^0(\Omega^2) = p_g \text{ (geometric genus)}$$
$$p_a = \chi_V - 1 \quad \text{(arithmetic genus).}$$
$$p_g - p_a = h^1(\mathcal{O}).$$

In characteristic zero *only,*

$$h^1(\Omega^0) = h^0(\Omega^1).$$
$$h^0(\Omega^1) = \text{number independent holomorphic 1-forms}$$
$$= q \text{ (irregularity} = \dim \text{Picard variety).}$$

Note that

$$p_g - p_a = q.$$

Grothendieck's Riemann-Roch theorem:

$$\chi_V = \frac{1}{12}((K^2) + [c_2]).$$

Here $[c_2]$ is the degree of the second Chern class of V. In characteristic zero, $[c_2]$ is the topological Euler characteristic, $\sum(-1)^i \dim_{\mathbb{Q}} H^i(V, \mathbb{Q})$ (any good cohomology theory and coefficient field $\mathbb{Q}$). This was proved in characteristic zero by Max Noether and in characteristic p by Grothendieck. (It would be nice to have a proof of the theorem just for surfaces.)

Proof of the weak weak Bertini theorem

We still have to prove:

THEOREM 11.45. *Let D be a divisor on a smooth projective variety V. Then $\mathcal{O}(D+C_n)$ has smooth zeros of sections, i.e., $|D + C_n|$ has smooth irreducible members, i.e., there exists a smooth irreducible (positive) divisor*

$$E \sim D + C_n, \quad n \gg 0.$$

(The weak Bertini theorem says that $D + C_m \sim E$, E smooth and irreducible (if $\dim(V) > 1$). Hence V has smooth hyperplane sections of all degrees.)

In fact, almost all members of $D + C_n$ have these properties.

We first show that it suffices to prove the theorem in the case of a hypersurface section. Use sections of $\mathcal{O}(D + C_n)$ to define a rational map

$$\varphi_{\mathcal{L}} : V \dashrightarrow \mathbb{P}^n.$$

Suppose that $f_0, \dots, f_N$ form a basis for the sections. Map

$$x \mapsto (f_0(x) : \ \cdots \ : f_N(x)).$$

We want to show that if $n \gg 0$, then $\varphi_{\mathcal{L}}$ is an isomorphism into $\mathbb{P}^n$ (then the hyperplane sections of $\varphi_{\mathcal{L}}(V)$ correspond to the divisors in $|D + C_n|$).

(An invertible sheaf $\mathcal{L}$ on V such that $\varphi_{\mathcal{L}} : V \dashrightarrow \mathbb{P}^n$ is an isomorphism from V onto its image is said to be "very ample" . Thus, we want to show that $\mathcal{L} \otimes \mathcal{O}(n)$ is very ample for $n >> 0$.)

STEP 1: $n \gg 0$, then $|D + C_n|$ has no base points, so $\varphi_{\mathcal{L}}$ is a morphism.

[Theorem: $\mathcal{O}(D + C_n)$ is generated by its global sections $\iff$ $|D + C_n|$ has no base points, i.e., there is no $x \in V$ common to all positive divisors $\sim D + C_n$; therefore $\varphi_{\mathcal{L}}$ is a rational map; elementary ZMT + normality implies that it is a morphism.]

STEP 2: For $n \geq n_0 + 1$, $|D + C_n|$ separates points (i.e., given x, y, there exists an E in the linear system such that $x \in E$ and $y \notin E$).

PROOF. Take $E' \sim D + C_n$, E' not containing x. Take a C_1 containing y but not x. Now $E' + C_1$ contains y but not x. □

Therefore, φ is bijective on points, $V(k) \simeq (\varphi V)(k)$. This is not enough to show that φ is an isomorphism, e.g.,

$$t \mapsto (t^2, t^3) : \mathbb{A}^1 \to \{Y^2 = X^3\}.$$

STEP 3: If $n \geq n_0 + 1$, then $|D + C_n|$ separates directions.

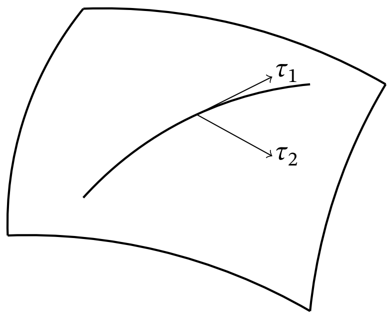
Consider the tangent space to V at x,

$$T_{x,V} = \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k), \quad \mathfrak{m} \subset \mathcal{O}_x.$$

To separate directions means that

$$T_{x,V} \to T_{\varphi x, \varphi V}$$

is injective. In other words, given two tangent vector τ_1 and τ_2 at x,there exists a divisor in $|D + C_n|$ through x such that its tangent space at x contains τ_1 but not τ_2:



If φ separates directions at x, then it is an isomorphism at x, because then $\mathfrak{m}' \to \mathfrak{m}$ is surjective, so $\mathcal{O}'_{\varphi x} \to \mathcal{O}_x$ is surjective, and it is certainly injective as φ is a surjective morphism.

Take $E' \sim D + C_n$ not through x. Take C_1 throught τ_1 and not through τ_2;now $E' + C_1$ works.

Second part of the proof: V is smooth projective $\subset \mathbb{P}^N$; then a general hyperplane section is smooth (and irreducible).

We can take V to be affine, say, $V = V(f_1, \dots, f_s)$ of dimension r in $\mathbb{A}^n$, with $f_i = f_i(X_1, \dots, X_n)$. Recall that the tangent space at x is defined by the system of linear equations

$$\sum_j \frac{\partial f_i}{\partial X_j}(X_j - x_j) = 0, \quad i = 1, \dots, r.$$

To say that V is smooth at x means that the tangent space has dimension $n - r$, or, equivalently, that the matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_x$ has rank $n - r$.

When is the hyperplane section defined by

$$h = \sum_j a_j(X_j - a_j) = 0$$

nonsingular at x? Its ideal is $(f_1, \dots, f_s, h)$, and we need

$$\mathrm{rank}\left(\frac{\partial f_i}{\partial X_j}, \frac{\partial h}{\partial X_j}\right) = n - r + 1,$$

i.e., the equation $h = 0$ should be independent from the equations defining the tangent space $T_{x,V}$ i.e., h is not zero on $T_{x,V}$.

We have to show that, in general, a hyperplane in $\mathbb{P}^n$ does not contain any tangent space to V.

Given $T_{x,V}$ of dimension r, how many hyperplanes H in $\mathbb{P}^n$ contain $T_{x,V}$? Note that

$$H \text{ contains } T_x \iff H \text{ contains } r+1 \text{ “independent” points of } T_x.$$

Heuristically, there are ∞^n hyperplanes in $\mathbb{P}^n$ (parametrized by the dual projective space), and so there are ∞^{n-r-1} hyperplanes containing T_x. But there are $\leq \infty^r$ tangent spaces T_x as x varies over V , i.e., a family of dimension $\leq r$. Hence the number of hyperplanes that do contain some T_x is $\infty^r + \infty^{n-r+1} \leq \infty^{n-1}$. Therefore, almost all hyperplanes in $\mathbb{P}^n$ do not conttain any $T_{x,V}$, and so give nonsingular sections. Irreducible? Generically they are irreducible (cf. Lang, IAG, p.213).

We now translate the Italian into English.[14] We are given V smooth of dimension r closed in $\mathbb{P}^n$. We want to show that “in general” a hyperplane of $\mathbb{P}^n$ does not contain any $T_{x,V}$. The hyperplanes $\sum a_j X_j = 0$ are parametrized by $\check{\mathbb{P}}^n = \{(a_0, \dots, a_n)\}$.

Claim: the hyperplanes containing some $T_{x,V}$ correspond to the points of $\check{\mathbb{P}}^n$ in some closed subset of dimension $\leq n-1$.

LEMMA 11.46. *The set of all hyperplanes containing a fixed L^r (linear space of dimension r) is represented in $\check{\mathbb{P}}^n$ by a linear space of dimension $n-r-1$.*

PROOF. A hyperplane H contains $L^r \iff H$ contains $r+1$ points that span L^r, say,

$$P_i = (x_0^i, \dots, x_n^i), \quad i = 1, \dots, r+1.$$

Thus,

$$H \supset \{P_1, \dots, P_{r+1}\} \iff \sum_j a_j x_j^i = 0, \quad i = 1, \dots, r+1.$$

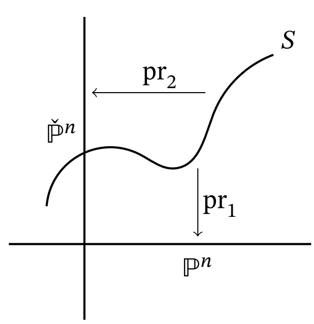
This is a system of equations in the variables a_j. The equations are independent, and so the solution space is a projective space of dimension $n-(r+1)$. □

LEMMA 11.47. *The set of tangent spaces $T_{x,V}$ is a “space of dimension $\leq r$”.*

PROOF. Let $S = \{(x, H) \mid x \in V, \quad H \supset T_{x,V}\} \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$.A point P of S has coordinates $(X_0, \dots, X_n, a_0, \dots, a_n)$, satisfying certain polynomial equations: the equations of V expressing that $x \in V$ and the vanishing of various subdeterminants of

$$\begin{pmatrix} a_0, \dots, a_n \\ \frac{\partial F_i}{\partial X_j} \end{pmatrix}$$

expressing that $\sum a_i X_i$ contains the tangent plane to V at x.



[14] According to Severi, modern algebraic geometers have feet of lead.

We have $\mathrm{pr}_1(S) = V$ and $\mathrm{pr}_1^{-1}(x)$ is a linear space of dimension $n-r-1$ (previous lemma). Hence $\dim(S) = n-1$. On the other hand, $\mathrm{pr}_2(S)$ is the set of hyperplanes containing some $T_{x,V}$. An $a \in \mathrm{pr}_2(S)$ came from a pair (X, a). It is a closed set because $\mathbb{P}^n$ is complete. As $\dim(S) = n-1$, we have $\dim \mathrm{pr}_2(S) \leq n-1$. □

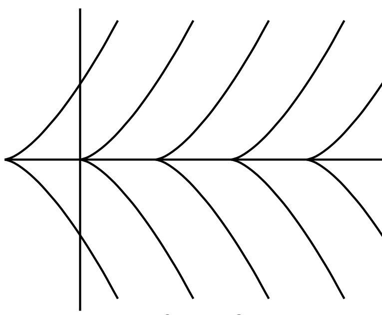
Contrast: Early Italian geometers, systems of objects ∞^r. Modern geometers, objects are parametrized by a projective space of dimension r, i.e., can be algebraized to such a space.

Let L be a linear system of positive divisors on V^r, projective. Assume that the general member is irreducible. Then it has no singularieties outside

(a) the singularities of V,

(b) base points of the system (characteristic zero).

EXAMPLE 11.48.



$Y^2 = X^3$

This family doesn't contradict Bertini because it is not a linear system (it is parametrized by t, $Y^2 = (X-t)^3$).

d. Proof of the Riemann hypothesis for curves

Let V be a smooth projective surface over an algebraically closed field k.

SOME ALGEBRA

Let V be a finite dimensional vector space over $\mathbb{Q}$ (or $\mathbb{R}$), and let $Q(x,y)$ be a symmetric bilinear form with values in $\mathbb{Q}$ (or $\mathbb{R}$),

$$Q(x,y) = \sum a_{ij} x_i x_j, \quad a_{ij} = a_{ji}.$$

The associated quadratic form is

$$Q(x,x) = \sum a_{ij} x_i x_j,$$

and we can recover $Q(x,y)$ from this by

$$Q(x,y) = \frac{1}{2}(Q(x+y, x+y) - Q(x,x) - Q(y,y)).$$

Such a form is diagonalizable by a rational orthogonal transformation of V,

$$Q(x,x) = d_1 x_1^2 + d_2 x_2^2 + \cdots + d_n x_n^2.$$

Law of inertia: the numbers of $d_i > 0$, $d_i < 0$, and $d_i = 0$ are invariants of Q. The index is the number of $d_i > 0$, and the signature is the family $(+,\cdots,+,-,\cdots,-,0,\cdots,0)$.

For example,

(a) index= n if and only if all $d_i > 0$, i.e., Q positive definite;

(b) index= 0 if and only if Q is negative semi-definite (Q is negative definite if all $d_i < 0$);

(c) index= 1 if and only if there exists an $x \in V$ such that (1) $Q(x,x) > 0$ and (2) $Q(y,y) \leq 0$ for all $y \in \langle x\rangle^{\perp}$.

(Proof of (c). $\Rightarrow$: $Q(\ ,\ ) = d_1x_1^2 - d_2x_2^2 - \cdots - d_nx_n^2$, $d_i \geq 0$, $d_1 > 0$. Take $x = (1,0,\dots,0)$. Then $y \in \langle x\rangle^{\perp}$ means that $y = (0,x_2,\dots,x_n)$. $\Leftarrow$: Choose coordinates so that $x = (1,0,\dots,0)$; choose the y_i to be an orthogonal basis of the complement $\langle x\rangle^{\perp}$ of $\langle x\rangle$. Then $d_1 = Q(x,x) > 0$ and $d_i = Q(y_i,y_i) \leq 0$ for $i = 2,\dots,n$.

DIVISORS

A divisor on V is a formal sum $D = \sum n_iC_i$ with $n_i \in \mathbb{Z}$ and C_i an irreducible curve on V. We say that D is positive, denoted $D \geq 0$, if all the $n_i \geq 0$. Every $f \in k(V)^{\times}$ has an associated divisor (f) of zeros and poles — these are the principal divisors. Two divisors D and D' are said to be linearly equivalent if

$$D' = D + (f) \text{ some } f \in k(V)^{\times}.$$

For a divisor D, let

$$L(D) = \{f \in k(V) \mid (f) + D \geq 0\}.$$

Then $L(C)$ is a finite-dimensional vector space over k, whose dimension we denote by $l(D)$. The map $g \mapsto gf$ is an isomorphism $L(D) \to L(D-(f))$, and so $l(D)$ depends only on the linear equivalence class of D.

ELEMENTARY INTERSECTION THEORY

Because V is smooth, a curve C on V has a local equation at every closed point P of V, i.e., there exists an f such that

$$C = (f) + \text{components not passing through } P.$$

If C and C' are distinct irreducible curves on V, then their intersection number at $P \in C \cap C'$ is

$$(C \cdot C')_P \stackrel{\text{def}}{=} \dim_k(\mathcal{O}_{V,P}/(f,f')),$$

where f and f' are local equations for C and C' at P, and their (global) intersection number is

$$(C \cdot C') = \sum_{P \in C \cap C'} (C \cdot C')_P.$$

This definition extends by linearity to pairs of divisors D, D' without common components. Now observe that $((f) \cdot C) = 0$, because it equals the degree of the divisor of $f|C$ on C, and so $(D \cdot D')$ depends only on the linear equivalence classes of D and D'. This allows us to define $(D \cdot D')$ for all pairs D, D' by replacing D with a linearly equivalent divisor that intersects D' properly. In particular, $(D^2) \stackrel{\text{def}}{=} (D \cdot D)$ is defined.

THE RIEMANN-ROCH THEOREM

Recall that the Riemann-Roch theorem for a curve C states that, for all divisors D on C,

$$l(D) - l(K_C - D) = \deg(D) + 1 - g,$$

where g is the genus of C and K_C is a canonical divisor (so $\deg K_C = 2g-2$ and $l(K_C) = g$). Better, in terms of cohomology,

$$\chi(\mathcal{O}(D)) = \deg(D) + \chi(\mathcal{O})$$
$$h^1(D) = h^0(K_C - D).$$

The Riemann-Roch theorem for a surface V states that, for all divisors D on V,

$$l(D) - \sup(D) + l(K_V - D) = p_a + 1 + \frac{1}{2}(D \cdot D - K_V),$$

where K_V is a canonical divisor and

$$p_a = \chi(\mathcal{O}) - 1 \quad \text{(arithmetic genus)},$$
$$\sup(D) = \text{superabundance of } D\ (\geq 0, \text{ and } = 0 \text{ for some divisors}).$$

Better, in terms of cohomology,

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}_V) + \frac{1}{2}(D \cdot D - K)$$
$$h^2(D) = h^0(K - D),$$

and so

$$\sup(D) = h^1(D).$$

We shall also need the adjunction formula: let C be a curve on V; then

$$K_C = (K_V + C) \cdot C.$$

THE HODGE INDEX THEOREM

Embed V in $\mathbb{P}^n$. A hyperplane section of V is a divisor of the form $H = V \cap H'$ with H' a hyperplane in $\mathbb{P}^n$ not containing V. Any two hyperplane sections are linearly equivalent (obviously).

LEMMA 11.49. *For a divisor D and hyperplane section H,*

$$l(D) > 1 \Rightarrow (D \cdot H) > 0. \tag{7}$$

PROOF. The hypothesis implies that there exists a $D_1 > 0$ linearly equivalent to D. If the hyperplane H' is chosen not to contain a component of D_1, then the hyperplane section $H = V \cap H'$ intersects D_1 properly. Now $D_1 \cap H = D_1 \cap H'$, which is nonempty by dimension theory, and so $(D_1 \cdot H) > 0$. □

THEOREM 11.50 (HODGE INDEX THEOREM). *For a divisor D and hyperplane section H,*

$$(D \cdot H) = 0 \Rightarrow (D \cdot D) \leq 0.$$

PROOF. We begin with a remark: suppose that $l(D) > 0$, i.e., there exists an $f \neq 0$ such that $(f) + D \geq 0$; then, for a divisor D',

$$l(D + D') = l((D + (f)) + D') \geq l(D'). \tag{8}$$

We now prove the theorem. To prove the contrapositive, it suffices to show that

$$(D \cdot D) > 0 \Rightarrow l(mD) > 1 \text{ for some integer } m,$$

because then

$$(D \cdot H) = \frac{1}{m}(mD \cdot H) \neq 0$$

by (7) above. Hence, suppose that $(D \cdot D) > 0$. By the Riemann-Roch theorem

$$l(mD) + l(K_V - mD) \geq \frac{(D \cdot D)}{2} m^2 + \text{lower powers of } m.$$

Therefore, for a fixed $m_0 \geq 1$, we can find an $m > 0$ such that

$$l(mD) + l(K_V - mD) \geq m_0 + 1$$
$$l(-mD) + l(K_V + mD) \geq m_0 + 1.$$

If both $l(mD) \leq 1$ and $l(-mD) \leq 1$, then both $l(K_V - mD) \geq m_0$ and $l(K_V + mD) \geq m_0$, and so

$$l(2K_V) = l(K_V - mD + K_V + mD) \overset{(8)}{\geq} l(K_V + mD) \geq m_0.$$

As m_0 was arbitrary, this is impossible. □

Let Q be a symmetric bilinear form on a finite-dimensional vector space W over $\mathbb{Q}$ (or $\mathbb{R}$). There exists a basis for W such that $Q(x, x) = a_1 x_1^2 + \cdots + a_n x_n^2$. The number of $a_i > 0$ is called the *index* (of positivity) of Q — it is independent of the basis. There is the following (obvious) criterion: Q has index 1 if and only if there exists an $x \in V$ such that $Q(x, x) > 0$ and $Q(y, y) \leq 0$ for all $y \in \langle x \rangle^{\perp}$.

Now consider a surface V as before, and let $\mathrm{Pic}(V)$ denote the group of divisors on V modulo linear equivalence. We have a symmetric bi-additive intersection form

$$\mathrm{Pic}(V) \times \mathrm{Pic}(V) \to \mathbb{Z}.$$

On tensoring with $\mathbb{Q}$ and quotienting by the kernels, we get a nondegenerate intersection form[15]

$$N(V) \times N(V) \to \mathbb{Q}.$$

COROLLARY 11.51. *The intersection form on $N(V)$ has index* 1.

PROOF. Apply the theorem and the criterion just stated. □

COROLLARY 11.52. *Let D be a divisor on V such that $(D^2) > 0$. If $(D \cdot D') = 0$, then $(D'^2) \leq 0$.*

PROOF. The form is negative definite on $\langle D \rangle^{\perp}$. □

[15]Here $N(V)$ is the Néron-Severi group of V.

THE INEQUALITY OF CASTELNUOVO-SEVERI

Now take V to be the product of two curves, $V = C_1 \times C_2$. Identify C_1 and C_2 with the curves $C_1 \times \mathrm{pt}$ and $\mathrm{pt} \times C_2$ on V, and note that

$$C_1 \cdot C_1 = 0 = C_2 \cdot C_2$$
$$C_1 \cdot C_2 = 1 = C_2 \cdot C_1.$$

Let D be a divisor on $C_1 \times C_2$ and set $d_1 = D \cdot C_1$ and $d_2 = D \cdot C_2$.

THEOREM 11.53 (CASTELNUOVO-SEVERI INEQUALITY). *Let D be a divisor on V; then*

$$(D^2) \leq 2d_1 d_2. \tag{9}$$

PROOF. We have

$$(C_1 + C_2)^2 = 2 > 0$$
$$(D - d_2C_1 - d_1C_2) \cdot (C_1 + C_2) = 0.$$

Therefore, by the Hodge index theorem,

$$(D - d_2C_1 - d_1C_2)^2 \leq 0.$$

On expanding this out, we find that $D^2 \leq 2d_1d_2$. □

Define the equivalence defect (*difetto di equivalenza*) of a divisor D by

$$\mathrm{def}(D) = 2d_1d_2 - (D^2) \geq 0.$$

COROLLARY 11.54. *Let D, D′ be divisors on V; then*

$$\left|(D \cdot D') - d_1d_2' - d_2d_1'\right| \leq \left(\mathrm{def}(D)\mathrm{def}(D')\right)^{1/2}. \tag{10}$$

PROOF. Let $m, n \in \mathbb{Z}$. On expanding out

$$\mathrm{def}(mD + nD') \geq 0,$$

we find that

$$m^2\mathrm{def}(D) - 2mn\left((D \cdot D') - d_1d_2' - d_2d_1'\right) + n^2\mathrm{def}(D') \geq 0.$$

As this holds for all m, n, it implies (10). □

EXAMPLE 11.55. Let f be a nonconstant morphism $C_1 \to C_2$, and let g_i denote the genus of C_i. The graph of f is a divisor Γ_f on $C_1 \times C_2$ with $d_2 = 1$ and d_1 equal to the degree of f. Now

$$K_{\Gamma_f} = (K_V + \Gamma_f) \cdot \Gamma_f \qquad \text{(adjunction formula)}.$$

On using that $K_V = K_{C_1} \times C_2 + C_1 \times K_{C_2}$, and taking degrees, we find that

$$2g_1 - 2 = (\Gamma_f)^2 + (2g_1 - 2) \cdot 1 + (2g_2 - 2)\deg(f).$$

Hence

$$\mathrm{def}(\Gamma_f) = 2g_2 \deg(f). \tag{11}$$

PROOF OF THE RIEMANN HYPOTHESIS FOR CURVES

Let C_0 be a projective smooth curve over a finite field k_0, and let C be the curve obtained by extension of scalars to the algebraic closure k of k_0. Let π be the Frobenius endomorphism of C. Then (see (11)), $\operatorname{def}(\Delta) = 2g$ and $\operatorname{def}(\Gamma_\pi) = 2gq$, and so (see (10)),

$$\left|(\Delta \cdot \Gamma_\pi) - q - 1\right| \leq 2gq^{1/2}.$$

As

$$(\Delta \cdot \Gamma_\pi) = \text{number of points on } C \text{ rational over } k_0,$$

we obtain Riemann hypothesis for C_0.

ASIDE 11.56. Note that, except for the last few lines, the proof is purely geometric and takes place over an algebraically closed field.[16] This is typical: study of the Riemann hypothesis over finite fields suggests questions in algebraic geometry whose resolution proves the hypothesis. This proof suggested to Grothendieck what have become known as the "standard conjectures", which apply to all projective smooth algebraic varieties, and which have the Riemann hypothesis for the variety as an immediate consequence when the ground field is finite.

CORRESPONDENCES

A divisor D on a product $C_1 \times C_2$ of curves is said to have *valence zero* if it is linearly equivalent to a sum of divisors of the form $C_1 \times \mathrm{pt}$ and $\mathrm{pt} \times C_2$. The group of correspondences $\mathcal{C}(C_1, C_2)$ is the quotient of the group of divisors on $C_1 \times C_2$ by those of valence zero. When $C_1 = C_2 = C$, the composite of two divisors D_1 and D_2 is

$$D_1 \circ D_2 \stackrel{\text{def}}{=} p_{13*}(p_{12}^* D_1 \cdot p_{23}^* D_2)$$

where the p_{ij} are the projections $C \times C \times C \to C \times C$; in general, it is only defined up to linear equivalence. When $D \circ E$ is defined, we have

$$d_1(D \circ E) = d_1(D) d_1(E), \quad d_2(D \circ E) = d_2(D) d_2(E), \quad (D \cdot E) = (D \circ E', \Delta) \qquad (12)$$

where, as usual, E' is obtained from E by reversing the factors. Composition makes the group $\mathcal{C}(C, C)$ of correspondences on C into a ring $\mathcal{R}(C)$.

Following Weil, we define the "trace" of a correspondence D on C by

$$\sigma(D) = d_1(D) + d_2(D) - (D \cdot \Delta).$$

Applying (12), we find that

$$\begin{aligned} \sigma(D \circ D') &\stackrel{\text{def}}{=} d_1(D \circ D') + d_2(D \circ D') - ((D \circ D') \cdot \Delta) \\ &= d_1(D) d_2(D) + d_2(D) d_1(D) - (D^2) \\ &= \operatorname{def}(D). \end{aligned}$$

Thus Weil's inequality $\sigma(D \circ D') \geq 0$ is a restatement of (9).

References.

Zariski, Algebraic surfaces 1935. (Reprinted with appendices by S. S. Abhyankar, J. Lipman and D. Mumford. Preface to the appendices by Mumford. Classics in Mathematics. Springer, 1995).

[16] I once presented this proof in a lecture. At the end, a listener at the back triumphantly announced that I couldn't have proved the Riemann hypothesis because I had only ever worked over an algebraically closed field.