Chapter 17

Lefschetz Pencils (Outline)

In this chapter, we see how to fibre a variety over $\mathbb{P}^1$ in such a way that the fibres have only very simple singularities. This result sometimes allows one to prove theorems by induction on the dimension of the variety. For example, Lefschetz initiated this approach in order to study the cohomology of varieties over $\mathbb{C}$.

Throughout this chapter, k is an algebraically closed field.

a. Definition

A linear form $H = \sum_{i=0}^{m} a_i T_i$ defines a hyperplane in $\mathbb{P}^m$, and two linear forms define the same hyperplane if and only if one is a nonzero multiple of the other. Thus the hyperplanes in $\mathbb{P}^m$ form a projective space, called the ***dual projective space*** $\check{\mathbb{P}}^m$.

A line D in $\check{\mathbb{P}}^m$ is called a ***pencil*** of hyperplanes in $\mathbb{P}^m$. If H_0 and H_∞ are any two distinct hyperplanes in D, then the pencil consists of all hyperplanes of the form $\alpha H_0 + \beta H_\infty$ with $(\alpha : \beta) \in \mathbb{P}^1(k)$. If $P \in H_0 \cap H_\infty$, then it lies on every hyperplane in the pencil — the ***axis*** A of the pencil is defined to be the set of such P. Thus

$$A = H_0 \cap H_\infty = \bigcap_{t \in D} H_t.$$

The axis of the pencil is a linear subvariety of codimension 2 in $\mathbb{P}^m$, and the hyperplanes of the pencil are exactly those containing the axis. Through any point in $\mathbb{P}^m$ not on A, there passes exactly one hyperplane in the pencil. Thus, one should imagine the hyperplanes in the pencil as sweeping out $\mathbb{P}^m$ as they rotate about the axis.

Let V be a nonsingular projective variety of dimension $d \geq 2$, and embed V in some projective space $\mathbb{P}^m$. By the square of an embedding, we mean the composite of $V \hookrightarrow \mathbb{P}^m$ with the Veronese mapping (6.23)

$$(x_0 : \ldots : x_m) \mapsto (x_0^2 : \ldots : x_i x_j : \ldots : x_m^2) : \mathbb{P}^m \to \mathbb{P}^{\frac{(m+2)(m+1)}{2}}.$$

DEFINITION 17.1. A line D in $\check{\mathbb{P}}^m$ is said to be a ***Lefschetz pencil*** for $V \subset \mathbb{P}^m$ if

(a) the axis A of the pencil $(H_t)_{t \in D}$ cuts V transversally;

(b) the hyperplane sections $V_t \stackrel{\text{def}}{=} V \cap H_t$ of V are nonsingular for all t in some open dense subset U of D;

This is Chapter 17 of Algebraic Geometry by J.S. Milne. Last revised November 4, 2024

(c) for $t \notin U$, V_t has only a single singularity, and the singularity is an ordinary double point.

Condition (a) means that, for every point $P \in A \cap V$, $\mathrm{Tgt}_P(A) \cap \mathrm{Tgt}_P(V)$ has codimension 2 in $\mathrm{Tgt}_P(V)$, the tangent space to V at P.

Condition (b) means that, except for a finite number of t, H_t cuts V transversally, i.e., for every point $P \in H_t \cap V$, $\mathrm{Tgt}_P(H_t) \cap \mathrm{Tgt}_P(V)$ has codimension 1 in $\mathrm{Tgt}_P(V)$.

A point P on a variety V of dimension d is an ***ordinary double point*** if the tangent cone at P (see 10.69) is isomorphic to the subvariety of $\mathbb{A}^{d+1}$ defined by a nondegenerate quadratic form $Q(T_1, \ldots, T_{d+1})$, or, equivalently, if

$$\hat{\mathcal{O}}_{V,P} \approx k[[T_1, \ldots, T_{d+1}]]/(Q(T_1, \ldots, T_{d+1})).$$

THEOREM 17.2. *There exists a Lefschetz pencil for V (after possibly replacing the projective embedding of V by its square).*

PROOF. (Sketch). Let $W \subset V \times \check{\mathbb{P}}^m$ be the closed variety whose points are the pairs (x, H) such that H contains the tangent space to V at x. For example, if V has codimension 1 in $\mathbb{P}^m$, then $(x, H) \in Y$ if and only if H is the tangent space at x. In general,

$$(x, H) \in W \iff x \in H \text{ and } H \text{ does not cut } V \text{ transversally at } x.$$

The image of W in $\check{\mathbb{P}}^m$ under the projection $V \times \check{\mathbb{P}}^m \to \check{\mathbb{P}}^m$ is called the ***dual variety*** $\check{V}$ of V. The fibre of $W \to V$ over x consists of the hyperplanes containing the tangent space at x, and these hyperplanes form an irreducible subvariety of $\check{\mathbb{P}}^m$ of dimension $m - (\dim V + 1)$; it follows that W is irreducible, complete, and of dimension $m - 1$ (see 9.11) and that V is irreducible, complete, and of codimension ≥ 1 in $\check{\mathbb{P}}^m$ (unless $V = \mathbb{P}^m$, in which case it is empty). The map $\varphi\colon\ W \to \check{V}$ is unramified at (x, H) if and only if x is an ordinary double point on $V \cap H$ (see SGA 7, XVII 3.7[1]). Either φ is generically unramified, or it becomes so when the embedding is replaced by its square (so, instead of hyperplanes, we are working with quadric hypersurfaces) (ibid. 3.7). We may assume this, and then (ibid. 3.5), one can show that for $H \in \check{V} \setminus \check{V}_{\mathrm{sing}}$, $V \cap H$ has only a single singularity and the singularity is an ordinary double point. Here $\check{V}_{\mathrm{sing}}$ is the singular locus of $\check{V}$.

By Bertini's theorem (Hartshorne 1977, II 8.18) there exists a hyperplane H_0 such that $H_0 \cap V$ is irreducible and nonsingular. Since there is an $(m-1)$-dimensional space of lines through H_0, and at most an $(m-2)$-dimensional family will meet V_{sing}, we can choose H_∞ so that the line D joining H_0 and H_∞ does not meet $\check{V}_{\mathrm{sing}}$. Then D is a Lefschetz pencil for V. □

THEOREM 17.3. *Let $D = (H_t)$ be a Lefschetz pencil for V with axis $A = \bigcap H_t$. Then there exists a variety V^* and maps*

$$V \leftarrow V^* \xrightarrow{\pi} D.$$

such that:

[1]Groupes de monodromie en géométrie algébrique. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7). Dirigé par A. Grothendieck. Lecture Notes in Mathematics, Vol. 288, 340. Springer-Verlag, Berlin-New York, 1972, 1973.

(a) *the map* $V^* \to V$ *is the blowing up of* V *along* $A \cap V$;

(b) *the fibre of* $V^* \to D$ *over* t *is* $V_t = V \cap H_t$.

Moreover, π *is proper, flat, and has a section.*

PROOF. (Sketch) Through each point x of $V \setminus A \cap V$, there will be exactly one H_x in D. The map

$$\varphi\colon V \setminus A \cap V \to D,\, x \mapsto H_x,$$

is regular. Take the closure of its graph Γ_φ in $V \times D$; this will be the graph of π. □

REMARK 17.4. The singular V_t may be reducible. For example, if V is a quadric surface in $\mathbb{P}^3$, then V_t is curve of degree 2 in $\mathbb{P}^2$ for all t, and such a curve is singular if and only if it is reducible (look at the formula for the genus). However, if the embedding $V \hookrightarrow \mathbb{P}^m$ is replaced by its cube, this problem will never occur.

REFERENCES

The only modern reference I know of is SGA 7, Exposé XVII.