Autoduality of the Jacobian

Lectures by

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1. Autoduality

Let C be a projective smooth curve over a field k, with chosen rational point θ. The Jacobian of C is an abelian variety A defined over k, such that for any field extension K of k we get a 1 - 1 correspondence

$$\left\{\begin{matrix}\text{points of A with}\\ \text{values in K}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{K-rational divisor}\\ \text{classes of degree 0 on C}\end{matrix}\right\}$$

Alternatively, if Pic C is the Picard scheme of C, then Pic C = A $\oplus$ Z and A = $\mathrm{Pic}^o C$ = the connected component of Pic C.

Suppose that C has genus g. Then dim A = g. In fact, let $S^g C$ be the g-fold symmetric product of C. $S^g C$ is by definition the quotient of Cx...xC(g-factors) under the obvious action of the symmetric group on g-letters. It is a non-singular complete variety of dimension g. Using the chosen point θ on C, define a map

$f: S^g C \longrightarrow A$ by $f((P_1, \ldots, P_g)) =$
the divisor class of $(P_1 + P_2 + \ldots + P_g - g \cdot \theta)$. It follows from the Riemann-Roch theorem on C that f is a birational morphism.

If $i: C \longrightarrow S^g C$ is defined by $i(P) = (P, \theta, \ldots, \theta)$, then we get the usual map $t = f \circ i: C \longrightarrow A$, namely t(P) = divisor class of $(P - \theta)$.

Proposition: (A,t) is the Albanese variety of (C, θ). That is, if B is any abelian variety and $h: C \longrightarrow B$ is a morphism such that $h(\theta)$ = the origin of B; then there exists a unique morphism of abelian varieties $\tilde{h}$: $A \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} & B \\ C \arrow[ur, "h"] \arrow[r, "t"'] & A \arrow[u, "\tilde{h}"'] \end{tikzcd}$$

<u>Proof</u>: There is a morphism $S^g h: S^g C \longrightarrow B$ given by $(S^g h)(P_1, \ldots, P_g) = h(P_1) + \ldots + h(P_g)$ (addition on the abelian variety B). Since A and $S^g C$ are birationally equivalent, there is at least a rational map $\tilde{h}$: $A \dashrightarrow B$ making the diagram commute. Also $\tilde{h}$ (origin of A) = $h(\sigma)$=origin of B, and by the theory of abelian varieties (cf. Lang [5]) $\tilde{h}$ is a homomorphism of abelian varieties.

If A is any abelian variety, let $\hat{A}$ = Pic $^\circ$ A = {divisors algebraically equivalent to 0}/ {divisors linearly equivalent to 0}. Then Pic and $\wedge$ are contravariant functors.

<u>Theorem</u> (Autoduality for the Jacobian of a curve): Let A be the jacobian of C. The map t: $C \longrightarrow A$ induces a map $\hat{t}$: $\hat{A} = \mathrm{Pic}^{\circ}A \longrightarrow \mathrm{Pic}^{\circ}C = A$, and this map is an isomorphism.

<u>Sketch of proof</u>: One first shows that $\mathrm{Pic}^{\circ}A \cong \mathrm{Pic}^{\circ}S^g C$ ("Pic is a birational invariant.") Now there is a map Pic $S^g C \longrightarrow$ Pic C as always. Moreover, given an invertible sheaf L on C, it determines in a canonical way an invertible sheaf $S^g L$ on $S^g C$ having the property that its pull back to Cx...xC (g-fold product) is $p_1^* L \otimes \ldots \otimes p_q^* C$, where $p_i: C^g \longrightarrow C$ is the i-th projection. Thus there is a map "$L \longrightarrow S^g L$"

from Pic C ——> Pic S^gC. That is:

$$\begin{tikzcd} \mathrm{Pic}\, C \arrow[r] & \mathrm{Pic}\, S^gC \arrow[r] & \mathrm{Pic}\, C \\ \cup & \cup & \cup \\ A \arrow[r] & \mathrm{Pic}^o S^gC \arrow[r] & A \\ & \hat{A} \arrow[u, phantom, "\wr\wr"] \arrow[ur, "\hat{t}"'] & \end{tikzcd}$$

This shows that $\hat{A} = A \oplus \ker \hat{t}$. Since $\hat{A}$ is connected, it suffices to prove $\ker \hat{t}$ finite. For use here and later we invoke the following consequence of the theorem of the square:

Deep Result: Suppose L is an invertible sheaf on the abelian variety A. Consider the three maps AxA ——> A given by P_1 (first projection), P_2 (second projection), and S (sum). Then L is algebraically equivalent to 0 $\Longleftrightarrow$ $S^*L \approx P_1^*L \otimes P_2^*L$. (Barsotti [2]).

Now consider the commutative diagram

$$\begin{tikzcd} C\times\ldots\times C \arrow[r, "t\times\ldots\times t"] \arrow[d, "\pi"'] & A\times\ldots\times A \arrow[ddl, "\mathrm{sum}"] \\ S^gC \arrow[d, "f"] & \\ A & \end{tikzcd} \qquad \text{(g-fold products)}$$

Suppose L is an invertible sheaf on A algebraically equivalent to zero. Then it follows from the deep result, applied to the above diagram that L pulls back to $t_1^*(L) \otimes \ldots \otimes t_g^*(L)$ on Cx...xC, where $t_i = P_i \circ t$. Thus if $L \in \ker \hat{t}$; that is L pulls back trivially to C, then $\pi^* f^*(L)$ is trivial. Now the map

$C\times\ldots\times C \xrightarrow{\pi} S^gC$ is finite, and of order $g!$, and so it follows from a usual norm argument that if F is an invertible sheaf on S^gC such that π^*F is trivial, then $F^{\otimes g!}$ is trivial on S^gC. Therefore $(f^*L)^{\otimes g!}$ is trivial, which shows that $g!$ annihilates $\ker \hat{t}$, whence $\ker \hat{t}$ is actually zero.

2. Interpretation of $\hat{A}$ as Ext (Serre [10], Ch. VII).

Definition: A sequence $G \longrightarrow E \longrightarrow A$ of commutative algebraic groups over a field is an extension of A by G if $E \longrightarrow A$ is a surjection with scheme-theoretic kernel G.

Isomorphism classes of extensions of A by G form a group $\mathrm{Ext}^1(A,G)$ under the usual Baer multiplication. The zero of $\mathrm{Ext}^1(A,G)$ is the "split extension" $G\longrightarrow G\times A\longrightarrow A$.

Theorem: (Serre) Let $G = \mathbb{G}_m$, let A be an abelian variety, and let $G \longrightarrow E \longrightarrow A$ be an extension. Then the sheaf $\underline{E}$ of germs of sections $A \longrightarrow E$ is a principal fibre space with group G. Thus the mapping $E \longmapsto \underline{E}$ defines a mapping $f\colon \mathrm{Ext}^1(A,G) \longrightarrow H^1(A,\mathbb{G}_m) = \mathrm{Pic}A$. This map is injective with image $\hat{A}$.

Proof: (see Serre [10], Ch. VII no. 5, 15, and 16).

a) f is injective. If $G \longrightarrow E \longrightarrow A$ is an extension such that $\underline{E}$ is the trivial sheaf, then there is a regular global section $S\colon A \longrightarrow E$. The obstruction cocycle to s being a homomorphism lies in Maps $(A\times A,G)$, which consists of constants, since $A\times A$ is complete and G is affine. Therefore if we choose s such that $s(0_A)= 0_E$, S must be a homomorphism.

b) Image $f \subseteq \hat{A}$: Let S, P_1, P_2: $A \times A \longrightarrow A$ be sum, first projection, and second projection. Now $\mathrm{Ext}^1(G,A)$ is a biadditive functor: $\mathrm{Ext}^1(G, A\times A) = \mathrm{Ext}^1(G,A) \oplus \mathrm{Ext}^1(G,A)$ and $S^*E = P_1^*E + P_2 E$ in $\mathrm{Ext}^1(G,A)$. Therefore $S^*\underline{E} = p_1^*\underline{E} \otimes p_2^*\underline{E}$ for the sheaves. The "Deep Result" stated earlier shows that $\underline{E}$ is algebraically equivalent to zero, so $\underline{E} \in \hat{A}$.

c) Image $f = \hat{A}$. Let F be an invertible sheaf on A. Let E be the "principal fibre space for $\mathbb{G}_m$" gotten by removing the zero section from the line bundle associated to F. We must show that if F is algebraically equivalent to zero, then E can be given the structure of an algebraic group so that $\mathbb{G}_m \longrightarrow E \longrightarrow A$ is an extension.

Suppose F is algebraically equivalent to zero. As in part (b) we have $S^*F = p_1^*F \otimes p_2^*F$, so also $s^*E = p_1^*E \otimes p_2^*E$. Consider the projection diagram

$$\begin{array}{ccc} E \times E & \rightrightarrows & E \\ \downarrow\ \ \downarrow & & \downarrow \\ A \times A & \rightrightarrows & A \end{array}$$

we get a map $E\times E \longrightarrow p_1^*(E) \otimes p_2^*(E)$, and composing with $S^*(E) \longrightarrow E$ we get a map $E \times E \longrightarrow E$ which defines the required group structure. See Serre. sec. 15 for details.

The point of the last theorem is that we can consider the autoduality $A \cong \hat{A}$ as a duality between A and $\mathrm{Ext}^1(A, \mathbb{G}_m)$. The category of commutative group schemes of finite type over k is an abelian category, so $\mathrm{Ext}^i(A, G_m)$ can be defined for all i. If k is algebraically closed, then $\mathrm{Ext}^i(A, \mathbb{G}_m) = 0$ for $i > 1$. (See Oort [9], p.II.12-1). Using

(Same for char = 0)

these results and standard results on $\mathbb{G}_m$, we can make the following table:

Let $f: C \longrightarrow \operatorname{Spec}(k) = Y$ be the structure morphism. Then:

$H^0(C, \mathbb{G}_m) = k^*$; that is, $f_* \mathbb{G}_{m_C} = \mathbb{G}_{m_Y}$

$R^1 f_* \mathbb{G}_{m_C} = \operatorname{Pic} C = A \oplus \mathbb{Z}$

$R^q f_* \mathbb{G}_{m_C} = 0$ if $q > 1$.

The Exts of these groups into $\mathbb{G}_m$ are the following:

$\operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$; $\operatorname{Ext}^q(\mathbb{G}_m, \mathbb{G}_m) = 0$ if $q > 0$.

$\operatorname{Hom}(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m$; $\operatorname{Ext}^q(\mathbb{Z}, \mathbb{G}_m) = 0$ if $q > 0$.

$\operatorname{Hom}(A, \mathbb{G}_m) = 0$; $\operatorname{Ext}^1(A, \mathbb{G}_m) \cong A$; $\operatorname{Ext}^q(A, \mathbb{G}_m) = 0$ if $q > 1$.

In other words, the same terms appear in the various Exts as in the first variable. It is therefore reasonable to expect that all of these facts should be combined in an isomorphism, in a suitable derived category, ([illegible]) as follows (See Hartshorne [3] for derived categories):

I expect [illegible]

$$Rf_* \mathbb{G}_m \xrightarrow{\ \sim\ } \underline{R\operatorname{Hom}}^{\bullet}(Rf_* \mathbb{G}_m, \mathbb{G}_m^1).$$

($\mathbb{G}_m^1$ is the complex with $\mathbb{G}_m$ in dimension <u>one</u> and zeros elsewhere). Section 4 below makes this viewpoint more precise in a more general setting.

[illegible] Ext^q([illegible]

$\Uparrow$

Ext^p([illegible])

3. Ideles.

Let C be a complete smooth curve over k with function field $K = k(C)$.

Definition: The local idele group at a closed point P of C is $I_P = K^*/(1 + \mathfrak{m}_P)$. (This is a slightly non-standard definition.) There is an exact sequence

$0 \longrightarrow k(P)^* \longrightarrow I_P \xrightarrow{\mathrm{ord}_P} \mathbb{Z} \longrightarrow 0$, where $k(P)^*$ is the multiplicative group of the residue class field of C at P.

<u>Definition:</u> The (global) idele group I of C is the restricted direct product of the I_p's with respect to the subgroups $k(P)^*$. In other words $I \subset \prod_p I_p$ is the subgroup of vectors $(\alpha_p), \alpha_p \in I_p$, $\alpha_p \in k(P)^*$ for all but a finite number of P.

There is a canonical injection $K^* \longrightarrow I$ given by $f \longrightarrow (\bar{f}_p)$, where $\bar{f}_p$ is the image of f in $K^*/(1+\mathfrak{m}_p)$. The relevant information can be collected in a wonderful commutative square:

$$\begin{tikzcd}
 & k^* \arrow[r] \arrow[d] & K^* \arrow[r] \arrow[d] & D_\ell \arrow[d] & \\
0 \arrow[r] & U \arrow[r] \arrow[d] \arrow[dr] & I \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & \bar{U} \arrow[r] & \mathcal{C} \arrow[r] & \mathrm{Pic}\,C &
\end{tikzcd}$$

(Tate),

where all rows and columns are short exact sequences; $U = \prod k(P)^*$ is the group of "unit ideles"; D = divisor group of C, D_ℓ = divisors linearly equivalent to zero, and $\mathcal{C}$ is (by definition) the <u>idele class group</u> of C.

<u>A Pairing:</u>

Suppose that v is any discrete valuation of K, with valuation ring R_v, maximal ideal $\mathfrak{M}_v$ and residue class field $R_v/\mathfrak{m}_v = k(v)$.

<u>Definition:</u> If $f, g \in K^*$, define the <u>local symbol</u> $(f,g)_v \in k(v)^*$ by $(f,g)_v$ = residue class of $(-1)^{mn} f^n/g^m \bmod \mathfrak{m}_v$, where $m = v(f)$, $n = v(g)$. For proofs see Serre [10], Chap. III, sec 4.)

Easy Properties:

(1) $(f_1 f_2, g)_v = (f_1, g)_v \cdot (f_2, g)_v$

(2) If f is a unit in R_v, $(f,g)_v \equiv f^n \bmod m_v$.

(3) $(\ ,\ )_v$: $K^* x K^* \longrightarrow k(v)^*$ annihilates $1+m_v$, giving a pairing $I_v \times I_v \longrightarrow k(v)^*$. (4) $(\ ,\ )$: $R_v^* \times R_v^* \longrightarrow k(v)^*$ is the trivial pairing.

Main Formula: (Gauss)

Suppose $K \subset K'$ is a finite algebraic extension, and that $v_1', \ldots, v_r'$ are the valuations of K' lying over v.

Suppose $f \in K'^*$, $g \in K^*$. Then

$(N_{K'/K}(f), g)_v = \prod_{i=1}^{r} N_{k'(v_i')/k(v)} (f,g)_{v_i'}$, where N denotes the norm mapping.

Product Formula:

If $K = k(C)$, and $f, g \in K^*$, then $\prod_{p \in C} (f,g)_p = 1$.

(Proof: This follows easily from the above formula. One checks the formula directly for the projective line, and then applies the Main Formula to the map g: $C \longrightarrow \mathbb{P}^1$, $k(\mathbb{P}^1) \subset k(C)$.)

One defines a "global symbol" or pairing of the idele group as follows:

If $\alpha, \beta \in I$, then $\langle \alpha, \beta \rangle = \prod_P N_{k(P)/k}(\alpha_p, \beta_p)$. (The factors are almost all 1). By (4) above $U \subset I$ is self-orthogonal, so from the wonderful square we get a pairing $U \times D \longrightarrow k^*$. By the product formula, K^* is self orthogonal, so we also have a pairing $K^* \times \mathcal{C} \longrightarrow k^*$.

Finally we note the following explicit consequence of the product formula:

Suppose that $d = \sum \nu_i P_i$ is any divisor on X. We define $f(d) = \prod N_{k(P_i)/k} f(P_i)^{\nu_i}$. Then if f and g have disjoint divisors (f) and (g), $\langle f,g \rangle = f((g))/g((f))$, whence $f((g)) = g((f))$.

1) check for $C = \mathbb{P}^1$, [illegible] linear factors [illegible] check for them directly
2) Consider [illegible] k, $C = \mathbb{P}^1$ (trivial)
Use [illegible] formula $\langle f,g \rangle_C = \langle N(f), g \rangle_{\mathbb{P}^1}$ 3) [illegible] $\mathbb{P}^1$ [illegible] finite field extension.

4. <u>Ideles in the relative case.</u> (The results of this section and the next were worked out jointly with B. Mazur).

Let Y be a reduced noetherian scheme. Let $f: X \longrightarrow Y$ be a projective morphism, flat, with fibres pure of dimension one and free of embedded components.

(e.g. $X = C$, $Y = \mathrm{Spec}\ k$ as in previous sections.)

Assume that $f_* \mathcal{O}_X = \mathcal{O}_Y$ universally, or equivalently (since $X \longrightarrow Y$ is flat) assume $H^0(X_y, \mathcal{O}_{X_y}) = k(y)$ for all $y \in Y$.

Consider arbitrary f.t. $Y \leftarrow Y'$, reduced. Work in Zariski topology on $X' = X \times_Y Y'$. Open pairs (Y', U'), $U' \subset X'$ Zar open. Y' étale. Sheaves for étale top'y / Y

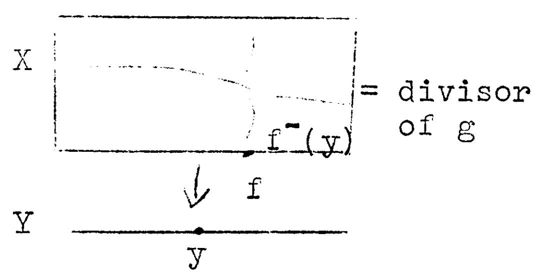
We work in the category on X defined by the Zariski topology on $X \times_Y Y'$ where $Y' \longrightarrow Y$ is any covering of <u>finite type</u> with Y' <u>reduced</u>. Alternatively, an "open set" of X is a pair (Y',U'), where $Y' \longrightarrow Y$ is of finite type and reduced, and U' is Zariski-open in $X' = X \times_Y Y'$. We consider functors on this category which are sheaves for some appropriate topology (étale or smooth, say . These two are in fact equivalent).

<u>Definition</u>: The (étale) sheaf of <u>relative rational functions on X/Y</u> is given by $R^*(U') = \{g \mid g$ is a rational function on X'; and there is an open subset $V \subseteq U'$ such that $g \in G_m(V)$ and $V \cap U'_y$ is dense in U'_y for all $y \in Y'\}$.

Suppose $x \in X$, so that $f(x)$ is its image in Y. Suppose $P_1, P_2, \ldots, P_r$ are the generic points of the components of the fibre $X_{f(x)}$ which pass through x. Then the stalk of R^* at x (for the Zariski topology) is $R^*_x = \prod_{i=1}^{r} \mathcal{O}^*_{X,P_i}$.

Roughly speaking, R^* is the sheaf of functions on X whose divisors contain no components of fibres of the map $f\colon X \longrightarrow Y$.

In other words, the divisor of a function $g \in R^*$ should be "horizontal".



<u>Definition:</u> The sheaf D of horizontal Cartier divisors of X/Y is defined as the cokernel of $\mathbb{G}_m \longrightarrow R^*$, so that the sequence $0 \longrightarrow \mathbb{G}_m \longrightarrow R^* \longrightarrow D \longrightarrow 0$ is exact.

<u>Proposition:</u> The following sequence of étale sheaves on $\underline{Y}$ is exact: (étale topy on Y, or horizontal is smooth)

$$0 \longrightarrow f_*\mathbb{G}_{m_{\underline{X}}} \overset{\simeq\, \mathbb{G}_{m_Y}}{\longrightarrow} f_*R^* \longrightarrow f_*D \longrightarrow R^1 f_*\mathbb{G}_{m_{\underline{X}}} \longrightarrow 0$$

By definition, $R^1 f_*\mathbb{G}_{m_X} = \underline{\mathrm{Pic}}(X/Y)$, the "relative Picard functor."

We define the complex $P^\bullet = \{0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow 0\}$ by $P^\bullet = \{0 \longrightarrow f_*R^* \longrightarrow f_*D \longrightarrow 0\}$, and we consider $P^\bullet$ as an object of the derived category of the category of abelian étale sheaves on Y. (See Hartshorne [3], Chapter I). Let $\bar{R}f_*\mathbb{G}_m$ denote the object of the derived category gotten by truncating $Rf_*\mathbb{G}_m$ above dimension 1. Then (almost by definition) we have $P^\bullet \simeq \bar{R}f_*\mathbb{G}_m$.

<u>Definition</u>: If x is a point of X', the <u>local relative idele group</u> is

$$I_x = R_x^*/(1+ m_x)$$

We have an exact sequence for every x:

$$0 \longrightarrow k(x)^* \longrightarrow I_x \longrightarrow D_x \longrightarrow 0 \quad ,$$

where D_x is the stalk at x of the sheaf of relative Cartier divisors.

<u>Definition</u>: The sheaf of <u>relative unit ideles</u> is defined by $U(V) = \prod_{x \varepsilon V} k(x)^*$ for every open set $V \subseteq X'$.

Note that $U(V) \subset \prod_{x \varepsilon V} I_x$. We also have $R^*(V) \subseteq \prod_{x \varepsilon V} I_x$.

<u>Definition</u>: The sheaf I_X of relative ideles is the subsheaf of $\prod_x I_x$ generated by R^* and U.

As in Section 3 we have a commutative square (this time, of sheaves/X)

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d] & R^* \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, "\Vert"] & 0 \\
 & U \arrow[r] \arrow[d] & I_X \arrow[r] \arrow[d] & D \arrow[d] & \\
 & \mathcal{C} \arrow[r, "="] \arrow[d] & \mathcal{C} \arrow[r] & 0 & \\
 & 0 & & &
\end{tikzcd}$$

[illegible]

where all rows and columns are exact and $\mathcal{C}$ is by definition the sheaf of <u>relative, idele classes on X/Y.</u>

[illegible], infinite product over pts

<u>Lemma</u>: U is acyclic for $f: X \longrightarrow Y$. Hence the sequence $\mathbb{G}_m \longrightarrow U \longrightarrow \mathcal{C}$ gives an exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow f_*U \longrightarrow f_*\mathcal{C} \longrightarrow \underline{\mathrm{Pic}}(X/Y) \longrightarrow 0.$$

We define the complex $U^\bullet = \{0 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow 0\}$

to be $\{0 \longrightarrow f_* U \longrightarrow f_* C \longrightarrow 0\}$. Just as with $P^\cdot$ we have $U^\cdot = \bar{R} f_* \mathbb{G}_m$, so that $U^\cdot \approx P^\cdot$ in the derived category of sheaves on Y.

<u>Some Auxiliary Sheaves on Y.</u>

Consider the sequence

$0 \longrightarrow \mathbb{G}_m Y \longrightarrow G^0 \longrightarrow G^1 \longrightarrow 0$, of sheaves on Y, where G^0 = sheaf of unit ideles, $G^0(Y') = \prod_{y \in Y'} k(y')^*$, and G^1 = sheaf of idele classes which is by definition $G^0/\mathbb{G}_m$. Define $f_* I_X = I$. We want to define a <u>pairing</u> $I \times I \longrightarrow G^0$ as follows: Suppose $\alpha, \beta \in I$. It will suffice to restrict α, β to each fiber X_y and define a pairing there. Write $X_y = \sum_{i=1}^{n} \nu_i C_i$ (as a cycle on X), where C_i is an irreducible curve. Here ν_i denotes the <u>length</u> of the local ring $\mathcal{O}_{X_{y}, c_i}$ = multiplicity [illegible] Λ_i of X_y at the generic point c_i of C_i. There is a pairing $\langle\ ,\ \rangle_i$ on C_i by the previous work on curves. We define

$\bar{C}_i$ = normalization of C_i, $\bar{\alpha}_i, \bar{\beta}_i$ restrictions of α, β to $\bar{C}_i$

$$\langle \alpha, \beta \rangle_y = \prod_{i=1}^{n} \langle \bar{\alpha}_i, \bar{\beta}_i \rangle_i^{\nu_i} ,$$

where now $\bar{\alpha}_i$ is the restriction of α to C_i. We glue these local pairing together to get a sheaf map $I \times I \longrightarrow G^0$. Since the local unit ideles and rational functions are self-orthogonal (see Section 3), $I \times I \longrightarrow G^0$ gives pairings

$U^0 \times P^1 \longrightarrow G^0$ and

$U^1 \times P^0 \longrightarrow G^0$.

These pairings give a commutative diagram

$$\begin{tikzcd} U^0 \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}\,(P^1, G^0) \arrow[d] \\ U^1 \arrow[r] & \underline{\mathrm{Hom}}\,(P^0, G^0) \end{tikzcd} .$$

Some notation: Define a bicomplex by

$$\underline{\mathrm{Hom}}^{p,q}(P^\bullet, G^{\bullet 1}) = \underline{\mathrm{Hom}}\,(P^{-p}, G^{q-1})$$

(Note the sign change in P and the dimension shift in G.) Let $\underline{\mathrm{Hom}}^\bullet(P^\bullet, G^{\bullet 1})$ be the associated total complex. The bicomplex looks like

	0	1	2
0	0	(P^0, G^0)	(P^0, G^1)
-1	0	(P^1, G^0)	(P^1, G^1)

and the associated total complex is

$$\underline{\mathrm{Hom}}^0(P^\bullet, G^{\bullet 1}) = \underline{\mathrm{Hom}}\,(P^1, G^0)$$

$$\underline{\mathrm{Hom}}^1(P^\bullet, G^{\bullet 1}) = \underline{\mathrm{Hom}}\,(P^0, G^0) \oplus \underline{\mathrm{Hom}}\,(P^1, G^1)$$

(We neglect $\underline{\mathrm{Hom}}^2$.) The boundary operator is $d = \underline{\mathrm{Hom}}\,(d_{P^\bullet}, G^0) \oplus \underline{\mathrm{Hom}}\,(P^1, d_{G^\bullet})$ where $d_{P^\bullet}$ and $d_{G^\bullet}$ are boundary operators for $P^\bullet$ and $G^\bullet$. We have already defined maps $U^0 \longrightarrow \underline{\mathrm{Hom}}\,(P^1, G^0)$ and $U^1 \longrightarrow \underline{\mathrm{Hom}}\,(P^0, G^0)$. We want a map $U^\bullet \longrightarrow \underline{\mathrm{Hom}}^\bullet(P^\bullet, G^{\bullet 1})$ of complexes, so we need a map $U^1 \xrightarrow{\varphi} \underline{\mathrm{Hom}}\,(P^1, G^1)$ and a commutative diagram

need map $U^1 \to \underline{\mathrm{Hom}}(P^1, G^1)$ s.t.

$$\begin{tikzcd}
U^1 \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(P^1, G^0) \arrow[d] \\
\underline{\mathrm{Hom}}(P^0, G^0) \arrow[r] & \underline{\mathrm{Hom}}(P^0, G^1)
\end{tikzcd}$$

$$\begin{tikzcd}
U^0 \arrow[r] \arrow[dd] & \underline{\mathrm{Hom}}(P^1, G^0) \arrow[dd] & & \\
& & \arrow[r, leftrightarrow] & \underline{\mathrm{Hom}}(P^1, d_{G^\bullet}) \\
U^1 \arrow[r, "\varphi"] & \underline{\mathrm{Hom}}(P^1, G^1) & &
\end{tikzcd}$$

Suppose for a moment that we have constructed the required map, so that we have a map of complexes $U^\bullet \longrightarrow \underline{\mathrm{Hom}}^\bullet(P^\bullet, G^{\bullet 1})$. There is always given a map $\underline{\mathrm{Hom}}^\bullet(P^\bullet, G^{\bullet 1}) \longrightarrow \underline{R\,\mathrm{Hom}}\,(P^\bullet, G^{\bullet 1})$ (in the derived category), we have

$$U^{\cdot} \simeq Rf_{*}\mathbb{G}_m, \quad P^{\cdot} \simeq Rf_{*}\mathbb{G}_m, \quad \text{and } G^{\cdot 1} \simeq \mathbb{G}_m^{\,1} = \{0 \to 0 \to \mathbb{G}_m \to 0\}$$

(all complexes truncated at dimension two). Therefore:

<u>Theorem</u>. There exists a map

($\tilde{\ell}$) $$Rf_{*}\mathbb{G}_m \longrightarrow \underline{R\ Hom}^{\cdot}(Rf_{*}\mathbb{G}_m, \mathbb{G}_m^{1}),$$ of complexes truncated at dimension two. In particular, there exists a map $\underline{\mathrm{Pic}}(X/Y) \longrightarrow \underline{\mathrm{Ext}}^1(\underline{\mathrm{Pic}}(X/Y), \mathbb{G}_m)$ which reduces to the auto-duality for a curve if $Y = \mathrm{Spec}\ k$.

<u>Remarks:</u> 1. The truncations at dimension two are possibly superfluous. In any case, $Rf_{*}\mathbb{G}_m$ is probably acyclic in dimension > 1, but this has not been proved.

2. The "in particular" part of the theorem follows in two steps. First the map in the derived category gives the map

$$\underline{\mathrm{Pic}}(X/Y) \longrightarrow \underline{\mathrm{Ext}}^1(\underline{\mathrm{Pic}}(X/Y), \mathbb{G}_m)$$ by general nonsense in derived category theory. Then in the case $Y = \mathrm{Spec}\ k$ this map of (representable) functors gives a map of schemes $A \longrightarrow \mathrm{Ext}^1(A, \mathbb{G}_m)$. To check that this is the same map as the one in Section 1, it suffices to check this on torsion points of A, and this can be done using known explicit formulas of Weil (cf. Weil [11], Lang [5]).

3. The map $\underline{\mathrm{Pic}}\ X/Y \longrightarrow \underline{\mathrm{Ext}}^1$ need not be an isomorphism, even on the connected component of $\underline{\mathrm{Pic}}$, when the fibres of X/Y are singular. However, we will see an important case in the next section where it is an isomorphism.

Finally, we sketch the construction of the map $U' \longrightarrow \underline{\mathrm{Hom}}\,(P', G')$. Actually we get a map $U' \times P' \longrightarrow G'$, or (idele classes on X) x (divisors on X) $\longrightarrow$ (idele classes on Y).

Suppose α is an idele on X, d a divisor on X. Suppose $x \varepsilon X$, and suppose g is a local equation for d at x. If g is regular at x, define

$\ell_x(d) = \dim_{k(x)}(\mathcal{O}_{X_y}/\bar{g})$, where $y = f(x)$, X_y is the fibre, and $\bar{g}$ is the restriction of g to X_y.

Suppose α is a unit idele at x, that is, $\alpha_x \varepsilon k(x)^*$. Then set

$$\alpha(d)_x = N_{k(x)/k(y)}\ \alpha_x^{\ell_x(d)} \qquad \text{and}$$

$$\alpha(d)_y = \prod_{f(x)=y} \alpha(d)_x \qquad \varepsilon\ k(y)^*$$

when these definitions are possible.

We have now defined a map $(\alpha, d) \longrightarrow \alpha(d)$ under certain special condition on α and d. This map can actually be extended to give the desired pairing $U' \times P' \longrightarrow G'$. That is given an idele class a and a divisor d, one can choose a good idele representative α and map $(\alpha, d) \longrightarrow$ image of $\alpha(d)$ in G'. We omit the details.

5. Applications.

Suppose Y is a regular one-dimensional scheme, X is a regular two dimensional scheme and f: $X \longrightarrow Y$ has one dimensional fibers. We also assume that the generic fibre is a non-singular curve with rational point. The basic example is gotten as follows (Kodaira [4], Lichtenbaum [6], Neron [8]):

Let C be a complete non-singular curve over the field K, and let $K = k(Y)$ be an algebraic function field in one variable with constant field k. Let X be a minimal non-singular model for the field K(C) over k. Then X is a surface and the field inclusion $K \subset K(C)$ induces a projection $f: X \longrightarrow Y$. In Kodaira's examples C is an elliptic curve, and at a point $y \in Y$ where C has "good reduction" the fiber X_y is just the reduced curve. If C has bad reduction at y', the X_y can be an elaborate concatenation of rational curves.

Let $f: X \longrightarrow Y$ be any morphism satisfying the conditions above, and let C be the generic fiber. Let A be the Jacobian of C. Then there exists a canonical "best" group scheme $N(A) \longrightarrow Y$ (the Neron minimal model). The following proposition describes Pic(X/Y).

<u>Proposition:</u> Let $Pic^{o}(X/Y)$ = {those invertible sheaves which have degree 0 on the generic fiber} so that $Pic(X/Y) = Pic^{o}(X/Y) \oplus \mathbb{Z}$. Then there is a sequence of group functors (and of group schemes /Y if the residue fields are separately closed) $0 \longrightarrow \Delta \longrightarrow Pic^{o}(X/Y) \longrightarrow N(A) \longrightarrow 0$. Δ is concentrated on the fibers of X over degenerate points of y, and if a finitely generated group at each of those.

The general auto-duality theorem gives a map $\varphi: Pic^{o}(X/Y) \longrightarrow Ext^{1}(Pic^{o}(X/Y), \mathbb{G}_m)$.

<u>Theorem</u>: φ is an isomorphism for sheaves in the <u>smooth</u> topology over Y.

The proof uses explicit descriptions of Yoneda extensions and explicit formulas for pairings.

<u>Open Questions:</u> Show that in a suitable context $\mathrm{Ext}^q(\mathrm{Pic}^o(X/Y), \mathbb{G}_m) = o$ $(q > 1)$ or that $\mathrm{Ext}^q(N(A), \mathbb{G}_m) = 0$ $(q > 1)$.

As a final application let $\mathcal{O}$ be the ring of integers of an algebraic number field K, and let $Y = \mathrm{Spec}\ \mathcal{O}$. Let C be a curve of genus $g > 0$ defined over K which has a point rational over K. Then there exists a non-singular proper model for C. That is, there is a 2-dimensional regular scheme X and a proper morphism $f: X \longrightarrow Y$ with C as generic fiber. Furthermore X is uniquely determined if there are no "exceptional curves" on X (Lichtenbaum [6]) .

Class field theory can be interpreted as the statement that Y is a "cohomological 3-manifold". (See Artin-Verdier [1], or Mazur [7]). Therefore X should be a "cohomological 5-manifold." Using the autoduality, one obtains

<u>Theorem</u> (Artin-Mazur): Let μ = n-th roots of unity, with $(2,n) = 1$ or K totally imaginary. Then there is a pairing $H^p(X,\mu) \times H^{5-p}(X,\mu) \longrightarrow H^3(X, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ which is a perfect duality, where the cohomology is taken with respect to the flat topologies on X and Y.

Analogue for Spec / finite field is easy, for étale cohomology, if $(n,p)=1$. For $n=p$ [illegible] [illegible] (not known)

$$H^r(X, \mu_n) \times H^{5-r}(X, \mu_n) \longrightarrow H^3(Y, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$$

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