The Fundamental Group of the Projective Line Minus Three Points

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Abstract

This is a translation of §5 and §6 of the classic article
Deligne, P. Le groupe fondamental de la droite projective moins trois points. Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.
It is available at https://www.jmilne.org/math/Documents/index.html. Corrections should be sent to the email address on that page.

5 Algebraic geometry in a tannakian category

5.1. Let X be a complex algebraic variety, $o \in X$, and Γ the largest torsion-free quotient of $\pi_1(X,o)$ of class N:

$$\Gamma = \pi_1(X,o)^{[N]} \quad (0.3).$$

For (X,o) defined over $k \subset \mathbb{C}$, we want to see $\mathrm{Lie}\,\Gamma$ as the Betti realization of a motive over k. In certain situations, we construct in every case a system of realizations over k of which $\mathrm{Lie}\,\Gamma$ is the Betti realization, relative to the inclusion $\sigma\colon\ k \hookrightarrow \mathbb{C}$. The Lie bracket must also be a morphism of motives.

Giving a nilpotent Lie algebra Γ is equivalent to giving a unipotent algebraic group $\Gamma^{\mathrm{alg,un}}$, or simply Γ^{alg}, whose Lie algebra it is (9.1, cf. also 9.5), and we can regard Γ^{alg} as being “motivic” . We will do it as follows: giving the algebraic group Γ^{alg} is equivalent to giving its Hopf algebra. The exponential $\mathrm{Lie}\,\Gamma \to \Gamma^{\mathrm{alg}}$ identifies this algebra with $\bigoplus_n \mathrm{Sym}^n((\mathrm{Lie}\,\Gamma)^\vee)$. This is the inductive limit over k of the algebras $\bigoplus_{n\le k} \mathrm{Sym}^n((\mathrm{Lie}\,\Gamma)^\vee)$. Each of these finite sums is the Betti realization, relative to σ, of an ind-object of the category of motives. The product and coproduct are induced by morphisms in this category.

For $x \in X$, the homotopy classes of paths from o to x form a torsor (0.6) under $\pi_1(X,o)$. From $\pi_1(X,o) \to \Gamma^{\mathrm{alg}}$, we deduce a Γ^{alg}-torsor $P^{(N)}_{x,o}$. For x defined over k, we want this torsor to be motivic over k. Interpretation: its affine algebra is the Betti realization relative to σ of an ind-object of the category of systems of Betti realizations relative to σ of an ind-object of the category of systems of realizations over k. In contrast to that which holds for Γ^{alg}, where we have the Lie algebra $\mathrm{Lie}\,\Gamma^{\mathrm{alg}}$, I know of no convenient way of expressing the motivic character of $P^{(N)}_{x,o}$ that avoids a detour through ind-motives. The purpose of this paragraph is to furnish a suitable language for these constructions.

5.2. Let k be a field and $\mathcal{T}$ a tannakian category over k. More generally (at least in 5.2–5.7), we could take $\mathcal{T}$ to be a rigid abelian tensor category with $\mathrm{End}(\mathbb{1}) = k$, i.e., a tensorial category over k. For our needs, it suffices to consider the case that $\mathcal{T}$ is equivalent (with its tensor product and associativity and commutativity constraints) to the category $\mathsf{Rep}(G)$ of linear representations of finite dimension of an affine group scheme G over k. We can paraphrase, in $\mathcal{T}$, the rudiments of algebraic geometry. Here is how.

5.3. The category $\mathrm{Ind}\,\mathcal{T}$ of ind-objects of $\mathcal{T}$ (4.1) is equipped with a tensor product deduced from that of $\mathcal{T}$. As in $\mathcal{T}$, it is exact.

A ***ring*** (always assumed to have a unity) A of $\mathrm{Ind}\,\mathcal{T}$ is an object A of $\mathrm{Ind}\,\mathcal{T}$ equipped with an associative product $\cdot\colon\ A \otimes A \to A$ and admitting a unity $1 \to A$ (which we denote also by 1) . "Associative" and "unity" are expressed by diagrams. If one prefers to express them by the usual formulas, one arrives at the following. Ind-objects of $\mathcal{T}$ can be identified with the ind-representable functors $\mathcal{T} \to \mathsf{Set}$ (see 4.1.1),

$$X \rightsquigarrow h_X\colon\ h_X(S) = \mathrm{Hom}(S, X).$$

The functor h_X even takes values in the category of k-vector spaces. Giving $X \otimes Y \to Z$ is equivalent to giving

$$h_X(S) \times h_Y(T) \to h_Z(S \otimes T),$$

bilinear and functorial in S and T. The associativity of $A \otimes A \to A$ becomes

$$(xy)z = x(yz) \text{ for } x \in h_A(S),\ y \in h_A(T),\ z \in h_A(U)$$

[therefore, $xy \in h_A(S \otimes T)$, $yz \in h_A(T \otimes U)$, $(xy)z$ and $x(yz) \in h_A(S \otimes T \otimes U)$]. That $1\colon\ 1 \to A$ is a unity becomes $1x = x1 = x$ for $x \in h_A(S)$ [we have $1 \in h_A(1)$, whence $1x \in h_A(1 \otimes S) = h_A(S)$, and even for $x \cdot 1$].

We define in an obvious way left and right A-modules, tensor products over A, and the commutativity of A. For example, a left A-module is an object of $\mathrm{Ind}\,\mathcal{T}$ equipped with a morphism $\cdot\colon\ A \otimes M \to M$ with $(ab)m = a(bm)$ and $1m = m$ (for the meaning of such formulas, cf. above). We have

$$M \otimes_A N = \mathrm{Coker}(M \otimes A \otimes N \rightrightarrows M \otimes N).$$

Let $f\colon\ A \to B$ be a morphism of commutative rings in $\mathrm{Ind}\,\mathcal{T}$. We say that B is ***faithfully flat*** over A if the functor $M \rightsquigarrow B \otimes_A M$ from A-modules to B-modules is exact and faithful. The formalism of faithfully flat descent of modules (SGA1, VIII, 1) applies: the functor $M \rightsquigarrow B \otimes M$ is an equivalence of categories of A-modules to that of B-modules N equipped with a desent datum

$$(B \otimes_A B) \otimes_B N \xrightarrow{\simeq} N \otimes_B (B \otimes_A B).$$

The proof in SGA 1, VIII, 1 still applies, or it can be reduced to the Barr-Beck theorem (cf., Deligne 1990, 4.1 and 4.2).

The structure morphism $\mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ makes $\mathbb{1}$ into a ring in $\mathrm{Ind}\,\mathcal{T}$ (even in $\mathcal{T}$), and for a ring (with unity) A of $\mathrm{Ind}\,\mathcal{T}$, there is a unique morphism of rings (with unity) $\mathbb{1} \to A$. If $A \neq 0$, this morphism is non null, therefore is a monomorphism (DM 1.17), and the exact functor $M \rightsquigarrow A \otimes_1 M = A \otimes M$ is faithful because $M \hookrightarrow A \otimes M$. If $A \neq 0$, A is therefore faithfully flat over $\mathbb{1}$.

5.4. In order to have a geometric language at our disposal, we define the ***category of affine schemes in*** $\mathcal{T}$ to be the dual of that of commutative rings with unity in Ind $\mathcal{T}$. We also say affine $\mathcal{T}$-scheme. We write $\mathrm{Sp}(A)$ for the affine $\mathcal{T}$-scheme defined by A. Fibre products exist: they correspond to tensor products. An A-module M will be called a module over $\mathrm{Sp}(A)$, and for $\mathrm{Sp}(B)$ over $\mathrm{Sp}(A)$, the functor $M \rightsquigarrow B \otimes_A M$ will be called the inverse image over $\mathrm{Sp}(B)$. The formalism (SGA 1, VIII, 2) of faithfully flat descent for affine schemes applies.

We have initial and final schemes, $\mathrm{Sp}(0)$ and $\mathrm{Sp}(1)$ — they will be called the empty and point schemes. We say that $S = \mathrm{Sp}(A)$ is nonempty if $A \neq 0$. If S is nonempty, S is faitfully flat over the point.

For X and S affine schemes in $\mathcal{T}$, the set $X(S)$ of S-***points*** of X is $\mathrm{Hom}(S, X)$.

An ***affine group*** $\mathcal{T}$***-scheme*** is a group object of the category of affine $\mathcal{T}$-schemes.

Let H be an affine group scheme in $\mathcal{T}$. An H-***torsor*** is a nonempty affine $\mathcal{T}$-scheme P equipped with a right action $\rho\colon P \times H \to P$ such that, for all S, $P(S)$ is either empty or a torsor under $H(S)$. The condition "empty or a torsor" means that, for all S, $(\mathrm{pr}_1, \rho)\colon P(S) \times H(S) \to P(S) \times P(S)$ is bijective, i.e., that $(\mathrm{pr}_1, \rho)\colon P \times H \to P \times P$ is an isomorphism.

EXAMPLE 5.5 (VECTORIAL $\mathcal{T}$-SCHEMES). For M in Ind $\mathcal{T}$, put $\Gamma(M) = \mathrm{Hom}(\mathbb{1}, M)$. For M a module over $S = \mathrm{Sp}(A)$, we have

$$\Gamma(M) = \mathrm{Hom}(\mathbb{1}, M) \xleftarrow{\simeq} \mathrm{Hom}_A(A, M),$$

and we call $\Gamma(M)$ the global sections of M over S. Take care that the functor Γ need not be exact: for $\mathcal{T} = \mathsf{Rep}(G)$, it is the functor of G-invariants.

An object X of $\mathcal{T}$ defines for each $S = \mathrm{Sp}(A)$ a module $X_S = A \otimes X$, the inverse image of X by $S \to (\mathrm{pt})$. The functor $S \rightsquigarrow \Gamma(X_S)$ is representable,

$$\mathrm{Hom}(\mathbb{1}, A \otimes X) = \mathrm{Hom}(X^\vee, A) = \mathrm{Hom}_{\mathrm{rings}}(\mathrm{Sym}(X^\vee), A).$$

We also call X the $\mathcal{T}$-scheme $\mathrm{Sp}(\mathrm{Sym}(X^\vee))$ representing this functor. This notation is parallel to the usage of identifying a finite-dimensional k-vector V with the scheme $\mathrm{Spec}(\mathrm{Sym}^*(V^\vee))$ that has V for its points over k.

The functor $S \rightsquigarrow \Gamma(X_S)$ is a functor to groups. The $\mathcal{T}$-scheme X is therefore a group scheme in $\mathcal{T}$. The group structure corresponds to the usual Hopf algebra structure on $\mathrm{Sym}^*(X^\vee)$.

EXAMPLE 5.6 (AN AFFINE k-SCHEME IS AN AFFINE $\mathcal{T}$-SCHEME). Since $\mathrm{End}(\mathbb{1}) = k$, the subcategory of $\mathcal{T}$ of sums of copies of $\mathbb{1}$ is naturally equivalent to that of vector spaces of finite dimension over k. We often identify the vector space V over k with the corresponding object of $\mathcal{T}$. When we need to be more precise, we write it $V \otimes \mathbb{1}$. The choice of a basis $e_1, \ldots, e_n$ of V identifies $V \otimes \mathbb{1}$ with $\mathbb{1}^n$.

Passing to the ind-objects, we obtain a functor from the category of (all) vector spaces over k to Ind $\mathcal{T}$. Under this functor, an affine scheme over k defines a scheme in $\mathcal{T}$. Similarly, for affine group schemes, torsors, ... The point $\mathrm{Spec}(k)$ defines the $\mathcal{T}$-scheme (pt).

5.7. Let G be an affine group $\mathcal{T}$-scheme and X an object of $\mathcal{T}$. To give an ***action*** of G on X is to give, for every S, an action of $G(S)$ on the S-module X_S, compatible with base changes S'/S. Such an action is defined by the action of $\mathrm{id}_G \in G(G)$ on X_G. For $G = \mathrm{Sp}(A)$, it is an A-linear morphism $A \otimes X \to A \otimes X$, defined by $X \to A \otimes X$. The morphism $X \to A \otimes X$ makes X a comodule over the Hopf algebra with counity A in $\mathrm{Ind}\,\mathcal{T}$.

5.8 (THE CASE OF $\mathsf{Rep}(G)$). Let G be an affine group scheme over k and $\mathcal{T} = \mathsf{Rep}(G)$.

The ind-objects of $\mathcal{T}$ are the linear representations — not necessarily of finite dimension — of G (4.3.2). The affine $\mathcal{T}$-schemes are the affine schemes over k equipped with an action of G, an affine group $\mathcal{T}$-scheme H is an affine group scheme over k equipped with an action of G, an H-torsor is an G-equivariant H-torsor (in the usual sense), a vectorial $\mathcal{T}$-scheme is the equivariant affine scheme of a finite-dimensional representation of G, and the inclusion of affine k-schemes into affine $\mathcal{T}$-schemes is "equip with the trivial action of G".

This interpretation allows us to routinely reduce questions on affine $\mathcal{T}$-schemes to questions in usual algebraic geometry.

5.9. Let $\mathcal{T}$ be a tannakian category over k. Recall that a fibre functor on $\mathcal{T}$ over a k-scheme S is a k-linear exact tensor functor from $\mathcal{T}$ to the vector bundles on S. For a scheme $\pi : S' \to S$ over S, the inverse image on S' of a fibre functor ω on S is the fibre functor $X \rightsquigarrow \pi^*\omega(X)$. Notation: $\omega_{S'}$ or $\pi^*\omega$.

If ω_1 and ω_2 are two fibre functors over S, the functor which to $\pi : S' \to S$ attaches the set of isomorphisms from $\pi^*\omega_1$ to $\pi^*\omega_2$ is representable by a scheme $\underline{\mathrm{Isom}}_S^{\otimes}(\omega_1, \omega_2)$ affine over S. For a fibre functor ω over S, we write $\underline{\mathrm{Aut}}_S^{\otimes}(\omega$ or $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ for the affine S-scheme $\underline{\mathrm{Isom}}_S(\omega, \omega)$.

The main result of Saavedra 1972 (cf., DM 2.11) is the following. If ω is a fibre functor on $\mathcal{T}$ over k (i.e., over $\mathrm{Spec}(k)$), ω induces an equivalence

$$\mathcal{T} \to \mathsf{Rep}(\underline{\mathrm{Aut}}(\omega)).$$

The interpretation 5.8 is then available. It has the following inconveniences.

(a) The group $\underline{\mathrm{Aut}}(\omega)$ is not often explicit, and to see the $\mathcal{T}$-schemes as equivariant affine k-schemes is scarcely illuminating. See §7 for other interpretations.

(b) If one uses 5.8 to construct affine $\mathcal{T}$-schemes, it may not be obvious that the $\mathcal{T}$-scheme constructed does not depend on the fibre functor chosen. For how to render it obvious, see 5.11.

EXAMPLE 5.10. Let G be an affine group scheme over k, X a linear representation of finite dimension of G, and let X also denote the corresponding vectorial group scheme $\mathrm{Spec}(\mathrm{Sym}^*(X^\vee))$. An extension

$$0 \to X \to E \to k \to 0$$

of the unity representation (k with the trivial action) by X defines an equivariant X-torsor, namely, the inverse image of $1 \in k$ in E. This construction is an equivalence of categories.

We want to deduce that for $\mathcal{T}$ as in 5.9 and X in $\mathcal{T}$, we have an equivalence from the category of extensions of $\mathbb{1}$ by X to that of X-torsors,

$$(\text{extensions of } \mathbb{1} \text{ by } X) \xrightarrow{\sim} (X\text{-torsors}).$$

We define a functor as follows. Let A be the vectorial $\mathcal{T}$-scheme defined by the identity object. It is also the image by 5.6 of the affine line $\operatorname{Spec} k[T]$ over k, and the point $T = 1$ defines a point $1\colon (pt) \to A$. An extension of E of $\mathbb{1}$ by X defines a vectorial scheme E mapping onto A. The action by translation of E by itself induces an action of X on E stabilizing the fibre at 1, P, of $E \to A$: $P \overset{\text{def}}{=} E \times_A (pt)$, relative to $1\colon (pt) \to A$. This fibre is the torsor sought.

This description is independent of the choice of a fibre functor. The interpretation 5.8 shows that it is an equivalence.

5.11. Let $\mathcal{T}$ be a tannakian category over k. The essential results of Deligne 1990, already announced in Saavedra 1972, but proved there only when $\mathcal{T}$ admits a fibre functor over k, i.e., is of the form $\mathsf{Rep}(G)$ (5.9), are the following.

(a) The fibre functors form a gerbe $\textsc{Fib}(\mathcal{T})$ over the k-schemes for the fpqc topology. This means that they form a stack: possibility of patching a fibre functor given locally on S to a fibre functor on S, that if ω_1 and ω_2 are two fibre functors on S, there exist a T faithfully flat and quasi-compact over S on which ω_1 and ω_2 become isomorphic, and that there exists on some $S \neq \emptyset$ a fibre functor.

(b) Each object X of $\mathcal{T}$ defines a morphism of stacks $\omega \rightsquigarrow \omega(X)$

$$\text{(fibre functors over } S \text{ variable)} \to \text{(vector bundles over } S\text{)}.$$

This construction is an equivalence of $\mathcal{T}$ with the category $\mathsf{Rep}(\textsc{Fib}\mathcal{T})$ of these functors: it "amounts to the same" to give X in $\mathcal{T}$ or to give, for each fibre functor ω over a k-scheme S, a vector bundle over S, functorially in ω, and compatible with base change $S' \to S$.

(c) By passage to ind-objects, a fibre functor ω on S defines a tensor functor, again denoted ω, from $\operatorname{Ind}\mathcal{T}$ into the category of quasi-coherent sheaves on S. Each object X of $\operatorname{Ind}\mathcal{T}$ defines a morphism of stacks

$$\text{(fibre functors over } S) \to \text{(sheaves quasi-coherent over} S).$$

This construction is an equivalence of $\operatorname{Ind}\mathcal{T}$ with the category of these functors.

It follows from (c) that it amounts to the same to give an affine $\mathcal{T}$-scheme X (resp. an affine group $\mathcal{T}$-scheme G, resp. a $\mathcal{T}$-torsor under G) or to give, for each fibre functor ω over a k-scheme S, an affine scheme X_ω over S (resp. an affine group scheme G_ω, resp. a torsor under G_ω) functorially in ω and compatible with changes of base $S' \to S$. To $X = \operatorname{Sp}(A)$, we attach the system $\omega(X) = \operatorname{Spec}(\omega(A))$.

In particular, to construct a morphism $F\colon X \to Y$ between affine $\mathcal{T}$-schemes, it suffices for every fibre functor ω to construct functorially in ω a morphism from $\omega(X)$ to $\omega(Y)$. If ω is a fibre functor over S, it suffices for that, for every S-scheme T, to construct functorially in T an map from $\omega(X)(T) \overset{\text{def}}{=} \operatorname{Hom}_S(T, \omega(X))$ into $\omega(Y)(T)$. To write such a construction, we "take the point of view of X", i.e., $x \in \omega(X)(T)$ and construct its image.

REMARK 5.12. For (X_ω) as above, each X_ω/S automatically has the following property (portant sur X/S).

(5.12.1) There exists an extension k' of k and $\pi\colon T \to S$ faithfully flat over S, such that the inverse image $\pi^*X = T \times_S X$ of X over T is the inverse image over T of a k'-scheme, by a morphism of T to k'.

Indeed, there exists a fibre functor ω_0 over an extension k' of k and, because $\mathrm{FIB}(\mathcal{T})$ is a gerbe, there exists T faithfully flat over $S \times \mathrm{Spec}(k')$ over which ω and ω_0 become isomorphic. Over this T, X_ω and X_{ω_0} have isomorphic inverse images.

A similar statement holds for schemes equipped with additional data.

5.13. Let Ξ be a construction of the following form: to affine schemes over a k-scheme S, equipped with suitable additional data, it attaches an affine scheme over S, equipped with additional data. It suffices that Ξ be defined for schemes, equipped with additional data, satisfying 5.12.1. We assume that Ξ is compatible with base change.

By 5.11, it then makes sense to apply Ξ to affine $\mathcal{T}$-schemes, equipped with additional data of the type required: to apply Ξ to the $\mathcal{T}$-schemes X_i, we apply it to the $\omega(X_i)$; the system $Y_\omega = \Xi(\omega(X_i))$ define by 5.11 a $\mathcal{T}$-scheme Y, which we call $\Xi(X_i)$.

Similarly, if P is a property of affine schemes over S equipped with additional structure, (satisfying 5.12.1 if one wishes) which is local for the fpqc topology, it makes sense to consider P "in $\mathcal{T}$",

Rather that make precise the sense of "construction", of "additional data", of "property", we give some examples.

EXAMPLE 5.14. (a) Let G be an affine group scheme over S, $\Xi(G)$ the Nth subgroup $Z^N(G)$ of G for the central descending series, or the quotient $G^{(N)} \overset{\text{def}}{=} G/Z^N(G)$. This construction is not compatible with arbitrary bases changes for G/S, but it is for an affine group scheme G over S satisfying (5.12.1).

(b) Let H be a normal subgroup of G and $\Xi(G,H)$ is G/H. Even if H is not normal, we can consider G/H when it is affine. The same remark as in (a) applies.

(c) Let G be an affine group scheme over S, and the property "G is unipotent".

APPLICATION 5.15. Over an arbitrary base S, giving an extension $\mathcal{E}$ of $\mathcal{O}$ by a vector bundle $\mathcal{V}$ is equivalent to giving a torsor under the vectorial group scheme defined by $\mathcal{V}$. This construction is compatible with base change. It follows that in every tannakian category, giving an extension E of $\mathbb{1}$ by an object V is equivalent to giving a torsor under the $\mathcal{T}$-vectorial scheme V. We have already proved this in 5.10 for a neutral $\mathcal{T}$.

5.16. Here is the relation between the points of view 5.8 and 5.11 for $\mathcal{T} = \mathsf{Rep}(G)$. Let ω_0 be the forgetful fibre functor. For ω a fibre functor over S, $\mathrm{Isom}(\omega_0, \omega)$ is a G-torsor P over S. Conversely, a G-torsor P defines a fibre functor

$$\omega_P : V \rightsquigarrow (V \text{ twisted by } P)$$

over S. If $P(S) \neq \emptyset$, the twisted V^P is a vector bundle over S equipped, for each $p \in P(S)$, with $\rho(p) : V \otimes \mathcal{O}_S \xrightarrow{\simeq} V^P$, with $\rho(pg) = \rho(p)\rho(g)$ for all $g \in G(S)$. The general case can be treated by descent. We have an equivalence

$$\mathrm{FIB}(\mathsf{Rep}(G)) \sim (G\text{-torsors over } S \text{ variable}).$$

If X is a $\mathcal{T}$-scheme, identified by 5.8 to a G-equivariant affine scheme, then for every fibre functor ω_P, $\omega_P(X)$ is the twist X^P of X by P.

We note for later use.

Lemma (5.16.1). $\underline{\mathrm{Aut}}(\omega_P) = \underline{\mathrm{Aut}}(P)$ is G^P for the inner action of G on itself.

Proof: When $P(S) \neq \emptyset$, each $p \in P(S)$ defines an isomorphism $\rho(p)$ of P with the trivial G-torsor G, therefore of $\mathrm{Aut}(P)$ with G (left translations of G). We have $\rho(pg) = \rho(p) \circ \mathrm{inn}(g)$: the automorphism of P which sends $p \cdot g$ to $p \cdot gh$ sends p to $p \cdot ghg^{-1}$. This satisfies 5.16.1 for $P(S) \neq \emptyset$, and the general case follows by descent.

5.17. The passage 5.11 from $\mathcal{T}$ to $\mathrm{FIB}(\mathcal{T})$ has an inverse (D1990, 1.12 and §3). Let G be a gerbe with affine band over k-schemes: we assume that, for an object ω of G over S, the functor that to $\pi : S' \to S$ attaches $\mathrm{Aut}(\pi^*\omega)$ is representable by an affine group scheme over S. Let $\mathsf{Rep}(\mathsf{G})$ be the category of morphisms of stacks

$$\mathsf{G} \to (\text{vector bundles over } S \text{ variable}).$$

Then $\mathsf{Rep}(\mathsf{G})$ is a tannakian category, and

$$\mathsf{G} \xrightarrow{\sim} \mathrm{FIB}(\mathsf{Rep}(\mathsf{G})).$$

5.18. From 5.11 and 5.17, we get a dictionary between tannakian categories over k and gerbes with affine band. We define the ***tensor product*** of two tannakian categories by

$$\mathrm{FIB}(\mathcal{T}_1 \otimes \mathcal{T}_2) \sim \mathrm{FIB}(\mathcal{T}_1) \times \mathrm{FIB}(\mathcal{T}_2).$$

Giving an object X of $\mathcal{T}_1 \otimes \mathcal{T}_2$ is equivalent to giving, for ω_1 and ω_2 fibre functors over S of $\mathcal{T}_1$ and $\mathcal{T}$, a vector bundle X_{ω_1,ω_2}on S, the formation of X_{ω_1,ω_2} being functorial in ω_1 and ω_2 and compatible with base change.

We have a tensor product

$$\boxtimes : \mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2,$$

such that, for fibre functor ω_1 and ω_2 on $\mathcal{T}_1$ and $\mathcal{T}_2$, there is a fibre functor on $\mathcal{T}_1 \otimes \mathcal{T}_2$ sending $X_1 \boxtimes X_2$ to $\omega_1(X_1) \otimes \omega_2(X_2)$. In D1990, §5, it is shown that $\mathcal{T}_1 \otimes \mathcal{T}_2$ is the universal target of such a tensor product with suitable properties.

If $\mathcal{T}_1, \mathcal{T}_2$ are $\mathsf{Rep}(G_1)$, $\mathsf{Rep}(G_2)$, then $\mathcal{T}_1 \otimes \mathcal{T}_2 \sim \mathsf{Rep}(G_1 \times G_2)$.

6 The fundamental group of a tannakian category

Let $\mathcal{T}$ be a tannakian category over k. For each fibre functor ω over a k-scheme S, $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$ (5.9) is an affine group scheme over S. Its formation is compatible with base change. By 5.11, the $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$ come from an affine $\mathcal{T}$-scheme.

DEFINITION 6.1. The ***fundamental group*** $\pi(\mathcal{T})$ of $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme satisfying functorially

$$\omega(\pi(\mathcal{T})) \simeq \underline{\mathrm{Aut}}^{\otimes}(\omega). \tag{1}$$

Let $X \in \mathrm{ob}\,\mathcal{T}$. For each fibre functor ω over S, $\omega(\pi(\mathcal{T})) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ acts on $\omega(X)$. We deduce an action (5.7) of $\pi(\mathcal{T})$ on X, functorial in X and compatible with tensor products. By passage to ind-objects, these actions furnish an action of $\pi(\mathcal{T})$ on all ind-objects. We deduce an action on all affine $\mathcal{T}$-schemes. The action of $\pi(\mathcal{T})$ on the $\mathcal{T}$-scheme $\pi(\mathcal{T})$ is the action of $\pi(\mathcal{T})$ on itself by inner automorphisms. Indeed, for any fibre functor ω, the action by functoriality of $\mathrm{Aut}^{\otimes}(\omega)$ on itself is its action by inner automorphisms.

6.2. Let X be a topological space, connected, locally connected, and locally simply connected. The vocabulary 6.1 provides the following analogy.

$\mathcal{T}$	X
object of $\mathcal{T}$	covering of X (=locally constant sheaf=local system on X)
fibre functor ω_0	point $x_0 \in X$
$\underline{\mathrm{Aut}}^{\otimes}(\omega_0)$	$\pi_1(X, x_0)$
$\pi(\mathcal{T})$	local system of the $\pi_1(X, x)$
action of $\pi(\mathcal{T})$ on Y in $\mathcal{T}$	action of the local system of the $\pi_1(X, x)$ on a locally constant sheaf.

This analogue, and that of Galois groups and π_1 (SGA 1, V, 8.1) led Grothendieck to define $\pi(\mathcal{T})$ and, for $\mathcal{T}$ the category of motives over k, he called it the ***motivic Galois group*** of k.

EXAMPLE 6.3. Let G be an affine group scheme over k and $\mathcal{T} = \mathsf{Rep}(G)$. After 5.16.1, the fundamental group $\pi(\mathcal{T})$, seen as an equivariant affine group scheme, is G equipped with the inner action on itself. The action of $\pi(\mathcal{T})$ on a representation V of G is the given action of G. It is G-equivariant,

$$h(gv) = hgh^{-1} \cdot hv.$$

6.4. Let $u\colon \mathcal{T}_1 \to \mathcal{T}$ be an exact k-linear tensor functor between tannakian categories over k. For any fibre functor ω on $\mathcal{T}$ over a k-scheme, $\omega \circ u$ is a fibre functor on $\mathcal{T}_1$ over S. We have

$$\underline{\mathrm{Aut}}^{\otimes}(\omega) \to \underline{\mathrm{Aut}}^{\otimes}(\omega \circ u) \tag{2}$$

The group $\mathcal{T}_1$-scheme $\pi(\mathcal{T}_1)$ defines, through the map u, a group $\mathcal{T}$-scheme $u\pi(\mathcal{T})$ and (2) is a morphism, functorial in ω, of $\omega(\pi(\mathcal{T}))$ into $\omega \circ u(\pi(\mathcal{T})) = \omega(u(\pi(\mathcal{T}_1)))$. By 5.11, it defines a morphism of $\mathcal{T}$-schemes

$$\pi(\mathcal{T}) \to u\pi(\mathcal{T}_1) \tag{3}$$

For any object X_1 of $\mathcal{T}_1$, the action 6.1 of $\pi(\mathcal{T}_1)$ on X_1 induces an action of $u\pi(\mathcal{T}_1)$ on uX_1. Via (3), this action induces the action of $\pi(\mathcal{T})$ on the object uX_1 of $\mathcal{T}$: this is indeed the case after the application of any fibre functor.

PROPOSITION 6.5. *With the preceding notation, u induces an equivalence of $\mathcal{T}_1$ with the category of objects of $\mathcal{T}$ equipped with an action of $u\pi(\mathcal{T}_1)$ extending the action of $\pi(\mathcal{T})$.*

PROOF. We give the proof only in the neutral case. The general case follows by Deligne 1990, 8.17

Let $\mathcal{T} = \mathsf{Rep}(G)$. Let ω be the forgetful functor. Let $G_1 = \underline{\mathrm{Aut}}^{\otimes}(\omega \circ u)$. The morphisms (2) define

$$f\colon G \to G_1, \tag{4}$$

equally deduced from (3) by applying ω. Via the equivalences $\mathcal{T} \sim \mathsf{Rep}(G)$, $\mathcal{T}_2 \sim \mathsf{Rep}(G_1)$, the functor u is the restriction to G (by f) of the action of G_1, and 6.5 reduces to the following triviality. For a vector space V, giving an action of G_1 on V is equivalent to giving an action of G plus a G-equivariant action of G_1 factoring through the action of G. □

6.6. While it is not necessary, assume again that $\mathcal{T}$ is neutal. After Saavedra 1972, II, 4.3.2 (g) , if u is fully faithful and identifies $\mathcal{T}_1$ to a full subcategory of $\mathcal{T}$ stable under subquotients, the morphisms (2) and (3) are epimorphisms (= are faithfully flat). If $H = \mathrm{Ker}(\pi(\mathcal{T}) \to u\pi(\mathcal{T}_1))$, 6.5 identifies $\mathcal{T}_1$ to the subcategory of $\mathcal{T}$ formed of the objects on which the action 6.1 of $\pi(\mathcal{T})$ induces the trivial action of H.

EXAMPLE 6.7. (i) For $\mathcal{T}_1$ the category of k-vector spaces, we have $\pi(\mathcal{T}_1) = \{e\}$ and the category of k-vector spaces can be identified by $V \rightsquigarrow V \otimes \mathbb{1}$ (5.6) with that of objects of $\mathcal{T}$ on which $\pi(\mathcal{T})$ acts trivially.

(ii) If k has characteristic 0 and ω_0 is a fibre functor with values in k, the semisimple objects of the abelian category of representations of the affine group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega_0)$ are the representations on which the unipotent radical $R_u\underline{\mathrm{Aut}}^{\otimes}(\omega_0)$ acts trivially. The subcategory $\mathcal{T}_1 \subset \mathcal{T}$ of semisimple objects is therefore stable under tensor products. The corresponding morphism (3) is

$$\pi(\mathcal{T}) \to \pi(\mathcal{T})/R_u\pi(\mathcal{T})$$

(for the definition of the second member, see 5.13).

(iii) Let T be an object of dimension 1 of $\mathcal{T}$. A representation ρ of $\mathbb{G}_m$ is the same thing as a graded vector space $V = \bigoplus V^j$, with $(\lambda)v^j = \lambda^j v^j$ for $v^j \in V^j$, and we define

$$u : \mathsf{Rep}(\mathbb{G}_m) \to \mathcal{T}$$

by $V \rightsquigarrow \bigoplus(V^j \otimes T^{\otimes j})$. From there, we get a morphism

$$\pi(\mathcal{T}) \to \mathbb{G}_m \tag{5}$$

such that the action of $\pi(\mathcal{T})$ on T factorizes through $\mathbb{G}_m$, with λ acting as multiplication by λ. In (5), we regard $\mathbb{G}_m$ as a group $\mathcal{T}$-scheme by 5.6.

If, for all $n > 0$, we have $\mathrm{Hom}(\mathbb{1}, T^{\otimes n}) = 0$, we can apply 6.6 to see that (5) is an epimorphism.

(iv) If the $T^{\otimes n}$ ($n \in \mathbb{Z}$) are the only simple objects of $\mathcal{T}$, and no two are isomorphic, we conclude from (ii) and (iii), at least in characteristic 0, that (5) makes $\pi(\mathcal{T})$ an extension of $\mathbb{G}_m$ by a unipotent group.

6.8. Let $\mathcal{T}$ be a tannakian category over a field k of characteristic 0 and, to simplify, suppose again that $\mathcal{T}$ is neutral. Let $\mathcal{T}^{\mathrm{ss}}$ be the category of semisimple objects of $\mathcal{T}$. The group $\mathcal{T}$-scheme $R_u\pi(\mathcal{T})$ acts trivially on $(R_u\pi(\mathcal{T}))^{\mathrm{ab}}$, which is a group $\mathcal{T}^{\mathrm{ss}}$-scheme. It is commutative and unipotent, and we can identify it with a pro-object $\mathcal{T}^{\mathrm{ss}}$ (either by Lie, cf. 4.8, or by writing it as a projective limit of vectorial group $\mathcal{T}$-schemes.

PROPOSITION 6.9. *With the preceding notation, for X semisimple in $\mathcal{T}$, we have*

$$\mathrm{Ext}^1(\mathbb{1}, X) \xrightarrow{\simeq} \mathrm{Hom}((R_u\pi(\mathcal{T}))^{\mathrm{ab}}, X). \tag{6}$$

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In (6), on the left X is an object of $\mathcal{T}$ and on the right it is the corresponding vectorial $\mathcal{T}$-scheme. We have

$$\mathrm{Hom}(R_u\pi(\mathcal{T}),X) \xrightarrow{\simeq} \mathrm{Hom}(R_u\pi(\mathcal{T}))^{\mathrm{ab}},X) \xrightarrow{\simeq} \mathrm{Hom}(\mathrm{Lie}(R_u\pi(\mathcal{T}))^{\mathrm{ab}},X).$$

If a group G acts on an extension E of A by B and acts trivially on A and B, the maps $\rho(g)-1: E \to E$ factor through morphisms from A to B. The principle 5.11, 5.13 allow us to repeat this "in $\mathcal{T}$".

If E is an extension of $\mathbb{1}$ by X, the action 6.1 of $R_u\pi(\mathcal{T}) \subset \pi(\mathcal{T})$ on E is trivial on $\mathbb{1}$ and X (6.7(ii)). It defines a morphism

$$R_u\pi(\mathcal{T}) \to \mathrm{Hom}(\mathbb{1},X) = X.$$

This construction defines the arrow (6).

PROOF. Injectivity: if the class of an extension E has trivial image under (6), the action of $R_u\pi(\mathcal{T})$ on E is trivial: E is semisimple and the extension is trivial.

Surjectivity: we may suppose that $\mathcal{T} = \mathsf{Rep}(G)$. Write G as a semi-direct product of a proreductive group scheme G^{ss} by R_uG (Levi decomposition). For (X,ρ) a representation of $G^{ss} = G/R_uG$ and a a G^{ss}-morphism of R_uG^{ab} into X, we define an extension E of the trivial representation by the representation X by making act $u \cdot g$ ($g \in G^{ss}$, $u \in R_uG$) on $\mathbb{1} \otimes X$ by $\begin{pmatrix} 1 & 0 \\ a(u) & \rho(g) \end{pmatrix}$. Its image by (6) is the morphism a. □

6.10 (NOTATION). For V a vector space over k and X in $\mathcal{T}$, $\underline{\mathrm{Hom}}(V,X)$ is the pro-object of $\mathcal{T}$, projective limit of the $W^\vee \otimes X$ for W a subspace of finite dimension of V.

Example: Let $\mathcal{T}$ be the category $\mathsf{Rep}(\mathbb{G}_m)$. Let $T(n)$ be the k-vector space on which $\lambda \in \mathbb{G}_m$ acts by multiplication by λ^n. For any pro-object X of $\mathcal{T}$, if we put $V(n) = \mathrm{Hom}(X,T(n))$, then we have

$$X = \prod_n \underline{\mathrm{Hom}}(V(n),T(n)). \tag{7}$$

6.11. Let $\mathcal{T}$ be a neutral tannakian category over k of characteristic 0 and $T \in \mathrm{ob}\,\mathcal{T}$. We assume that T has dimension 1 and we put $T(n) = T^{\otimes n}$. We assume that the morphism 6.7(iii) of $\pi(\mathcal{T})$ into $\mathbb{G}_m$ is an epimorphism with unipotent kernel, i.e., that the conditions of 6.7(iv) are fulfilled. Let $U = \mathrm{Ker}(\pi(\mathcal{T}) \to \mathbb{G}_m)$. Applying 6.9 and 6.10 and identifying U^{ab} to its Lie algebra, we find,

6.12. With the hypotheses and notation of 6.11

$$U/U^{\mathrm{ab}} = \prod \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Ext}^1(\mathbb{1},T(n)),$$

[Should be U^{ab}.]

6.13. To two fibre functors ω_1, ω_2 of $\mathcal{T}$ over S, we attach the affine scheme over S, $\mathrm{Isom}_S^{\otimes}(\omega_2,\omega_1)$. This construction is compatible with change of base. By 5.18 and 5.11, it defines a $\mathcal{T} \otimes \mathcal{T}$-scheme $G(\mathcal{T})$, with

$$(\omega_1 \otimes \omega_2)(G(\mathcal{T})) = \underline{\mathrm{Isom}}_S^{\otimes}(\omega_2,\omega_1).$$

It is the ***fundamental groupoid*** of $\mathcal{T}$.

For any mapping between finite sets $\varphi : I \to J$, we define $T(\varphi) : \mathcal{T}^{\otimes I} \to \mathcal{T}^{\otimes J}$ by

$$T(\varphi)(\boxtimes X_i) = \boxtimes_j \Big(\bigotimes_{\varphi(i)=j} X_i \Big),$$

where the tensor product is over the $i \in \varphi^{-1}(j)$ is taken in $\mathcal{T}$, and is $\mathbb{1}$ if $\varphi^{-1}(j) = \phi$.

Put $j_{a,b} = T(\varphi)$ for

$$\varphi : \{1,2\} \to \{1,2,3\}, \quad 1 \mapsto a,\ 2 \mapsto b.$$

Composition of isomorphisms defines

$$j_{1,2}(G(\mathcal{T})) \times j_{2,3}(G(\mathcal{T})) \to j_{1,3}(G(\mathcal{T})) \tag{8}$$

in $\mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T}$. For $\varphi : \{1,2\} \to \{1\}$, $T(\varphi)$ is $T : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$, $X \boxtimes_j Y \mapsto X \otimes_{\mathcal{T}} Y$. We have

$$T(G(\mathcal{T})) = \pi(\mathcal{T}). \tag{9}$$

For any fibre functor ω over S, $(\mathrm{pr}_1^* \omega, \mathrm{pr}_2^* \omega)$ defines a fibre functor $\omega \boxtimes \omega$ on $\mathcal{T} \otimes \mathcal{T}$ over $S \times S$. The image of $G(\mathcal{T})$ by $\omega \boxtimes \omega$ is the groupoid $\underline{\mathrm{Aut}}_k^{\otimes}(\omega) \overset{\mathrm{def}}{=} \underline{\mathrm{Isom}}_{S\times S}(\mathrm{pr}_2^* \omega, \mathrm{pr}_2^* \omega)$ over S, and the groupoid structure is deduced from (8).

6.14. We give in Deligne 1990, the following deseription of the algebra Λ of $\mathrm{Ind}(\mathcal{T} \otimes \mathcal{T})$ of which $G(\mathcal{T})$ is the spectrum (0.5): as ind-object, it is the target of the universal morphism

$$X^\vee \otimes_k X \to \Lambda \quad (X \text{ in } \mathcal{T}) \tag{10}$$

rendering, for all $f : X \to Y$ the following diagram commutative

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{f^t \otimes 1} & X^\vee \otimes X \\ \big\downarrow{\scriptstyle 1\otimes f} & & \big\downarrow \\ Y^\vee \otimes Y & \longrightarrow & \Lambda \end{array} \tag{6.14.2}$$

For any fibre functor ω over S, the groupoid $\underline{\mathrm{Aut}}_k^{\otimes}(\omega) \overset{\mathrm{def}}{=} \underline{\mathrm{Isom}}_{S\times S}^{\otimes}(\mathrm{pr}_2^* \omega, \mathrm{pr}_1^* \omega)$ is therefore the spectrum of $\omega \boxtimes \omega(\Lambda)$: the quasi-coherent sheaf of algebras L on $S \times S$ which, as a quasi-coherent sheaf, is the universal target of morphisms

$$\mathrm{pr}_1^* \omega(X)^\vee \otimes \mathrm{pr}_2^* \omega(X) \to L$$

(X in $\mathcal{T}$), satisfying a compactibility analogous to (6.14.2) for all $f : X \to Y$.