I HODGE CYCLES ON ABELIAN VARIETIES

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Introduction

The main result proved in these notes is that any Hodge cycle on an abelian variety (in characteristic zero) is an absolute Hodge cycle -- see §2 for definitions and (2.11) for a precise statement of the result.

The proof is based on the following two principles.

A. Let $t_1,\ldots,t_N$ be absolute Hodge cycles on a projective smooth variety X and let G be the largest algebraic subgroup of $GL(H^*(X,\mathbb{Q})) \times GL(\mathbb{Q}(1))$ fixing the t_i; any t on X fixed by G is an absolute Hodge cycle (see 3.8).

B. If $(X_s)_{s\in S}$ is an algebraic family of projective smooth varieties with S connected, and t_s is a family of rational cycles (i.e. a global section of ...) such that t_s is an absolute Hodge cycle for one s, then t_s is an absolute Hodge cycle for all s (see 2.12, 2.15).

Using B and the families of abelian varieties parametrized by Shimura varieties, one shows that it suffices to prove the main result for A an abelian variety of CM-type (see §6). Fix a CM-field E , which we can assume to be Galois over $\mathbb{Q}$, and let (A_α) be the family of all abelian varieties, up to E-isogeny, over $\mathbb{C}$ with complex multiplication by E . Principle B is used to construct some absolute Hodge cycles on varieties of the form $\bigoplus_{\alpha=1}^{d} A_\alpha$ -- the principle allows us to replace $\oplus A_\alpha$ by an abelian variety of the form $A_0 \otimes_{\mathbb{Q}} E$ (see §4). Let $G \subset GL(\oplus H_1(A_\alpha,\mathbb{Q})) \times GL(\mathbb{Q}(1))$ be the subgroup fixing the absolute Hodge cycles just constructed plus some other (obvious) absolute Hodge cycles. It is shown that G fixes every Hodge cycle on an A_α , and Principle A therefore completes the proof (see §5).

On analyzing which properties of absolute Hodge cycles are used in the above proof, one arrives at a slightly stronger result. Call a rational cohomology class c on a projective smooth variety X <u>accessible</u> if it belongs to the smallest family of rational cohomology classes such that:

(a) the cohomology class of any algebraic cycle is accessible;

(b) the pull-back by a map of varieties of an accessible class is accessible;

(c) if $t_1,\ldots,t_N \in H^*(X,\mathbb{Q})$ are accessible, and if a rational class t in some $H^{2p}(X,\mathbb{Q})$ is fixed by the algebraic subgroup G of $\mathrm{Aut}(H^*(X,\mathbb{Q}))$ (automorphisms of $H^*(X,\mathbb{Q})$ as a graded algebra) fixing the t_i , then t is accessible;

(d) Principle B , with "absolute Hodge" replaced by accessible, holds.

Sections 4, 5, 6 of these notes can be interpreted as proving that, when X is an abelian variety, any Hodge cycle (i.e., rational (p,p)-cycle) in $H^{2p}(X,\mathbb{Q})$ is accessible. Sections 2,3 define the notion of an absolute Hodge cycle and show that the family of absolute Hodge cycles satisfies (a), (b), (c), and (d) ; therefore an accessible class is absolutely Hodge. We have the implications:

$$\text{Hodge} \xRightarrow{\text{ab. var.}} \text{accessible} \Rightarrow \text{absolute Hodge} \Rightarrow \text{Hodge} .$$

Only the first implication is restricted to abelian varieties.

The remaining two sections, §1 and §7 , serve respectively to review the different cohomology theories and to give some applications of the main result to the algebraicity of certain products of special values of the Γ-function.

<u>Notations</u>: All algebraic varieties are complete and smooth over fields of characteristic zero unless stated otherwise. (The reader will lose little if he takes all varieties to be projective.) $\mathbb{C}$ denotes an algebraic closure of $\mathbb{R}$ and $i \in \mathbb{C}$ a square root of -1 ; thus i is defined only up to sign. A choice of i determines an orientation of $\mathbb{C}$ as a real manifold -- we take that for which $1 \wedge i > 0$ -- and hence an orientation of any complex manifold. Complex conjugation on $\mathbb{C}$ is denoted by ι or by $z \mapsto \bar{z}$. Recall that the category of abelian varieties up to isogeny is obtained from the category of abelian varieties by taking the same class of objects but replacing

Hom(A,B) by Hom(A,B) ⊗ $\mathbb{Q}$. We shall always regard an abelian variety as an object in the category of abelian varieties up to isogeny: thus Hom(A,B) is a vector space over $\mathbb{Q}$.

If (V_α) is a family of rational representations of an algebraic group G over k and $t_{\alpha,\beta} \in V_\alpha$, then the <u>subgroup of G fixing the</u> $t_{\alpha,\beta}$ is the algebraic subgroup H of G such that, for all k-algebras R, $H(R) = \{g \in G(R) \mid g(t_{\alpha,\beta} \otimes 1) = t_{\alpha,\beta} \otimes 1 \text{ , all } \alpha,\beta\}$. Linear duals are denoted by a superscript $\vee$. If X is a variety over a field k and σ is an embedding $\sigma\colon k \hookrightarrow k'$, then σX denotes $X \otimes_{k,\sigma} k'$ $(= X \times_{\mathrm{spec}(k)} \mathrm{spec}(k'))$.

1. Review of cohomology

Let X be a topological manifold and F a sheaf of abelian groups on X . We define

$$H^n(X,F) = H^n(\Gamma(X,F^\bullet))$$

where $F \to F^\bullet$ is any acyclic resolution of F ; thus $H^n(X,F)$ is uniquely defined, up to a unique isomorphism.

When F is the constant sheaf defined by a field K , these groups can be identified with singular cohomology groups as follows. Let $S_\bullet(X,K)$ be the complex in which $S_n(X,K)$ is the vector space over K with basis the singular n-simplices in X and the boundary map sends a simplex to the (usual) alternating sum of its faces. Set $S^\bullet(X,K) = \mathrm{Hom}(S_\bullet(X,K),K)$ with the boundary map for which

$$(\alpha,\sigma) \longmapsto \alpha(\sigma): \quad S^{\cdot}(X,K) \otimes S_{\cdot}(X,K) \to K$$

is a morphism of complexes, namely that defined by $(d\alpha)(\sigma) = (-1)^{\deg(\alpha)+1}\alpha(d\sigma)$.

<u>Proposition</u> 1.1. There is a canonical isomorphism $H^n(S^{\cdot}(X,K)) \xrightarrow{\approx} H^n(X,K)$.

<u>Proof</u>: If U is a unit ball, then $H^0(S^{\cdot}(U,K)) = K$ and $H^n(S^{\cdot}(U,K)) = 0$ for $n > 0$. Thus $K \to S^{\cdot}(U,K)$ is a resolution of the group K . Let $\underline{S}^n$ be the sheaf on X associated with the presheaf $V \mapsto S^n(V,K)$. The last remark shows that $K \to \underline{S}^{\cdot}$ is a resolution of the sheaf K . As each $\underline{S}^n$ is fine (Warner [1,5.32]), $H^n(X,K) = H^n(\Gamma(X,\underline{S}^{\cdot}))$. But the obvious map $S^{\cdot}(X,K) \to \Gamma(X,\underline{S}^{\cdot})$ is surjective with an exact complex as kernel (loc. cit.), and so

$$H^n(S^{\cdot}(X,K)) \xrightarrow{\approx} H^n(\Gamma(X,\underline{S}^{\cdot})) = H^n(X,K) .$$

Now assume X is a differentiable manifold. On replacing "singular n-simplex" by "differentiable singular n-simplex" in the above definitions, one obtains complexes $S^{\infty}_{\cdot}(X,K)$ and $S^{\cdot}_{\infty}(X,K)$. The same argument shows there is a canonical isomorphism $H^n_{\infty}(X,K) \overset{df}{=} H^n(S^{\infty}_{\cdot}(X,K)) \xrightarrow{\approx} H^n(X,K)$ (loc. cit.).

Let $O_{X^{\infty}}$ be the sheaf of C^{∞} real-valued functions on X , $\Omega^n_{X^{\infty}}$ the $O_{X^{\infty}}$-module of C^{∞} differential n-forms on X , and $\Omega^{\cdot}_{X^{\infty}}$ the complex

$$O_{X^{\infty}} \xrightarrow{d} \Omega^1_{X^{\infty}} \xrightarrow{d} \Omega^2_{X^{\infty}} \xrightarrow{d} \cdots$$

The de Rham cohomology groups of X are defined to be

$$H^n_{DR}(X) = H^n(\Gamma(X,\Omega^{\cdot}_{X^\infty})) = \{\text{closed n-forms}\}/\{\text{exact n-forms}\} .$$

If U is the unit ball, Poincaré's lemma shows that $H^0_{DR}(U) = \mathbb{R}$ and $H^n_{DR}(U) = 0$ for $n > 0$. Thus $\mathbb{R} \to \Omega^{\cdot}_{X^\infty}$ is a resolution of the constant sheaf $\mathbb{R}$, and as the sheaves $\Omega^n_{X^\infty}$ are fine (Warner [1,5.28]), we have $H^n(X,\mathbb{R}) = H^n_{DR}(X)$.

For $\omega \in \Gamma(X,\Omega^n_{X^\infty})$ and $\sigma \in S^\infty_n(X,\mathbb{R})$, define $<\omega,\sigma> = (-1)^{\frac{n(n+1)}{2}} \int_\sigma \omega \in \mathbb{R}$. Stokes's theorem states that $\int_\sigma d\omega = \int_{d\sigma}\omega$, and so $<d\omega,\sigma> + (-1)^n<\omega,d\sigma> = 0$. The pairing $<\ ,\ >$ therefore defines a map of complexes $f\colon \Gamma(X,\Omega^{\cdot}_{X^\infty}) \to S^{\cdot}_\infty(X,\mathbb{R})$.

<u>Theorem</u> 1.2 (de Rham): The map $H^n_{DR}(X) \to H^n_\infty(X,\mathbb{R})$ defined by f is an isomorphism for all n.

<u>Proof</u>: The map is inverse to the map $H^n_\infty(X,\mathbb{R}) \xrightarrow{\approx} H^n(X,\mathbb{R}) = H^n_{DR}(X)$ defined in the previous two paragraphs (Warner [1,5.36]). (Our signs differ from the usual because the standard sign conventions $\int_\sigma d\omega = \int_{d\sigma}\omega$, $\int_{X\times Y} pr_1^*\omega \wedge pr_2^*\eta = \int_X \omega \int_Y \eta$ etc. violate the standard sign conventions for complexes.)

A number $\int_\sigma \omega$, $\sigma \in H_n(X,\mathbb{Q})$, is called a <u>period</u> of ω. The map in (1.2) identifies $H^n(X,\mathbb{Q})$ with the space of classes of closed forms whose periods are all rational. Theorem 1.2 can be restated as follows: a closed differential form is exact if all its periods are zero; there exists a closed differential form having arbitrarily assigned periods on an independent set of cycles.

<u>Remark</u> 1.3 (Singer-Thorpe [1,6.2]). If X is compact then it has a smooth triangulation T . Define $S.(X,T,K)$ and $S^{\cdot}(X,T,K)$ as before, but using only simplices in T . Then the map $\Gamma(X,\Omega^{\cdot}_{X^\infty}) \to S^{\cdot}(X,T,K)$, defined by the same formulas as f above, induces isomorphisms $H^n_{DR}(X) \xrightarrow{\approx} H^n(S^{\cdot}(X,T,K))$.

Next assume that X is a complex manifold, and write $\Omega^{\cdot}_{X^{an}}$ for the complex

$$O_{X^{an}} \xrightarrow{d} \Omega^1_{X^{an}} \xrightarrow{d} \Omega^2_{X^{an}} \xrightarrow{d} \cdots$$

in which $\Omega^n_{X^{an}}$ is the sheaf of holomorphic differential n-forms. (Thus locally a section of $\Omega^n_{X^{an}}$ is of the form $\omega = \sum \alpha_{i_1\ldots i_n} dz_{i_1} \wedge \cdots \wedge dz_{i_n}$ with $\alpha_{i_1\ldots i_n}$ a holomorphic function and the z_i local coordinates.) The complex form of Poincaré's lemma shows that $\mathbb{C} \to \Omega^{\cdot}_{X^{an}}$ is a resolution of the constant sheaf $\mathbb{C}$, and so there is a canonical isomorphism $H^n(X,\mathbb{C}) \xrightarrow{\approx} \mathbb{H}^n(X,\Omega^{\cdot}_{X^{an}})$ (hypercohomology) .

If X is a compact Kähler manifold, the spectral sequence

$$E_1^{p,q} = H^q(\Omega^p_{X^{an}}) \Longrightarrow \mathbb{H}^{p+q}(\Omega^{\cdot}_{X^{an}})$$

degenerates, and so provides a canonical splitting $H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(X,\Omega^p_{X^{an}})$ (the Hodge decomposition); moreover $H^{p,q} \overset{\text{df}}{=\!=} H^q(X,\Omega^p_{X^{an}})$ is the complex conjugate of $H^{q,p}$ relative to the real structure $H^n(X,\mathbb{R}) \otimes \mathbb{C} \xrightarrow{\approx} H^n(X,\mathbb{C})$ (Weil [2]).

The decomposition has the following explicit description: the complex $\Omega^{\cdot}_{X^{\infty}} \otimes \mathbb{C}$ of sheaves of complex-valued differential forms on the underlying differentiable manifold is an acyclic resolution of $\mathbb{C}$, and so $H^n(X,\mathbb{C}) = H^n(\Gamma(X,\Omega^{\cdot}_{X^{\infty}} \otimes \mathbb{C}))$; Hodge theory shows that each element of the second group is represented by a unique harmonic n-form, and the decomposition corresponds to the decomposition of harmonic n-forms into sums of harmonic (p,q)-forms, $p + q = n$.

Finally, let X be an algebraic variety over a field k . If $k = \mathbb{C}$ then X defines a compact complex manifold X^{an} , and there are therefore groups $H^n(X^{an},\mathbb{Q})$, depending on the map $X \to \mathrm{spec}(\mathbb{C})$, that we shall write $H^n_B(X)$ (here B abbreviates Betti). There exist canonical Hodge decompositions: $H^n_B(X) = \bigoplus_{p+q=n} H^{p,q}(X)$, $\overline{H^{p,q}} = H^{q,p}$. If X is projective, then the choice of a projective embedding determines a Kähler structure on X^{an} , and hence a Hodge decomposition (which is independent of the choice of the embedding because it is determined by the Hodge filtration, and the Hodge filtration depends only on X ; see 1.4). In the general case we refer to Deligne [1,5.3,5.5] for the existence of the decompositions.

For an arbitrary k and an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ we write $H^n_\sigma(X)$ for $H^n_B(\sigma X)$ and $H^{p,q}_\sigma(X)$ for $H^{p,q}(\sigma X)$. As ι defines a homeomorphism $\sigma X^{an} \to \iota\sigma X^{an}$, it induces an isomorphism $H^n_{\iota\sigma}(X) \xrightarrow{\approx} H^n_\sigma(X)$.

Let $\Omega^{\cdot}_{X/k}$ be the complex in which $\Omega^n_{X/k}$ is the sheaf of algebraic differential n-forms, and define the (algebraic) de Rham cohomology group $H^n_{DR}(X/k)$ to be $\mathbb{H}^n(X_{Zar},\Omega^{\bullet}_{X/k})$ (hypercohomology relative to the Zariski topology). For any map $\sigma\colon k \hookrightarrow k'$ there is a canonical isomorphism $H^n_{DR}(X/k) \otimes_{k,\sigma} k' \xrightarrow{\approx} H^n_{DR}(X \otimes_k k'/k')$. The spectral sequence

$$E_1^{p,q} = H^q(X_{Zar},\Omega^p_{X/k}) \Longrightarrow \mathbb{H}^{p+q}(X_{Zar},\Omega^{\bullet}_{X/k})$$

defines a filtration (the Hodge filtration) $F^pH^n_{DR}(X)$ on $H^n_{DR}(X)$ which is stable under base change.

<u>Theorem</u> 1.4. If $k = \mathbb{C}$ the obvious maps $X^{an} \to X_{Zar}, \Omega^{\bullet}_{X^{an}} \leftarrow \Omega^{\bullet}_X$, induce an isomorphism $H^n_{DR}(X) \xrightarrow{\approx} H^n_{DR}(X^{an}) = H^n(X^{an},\mathbb{C})$ under which $F^pH^n_{DR}(X)$ corresponds to $F^pH^n(X^{an},\mathbb{C}) \overset{df}{=} \bigoplus_{p' \geq p} H^{p',q'}$.

<u>Proof</u>: The initial terms of the spectral sequences

$$E_1^{p,q} = H^q(X_{Zar},\Omega^p_X) \Longrightarrow H^{p+q}_{DR}(X)$$

$$E_1^{p,q} = H^q(X^{an},\Omega^p_{X^{an}}) \Longrightarrow H^{p+q}_{DR}(X^{an})$$

are isomorphic. (See Serre [1] for the projective case, and Grothendieck [2] for the general case.) The theorem follows from this because, by definition of the Hodge decomposition, the filtration of $H^n_{DR}(X^{an})$ defined by the above spectral sequence is equal to the filtration of $H^n(X^{an},\mathbb{C})$ defined in the statement of the theorem.

It follows from the theorem and the discussion preceding it that any embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ defines an isomorphism $H^n_{DR}(X) \otimes_{k,\sigma} \mathbb{C} \xrightarrow{\approx} H^n_{\sigma}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ and, in particular, a k-structure on

$H^n_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{C}$. When $k = \mathbb{Q}$, this structure should be distinguished from the $\mathbb{Q}$-structure defined by $H^n_\sigma(X)$: the two are related by the periods (see below).

When k is algebraically closed we write $H^n(X, \mathbb{A}^f)$, or $H^n_{et}(X)$, for $H^n(X_{et}, \hat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $H^n(X_{et}, \hat{\mathbb{Z}}) = \varprojlim_m H^n(X_{et}, \mathbb{Z}/m\mathbb{Z})$ (étale cohomology). If X is connected, $H^0(X, \mathbb{A}^f) = \mathbb{A}^f$, the ring of finite adèles for $\mathbb{Q}$, which justifies the first notation. By definition, $H^n_{et}(X)$ depends only on X (and not on the map $X \to \operatorname{spec} k$). The map $H^n_{et}(X) \to H^n_{et}(X \otimes k')$ defined by an inclusion of algebraically closed fields $k \hookrightarrow k'$ is an isomorphism (special case of the proper base change theorem, Artin et.al.[1, XII]). The comparison theorem (ibid. XI) shows that, when $k = \mathbb{C}$, there is a canonical isomorphism $H^n_B(X) \otimes \mathbb{A}^f \xrightarrow{\approx} H^n_{et}(X)$. It follows that $H^n_B(X) \otimes \mathbb{A}^f$ is independent of the map $X \to \operatorname{spec} \mathbb{C}$, and that, over any (algebraically closed) field, $H^n_{et}(X)$ is a free $\mathbb{A}^f$-module.

$H^n(X, \mathbb{A}^f)$ can also be described as the restricted product of the spaces $H^n(X, \mathbb{Q}_\ell)$, ℓ prime, with respect to the subspaces $H^n(X, \mathbb{Z}_\ell)$.

Next we define the notion of a "Tate twist" in each of our three cohomology theories. For this we shall define objects $\mathbb{Q}(1)$, and set $H^n(X)(m) = H^n(X) \otimes \mathbb{Q}(1)^{\otimes m}$. We want $\mathbb{Q}(1)$ to be $H_2(\mathbb{P}^1)$ (realization of the Tate motive in the cohomology theory) but to avoid the possibility of introducing sign ambiguities, we shall define it directly:

$$\mathbb{Q}_B(1) = 2\pi i\ \mathbb{Q}$$

$$\mathbb{Q}_{et}(1) = \mathbb{A}^f(1) \overset{df}{=} (\varprojlim_m \mu_m(k)) \otimes_{\mathbb{Z}} \mathbb{Q},\ \mu_m(k) = \{\zeta \in k \mid \zeta^m = 1\}$$

$$\mathbb{Q}_{DR}(1) = k\ ,$$

and so

$$H^n_B(X)(m) = H^n_B(X) \otimes_{\mathbb{Q}} (2\pi i)^m \mathbb{Q} = H^n(X^{an}, (2\pi i)^m \mathbb{Q}) \qquad (k = \mathbb{C})$$

$$H^n_{et}(X)(m) = H^n_{et}(X) \otimes_{\mathbb{A}^f} (\mathbb{A}^f(1))^{\otimes m} = (\varprojlim_r H^n(X_{et}, \mu_r(k)^{\otimes m})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

(k alg. cl.)

$$H^n_{DR}(X)(m) = H^n_{DR}(X)\ .$$

These definitions extend in an obvious way to negative m ; for example we set $\mathbb{Q}_{et}(-1) = \mathrm{Hom}_{\mathbb{A}^f}(\mathbb{A}^f(1), \mathbb{A}^f)$ and define $H^n_{et}(X)(-m) = H^n_{et}(X) \otimes \mathbb{Q}_{et}(-1)^{\otimes m}$.

There are canonical isomorphisms

$$\mathbb{Q}_B(1) \otimes_{\mathbb{Q}} \mathbb{A}^f \xrightarrow{\approx} \mathbb{Q}_{et}(1) \qquad (k \text{ alg. cl., } k \subset \mathbb{C})$$

$$\mathbb{Q}_B(1) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\approx} \mathbb{Q}_{DR}(1) \otimes_k \mathbb{C} \qquad (k \subset \mathbb{C})$$

and hence canonical isomorphisms (the <u>comparison</u> isomorphisms)

$$H^n_B(X)(m) \otimes \mathbb{A}^f \xrightarrow{\approx} H^n_{et}(X)(m) \qquad (k \text{ alg. cl., } k \subset \mathbb{C})$$

$$H^n_B(X)(m) \otimes \mathbb{C} \xrightarrow{\approx} H^n_{DR}(X)(m) \otimes_k \mathbb{C} \qquad (k \subset \mathbb{C})\ .$$

To define the first, we note that exp defines an isomorphism $2\pi i\ \mathbb{Z}/m2\pi i\ \mathbb{Z} \xrightarrow{\approx} \mu_m(k)$; after passing to the limit over m, and tensoring with $\mathbb{Q}$, we obtain the required isomorphism $2\pi i\ \mathbb{A}^f \xrightarrow{\approx} \mathbb{A}^f(1)$. The second isomorphism is induced by the inclusions $2\pi i\,\mathbb{Q} \hookrightarrow \mathbb{C} \hookleftarrow k$. Although the Tate twist for de Rham cohomology is trivial, it should not be ignored. For example,

$$\begin{array}{ccc} H_B^n(X) \otimes \mathbb{C} & \xrightarrow{\approx} & H_B^n(X)(m) \otimes \mathbb{C} \quad (1 \mapsto (2\pi i)^m;\ \text{defined up to sign}) \\ \approx \Big\downarrow \text{canon.} & & \approx \Big\downarrow \text{canon.} \\ H_{DR}^n(X) & \xrightarrow{=} & H_{DR}^n(X)(m) \quad (k = \mathbb{C}) \end{array}$$

fails to commute by a factor of $(2\pi i)^m$.

In each cohomology theory there is a canonical way of associating a class $cl(Z)$ in $H^{2p}(X)(p)$ with an algebraic cycle Z on X of pure codimension p . Since our cohomology groups are without torsion, we can do this using Chern classes (Grothendieck [1]). Starting with a functorial homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X)(1)$, one use the splitting principle to define the Chern polynomial $c_t(E) = \Sigma\, c_p(E)t^p$, $c_p(E) \in H^{2p}(X)(p)$, of a vector bundle E on X. The map $E \longmapsto c_t(E)$ is additive, and therefore factors through the Grothendieck group of the category of vector bundles on X . But, as X is smooth, this group is the same as the Grothendieck group of the category of coherent O_X-modules, and we can therefore define

$$cl(Z) = \frac{1}{(p-1)!}\, c_p(O_Z)$$

(loc. cit 4.3).

In defining c_1 for the Betti and étale theories, we begin with the maps

$$\mathrm{Pic}(X) \longrightarrow H^2(X^{an}, 2\pi i\, \mathbb{Z})$$

$$\mathrm{Pic}(X) \longrightarrow H^2(X_{et}, \mu_m(k))$$

arising (as boundary maps) from the sequences

$$0 \longrightarrow 2\pi i\,\mathbb{Z} \longrightarrow O_{X^{an}} \xrightarrow{\exp} O^{\times}_{X^{an}} \longrightarrow 0$$

$$0 \longrightarrow \mu_m \longrightarrow O^{\times}_X \xrightarrow{m} O^{\times}_X \longrightarrow 0 .$$

For the de Rham theory, we note that the dlog map , $f \longmapsto \frac{df}{f}$, defines a map of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & O^{\times}_S & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & & \downarrow \mathrm{dlog} & & \downarrow & & \\ O_X & \xrightarrow{d} & \Omega^1_X & \xrightarrow{d} & \Omega^2_X & \xrightarrow{d} & \cdots \end{array}$$

and hence a map

$$c_1 : \mathrm{Pic}(X) = H^1(X, O^{\times}_X) = \mathbb{H}^2(X, 0 \to O^{\times}_X \to \cdots) \longrightarrow \mathbb{H}^2(X, \Omega^{\bullet}_X)$$

$$= H^2_{DR}(X) = H^2_{DR}(X)(1) .$$

It can be checked that the three maps c_1 are compatible with the comparison isomorphisms and it follows formally that the maps cl are also compatible. (At least, it does once one has checked that Gysin maps and multiplicative structures are compatible with the comparison isomorphisms.)

When $k = \mathbb{C}$, there is a direct way of defining a class $cl(Z) \in H_{2d-2p}(X(\mathbb{C}), \mathbb{Q})$ (singular homology; $d = \dim(X)$, $p = \mathrm{codim}(Z)$): the choice of an i determines an orientation

X and of the smooth part of Z , and there is therefore a topologically defined class $cl(Z) \in H_{2d-2p}(X(\mathbb{C}),\mathbb{Q})$. This class has the property that for $[\omega] \in H^{2d-2p}(X^{\infty},\mathbb{R}) = H^{2d-2p}(\Gamma(X,\Omega^{\cdot}_{X^{\infty}}))$ represented by the closed form ω ,

$$< cl(Z), [\omega] > = \int_Z \omega .$$

By Poincaré duality, $cl(Z)$ corresponds to a class $cl_{top}(Z) \in H_B^{2p}(X)$, whose image in $H_B^{2p}(X)(p)$ under the map induced by $1 \longmapsto 2\pi i: \mathbb{Q} \longrightarrow \mathbb{Q}(1)$ is known to be $cl_B(Z)$. The above formula becomes

$$\int_X cl_{top}(Z) \cup [\omega] = \int_Z \omega .$$

There are trace maps $(d = \dim X)$

$$Tr_B: \ H_B^{2d}(X)(d) \xrightarrow{\approx} \mathbb{Q}$$

$$Tr_{et}: \ H_{et}^{2d}(X)(d) \xrightarrow{\approx} \mathbb{A}^f$$

$$Tr_{DR}: \ H_{DR}^{2d}(X)(d) \xrightarrow{\approx} k$$

that are determined by the requirement $Tr(cl(point)) = 1$; they are compatible with the comparison isomorphisms. When $k = \mathbb{C}$, Tr_B and Tr_{DR} are equal to the composites

$$Tr_B: \ H_B^{2d}(X)(d) \xrightarrow{\approx} H_B^{2d}(X) \longrightarrow H^{2d}(\Gamma(\Omega^{\cdot}_{X^{\infty}})) \longrightarrow \mathbb{C}$$

$$1 \longmapsto 2\pi i \qquad\qquad [\omega] \longmapsto \int_X \omega$$

$$Tr_{DR}: \ H_{DR}^{2d}(X)(d) = H_{DR}^{2d}(X) \xrightarrow{\approx} H^{2d}(\Gamma(\Omega^{\cdot}_{X^{\infty}})) \longrightarrow \mathbb{C}$$

$$[\omega] \longmapsto \frac{1}{(2\pi i)^d} \int_X \omega$$

where we have chosen an i and used it to orientate X. (Note that the composite maps are obviously independent of the choice of i.) The formulas in the last paragraph show that

$$\mathrm{Tr}_{DR}(\mathrm{cl}_{DR}(Z) \cup [\omega] = \frac{1}{(2\pi i)^{\dim Z}} \int_Z \omega \quad .$$

A definition of Tr_{et} can be found in (Milne [1, VI.11]).

We now deduce some consequences concerning periods.

<u>Proposition</u> 1.5. Let X be a variety over an algebraically closed field $k \subset \mathbb{C}$ and let Z be an algebraic cycle on $X_{\mathbb{C}}$ of dimension r. For any C^{∞} differential r-form ω on $X_{\mathbb{C}}$ whose class $[\omega]$ in $H^{2r}_{DR}(X_{\mathbb{C}})$ lies in $H^{2r}_{DR}(X)$,

$$\int_Z \omega \in (2\pi i)^r k \ .$$

<u>Proof</u>: We first note that Z is algebraically equivalent to a cycle Z_0 defined over k. In proving this we can assume Z to be prime. There exists a smooth (not necessarily complete) variety T over k, a subvariety $\tilde{Z} \subset X \times T$ that is flat over T, and a point $\mathrm{spec}\,\mathbb{C} \to T$ such that $Z = \tilde{Z} \times_T \mathrm{spec}\,\mathbb{C}$ in $X \times_T \mathrm{spec}\,\mathbb{C} = X_{\mathbb{C}}$. We can therefore take Z_0 to be $Z \times_T \mathrm{spec}\,k \subset X \times_T \mathrm{spec}\,k = X$ for any point $\mathrm{spec}\,k \to T$. From this it follows that $\mathrm{cl}_{DR}(Z) = \mathrm{cl}_{DR}(Z_0) \in H^{2r}_{DR}(X)(r)$ and $\mathrm{Tr}_{DR}(\mathrm{cl}_{DR}(Z) \cup [\omega]) \in k$. But we saw above that $\int_Z \omega = (2\pi i)^r\, \mathrm{Tr}_{DR}(\mathrm{cl}_{DR}(Z) \cup [\omega])$.

We next derive a classical relation between the periods of an elliptic curve. For a complete smooth curve X and an open affine subset U, the map

$$H^1_{DR}(X) \longrightarrow H^1_{DR}(U) = \Gamma(U,\Omega^1_X)/d\Gamma(U,\mathcal{O}_X)$$

$$= \frac{\{\text{meromorphic differentials, holomorphic on } U\}}{\{\text{exact differentials}\}}$$

is injective with image the set of classes represented by forms whose residues are all zero (such forms are said to be of the second kind). When $k = \mathbb{C}$, $Tr_{DR}([\alpha] \cup [\beta])$, where α and β are differential 1-forms of the second kind, can be computed as follows. Let Σ be the finite set of points where α or β has a pole. For z a local parameter at $P \in \Sigma$, α can be written $\alpha = (\sum_{-\infty \ll i}^{\infty} a_i z^i)dz$, with $a_{-1} = 0$. There therefore exists a meromorphic function a defined near P such that $da = \alpha$. We write $\int\alpha$ for any such function; it is defined up to a constant. As $Res_P \beta = 0$, $Res_P(\int\alpha)\beta$ is well-defined, and one proves that

$$Tr_{DR}([\alpha] \cup [\beta]) = \sum_{P \in \Sigma} Res_P(\int\alpha)\beta .$$

Now let X be the elliptic curve

$$y^2 z = 4x^3 - g_2 xz^2 - g_3 z^3$$

There is a lattice Λ in $\mathbb{C}$ and corresponding Weierstrass function $\wp(z)$ such that $z \mapsto (\wp(z), \wp'(z), 1)$ defines an isomorphism $\mathbb{C}/\Lambda \xrightarrow{\approx} X(\mathbb{C})$. Let γ_1 and γ_2 be generators of Λ such that the bases $\{\gamma_1, \gamma_2\}$ and $\{1, i\}$ of $\mathbb{C}$ have the same orientation. We can regard γ_1 and γ_2 as elements of $H_1(X, \mathbb{Z})$, and then $\gamma_1 \cdot \gamma_2 = 1$. The differentials

$\omega = \frac{dx}{y}$ and $\eta = \frac{xdx}{y}$ on X pull back to dz and $\wp(z)dz$ respectively on $\mathbb{C}$; the first is therefore holomorphic and the second has a single pole at $\infty = (0,1,0)$ on X with residue zero (because $0 \in \mathbb{C}$ maps to $\infty \in X$ and $\wp(z) = \frac{1}{z^2} + a_2z^2 + \ldots$). We find that

$$\mathrm{Tr}_{DR}([\omega] \cup [\eta]) = \mathrm{Res}_0(\textstyle\int dz)\wp(z)dz = \mathrm{Res}_0(z\wp(z)dz) = 1 .$$

Let $\int_{\gamma_i} \frac{dx}{y} = \int_{\gamma_i} \frac{dx}{\sqrt{4x^3-g_2x-g_3}} = \omega_i \qquad (i=1,2)$

and $\int_{\gamma_i} \frac{xdx}{y} = \int_{\gamma_i} \frac{xdx}{\sqrt{4x^3-g_2x-g_3}} = \eta_i \qquad (i=1,2)$

be the periods of ω and η. Under the map $H^1_{DR}(X) \to H^2(X,\mathbb{C})$, ω maps to $\omega_1\gamma_1' + \omega_2\gamma_2'$ and η maps to $\eta_1\gamma_1' + \eta_2\gamma_2'$, where $\{\gamma_1',\gamma_2'\}$ is the basis dual to $\{\gamma_1, \gamma_2\}$. Thus

$$\begin{aligned} 1 = \mathrm{Tr}_{DR}([\omega] \cup [\eta]) &= \mathrm{Tr}_B((\omega_1\gamma_1' + \omega_2\gamma_2') \cup (\eta_1\gamma_1' + \eta_2\gamma_2')) \\ &= (\omega_1\eta_2 - \omega_2\eta_1)\mathrm{Tr}_B(\gamma_1' \cup \gamma_2') \\ &= \frac{1}{2\pi i}(\omega_1\eta_2 - \omega_2\eta_1) . \end{aligned}$$

Hence $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$. This is the Legendre relation.

The next proposition shows how the existence of algebraic cycles can force algebraic relations between the periods of abelian integrals. Let X be an abelian variety over a subfield k of $\mathbb{C}$. Recall that $H^r(X) = \Lambda^r(H^1(X))$ and $H^1(X \times X \times \ldots) = H^1(X) \oplus H^1(X) \oplus \ldots$ (any cohomology theory). Let $\nu \in \mathbb{G}_m(\mathbb{Q})$ act on $\mathbb{Q}_B(1)$ as ν^{-1}; there is then a natural action of $GL(H^1_B(X)) \times \mathbb{G}_m$ on $H^r_B(X^n)(m)$ for any r,n, and m. We define G to be the subgroup of $GL(H^1_B(X)) \times \mathbb{G}_m$ fixing all tensors of the form $cl_B(Z)$, Z an algebraic cycle on some X^n. (See Notations for a precise description of what this means.)

Consider the canonical isomorphisms

$$H^1_{DR}(X) \otimes_k \mathbb{C} \xrightarrow{\approx} H^1(X^{an},\mathbb{C}) \xleftarrow{\approx} H^1_B(X) \otimes_{\mathbb{Q}} \mathbb{C} .$$

The periods p_{ij} of X are defined by the equations

$$\alpha_i = \sum p_{ji} a_j$$

where $\{\alpha_i\}$ and $\{a_i\}$ are bases for $H^1_{DR}(X)$ and $H^1_B(X)$ over k and $\mathbb{Q}$ respectively. The field $k(p_{ij})$ generated over k by the p_{ij} is independent of the bases chosen.

Proposition 1.6. With the above definitions

$$\text{tr.deg}_k\, k(p_{ij}) \leq \dim(G) .$$

Proof: We can replace k by its algebraic closure in $\mathbb{C}$, and hence assume that each algebraic cycle on $X_{\mathbb{C}}$ is equivalent to an algebraic cycle on X (see the proof of 1.5). Define P to be the functor of k-algebras such that an element of $P(A)$ is an isomorphism $p\colon H^1_B \otimes_{\mathbb{Q}} A \xrightarrow{\approx} H^1_{DR} \otimes_k A$ mapping $cl_B(Z) \otimes 1$ to $cl_{DR}(Z) \otimes 1$ for all algebraic cycles Z on a power of X. When $A = \mathbb{C}$, the comparison isomorphism is such a p, and so $P(\mathbb{C})$ is not empty. It is easily seen that P is represented by an algebraic variety that becomes a G_k-torsor under the

the obvious action. The bases $\{\alpha_i\}$ and $\{a_i\}$ can be used to identify the points of P with matrices. The matrix (p_{ij}) is a point of P with coordinates in $\mathbb{C}$, and so the proposition is a consequence of the following well-known lemma.

Lemma 1.7. Let $\mathbb{A}^N$ be an affine space over k , and let $z \in \mathbb{A}^N(\mathbb{C})$; the transcendence degree of $k(z_1,\ldots,z_N)$ over k is the dimension of the Zariski closure of $\{z\}$.

Remark 1.8. If X is an elliptic curve then dim G is 2 or 4 according as X has complex multiplication or not. Chudnovsky has shown that $\operatorname{tr\,deg}_k k(p_{ij}) = \dim(G)$ when X is an elliptic curve with complex multiplication. Does equality hold for all abelian varieties?

One of the main purposes of the seminar was to show (1.5) and (1.6) make sense, and remain true. if "algebraic cycle" is replaced by "Hodge cycle" (in the case the X is an abelian variety).

2. Absolute Hodge cycles; principle B.

Let k be an algebraically closed field of finite transcendence degree over $\mathbb{Q}$, and let X be a variety over k . Write $H^n_{\mathbb{A}}(X)(m) = H^n_{DR}(X)(m) \times H^n_{et}(X)(m)$; it is a free $k \times \mathbb{A}^f$-module. Corresponding to an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ there are canonical isomorphisms

$$\sigma^*_{DR}\colon\ H^n_{DR}(X)(m) \otimes_{k,\sigma} \mathbb{C} \xrightarrow{\approx} H^n_{DR}(\sigma X)(m)$$

$$\sigma^*_{et}\colon\ H^n_{et}(X)(m) \xrightarrow{\approx} H^n_{et}(\sigma X)(m)$$

whose product we write σ^* . The diagonal embedding $H^n_\sigma(X)(m) \hookrightarrow H^n_{DR}(\sigma X)(m) \times H^n_{et}(\sigma X)(m)$ induces an isomorphism $H^n_\sigma(X)(m) \otimes (\mathbb{C} \times \mathbb{A}^f) \xrightarrow{\approx} H^n_{DR}(\sigma X)(m) \times H^n_{et}(\sigma X)(m)$ (product of the comparison isomorphisms, §1). An element $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is a <u>Hodge cycle relative to σ</u>, if

(a) t is rational relative to σ , i.e., $\sigma^*(t)$ lies in the rational subspace $H^{2p}_\sigma(X)(p)$ of $H^{2p}_{DR}(\sigma X)(p) \times H^{2p}_{et}(\sigma X)(p)$;

(b) the first component of t lies in $F^0 H^{2p}_{DR}(X)(p) \overset{df}{=} F^p H^{2p}_{DR}(X)$.

Under the assumption (a), condition (b) is equivalent to requiring that the image of t in $H^{2p}_{DR}(X)(p)$ is of bidegree $(0,0)$. If t is a Hodge cycle relative to every embedding $\sigma: k \hookrightarrow \mathbb{C}$ then it is called an <u>absolute Hodge cycle</u>.

<u>Example</u> 2.1 (a) For any algebraic cycle Z on X , $t = (cl_{DR}(Z), cl_{et}(Z))$ is an absolute Hodge cycle. (The Hodge conjecture asserts there are no others.) Indeed, for any $\sigma: k \hookrightarrow \mathbb{C}$, $\sigma^*(t) = cl_B(Z)$, and is therefore rational, and it is well-known that $cl_{DR}(\sigma Z)$ is of bidegree (p,p) in $H^{2p}_{DR}(\sigma X)$.

(b) Let X be a variety of dimension d , and consider the diagonal $\Delta \subset X \times X$. Corresponding to the decomposition

$$H^{2d}(X \times X)(d) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$

(Künneth formula)

we have $cl(\Delta) = \sum_{i=0}^{2d} \pi^i$. The π^i are absolute Hodge cycles.

(c) Suppose that X is given with a projective embedding, and let $\gamma \in H^2_{DR}(X)(1) \times H^2_{et}(X)(1)$ be the class of a hyperplane section. The hard Lefschetz theorem shows that

$$H^{2p}(X)(p) \longrightarrow H^{2d-2p}(X)(d-p), \quad x \longmapsto \gamma^{d-2p}\cdot x$$

is an isomorphism. The class x is an absolute Hodge cycle if and only if $\gamma^{d-2p}\cdot x$ is an absolute Hodge cycle.

(d) Loosely speaking, any cycle that is constructed from a set of absolute Hodge cycles by a canonical rational process will again be an absolute Hodge cycle.

<u>Open Question</u> 2.2. Does there exist a cycle rational relative to every σ but which is not absolutely Hodge?

More generally, consider a family $(X_\alpha)_{\alpha\in A}$ of varieties over a field k (as above). Choose $(m(\alpha)) \in \mathbb{N}^{(A)}$, $(n(\alpha)) \in \mathbb{N}^{(A)}$, and $m \in \mathbb{Z}$, and write

$$T_{DR} = \otimes_\alpha H^{m(\alpha)}_{DR}(X_\alpha)\otimes\otimes_\alpha H^{n(\alpha)}_{DR}(X_\alpha)^\vee \quad (m)$$

$$T_{et} = \otimes_\alpha H^{m(\alpha)}_{et}(X_\alpha)\otimes\otimes_\alpha H^{n(\alpha)}_{et}(X_\alpha)^\vee \quad (m)$$

$$T_{\mathbb{A}} = T_{DR} \times T_{et}$$

$$T_\sigma = \otimes_\alpha H^{m(\alpha)}_\sigma(X_\alpha)\otimes\otimes_\alpha H^{n(\alpha)}_\sigma(X_\alpha)^\vee \quad (m) \;, \quad \sigma\colon \; k \hookrightarrow \mathbb{C} \;.$$

Then we say that $t \in T_{\mathbb{A}}$ is <u>rational relative to σ</u> if its image in $T_{\mathbb{A}} \otimes_{k \times \mathbb{A}^f, (\sigma,1)} \mathbb{C} \times \mathbb{A}^f$ is in T_σ, that it is a <u>Hodge cycle relative to σ</u> if it is rational relative to σ and its first component lies in F^0, and that it is an <u>absolute Hodge cycle</u> if it is a Hodge cycle relative to every σ.

Note that, for there to exist Hodge cycles in $T_{\mathbb{A}}$ it is necessary that $\Sigma\, m(\alpha) - \Sigma\, n(\alpha) = 2m$.

<u>Example</u> 2.3. Cup-product defines a map $T_{\mathbb{A}}^{m,n}(p) \times T_{\mathbb{A}}^{m',n'}(p') \longrightarrow T_{\mathbb{A}}^{m+m',n+n'}(p+p')$, and hence an element of $T_{\mathbb{A}}^{\vee} \otimes T_{\mathbb{A}}^{\vee} \otimes T_{\mathbb{A}}$; this element is an absolute Hodge cycle.

<u>Open Question</u> 2.4. Let $t \in H_{DR}^{2p}(X)(p)$ and suppose that $t \in F^0H_{DR}^{2p}(X)(p)$ and that, for all $\sigma\colon k \hookrightarrow \mathbb{C}$, $\sigma_{DR}^*(t) \in H_\sigma^{2p}(X)(p)$. Do these conditions imply that t is the first component of an absolute Hodge cycle?

In order to develop the theory of absolute Hodge cycles, we shall need to use the Gauss-Manin connection (Katz-Oda [1], Katz [1], Deligne [2]). Let k_0 be a field of characteristic zero and S a smooth k_0-scheme (or the spectrum of a finitely generated field over k_0). A <u>k_0-connection</u> on a coherent O_S-module $\mathcal{E}$ is a homomorphism of sheaves of abelian group

$$\nabla\colon\ \mathcal{E} \longrightarrow \Omega^1_{S/k_0} \otimes_{O_S} \mathcal{E}$$

such that

$$\nabla(fe) = f\nabla(e) + df \otimes e$$

for sections f of O_S and e of $\mathcal{E}$. The kernel of ∇, $\mathcal{E}^{\nabla}$, is the sheaf of horizontal sections of $(\mathcal{E},\nabla)$. Such a ∇ can be extended to a homomorphism of abelian sheaves,

$$\nabla_n\colon\ \Omega^n_{S/k_0} \otimes_{O_S} \mathcal{E} \longrightarrow \Omega^{n+1}_{S/k_0} \otimes_{O_S} \mathcal{E}$$

$$\omega \otimes e \longmapsto d\omega \otimes e + (-1)^n\, \omega \wedge \nabla(e)\ ,$$

and ∇ is said to be integrable if $\nabla_1 \circ \nabla = 0$. Moreover ∇ gives rise to an O_S-linear map

$$D \longmapsto \nabla_D\colon\ \mathrm{Der}(S/k_0) \longrightarrow \mathrm{End}_{k_0}(\mathcal{E})$$

$$\nabla_D = (\mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/k_0} \otimes_{O_S} \mathcal{E} \xrightarrow{D \otimes 1} O_S \otimes_{O_S} \mathcal{E} = \mathcal{E})\ .$$

Note that $\nabla_D(fe) = D(f)e + f\nabla_D(e)$. One checks that ∇ is integrable if and only if $D \mapsto \nabla_D$ is a Lie algebra homomorphism.

Now consider a proper smooth morphism $\pi\colon X \to S$, and write $\underline{H}^n_{DR}(X/S)$ for $\mathbb{R}^n\pi_*(\Omega^{\cdot}_{X/S})$. This is a locally free sheaf of O_S-modules and has a canonical connection ∇, the Gauss-Manin connection, which is integrable. It therefore defines a Lie algebra homomorphism $\mathrm{Der}(S/k_0) \longrightarrow \mathrm{End}_{k_0}(\underline{H}^n_{DR}(X/S))$. If $k_0 \hookrightarrow k_0'$ is an inclusion of fields and $X'/S' = (X/S) \otimes_{k_0} k_0'$, then the Gauss-Manin connection on $H^n_{DR}(X'/S')$ is $\nabla \otimes 1$. In the case that $k_0 = \mathbb{C}$, the relative form of Serre's GAGA theorem [1] shows that $\underline{H}^n_{DR}(X/S)^{an} = \underline{H}^n_{DR}(X^{an}/S^{an})$

and ∇ gives rise to a connection ∇^{an} on $\underline{H}^n_{DR}(X^{an}/S^{an})$. The relative Poincaré lemma shows that $(R^n\pi_* \mathbb{C}) \otimes O_{S^{an}} \xrightarrow{\approx} \underline{H}^n_{DR}(X^{an}/S^{an})$, and it is known that ∇^{an} is the unique connection such that

$$R^n \pi_*(\mathbb{C}) \xrightarrow{\approx} \underline{H}^n_{DR}(X^{an}/S^{an})^{\nabla^{an}} .$$

Proposition 2.5. Let $k_0 \subset \mathbb{C}$ have finite transcendence degree over $\mathbb{Q}$, let k be a field which is finitely generated over k_0 , let X be a variety over k , and let ∇ be the Gauss-Manin connection on $H^n_{DR}(X)$ relative to $X \to \operatorname{spec} k \to \operatorname{spec} k_0$. If $t \in H^n_{DR}(X)$ is rational relative to all embeddings of k into $\mathbb{C}$ then $\nabla t = 0$.

Proof: Choose a regular k_0-algebra A of finite type and a smooth projective map π: $X_A \to \operatorname{spec} A$ whose generic fibre is $X \to \operatorname{spec} k$ and which is such that t extends to an element of $\Gamma(\operatorname{spec} A, \underline{H}^n_{DR}(X/\operatorname{spec} A))$. After a base change relative to $k_0 \hookrightarrow \mathbb{C}$, we obtain maps

$$X_S \longrightarrow S \longrightarrow \operatorname{spec} \mathbb{C} \ , \ S = \operatorname{spec} A_{\mathbb{C}} \ ,$$

and a global section $t' = t \otimes 1$ of $\underline{H}^n_{DR}(X_S^{an}/S^{an})$. We have to show that $(\nabla \otimes 1)\, t' = 0$ or, equivalently, that t' is a global section of $\underline{H}^n(X_S^{an},\mathbb{C}) \overset{df}{=} R^n\pi_*^{an} \mathbb{C}$.

An embedding σ: $k \hookrightarrow \mathbb{C}$ gives rise to an injection $A \hookrightarrow \mathbb{C}$ (i.e. a generic point of spec A in the sense of Weil) and hence a point s of S . The hypotheses show that, at each of these points, $t(s) \in H^n(X_s^{an},\mathbb{Q}) \subset H^n_{DR}(X_s^{an})$. Locally on S, $\underline{H}^n_{DR}(X_S^{an}/S^{an})$ will be the sheaf of holomorphic sections of the trivial bundle, $S \times \mathbb{C}^m$, and $\underline{H}^n(X^{an},\mathbb{C})$ the sheaf of locally constant sections. Thus, locally, t' is a function $S \to S \times \mathbb{C}^m$, $s \longmapsto (t_1(s),\ldots,t_m(s))$. Each $t_i(s)$ is a holomorphic function which, by hypothesis, takes real (even rational) values on a dense subset of S. It is therefore constant.

Remark 2.6. In the situation of (2.5), assume that $t \in H^n_{DR}(X)$ is rational relative to one σ and horizontal for ∇. An argument similar to the above then shows that t is rational relative to all embeddings that agree with σ on k_0.

Corollary 2.7. Let $k_0 \subset k$ be algebraically closed fields of finite transcendence degree over $\mathbb{Q}$, and let X be a variety over k_0. If $t \in H^n_{DR}(X_k)$ is rational relative to all $\sigma\colon k \hookrightarrow \mathbb{C}$ then it is defined over k_0, i.e. it is in the image of $H^n_{DR}(X) \longrightarrow H^n_{DR}(X_k)$.

Proof: Let k' be a subfield of k which is finitely generated over k_0 and such that $t \in H^n_{DR}(X \otimes_{k_0} k')$. The hypothesis shows that $\nabla t = 0$, where ∇ is the Gauss-Manin connection for $X_{k'} \to \operatorname{spec} k' \to \operatorname{spec} k_0$. Thus, for any $D \in \operatorname{Der}(k'/k_0)$, $\nabla_D(t) = 0$. But $X_{k'}$ arises from a variety over k_0, and so the action of $\operatorname{Der}(k'/k_0)$ on $H^n_{DR}(X_{k'}) = H^n_{DR}(X) \otimes_{k_0} k'$ is through k': $\nabla_D = 1 \otimes D$. Thus the corollary follows from the next well-known lemma.

Lemma 2.8. Let $k_0 \subset k'$ be as above, and let $V = V_0 \otimes_{k_0} k'$ where V_0 is a vector space over k_0. If $t \in V$ is fixed (i.e. killed) by all derivations of k/k_0, then $t \in V_0$.

Let $C^p_{AH}(X)$ denote the subset of $H^{2p}_{\mathbb{A}}(X)(p)$ of absolute Hodge cycles; it is a finite-dimensional vector space over $\mathbb{Q}$.

Proposition 2.9 (a) Let X be a variety over an algebraically closed field k_0, let k be an algebraically closed field containing k_0, and assume that k_0 and k have finite transcendence degree over $\mathbb{Q}$. Then the canonical map

$$H^{2p}_{\mathbb{A}}(X)(p) \longrightarrow H^{2p}_{\mathbb{A}}(X_k)(p)$$

induces an isomorphism

$$C^p_{AH}(X) \xrightarrow{\approx} C^p_{AH}(X_k) .$$

(b) Let k be an algebraically closed field of finite transcendence degree over $\mathbb{Q}$, and let X_0 be a variety defined over a subfield k_0 of k whose algebraic closure is k; write $X = X_0 \otimes_{k_0} k$. Then $\mathrm{Gal}(k/k_0)$ acts on $C^p_{AH}(X)$ through a finite quotient.

Proof (a) The map is injective, and a cycle on X is absolutely Hodge if and only if it is absolutely Hodge on X_k, and so the only non-obvious step is to show that an absolute Hodge cycle t on X_k arises from a cycle on X. But (2.8) shows that t_{DR} arises from an element of $H^{2p}_{DR}(X)(p)$, and

$H^{2p}_{et}(X)(p) \to H^{2p}_{et}(X_k)(p)$ is an isomorphism .

(b) It is obvious that the action of $\mathrm{Gal}(k/k_0)$ on $H^{2p}_{DR}(X)(p) \times H^{2p}_{et}(X)(p)$ stabilizes $C^p_{AH}(X)$. We give three proofs that it factors through a finite quotient.

(i) Note that $C^p_{AH}(X) \to H^{2p}_{DR}(X)$ is injective. Clearly $H^{2p}_{DR}(X) = \cup H^{2p}_{DR}(X_0 \otimes k_i)$, where the k_i run through the finite extensions of k_0 contained in k , and hence all elements of a basis for $C^p_{AH}(X)$ lie in $H^{2p}_{DR}(X_0 \otimes k_i)$ for some i .

(ii) Note that $C^p_{AH}(X) \to H^{2p}(X_{et}, \mathbb{Q}_\ell)(p)$ is injective for any ℓ . The subgroup H of $\mathrm{Gal}(k/k_0)$ fixing $C^p_{AH}(X)$ is closed. Thus $\mathrm{Gal}(k/k_0)/H$ is a profinite group, which is countable since it is a subgroup of $GL_m(\mathbb{Q})$ for some m . It follows that it is finite.

(iii) A polarization on X gives a positive definite form on $C_{AH}(X)$, which is stable under $\mathrm{Gal}(k/k_0)$. This shows that the action factors through a finite quotient.

<u>Remark</u> 2.10 (a) All of the above is still valid if we work with a family of varieties (X_α) rather than a single X .
(b) Proposition (2.9) would remain true if we had defined an absolute Hodge cycle to be an element t of $F^0H^{2p}_{DR}(X)(p)$ such that, for all $\sigma\colon k \hookrightarrow \mathbb{C}$, $\sigma^*_{DR}(t) \in H^{2p}_{\sigma}(X)$.

Proposition (2.9) allows us to define the notion of an absolute Hodge cycle on any (complete smooth) variety X over a field k (of characteristic zero). If k is algebraically closed then we choose an algebraically closed subfield k_0 that is of finite transcendence degree over $\mathbb{Q}$ and such that

X has a model X_0 over k_0 ; then $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is an <u>absolute Hodge cycle</u> if it lies in the subspace $H^{2p}_{\mathbb{A}}(X_0(p))$ and is an absolute Hodge cycle there. The proposition shows that this definition is independent of the choice of k_0 and X_0 . (This definition is forced on us if we want (2.9a) to hold without restriction on the transcendence degrees of k and k_0 .) If k is not algebraically closed we choose an algebraic closure $\bar{k}$ of k and define an <u>absolute Hodge cycle</u> on X to be an absolute Hodge cycle on $X \otimes_k \bar{k}$ that is fixed by $\mathrm{Gal}(\bar{k}/k)$.

One can show that if k is algebraically closed and of cardinality not greater than that of $\mathbb{C}$, then $t \in H^{2p}_{DR}(X)(p) \times H^{2p}_{et}(X)(p)$ is an absolute Hodge cycle if it is rational relative to all embeddings $\sigma\colon k \hookrightarrow \mathbb{C}$ and $t_{dR} \in F^0H^{2p}_{DR}(X)(p)$. If $k = \mathbb{C}$ then the first condition has to be checked only for isomorphisms σ . (Provided the axiom of choice is assumed!) When $k = \mathbb{C}$ we define a <u>Hodge cycle</u> to be a cycle that is Hodge relative to $\sigma = \mathrm{id}\colon \mathbb{C} \hookrightarrow \mathbb{C}$.

<u>Main Theorem</u> 2.11. If X is an abelian variety over an algebraically closed field k , and t is a Hodge cycle on X relative to one embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, then it is an absolute Hodge cycle.

The proof will occupy most of the rest of these notes. We begin with a result concerning families of varieties parametrized by smooth (not necessarily complete) algebraic varieties over $\mathbb{C}$. Let S be such a parameter variety and let $\pi\colon X \to S$ be a smooth proper map. We write $\underline{H}^n_{et}(X)(r)$ for $\varprojlim_m (R^n\pi^{et}_* \mu_m^{\otimes r}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 2.12 (Principle B). Let t be a global section of $\underline{H}^{2p}_{DR}(X/S)(p) \times \underline{H}^{2p}_{et}(X)(p)$ such that $\nabla t_{DR} = 0$ and $(t_{DR})_s \in F^o H^{2p}_{DR}(X_s)(p)$ for all $s \in S$. If $t_s \in H^{2p}_{\mathbb{A}}(X_s)(p)$ is an absolute Hodge cycle for one s, it is an absolute Hodge cycle for all s.

Proof: Suppose that t_s is an absolute Hodge cycle for $s = s_1$, and let s_2 be a second point of S. We have to show that t_{s_2} is rational relative to an isomorphism $\sigma: \mathbb{C} \xrightarrow{\approx} \mathbb{C}$. On applying σ we obtain a map $\sigma X \to \sigma S$ and a global section σt of $\underline{H}^{2p}_{DR}(\sigma X/\sigma S)(p) \times \underline{H}^{2p}_{et}(\sigma X)(p)$. We know that $\sigma(t)_{\sigma(s_1)}$ is rational and have to show $\sigma(t)_{\sigma(s_2)}$ is rational. Clearly σ only translates the problem, and so we can omit it.

First consider the component t_{DR} of t. By assumption $\nabla t_{DR} = 0$, and so t_{DR} is a global section of $\underline{H}^{2p}(X^{an}, \mathbb{C})$. Since it is rational at one point, it must be rational at every point.

Next consider t_{et}. As $\underline{H}^{2p}_B(X) \overset{df}{=} R^{2p}\pi_*^{an}\mathbb{Q}$ and $\underline{H}^{2p}_{et}(X)$ are local systems (i.e. are locally constant), for any point s of S there are isomorphisms

$$\Gamma(S, \underline{H}^{2p}_B(X)(p)) \xrightarrow{\approx} H^{2p}_B(X_s)(p)^{\pi_1(S,s)} \quad \text{and}$$
$$\Gamma(S, \underline{H}^{2p}_{et}(X)(p)) \xrightarrow{\approx} H^{2p}_{et}(X_s)(p)^{\pi_1(S,s)} .$$

Consider,

$$\begin{array}{ccccc}
\Gamma(S,\underline{H}_B^{2p}(X)(p)) & \hookrightarrow & \Gamma(S,\underline{H}_B^{2p}(X)(p)) \otimes \mathbb{A}^f & = & \Gamma(S,\underline{H}_{et}^{2p}(X)(p)) \\
\downarrow \approx & & \downarrow \approx & & \\
H_B^{2p}(X_s)(p)^{\pi_1} & \hookrightarrow & H_B^{2p}(X_s)(p)^{\pi_1} \otimes \mathbb{A}^f & = & H_{et}^{2p}(X_s)(p)^{\pi_1} \\
\cap\downarrow & & \cap\downarrow & & \\
H_B^{2p}(X_s)(p) & \hookrightarrow & H_B^{2p}(X_s)(p) \otimes \mathbb{A}^f & = & H_{et}^{2p}(X_s)(p) .
\end{array}$$

We are given $t_{et} \in \Gamma(S,\underline{H}_{et}^{2p}(X)(p))$ and are told that its image in $H_{et}^{2p}(X_s)(p)$ is in $H_B^{2p}(X_s)(p)$ if $s = s_1$. The next easy lemma shows that t_{et} lies in $\Gamma(S,\underline{H}_B^{2p}(X)(p))$, and therefore is in $H_B^{2p}(X_s)(p)$ for all s.

Lemma 2.13. Let $V \hookrightarrow W$ be an inclusion of vector spaces, and let Z be a third vector space. Then $V \otimes Z \hookrightarrow W \otimes Z$, and $(V \otimes Z) \cap W = V$.

Remark 2.14. The assumption in the theorem that $(t_{dR})_s \in F^0H_{dR}^{2p}(X_s)(p)$ for all s is unnecessary; it is implied by the condition that $\nabla t_{DR} = 0$ (Deligne [4, 4.1.2, Théorème de la partie fixe]).

We shall also need a slight generalization of (2.12).

Theorem 2.15. Let $\pi: X \to S$ be as in (2.12), and let V be a local subsystem of $R^{2p}\pi_*\mathbb{Q}(p)$ such that V_s consists of (0,0)-cycles for all s and of absolute Hodge cycles for at least one s. Then V_s consists of absolute Hodge cycles for all s.

Proof. If V is constant, so that every element of V_s extends to a global section, then this follows from (2.12). The following argument reduces the general case to that case.

At each point $s \in S$, $R^{2p}\pi_* \mathbb{Q}(p)_s$ has a Hodge structure. Moreover $R^{2p}\pi_*\mathbb{Q}(p)$ has a polarization, i.e., there is a form ψ: $R^{2p}\pi_* \mathbb{Q}(p) \times R^{2p}\pi_* \mathbb{Q}(p) \to \mathbb{Q}$ which at each point s defines a polarization on the Hodge structure $R^{2p}\pi_* \mathbb{Q}(p)_s$. On $R^{2p}\pi_* \mathbb{Q}(p) \cap (R^{2p}\pi_* \mathbb{C}(p))^{0,0}$ the form is symmetric, bilinear, rational, and positive definite. Since the action of $\pi_1(S,s_0)$ preserves the form, the image of $\pi_1(S,s_0)$ in $\mathrm{Aut}(V_{s_0})$ is finite. Thus, after passing to a finite covering we can assume that V is constant.

Remark 2.16. Both (2.12) and (2.15) generalize, in an obvious way, to families π_α: $X_\alpha \to S$.

3. Mumford-Tate groups; principle A.

Let G be a reductive algebraic group over a field k of characteristic zero, and let $(V_\alpha)_{\alpha \in A}$ be a faithful family of finite-dimensional representations over k of G, so that the map $G \hookrightarrow \Pi GL(V_\alpha)$ is injective. For any $m \in \mathbb{N}^{(A)}$, $n \in \mathbb{N}^{(A)}$ we can form $T^{m,n} = \otimes V_\alpha^{\otimes m(\alpha)} \otimes \check{V}_\alpha^{\otimes n(\alpha)}$, which is again a representation of G. For any subgroup H of G we write H' for the subgroup of G fixing all tensors, occurring in some $T^{m,n}$, that are fixed by H. Clearly $H \subset H'$, and we shall need criteria guaranteeing their equality.

Proposition 3.1. The notations are as above.

(a) Any finite-dimensional representation of G is contained in a direct sum of representations $T^{m,n}$.

(b) (Chevalley's theorem). Any subgroup H of G is the stabilizer of a line D in some finite-dimensional representation of G .

(c) If H is reductive, or if $X_k(G) \to X_k(H)$ is surjective (or has finite cokernel), then $H = H'$. (Here $X_k(G)$ denotes $\mathrm{Hom}_k(G,\mathbb{G}_m)$).

Proof. (a) Let W be a representation of G , and let W_0 be the trivial representation (meaning $gw = w$, all $g \in G, w \in W$) with the same underlying vector space as W . Then $G \times W \to W$ defines a map $W \to W_0 \otimes k[G]$ which is G-equivariant (Waterhouse [1, 3.5]). Since $W_0 \otimes k[G] \approx k[G]^{\dim W}$, it suffices to prove (a) for the regular representation. There is a finite sum $V = \oplus V_\alpha$ such $G \to GL(V)$ is injective (because G is Noetherian). The map $GL(V) \to \mathrm{End}(V) \times \mathrm{End}(V^\vee)$ identifies $GL(V)$ (and hence G) with a closed subvariety of $\mathrm{End}(V) \times \mathrm{End}(V^\vee)$ (loc. cit.). There is therefore a surjection $\mathrm{Sym}(\mathrm{End}(V)) \times \mathrm{Sym}(\mathrm{End}(V^\vee)) \twoheadrightarrow k[G]$, where Sym denotes a symmetric algebra, and (a) now follows from the fact that representations of reductive groups are semisimple (see II.2).

(b) Let I be the idea of functions on G which are zero on H . Then, in the regular representation of G on $k[G]$, H is the stabilizer of I . Choose a finite-dimensional subspace W of $k[G]$ that is G-stable and contains a generating set for the ideal I . Then H is the stabilizer of the subspace $I \cap W$ of W , and of $\Lambda^d(I \cap W)$ in $\Lambda^d W$, where d is the dimension of $I \cap W$ (Borel [1, 5.1]).

(c) According to (b), H is the stabilizer of a line D in some representation V of G and it follows from (a) that V can be taken to be a direct sum of $T^{m,n}$ s .

Assume that H is reductive. Then $V = V' \oplus D$ for some H-stable V' and $\check{V} = \check{V}' \oplus \check{D}$. Thus H is the group fixing a generator of $D \otimes \check{D}$ in $V \otimes \check{V}$.

Assume that $X_k(G) \to X_k(H)$ is surjective, i.e. that any character of H extends to a character of G . The one-dimensional representation of H on D can be regarded as the restriction to H of a representation of G . Now H is the group fixing a generator of $D \otimes \check{D}$ in $V \otimes \check{D}$.

Remark 3.2 (a) It is clearly necessary to have some condition on H in order to have $H' = H$. For example, let B be a Borel subgroup of the reductive group G and let $v \in V$ be fixed by B . Then $g \longmapsto gv$ defines a map of algebraic varieties $G/B \to V$ which is constant because G/B is complete. Thus v is fixed by G , and $B' = G$.

However, the above argument shows the following: let H' be the group fixing all tensors occurring in subquotients of $T^{m,n}$s that are fixed by H ; then $H = H'$.

(b) In fact, in all our applications of (3.1c), H will be the Mumford-Tate group of a polarizable Hodge structure, and hence will be reductive. However the Mumford-Tate groups of mixed Hodge structure (even polarizable) need not be reductive, but may satisfy the second condition of (3.1c) (with $G = GL$) .

(c) The Theorem of Haboush (Demazure [1]) can be used to show that the second from of (3.1c) holds when k has characteristic p .

Let V be a finite-dimensional vector space over $\mathbb{Q}$. A $\mathbb{Q}$-rational <u>Hodge structure</u> of weight n on V is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$. Such a structure determines a map $\mu\colon \mathbb{G}_m \to GL(V_{\mathbb{C}})$ such that $\mu(\lambda)v^{p,q} = \lambda^{-p}v^{p,q}$, $v^{p,q} \in V^{p,q}$. The complex conjugate $\bar{\mu}$ of μ satisfies $\bar{\mu}(\lambda)v^{p,q} = \bar{\lambda}^{-q}v^{p,q}$. Since μ and $\bar{\mu}$ commute, their product determines a map of real algebraic groups $h\colon \mathbb{C}^{\times} \to GL(V_{\mathbb{R}})$, $h(\lambda)v^{p,q} = \lambda^{-p}\bar{\lambda}^{-q}v^{p,q}$. Conversely, a homomorphism $h\colon \mathbb{C}^{\times} \to GL(V_{\mathbb{R}})$ such that $\mathbb{R}^{\times} \hookrightarrow \mathbb{C}^{\times} \longrightarrow GL(V_{\mathbb{R}})$ is $\lambda \longmapsto \lambda^{-n}.\mathrm{id}$ defines a Hodge structure of weight n on V .

We write $F^pV = \bigoplus_{p' \geq p} V^{p',q'}$, so that $\ldots \supset F^pV \supset F^{p+1}V \supset \ldots$ is a decreasing filtration on V .

Let $\mathbb{Q}(1)$ denote the vector space $\mathbb{Q}$ with the unique Hodge structure such that $\mathbb{Q}(1)_{\mathbb{C}} = \mathbb{Q}(1)^{-1,-1}$; it has weight -2 and $h(\lambda)1 = \lambda\bar{\lambda}1$. For any integer m, $\mathbb{Q}(1)^{\otimes m} \stackrel{df}{=} \mathbb{Q}(m)$ $\mathbb{Q}(m)^{-m,-m}$ has weight $-2m$. (Strictly speaking, we should define $\mathbb{Q}(1) = 2\pi i\,\mathbb{Q}$ ....)

<u>Remark</u> 3.3. The notation $h(\lambda)\, v^{p,q} = \lambda^{-p}\bar{\lambda}^{-q}v^{p,q}$ is the negative of that used in Deligne [2] and Saavedra [1]. It is perhaps justified by the following. Let A be an abelian variety over $\mathbb{C}$. The exact sequences,

$$0 \longrightarrow \mathrm{Lie}(A^{\vee})^{\vee} \longrightarrow H_1(A,\mathbb{C}) \longrightarrow \mathrm{Lie}(A) \longrightarrow 0$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1H^1(A,\mathbb{C}) & \longrightarrow & H^1(A,\mathbb{C}) & \longrightarrow & F^1/F^2 & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & H^{1,0} = H^0(\Omega^1) & & & & H^{0,1} = H^1(O_X) & & \end{array}$$

are canonically dual. Since $H^1(A,\mathbb{C})$ has a natural Hodge structure of weight 1 with (1,0)-component $H^0(\Omega^1)$, $H_1(A,\mathbb{C})$ has a natural Hodge structure of weight -1 with (-1,0)-component Lie(A) . Thus $h(\lambda)$ acts as λ on Lie(A) , the tangent space to A at zero.

Let V be a vector space over $\mathbb{Q}$ with Hodge structure h of weight n . For $m_1, m_2 \in \mathbb{N}$ and $m_3 \in \mathbb{Z}$, $T = V^{\otimes m_1} \otimes V^{\vee \otimes m_2} \otimes \mathbb{Q}(1)^{\otimes m_3}$ has a Hodge structure of weight $(m_1-m_2)n - 2m_3$. An element of $T_{\mathbb{C}}$ is said to be <u>rational of bidegree</u> (p,q) if it lies in $T \cap T^{p,q}$. We let $\nu \in \mathbb{G}_m$ act on $\mathbb{Q}(1)$ as ν^{-1} ; there is then a canonical action of $GL(V) \times \mathbb{G}_m$ on T . The <u>Mumford-Tate</u> group G of (V,h) is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing all rational tensors of bidegree (0,0) belonging to any T . Thus projection on the first factor identifies $G(\mathbb{Q})$ with the set of $g \in GL(V)$ for which there exists a $\nu(g) \in \mathbb{Q}^{\times}$ with the property that $gt = \nu(g)^p t$ for any $t \in V^{\otimes m_1} \otimes V^{\vee \otimes m_2}$ of bidegree (p,p) .

<u>Proposition</u> 3.4. The group G is the smallest algebraic subgroup of $GL(V) \times \mathbb{G}_m$ defined over $\mathbb{Q}$ for which $\mu(\mathbb{G}_m) \subset G_{\mathbb{C}}$.

Proof: Let $cl(\mu)$ be the intersection of all $\mathbb{Q}$-rational algebraic subgroups of $GL(V) \times \mathbb{G}_m$ which, over $\mathbb{C}$, contain $\mu(\mathbb{G}_m)$. For any $t \in T$, t is of type $(0,0)$ if and only if it is fixed by $\mu(\mathbb{G}_m)$ or, equivalently, it is fixed by $cl(\mu)$. Thus $G = cl(\mu)'$ in the notation of (3.1) and the next lemma completes the proof.

Lemma 3.5. Any $\mathbb{Q}$-rational character of $cl(\mu)$ extends to a $\mathbb{Q}$-rational character of $GL(V) \times \mathbb{G}_m$.

Proof: Let χ: $cl(\mu) \to GL(W)$ be a representation of dimension one defined over $\mathbb{Q}$, i.e. a $\mathbb{Q}$-rational character. The restriction of the representation to $\mathbb{G}_m$ is isomorphic to $\mathbb{Q}(n)$ for some n. After tensoring W with $\mathbb{Q}(-n)$, we can assume that $\chi \circ \mu = 1$, i.e. $\mu(\mathbb{G}_m)$ acts trivially. But then $cl(\mu)$ must act trivially, and the trivial character extends to the trivial character.

Proposition 3.6. If V is polarizable then G is reductive.

Proof: Choose an i and write $C = h(i)$. (C is often called the Weil operator.) For $v^{p,q} \in V^{p,q}$, $Cv^{p,q} = i^{-p+q} v^{p,q}$, and so C^2 acts as $(-1)^n$ on V ($n = p+q$ is the weight of (V,h)).

We choose a polarization ψ for V. Recall that ψ is a morphism ψ: $V \otimes V \to \mathbb{Q}(-n)$ of Hodge structures such that the real-valued form $\psi(x,Cy)$ on $V_{\mathbb{R}}$ is symmetric and positive definite. Under the canonical isomorphism $\mathrm{Hom}(V \otimes V, \mathbb{Q}(-n)) \xrightarrow{\approx} V^{\vee} \otimes V^{\vee}(-n)$, ψ corresponds to a tensor of bidegree $(0,0)$ (because it is a morphism of Hodge structures) and therefore it is fixed by G: $\psi(g_1 v, g_1 v') = g_2^n \psi(v,v')$ for $(g_1,g_2) \in G \subset GL(V) \times \mathbb{G}_m$ and $v, v' \in V$.

Recall that if H is a real algebraic group and σ is an involution of $H_{\mathbb{C}}$, then the <u>real-form</u> of H defined by σ is a real algebraic group H_σ together with an isomorphism $H_{\mathbb{C}} \xrightarrow{\approx} (H_\sigma)_{\mathbb{C}}$ under which complex conjugation on $H(\mathbb{C})$ corresponds to $\sigma \circ$ (complex conjugation) on $H_\sigma(\mathbb{C})$. We are going to use the following criterion: a connected algebraic group H over $\mathbb{R}$ is reductive if it has a compact real-form H_σ. To prove the criterion it suffices to show that H_σ is reductive. On any finite-dimensional representation of V of H there is an H_σ-invariant positive-definite symmetric form, namely $\langle u,v \rangle_0 = \int_{H_\sigma} \langle hu,hv \rangle \, dh$ where $\langle\ ,\ \rangle$ is any positive-definite symmetric form on V. If W is an H_σ-stable subspace of V, then its orthogonal complement is also H_σ-stable. Thus every finite-dimensional representation of H_σ is semisimple, and this implies H_σ is reductive (see [II.2]).

We shall apply the criterion to the <u>special Mumford-Tate group</u> of (V,h), $G^0 \overset{df}{=} \mathrm{Ker}(G \to \mathbb{G}_m)$. Let G^1 be the smallest $\mathbb{Q}$-rational subgroup of $GL(V) \times \mathbb{G}_m$ such that $G^1_{\mathbb{R}}$ contains $h(U^1)$, where $U^1(\mathbb{R}) = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\}$. Then $G^1 \subset G$, and in fact $G^1 \subset G^0$. Since $G^1_{\mathbb{R}} \cdot h(\mathbb{C}^\times) = G_{\mathbb{R}}$ and $h(U^1) = \mathrm{Ker}(h(\mathbb{C}^\times) \to \mathbb{G}_m)$, it follows that $G^0 = G^1$, and therefore G^0 is connected.

Since $C = h(i)$ acts as $i\bar{i} = 1$ on $\mathbb{Q}(1)$, $C \in G^0(\mathbb{R})$. Its square C^2 acts as $(-1)^n$ on V and therefore lies in the centre of $G^0(\mathbb{R})$. The inner automorphism ad C of $G_{\mathbb{R}}$ defined by C is therefore an involution. For $u,v \in V_{\mathbb{C}}$ and $g \in G^0(\mathbb{C})$ we have

$$\psi(u,C\bar{v}) = \psi(gu,gC\bar{v}) = \psi(gu,CC^{-1}gC\bar{v}) = \psi(gu,C\overline{g^*v})$$

where $g^* = C^{-1}\bar{g}C = (\text{ad } C)(\bar{g})$. Thus the positive definite form $\phi(u,v) \overset{\text{df}}{=} \psi(u,Cv)$ on $V_{\mathbb{R}}$ is invariant under the real-form of G^0 defined by ad C, and so the real-form is compact.

<u>Example</u> 3.7. (Abelian variety of CM-type). Let F be a finite product of totally real number fields F_i, and E a product of fields, each of which is a quadratic imaginary extension of exactly one of the fields F_i. Let $S = \text{Hom}(E,\mathbb{C}) = \text{Hom}(E,\bar{\mathbb{Q}}) = \text{spec } E_{\mathbb{C}}$. $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on S and for any $\sigma \in S, \iota\sigma = \sigma\iota_{E/F}$, where $\iota_{E/F}$ is the canonical involution of E with fixed algebra F. A <u>CM-type</u> for E is a subset $\Sigma \subset S$ such that $S = \Sigma \cup \iota\Sigma$ (disjoint union). Correspondingly we define A to be $\mathbb{C}^{\Sigma}/\Sigma(O_E)$ where O_E, the ring of integers in E, is embedded in $\mathbb{C}^{\Sigma}$ by $u \longrightarrow (\sigma u)_{\sigma\in\Sigma}$. Obviously E acts on A; moreover $H_1(A,\mathbb{Q}) = E$, and

$$H_1(A) \otimes \mathbb{C} = E \otimes \mathbb{C} \xrightarrow{\approx} \mathbb{C}^S = \mathbb{C}^{\Sigma} \oplus \mathbb{C}^{\iota\Sigma}, \quad \text{with } \mathbb{C}^{\Sigma} \text{ the}$$
$$u \otimes 1 \longrightarrow (\sigma u)_{\sigma\in S}$$

(-1,0)-component of $H_1(A) \otimes \mathbb{C}$ and $\mathbb{C}^{\iota\Sigma}$ the (0,-1)-component. Thus $\mu(\lambda)$ acts as λ on $\mathbb{C}^{\Sigma}$ and 1 on $\mathbb{C}^{\iota\Sigma}$.

Let G be the Mumford-Tate group of $H_1(A)$. The actions of $\mu(\mathbb{C}^\times)$ and $E^\times$ on $H_1(A) \otimes \mathbb{C}$ commute. As $E^\times$ is its own commutant in $GL(H_1(A))$ this means that $\mu(\mathbb{C}^\times) \subset (E \otimes \mathbb{C})^\times$ and $G = cl(\mu) \subset E^\times$. In particular G is a torus, and can be described by its cocharacter group $Y(G) \overset{df}{=} Hom_{\overline{\mathbb{Q}}}(\mathbb{G}_m, G)$.

Clearly $Y(G) \subset Y(E^\times) \times Y(\mathbb{G}_m) = \mathbb{Z}^S \times \mathbb{Z}$. Note that $\mu \in Y(G)$ is equal to $\sum_{s \in \Sigma} e_s + e_0$, where $\{e_s\} \subset \mathbb{Z}^S$ is the basis dual to $S = \{s\} \subset X(E^\times)$ and e_0 is the element 1 of the last copy of $\mathbb{Z}$. The following are obvious:

(a) $(\mathbb{Z}^S \times \mathbb{Z})/Y(G)$ is torsion-free.

(b) $\mu \in Y(G)$.

(c) $Y(G)$ is stable under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$; thus $Y(G)$ is the Gal-module generated by μ.

(d) Since $\mu + \iota\mu = 1$ on S, $Y(G) \subset \{y \in \mathbb{Z}^S \times \mathbb{Z} \mid y = \sum n_s e_s + n_0 e_0, n_s + n_{\iota s} = cnst\}$.

This means $G(\mathbb{Q}) \subset Ker(N_{E/F}\colon E^\times \to F^\times/\mathbb{Q}^\times) \times \mathbb{Q}^\times$.

<u>Theorem</u> 3.8 (Principle A) Let $(X_\alpha)_\alpha$ be a family of varieties over $\mathbb{C}$ and consider spaces T obtained by tensoring spaces of the form $H_B^{n_\alpha}(X_\alpha)$, $H_B^{n_\alpha}(X_\alpha)^\vee$, and $\mathbb{Q}(1)$. Let $t_i \in T_i$, $i = 1,\ldots,N$, (T_i of the above type) be absolute Hodge cycles and let G be the subgroup of $\prod_{\alpha, n_\alpha} GL(H_B^{n_\alpha}(X_\alpha)) \times \mathbb{G}_m$ fixing the t_i. If t belongs to some T and is fixed by G, then it is an absolute Hodge cycle.

<u>Proof</u>: We remove the identification of the ground field k with $\mathbb{C}$. Let $\sigma\colon k \xrightarrow{\approx} \mathbb{C}$ be the isomorphism implicit in

statement of the theorem and let $\tau: k \xrightarrow{\approx} \mathbb{C}$ be a second isomorphism. We can assume that the t_i and t all belong to the same space T. The canonical inclusions of cohomology groups

$$H_\sigma(X_\alpha) \hookrightarrow H_\sigma(X_\alpha) \otimes (\mathbb{C} \times \mathbb{A}^f) \hookleftarrow H_\tau(X_\alpha)$$

induce maps

$$T_\sigma \hookrightarrow T \otimes (\mathbb{C} \times \mathbb{A}^f) \hookleftarrow T_\tau \; .$$

We shall regard these maps as inclusions. Thus $\{t_1,\ldots,t_N,t\} \subset T_\sigma \subset T \otimes (\mathbb{C} \times \mathbb{A}^f)$ and $\{t_1,\ldots,t_N\} \subset T_\tau$. To show that t is rational we have to show that $t \in T_\tau$.

Let P be the functor of $\mathbb{Q}$-algebras such that

$$P(R) = \{p\colon \; H_\sigma \otimes R \xrightarrow{\approx} H_\tau \otimes R \,|\, p \text{ maps } t_i (\text{in } T_\sigma) \text{ to } t_i (\text{in } T_\tau),\ i = 1,\ldots,N\} \; .$$

The existence of the canonical inclusions mentioned above shows that $P(\mathbb{C} \times \mathbb{A}^f)$ is non-empty, and it is easily checked that P is a G-torsor.

<u>Lemma</u> 3.9 Let P be a $\mathbb{Q}$-rational G-torsor of maps $H_\sigma^\alpha \xrightarrow{\approx} H_\tau^\alpha$ where $(H_\sigma^\alpha)_\alpha$ and $(H_\tau^\alpha)_\alpha$ are families of $\mathbb{Q}$-rational representations of G. Let T_σ and T_τ be like spaces of tensors constructed out of H_σ and H_τ respectively. Then P defines a map $T_\sigma^G \to T_\tau$.

Proof: Locally for the étale topology on spec($\mathbb{Q}$), points of P define maps $T_\sigma \xrightarrow{\approx} T_\tau$. The restriction to T_σ^G of such a map is independent of the point. Thus, by étale descent theory, they define a map of vector spaces $T_\sigma^G \longrightarrow T_\tau$.

On applying the lemma (and its proof) in the above situation we obtain a map $T_\sigma^G \longrightarrow T_\tau$ such that

$$\begin{array}{ccc} T_\sigma^G & \longrightarrow & T_\tau \\ \downarrow & & \downarrow \\ T_\sigma & \longrightarrow & T\otimes(\mathbb{C}\times \mathbb{A}^f) \end{array}$$

commutes. This means that $T_\sigma^G \subset T_\tau$, and therefore $t \in T_\tau$.

It remains to show that the first component t_{DR} of t, lying in $T\otimes\mathbb{C} = T_{DR}$, is in F^0T_{DR}. But in general, if s is rational and $s \in T_{DR}$, where T_{DR} is constructed out of spaces $H_{DR}^{n_\alpha}(X_\alpha)$, $H_{DR}^{n_\alpha}(X_\alpha)^\vee$, $\mathbb{Q}(1)$, then $s \in F^0T_{DR}$ is equivalent to s being fixed by $\mu(\mathbb{C}^\times)$. Thus $(t_i)_{DR} \in F^0$, $i = 1,\ldots,N$, implies $G \supset \mu(\mathbb{C}^\times)$, which implies $t_{DR} \in F^0$.

4. Construction of some absolute Hodge cycles

Recall that a number field E is a CM-field if, for any embedding $E \hookrightarrow \mathbb{C}$, complex conjugation induces a nontrivial automorphism $e \longmapsto \bar{e}$ of E that is independent of the embedding. The fixed field of the automorphism is then a totally real field F over which E has degree two.

A bi-additive form

$$\phi: \ V\times V \longrightarrow E$$

on a vector space V over such a field E is <u>Hermitian</u> if $\phi(ev,w) = e\phi(v,w)$ and $\phi(v,w) = \overline{\phi(w,v)}$ for $v,w \in V$, $e \in E$. For any embedding $\tau\colon F \hookrightarrow \mathbb{R}$ we obtain a Hermitian form ϕ_τ in the usual sense on the vector space $V_\tau = V \otimes_{F,\tau} \mathbb{R}$, and we let a_τ and b_τ be the dimensions of maximal subspaces of V_τ on which ϕ_τ is positive definite and negative definite respectively. If $d = \dim V$ then ϕ defines a Hermitian form on $\Lambda^d V$ that, relative to some basis vector, is of the form $(x,y) \longmapsto fx\bar{y}$. The element f is in F, and is independent of the choice of the basis vector up to multiplication by an element of $N_{E/F}E^\times$. It is called the discriminant of ϕ. Let $\{v_1,\ldots,v_d\}$ be an orthogonal basis for ϕ and let $\phi(v_i,v_i) = c_i$; then a_τ is the number of i for which $\tau c_i > 0$, b_τ the number of i for which $\tau c_i < 0$, and $f = \Pi c_i \pmod{N_{E/F}E^\times}$. Note that $\operatorname{sign}(\tau f) = (-1)^{b_\tau}$.

<u>Proposition</u> 4.1. Suppose given integers (a_τ, b_τ) for each τ, and an element $f \in F^\times/N_{E/F}E^\times$, such that $a_\tau + b_\tau = d$ all τ and $\operatorname{sign}(\tau f) = (-1)^{b_\tau}$. Then there exists a non-degenerate Hermitian form ϕ on a vector space V of dimension d with invariants (a_τ, b_τ) and f; moreover (V,ϕ). is unique up to isomorphism.

<u>Proof</u>: This result is due to Landherr [1]. Today one prefers to regard it as a consequence of the Hasse principle for simply-connected semisimple algebraic groups and the classification of Hermitian forms over local fields.

Corollary 4.2. Assume that the Hermitian space (V,ϕ) is non-degenerate and let $d = \dim(V)$. The following are equivalent:

(a) $a_\tau = b_\tau$ for all τ , and $\mathrm{disc}(f) = (-1)^{d/2}$;

(b) there is a totally isotropic subspace of V of dimension $d/2$.

Proof: Let W be a totally isotropic subspace of V of dimension $d/2$. The map $v \longmapsto \phi(-,v)\colon V \to W^\vee$ induces an anti-linear isomorphism $V/W \xrightarrow{\approx} W$. Thus a basis $v_1,\ldots,v_{d/2}$ of W can be extended to a basis $\{v_i\}$ of V such that

$$\phi(v_i,v_{d/2+i}) = 1,\ 1 \leq i \leq d/2 ,$$

$$\phi(v_i,v_j) = 0,\quad j \neq i \pm d/2 .$$

It is now easily checked that (V,ϕ) satisfies (a). Conversely (E^d,ϕ), where

$$\phi((a_i),(b_i)) = \sum_{1\leq i\leq d/2} a_i\bar{b}_{d/2+i} + \sum_{d/2<i\leq d} a_{d/2+i}\bar{b}_i ,$$

is, up to isomorphism, the only Hermitian space satisfying (a), and it also satisfies (b).

A Hermitian form satisfying the equivalent conditions of the corollary will be said to be split (because then $\mathrm{Aut}_E(V,\phi)$ is an E-split algebraic group).

We shall need the following (trivial) lemma.

Lemma 4.3. Let k be a field, let k' be an étale k-algebra (i.e., a finite product of separable finite field extensions of k), and let V be a free k'-module of finite rank.

(a) For any k'-linear map $f\colon V \to k'$, define $\mathrm{Tr}_{k'/k} f$ to be the k-linear map $v \longmapsto \mathrm{Tr}_{k'/k}(f(v))\colon V \to k$; then $f \longmapsto \mathrm{Tr}_{k'/k} f\colon \mathrm{Hom}_{k'}(V,k') \to \mathrm{Hom}_k(V,k)$ is an isomorphism.

(b) $\Lambda^n_{k'} V$ is, in a natural way, a direct summand of $\Lambda^n_k V$.

<u>Proof</u>: (a) Since the pairing $\mathrm{Tr}_{k'/k}\colon k' \times k' \to k$ is non-degenerate, it is obvious that $f \longmapsto \mathrm{Tr}_{k'/k} f$ is injective, and the two spaces have the same dimension over k.

(b) There are obvious maps $\Lambda^n_{k'} V \longrightarrow \Lambda^n_k V$

and $\Lambda^n_{k'} \check{V} \longrightarrow \Lambda^n_k \check{V}$

where $\check{V}$ is the k'-linear dual of V. But $(\Lambda \check{V}) = (\Lambda V)^{\vee}$, and so the second map gives rise to a map $\Lambda^n_k V \longrightarrow \Lambda^n_{k'} V$, which is inverse to the first. (More elegantly, descent theory shows that it suffices to prove the proposition with $k' = k^S$, $S = \mathrm{Hom}_k(k', \bar{k})$. Then $V = \bigoplus_{s\in S} V_s$ and the map in (a) sends $f = (f_s)$ to Σf_s, which is obviously an isomorphism. For (b), note that

$$\Lambda^n_k V = \bigoplus_{\Sigma n_s = n} \left(\bigotimes_{s\in S} \Lambda^{n_s} V_s \right) \supset \bigoplus_{s\in S} \Lambda^n V_s = \Lambda^n_{k'} V\,.)$$

Let A be an abelian variety over $\mathbb{C}$, E a CM-field, and $\nu\colon E \longrightarrow \mathrm{End}(A)$ a homomorphism (so, in particular, $\nu(1) = \mathrm{id}$). Let d be the dimension of $H_1(A,\mathbb{Q})$ over E, so that $d[E:\mathbb{Q}] = 2\dim(A)$. When $H_1(A,\mathbb{R})$ is identified with the tangent space to A at zero it acquires a complex structure; we denote by J the $\mathbb{R}$-linear endomorphism "multiplication by i" of $H_1(A,\mathbb{R})$. If $h\colon \mathbb{C}^\times \to GL(H^1(A,\mathbb{R})) = GL(H_1(A,\mathbb{R}))$ is the homomorphism determined by the Hodge structure on $H^1(A,\mathbb{R})$ then $h(i) = J$.

Corresponding to the decomposition

$$e \otimes z \longmapsto (\ldots,\sigma(e)z,\ldots)\colon E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\approx} \bigoplus_{\sigma \in S} \mathbb{C}, \; S = \mathrm{Hom}(E,\mathbb{C})$$

there is a decomposition

$$H_B^1(A) \otimes \mathbb{C} \xrightarrow{\approx} \bigoplus_{\sigma \in S} H_{B,\sigma}^1 \qquad \text{(E-linear isomorphism)}$$

such that $e \in E$ acts on the complex vector space $H_{B,\sigma}$ as $\sigma(e)$. Each $H_{B,\sigma}^1$ has dimension d, and (as E respects the Hodge structure on $H_B^1(A)$) acquires a Hodge structure,

$$H_{B,\sigma}^1 = H_{B,\sigma}^{1,0} \oplus H_{B,\sigma}^{0,1}.$$

Let $a_\sigma = \dim H_{B,\sigma}^{1,0}$ and $b_\sigma = \dim H_{B,\sigma}^{0,1}$; thus $a_\sigma + b_\sigma = d$.

<u>Proposition</u> 4.4: The subspace $\bigwedge_E^d H_B^1(A)$ of $H^d(A,\mathbb{Q})$ is purely of bidegree $(\frac{d}{2},\frac{d}{2})$ if and only if $a_\sigma = \frac{d}{2} = b_\sigma$.

<u>Proof</u>: Note that $H^d(A,\mathbb{Q}) = \bigwedge_{\mathbb{Q}}^d H^1(A,\mathbb{Q})$, and so (4.3) canonically identifies $\bigwedge_E^d H_B^1(A)$ with a subspace of $H_B^d(A)$.

As in the last line of the proof of (4.3) we have

$$(\Lambda_E^d H_B^1) \otimes \mathbb{C} = \Lambda_{E\otimes\mathbb{C}}^d \; H_B^1 \otimes \mathbb{C} = \bigoplus_{\sigma\in S} \Lambda^d H_{B,\sigma}^1 = \bigoplus_{\sigma\in S} \Lambda^d (H_{B,\sigma}^{1,0} \oplus H_{B,\sigma}^{0,1})$$

$$= \bigoplus_{\sigma\in S} \Lambda^{a_\sigma} H_{B,\sigma}^{1,0} \otimes \Lambda^{b_\sigma} H_{B,\sigma}^{0,1} ,$$

and $\Lambda^{a_\sigma} H_{B,\sigma}^{1,0}$ and $\Lambda^{b_\sigma} H_{B,\sigma}^{0,1}$ are purely of bidegree $(a_\sigma, 0)$ and $(0, b_\sigma)$ respectively.

Thus, in this case, $(\Lambda_E^d H_B^1(A))(\frac{d}{2})$ consists of Hodge cycles, and we would like to show that it consists of absolute Hodge cycles. In one special case this is easy.

<u>Lemma</u> 4.5. Let A_0 be an abelian variety of dimension $\frac{d}{2}$ and let $A = A_0 \otimes_{\mathbb{Q}} E$. Then $\Lambda_E^d H^1(A,\mathbb{Q})(\frac{d}{2}) \subset H^d(A,\mathbb{Q})(\frac{d}{2})$ consists of absolute Hodge cycles.

<u>Proof</u>: There is a commutative diagram

$$\begin{array}{ccc} H_B^d(A_0)(\frac{d}{2}) \otimes_{\mathbb{Q}} E & \longrightarrow & H_{\mathbb{A}}^d(A_0)(\frac{d}{2}) \otimes_{\mathbb{Q}} E \\ \downarrow \approx & & \downarrow \approx \\ (\Lambda_E^d H_B^1(A_0 \otimes_{\mathbb{Q}} E))(\frac{d}{2}) & \longrightarrow & (\Lambda_{E\otimes\mathbb{A}}^d H_{\mathbb{A}}^1(A_0 \otimes E))(\frac{d}{2}) \subset H_{\mathbb{A}}^d(A_0 \otimes E)(\frac{d}{2}) \end{array}$$

in which the vertical maps are induced by $H^1(A_0) \otimes E \xrightarrow{\approx} H^1(A_0 \otimes E)$. From this, and similar diagrams corresponding to isomorphisms $\sigma: \mathbb{C} \xrightarrow{\approx} \mathbb{C}$, one seesthat $H_{\mathbb{A}}^d(A_0)(\frac{d}{2}) \otimes E \hookrightarrow H_{\mathbb{A}}^d(A_0 \otimes E)(\frac{d}{2})$ induces an inclusion $C_{AH}^d(A_0) \otimes E \hookrightarrow C_{AH}^d(A_0 \otimes E)$. But $C_{AH}^d(A_0) = H_B^d(A_0)(\frac{d}{2})$ since $H_B^d(A_0)(\frac{d}{2})$ is a one-dimensional space generated by the class of any point on A_0 .

In order to prove the general result we need to consider families of abelian varieties (ultimately, we wish to apply (2.15)), and for this we need to consider polarized abelian varieties. A polarization θ on A is determined by a Riemann form, i.e. a $\mathbb{Q}$-bilinear alternating form ψ on $H_1(A,\mathbb{Q})$ such that the form $(z,w) \longmapsto \psi(z,Jw)$ on $H_1(A,\mathbb{R})$ is symmetric and definite; two Riemann forms ψ and ψ' on $H_1(A,\mathbb{Q})$ correspond to the same polarizaton if and only if there is an $a \in \mathbb{Q}^\times$ such that $\psi' = a\psi$. We shall consider only triples (A,θ,ν) in which the Rosati involution defined by θ induces complex conjugation on E. (The Rosati involution $e \longmapsto {}^te: \mathrm{End}(A) \to \mathrm{End}(A)$ is determined by the condition $\psi(ev,w) = \psi(v,{}^tew)$, $v,w \in H_1(A,\mathbb{Q})$.)

Lemma 4.6. Let $f \in E^\times$ be such that $\bar{f} = -f$, and let ψ be a Riemann form for A. There exists a unique E-Hermitian form ϕ on $H_1(A,\mathbb{Q})$ such that $\psi(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}(f\phi(x,y))$.

Proof: We first need:

Sublemma 4.7. Let V and W be finite-dimensional vector spaces over E, and let $\psi: V \times W \to \mathbb{Q}$ be a $\mathbb{Q}$-bilinear form such that $\psi(ev,w) = \psi(v,ew)$. Then there exists a unique E-bilinear form ϕ such that $\psi(v,w) = \mathrm{Tr}_{E/\mathbb{Q}}\phi(v,w)$.

Proof: ψ defines a $\mathbb{Q}$-linear map $V \otimes_E W \to \mathbb{Q}$, i.e. an element of $(V \otimes_E W)^\vee$. But $\mathrm{Tr}_{E/\mathbb{Q}}$ identifies the $\mathbb{Q}$-linear dual of $V \otimes_E W$ with the E-linear dual, and ψ with a ϕ.

To prove (4.6), we take V to be $H_1(A,\mathbb{Q})$ and W to be $H_1(A,\mathbb{Q})$ with E acting through complex conjugation, and apply (4.7). This shows that $\psi(x,y) = Tr_{E/\mathbb{Q}}\phi_1(x,y)$ with ϕ_1 sesquilinear. Let $\phi = f^{-1}\phi_1$, so that $\psi(x,y) = Tr_{E/\mathbb{Q}}(f\phi(x,y))$. Since ϕ is sesquilinear it remains to show that $\phi(x,y) = \overline{\phi(y,x)}$. As $\psi(x,y) = -\psi(y,x)$ for all $x,y \in H_1(A,\mathbb{Q})$, $Tr(f\phi(x,y)) = -Tr(f\phi(y,x)) = Tr(\bar{f}\phi(y,x))$.
On replacing x by ex with $e \in E$, we find that $Tr(fe\phi(x,y)) = Tr(\overline{fe}\phi(y,x))$. On the other hand $Tr(fe\phi(x,y)) = Tr(\overline{fe}\,\overline{\phi(x,y)})$ and, as $\overline{fe}$ is an arbitrary element of E, the non-degeneracy of the trace implies $\overline{\phi(x,y)} = \phi(y,x)$.
The uniqueness of ϕ is obvious from (4.7).

Theorem 4.8. Let A be an abelian variety over $\mathbb{C}$, and let $\nu\colon E \longrightarrow End(A)$ be a homomorphism, where E is a CM-field. Assume there exists a polarization θ for A such that:

(a) the Rosati involution of θ induces complex conjugation on E;

(b) there exists a split E-Hermitian form ϕ on $H_1(A,\mathbb{Q})$ and on $f \in E^{\times}$, with $\bar{f} = -f$, such that $\psi(x,y) \stackrel{df}{=} Tr_{E/\mathbb{Q}}(f\phi(x,y))$ is a Riemann form for θ.

Then the subspace $(\Lambda_E^d H^1(A,\mathbb{Q}))(\frac{d}{2}) \subset H^d(A,\mathbb{Q})(\frac{d}{2})$, where $d = \dim_E H_1(A,\mathbb{Q})$, consists of absolute Hodge cycles.

Proof: In the course of the proof we shall see that (b) implies that A satisfies the equivalent statements of (4.4). Thus the theorem will follow from (2.15), (4.4), and (4.5) once we have show there exists a connected smooth (not necessarily complete) variety S over $\mathbb{C}$ and an abelian scheme Y over S

together with an action ν of E on Y/S such that:

(a) for all $s \in S$, (Y_s, ν_s) satisfies the equivalent statements in (4.4);

(b) for some $s_0 \in S$, $Y_{s_0} = A_0 \otimes_{\mathbb{Q}} E$, with $e \in E$ acting as $\mathrm{id} \otimes e$;

(c) for some $s_1 \in S$, $(Y_s, \nu_{s_1}) = (A, \nu)$.

We shall first construct an analytic family of abelian varieties satisfying these conditions, and then pass to the quotient by a discrete group to obtain an algebraic family.

Let $H = H_1(A, \mathbb{Q})$, regarded as an E-space, and choose a θ, ϕ, f, and ψ as in the statement of the theorem. We choose i such that $\psi(x, h(i)y)$ is positive definite.

Consider the set of all quadruples $(A_1, \theta_1, \nu_1, k_1)$ in which A_1 is an abelian variety over $\mathbb{C}$, ν_1 is an action of E on A_1, θ_1 is a polarization of A, and k_1 is an E-linear isomorphism $H_1(A_1, \mathbb{Q}) \xrightarrow{\approx} H$ carrying a Riemann form for θ_1 into $c\psi$ for some $c \in \mathbb{Q}^\times$. From such a quadruple we obtain a complex structure on $H(\mathbb{R})$ (corresponding via k_1 to the complex structure on $H_1(A_1, \mathbb{R}) = \mathrm{Lie}(A_1)$) such that:

(a) the action of E commutes with the complex structure;

(b) ψ is a Riemann form relative to the complex structure.

Conversely, a complex structure on $H \otimes \mathbb{R}$ satisfying (a) and (b) determines a quadruple $(A_1, \theta_1, \nu_1, k_1)$ with $H_1(A_1, \mathbb{Q}) = H$

(as an E-module), $\mathrm{Lie}(A_1) = H \otimes \mathbb{R}$ (provided with the given complex structure), θ_1 the polarization with Riemann form ψ, and k_1 the identity map. Moreover two quadruples (A_1,θ_1,ν_1,k_1) and (A_2,θ_2,ν_2,k_2) are isomorphic if and only if they define the same complex structure on H. Let X be the set of complex structures on H satisfying (a) and (b). Our first task will be to turn X into an analytic manifold in such a way that the family of abelian varieties that it parametrizes becomes an analytic family.

A point of X is determined by an $\mathbb{R}$-linear map $J\colon H \otimes \mathbb{R} \to H \otimes \mathbb{R}$, $J^2 = -1$, such that

(a') J is E-linear, and

(b') $\psi(x,Jy)$ is symmetric and definite.

Note that $\psi(x,Jy)$ is symmetric if and only if $\psi(Jx,Jy) = \psi(x,y)$. Fix an isomorphism

$$E \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\approx} \bigoplus_{\tau\in T} \mathbb{C} \qquad (T = \mathrm{Hom}(F,\mathbb{R}),\ F = \text{real subfield of } E)$$

such that $f \otimes 1 \longmapsto (if_\tau)$ with $f_\tau \in \mathbb{R}$, $f_\tau > 0$. Corresponding to this isomorphism there is a decomposition

$$H \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\approx} \bigoplus_{\tau\in T} H_\tau$$

in which each H_τ is a complex vector space. Condition (a') implies $J = \oplus J_\tau$, where J_τ is a $\mathbb{C}$-linear isomorphism

$H_\tau \xrightarrow{\approx} H_\tau$ such that $J_\tau^2 = -1$. Let

$$H_\tau = H_\tau^+ \oplus H_\tau^-$$

where H_τ^+ and H_τ^- are the eigenspaces of J_τ with eigenvalues $+i$ and $-i$ respectively. The compatibility of ψ and ν implies

$$(H,\psi) \otimes \mathbb{R} \xrightarrow{\approx} \bigoplus_{\tau \in T} (H_\tau, \psi_\tau)$$

with ψ_τ an $\mathbb{R}$-bilinear alternating from on H_τ such that $\psi_\tau(zx,y) = \psi_\tau(x,\bar{z}y)$ for $z \in \mathbb{C}$. The condition $\psi(Jx,Jy) = \psi(x,y)$ implies that H_τ^+ is the orthogonal complement of H_τ^- relative to ψ_τ: $H_\tau = H_\tau^+ \perp H_\tau^-$. We also have

$$(H,\phi) \otimes \mathbb{R} \xrightarrow{\approx} \bigoplus_{\tau \in T} (H_\tau, \phi_\tau)$$

and $\psi_\tau(x,y) = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(if_\tau \phi_\tau(x,y))$. As $\psi(x,y) = \sum_\tau \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(if_\tau \phi_\tau(x_\tau,y_\tau))$, we find

$$\begin{aligned}
\psi(x,Jx) > 0, \text{ all } x &\iff \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(if_\tau \phi_\tau(x_\tau,Jx_\tau)) > 0, \\
&\qquad \text{all } x_\tau,\ \tau, \\
&\iff \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(i\phi_\tau(x_\tau,Jx_\tau)) > 0 \text{ all } x_\tau,\ \tau, \\
&\iff \begin{cases} \phi_\tau \text{ is positive definite on } H_\tau^+, \text{ and} \\ \phi_\tau \text{ is negative definite on } H_\tau^- . \end{cases}
\end{aligned}$$

This shows, in particular, that $H_\tau^+ = H_\tau^{-1,0}$ and $H_\tau^- = H_\tau^{0,-1}$ each have dimension $d/2$ (cf. 4.4). Let X^+ and X^- be the sets of $J \in X$ for which $\psi(x,Jy)$ is positive definite and negative definite respectively. Then X is a disjoint union $X = X^+ \cup X^-$. As J is determined by its $+i$ eigenspace we see that X^+ can be identified with

$$\{(V_\tau)_{\tau \in T} | V_\tau \text{ a maximal subspace of } H_\tau \text{ such that } \phi_\tau > 0 \text{ on } V_\tau\} .$$

This is an open connected complex submanifold of a product of Grassman manifolds

$$X^+ \subset \prod_{\tau \in T} \mathrm{Grass}_{d/2}(V_\tau) .$$

Moreover, there is an analytic structure on $X^+ \times V(\mathbb{R})$ such that $X^+ \times V(\mathbb{R}) \to X^+$ is analytic and the inverse image of $J \in X^+$ is $V(\mathbb{R})$ with the complex structure provided by J . On dividing $V(\mathbb{R})$ by an O_E-stable lattice $V(\mathbb{Z})$ in V , we obtain the sought analytic family B of abelian varieties.

Note that A is a member of the family. We shall next show that there is also an abelian variety of the form $A_0 \otimes E$ in the family. To do this we only have to show that there exists a quadruple $(A_1, \theta_1, \nu_1, k_1)$ of the type discussed above with $A_1 = A_0 \otimes E$. Let A_0 be any abelian variety of dimension $d/2$ and define $\nu_1 \colon E \to \mathrm{End}(A_0 \otimes E)$ so that $e \in E$ acts on $H_1(A_0 \otimes E) = H_1(A_0) \otimes E$ through its action on E . A Riemann form ψ_0 on A_0 extends in an obvious way to a Riemann form ψ_1 on A_1 that is compatible with the action of E . We define θ_1 to be the corresponding polarization, and let ϕ_1 be the Hermitian form on $H_1(A_0 \otimes E, \mathbb{Q})$ such that $\psi_1 = Tr_{E/\mathbb{Q}}(f\phi_1)$ (see 4.6) . If $I_0 \subset H_1(A_0, \mathbb{Q})$ is a totally isotropic subspace of $H_1(A_0, \mathbb{Q})$ of (maximum) dimension $d/2$ then $I_0 \otimes E$ is a totally isotropic subspace of dimensiona $d/2$ over E, which (by 4.2) shows that the Hermitian space $(H_1(A_0 \otimes E, \mathbb{Q}), \phi_1)$ is split. There is therefore an E-linear isomorphism k_1: $(H_1(A_0 \otimes E, \mathbb{Q}), \phi_1) \xrightarrow{\approx} (H, \phi)$, which carries $\psi_1 = Tr_{E/\mathbb{Q}}(f\phi_1)$ to $\psi = Tr_{E/\mathbb{Q}}(f\phi)$ This completes this part of the proof.

Let n be an integer ≥ 3, and let Γ be the set of O_E-isomorphisms g: $V(\mathbb{Z}) \to V(\mathbb{Z})$ preserving ψ and such that $(g-1)V(\mathbb{Z}) \subset nV(\mathbb{Z})$. Then Γ acts on X^+ by $J \longmapsto g \circ J \circ g^{-1}$ and (compatibly) on B . On forming the quotients, we obtain a map $\Gamma \backslash B \to \Gamma \backslash X^+$ which is an algebraic family of abelian varieties. In fact $\Gamma \backslash X^+$ is the moduli variety for quadruples $(A_1, \theta_1, \nu_1, k_1)$ in which A_1, θ_1 and ν_1 are essentially as before, but now k_1 is a level n structure k_1: $A_n(\mathbb{C}) = H_1(A, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\approx} V(\mathbb{Z})/nV(\mathbb{Z})$; the map $X^+ \to \Gamma \backslash X^+$ can be interpreted as "regard k_1 modulo n" . To prove these facts, one can use the theorem of Baily-Borel [1] to show that $\Gamma \backslash X^+$ is algebraic and a theorem of Borel [2] to show that $\Gamma \backslash B$ is algebraic — see §6 where we discuss a similar question in greater detail.

Remark 4.9. With the notations of the theorem, let G be the $\mathbb{Q}$-rational algebraic group such that

$G(\mathbb{Q}) = \{g \in GL_E(H) \mid \exists \nu(g) \in \mathbb{Q}^\times$ such that $\psi(gx,gy) = \nu(g)\psi(x,y),\ \forall x,y \in H\}$. The homomorphism $h: \mathbb{C}^\times \to GL(H \otimes \mathbb{R})$ defined by the Hodge structure on $H_1(A,\mathbb{Q})$ factors through $G_{\mathbb{R}}$, and X can be identified with the $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{C}^\times \to G_{\mathbb{R}}$ containing h . Let K be the compact open subgroup of $G(\mathbb{A}^f)$ of g such that $(g-1)V(\hat{\mathbb{Z}}) \subset nV(\hat{\mathbb{Z}})$. Then $\Gamma\backslash X^+$ is a connected component of the Shimura variety $Sh_K(G,X)$. The general theory shows that $Sh_K(G,X)$ is a fine moduli scheme (see Deligne [3,§4] or V.2 below) and so, from this point of view, the only part of the above proof that is not immediate is that the connected component of $Sh_K(G,X)$ containing A also contains a variety $A_0 \otimes E$.

<u>Remark</u> 4.10. It is easy to construct algebraic cycles on $A_0 \otimes E$; any $\mathbb{Q}$-linear map $\lambda: E \to \mathbb{Q}$ defines a map $A_0 \otimes E \to A_0 \otimes \mathbb{Q} = A_0$, and we can take $cl(\lambda)$ = image of the class of a point in $H^d(A_0) \to H^d(A_0 \otimes E)$. More generally we have $\mathrm{Sym}^*(\mathrm{Hom}_{\mathbb{Q}\text{-linear}}(E,\mathbb{Q})) \to \{\text{algebraic cycles on } A_0 \otimes E\}$. If $E = \mathbb{Q}^r$, this gives the obvious cycles.

<u>Remark</u> 4.11. The argument in the proof of (4.8) is similar to, and was suggested by, an argument of B. Gross [1].

5. <u>Completion of the proof for abelian varieties of CM-type</u>.

The <u>Mumford-Tate,</u> or <u>Hodge</u>, group of an abelian variety A over $\mathbb{C}$ is defined to be the Mumford-Tate group of the rational Hodge structure $H_1(A,\mathbb{Q})$: it is therefore the subgroup of $GL(H_1(A,\mathbb{Q})) \times \mathbb{G}_m$ fixing all Hodge cycles (see §3). In the language of the next article, the category of rational Hodge structures is Tannakian with an obvious fibre functor, and the Mumford-Tate group of A is the group associated with the subcategory generated by $H_1(A,\mathbb{Q})$ and $\mathbb{Q}(1)$.

An abelian variety A is said to be of <u>CM-type</u> if its Mumford-Tate group is commutative. Since any abelian variety A is a product $A = \Pi A_\alpha$ of simple abelian varieties (up to isogeny) and A is of CM-type if and only if each A_α is of CM-type (the Mumford-Tate group of A is contained in the product of those of the A_α) , in understanding this concept we can assume A is simple.

<u>Proposition</u> 5.1. A simple abelian variety A over $\mathbb{C}$ is of CM-type if and only if $E = \text{End } A$ is a commutative field over which $H_1(A,\mathbb{Q})$ has dimension 1. Then E is a CM-field, and the Rosati involution on $E = \text{End}(A)$ defined by any polarization of A is complex conjugation.

<u>Proof:</u> Let A be simple and of CM-type, and let $\mu: \mathbb{G}_m \to GL(H_1(A,\mathbb{C}))$ be defined by the Hodge structure on $H_1(A,\mathbb{C})$ (see §3). As A is simple, $E = \text{End}(A)$ is a field (possibly noncommutative) of degree $\leq \dim H_1(A,\mathbb{Q})$ over $\mathbb{Q}$. As for any abelian variety, End(A) is the subalgebra of $\text{End}(H_1(A,\mathbb{Q}))$ of elements commuting with the Hodge structure or, equivalently that commute with $\mu(\mathbb{G}_m)$ in $GL(H_1(A,\mathbb{C}))$.

If G is the Mumford-Tate group of A then $G_{\mathbb{C}}$ is generated by the groups $\{\sigma\mu(\mathbb{G}_m) \mid \sigma \in \mathrm{Aut}(\mathbb{C})\}$ (see 3.4). Therefore E is the commutant of G in $\mathrm{End}(H_1(A,\mathbb{Q}))$. By assumption G is a torus, and so $H_1(A,\mathbb{C}) = \bigoplus_{\chi \in X(G)} H_\chi$. The commutant of G therefore contains étale commutative algebras of rank $\dim H_1(A,\mathbb{Q})$ over $\mathbb{Q}$. It follows that E is a commutative field of degree $\dim H_1(A,\mathbb{Q})$ over $\mathbb{Q}$ (and that it is generated, as a $\mathbb{Q}$-algebra, by $G(\mathbb{Q})$; in particular, $h(i) \in E \otimes \mathbb{R}$).

Let ψ be a Riemann form corresponding to some polarization on A. The Rosati involution $e \longmapsto e^*$ on $\mathrm{End}(A) = E$ is determined by the condition $\psi(x,ey) = \psi(e^*x,y)$, $x,y \in H_1(A,\mathbb{Q})$. It follows from $\psi(x,y) = \psi(h(i)x,h(i)y)$ that $h(i)^* = h(i)^{-1} (= -h(i))$. The Rosati involution therefore is non-trivial on E, and E has degree 2 over its fixed field F. We can write $E = F[\sqrt{\alpha}]$, $\alpha \in F$, $\sqrt{\alpha}^* = -\sqrt{\alpha}$; α is uniquely determined up to multiplication by a square in F. If E is identified with $H_1(A,\mathbb{Q})$ through the choice of an appropriate basis vector, then $\psi(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}\, \alpha x y^*$, $x,y \in E$ (cf. 4.6). The positivity condition on ψ implies $\mathrm{Tr}_{E\otimes\mathbb{R}/\mathbb{R}} (\alpha h(i)^{-1} x x^*) > 0$, $x \neq 0$, $x \in E \otimes_{\mathbb{Q}} \mathbb{R}$. In particular, $\mathrm{Tr}_{F\otimes\mathbb{R}/\mathbb{R}}(fx^2) > 0$, $x \neq 0$, $x \in F \otimes \mathbb{R}$, $f = \alpha/h(i)$ which implies that F is totally real. Moreover, for every embedding $\sigma\colon F \hookrightarrow \mathbb{R}$ we must have $\sigma(\alpha) < 0$, for otherwise $E \otimes_{F,\sigma} \mathbb{R} = \mathbb{R} \times \mathbb{R}$ with $(r_1,r_2)^* = (r_2,r_1)$, and the positivity condition is impossible. Thus $\sigma(\alpha) < 0$, and $*$ is complex conjugation relative to any embedding of E in $\mathbb{C}$.

For the converse we only have to observe that $\mu(\mathbb{G}_m)$ commutes with $E \otimes \mathbb{R}$ in $\mathrm{End}(H_1(A,\mathbb{R}))$, and so if $H_1(A,\mathbb{Q})$ is of dimension 1 over E then $\mu(\mathbb{G}_m) \subset (E \otimes \mathbb{R})^{\times}$ and $G \subset E^{\times}$.

Let (A_α) be a finite family of abelian varieties over $\mathbb{C}$ of CM-type. We shall show that every element of a space

$$T_{\mathbb{A}} = (\otimes_\alpha H^1_{\mathbb{A}}(X_\alpha)^{\otimes m_\alpha}) \otimes (\otimes_\alpha H^1_{\mathbb{A}}(X_\alpha)^{\vee\otimes n_\alpha})(m)$$

that is a Hodge cycle (relative to $\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C}$) is an absolute Hodge cycle. According to (3.8) (Principle A) to do this it suffices to show that the following two subgroups of $GL(\Pi H_1(A_\alpha,\mathbb{Q})) \times \mathbb{G}_m$ are equal:

G^H = group fixing all Hodge cycles;

G^{AH} = group fixing all absolute Hodge cycles.

Obviously $G^H \subset G^{AH}$.

After breaking up each A_α into its simple factors, we can assume A_α itself is simple. Let E_α be the CM-field $\mathrm{End}(A_\alpha)$ and let E be the smallest Galois extension of $\mathbb{Q}$ containing all E_α; it is again a CM-field. Let $B_\alpha = A_\alpha \otimes_{E_\alpha} E$. It suffices to prove the theorem for the family (B_α) (because the Tannakian category generated by the $H_1(B_\alpha)$ and $\mathbb{Q}(1)$ contains every $H_1(A_\alpha)$; cf. the next article).

In fact we consider an even larger family. Fix E, a CM-field Galois over $\mathbb{Q}$, and consider the family (A_α) of all abelian varieties with complex multiplication by E

(so $H_1(A_\alpha)$ has dimension 1 over E) up to E-isogeny. This family is indexed by $\mathcal{S}$, the set of CM-types for E. Thus, if $S = \mathrm{Hom}(E,\mathbb{C})$ then each element of $\mathcal{S}$ is a set $\Phi \subset S$ such that $S = \Phi \cup \iota\Phi$ (disjoint union). We often identify Φ with the characteristic function of Φ, i.e. we write

$$\Phi(s) = 1 \; , \; s \in \Phi$$
$$\Phi(s) = 0 \; , \; s \notin \Phi \; .$$

With each Φ we associate the isogeny class of abelian varieties containing the abelian variety $\mathbb{C}^\Phi/\Phi(O_E)$ where O_E = ring of integers in E and $\Phi(O_E) = \{(\sigma e)_{\sigma\in\Phi} \in \mathbb{C}^\Phi \mid \sigma \in O_E\}$.

With this new family we have to show that $G^H = G^{AH}$. We begin by determining G^H (cf. 3.7). The Hodge structure on each $H_1(A_\Phi,\mathbb{Q})$ is compatible with the action of E. This implies that

$$G^H \subset \prod_{\Phi\in\mathcal{S}} GL(H_1(A_\Phi)) \times \mathbb{G}_m$$

commutes with $\prod_{\Phi\in\mathcal{S}} E^\times$. It is therefore contained in $\prod E^\times \times \mathbb{G}_m$. In particular G^H is a torus, and can be described by its group of cocharacters $Y(G^H) \overset{\mathrm{df}}{=} \mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathbb{G}_m,G^H)$ or its group of characters $X(G^H)$. Note that $Y(G^H) \subset Y(\prod_{\Phi\in\mathcal{S}} E^\times \times \mathbb{G}_m) = \mathbb{Z}^{S\times\mathcal{S}} \times \mathbb{Z}$. There is a canonical basis for $X(E^\times)$, namely S, and therefore a canonical basis for $X(\prod_{\Phi\in\mathcal{S}} E^\times \times \mathbb{G}_m)$ which we denote $((x_{s,\Phi}),x_0)$. We denote the dual basis for $Y(\Pi E^{\times} \times \mathbb{G}_m)$ by $(y_{s,\Phi}, y_0)$. The element $\mu \in Y(G^H)$ equals $\sum_{s,\Phi} \Phi(s) y_{s,\Phi} + y_0$ (see 3.7) . As $G^H_{\mathbb{C}}$ is generated by $\{\sigma\mu(\mathbb{G}_m) | \sigma \in \text{Aut } \mathbb{C}\}$, $Y(G^H)$ is the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-submodule of $Y(\Pi E^{\times} \times \mathbb{G}_m)$ generated by μ . ($\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on S by $\sigma s = s \circ \sigma^{-1}$; it acts on $Y(\Pi E^{\times} \times \mathbb{G}_m) = \mathbb{Z}^{S \times \mathcal{S}} \times \mathbb{Z}$ through its action on S: $\sigma y_{s,\Phi} = y_{\sigma s,\Phi}$; these actions factor through $\text{Gal}(E/\mathbb{Q})$).

To begin the computation of G^{AH} , we make a list of tensors that we know to be absolute Hodge cycles on the A_α .

(a) The endomorphisms $E \subset \text{End}(A_\Phi)$ for each Φ . (More precisely we mean the classes $cl_{\mathbb{A}}(\Gamma_e) \in H_{\mathbb{A}}(A_\Phi) \otimes H_{\mathbb{A}}(A_\Phi)$, Γ_e = graph of e , $e \in E$.)

(b) Let $(A_\Phi, \nu: E \hookrightarrow \text{End}(A_\Phi))$ correspond to $\Phi \in \mathcal{S}$, and let $\sigma \in \text{Gal}(E/\mathbb{Q})$. Define $\sigma\Phi = \{\sigma s | s \in \Phi\}$. There is an isomorphism $A_\Phi \to A_{\sigma\Phi}$ induced by

$$\begin{array}{ccc} \mathbb{C}^{\Phi} & \longrightarrow & \mathbb{C}^{\Phi} \\ \downarrow & & \downarrow \\ \mathbb{C}^{\Phi}/\Phi(O_E) & \longrightarrow & \mathbb{C}^{\sigma\Phi}/\sigma\Phi(O_E) \end{array} \qquad (\ldots, z(\tau), \ldots) \longmapsto (\ldots, z(\sigma\tau), \ldots)$$

whose graph is an absolute Hodge cycle. (Alternatively, we could have used the fact that $(A_\Phi, \sigma\nu: E \to \text{End}(A_\Phi))$, where $\sigma\nu = \nu \circ \sigma^{-1}$, is of type $\sigma\Phi$ to show that A_Φ and $A_{\sigma\Phi}$ are isomorphic.)

(c) Let $(\Phi_i)_{1 \le i \le d}$ be a family of elements of $\mathcal{S}$ and let $A = \bigoplus_{i=1}^{d} A_i$ where $A_i = A_{\Phi_i}$. Then E acts on A and

$H_1(A,\mathbb{Q}) = \bigoplus_{i=1}^{d} H_1(A_i,\mathbb{Q})$ has dimension d over E. Under the assumption that $\sum_i \Phi_i$ = constant (so that $\sum_i \Phi_i(s) = d/2$, all $s \in S$) we shall apply (4.8) to construct absolute Hodge cycles on A.

For each i, there is an E-linear isomorphism

$$H_1(A_i,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\approx} \bigoplus_{s \in S} H_1(A_i)_s$$

such that $s \in E$ acts on $H_1(A_i)_s$ as $s(e)$. From the definitions one sees that

$$\begin{aligned} H_1(A_i)_s &= H_1(A_i)_s^{-1,0}, & s \in \Phi_i \\ &= H_1(A_i)_s^{0,-1} & s \notin \Phi_i . \end{aligned}$$

Thus, with the notations of (4.4),

$$\begin{aligned} a_s &= \sum_i \Phi_i(s) \\ b_s &= \sum_i (1-\Phi_i(s)) = \sum_i \Phi_i(\iota s) = a_{\iota s} . \end{aligned}$$

The assumption that $\sum \Phi_i$ = constant therefore implies $a_s = b_s = d/2$, all s.

For each i, choose a polarization θ_i for A_i whose Rosati involution stablizes E, and let ψ_i be the corresponding Riemann form. For any totally positive elements f_i in F (the maximal totally real subfield of E) $\theta = \oplus f_i \theta_i$ is a polarization for A. Choose $v_i \neq 0$, $v_i \in H_1(A_i,\mathbb{Q})$; then $\{v_i\}$ is a basis for $H_1(A_i,\mathbb{Q})$ over E. There exists a $\zeta_i \in E^\times$ such that $\bar{\zeta}_i = -\zeta_i$ and $\psi_i(xv_i,yv_i) = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta_i x\bar{y})$

for all $x,y \in E$. Thus ϕ_i, where $\phi_i(xv_i,yv_i) = (\zeta_i/\zeta_1)x\bar{y}$, is an E-Hermitian form on $H_1(A_i,\mathbb{Q})$ such that $\psi(v,w) = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta_1\phi_i(v,w))$. The E-Hermitian form on $H_1(A,\mathbb{Q})$

$$\phi(\sum x_iv_i,\sum y_iv_i) = \sum_i f_i\phi_i(x_iv_i,y_iv_i)$$

is such that $\psi(v,w) = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta_1\phi(v,w))$ is the Riemann form of θ . The discriminant of ϕ is $\prod_i f_i(\zeta_i/\zeta_1)$. On the other hand, if $s \in S$ restricts to τ on F , then $\mathrm{sign}(\tau\mathrm{disc}(\phi)) = (-1)^{b_s} = (-1)^{d/2}$. Thus $\mathrm{disc}\ \phi = f(-1)^{d/2}$ for some totally positive element f of F . After replacing one f_i with f_i/f , we have that $\mathrm{disc}(f) = (-1)^{d/2}$, and that ϕ is split. Hence (4.8) applies.

In summary: let $A = \bigoplus_{i=1}^{d} A_{\Phi_i}$ be such that $\sum_i \Phi_i$ = constant; then $(\Lambda_E^d H^1(A,\mathbb{Q}))(d/2) \subset H^d(A,\mathbb{Q})(d/2)$ consists of absolute Hodge cycles.

Since G^{AH} fixes the absolute Hodge cycles of type (a), $G^{AH} \subset \prod_\Phi E^\times \times \mathbb{G}_m$. It is therefore a torus, and we have an inclusion

$$Y(G^{AH}) \subset Y(\Pi E^\times \times \mathbb{G}_m) = \mathbb{Z}^{S \times \mathcal{S}} \times \mathbb{Z}$$

and a surjection,

$$X(\Pi E^\times \times \mathbb{G}_m) = \mathbb{Z}^{S \times \mathcal{S}} \times \mathbb{Z} \twoheadrightarrow X(G^{AH}).$$

Let W be a space of absolute Hodge cycles. Under the action of the torus $\Pi E^\times \times \mathbb{G}_m$, $W \otimes \mathbb{C} \approx \oplus W_\chi$ where the sum is over $\chi \in X(\Pi E^\times \times \mathbb{G}_m)$ and the torus acts on W_χ through χ.

Since G^{AH} fixes the elements of W, the χ for which $W_\chi \neq 0$ map to zero in $X(G^{AH})$.

On applying this remark with W equal to the space of absolute Hodge cycles described in (b), we find that $x_{s,\Phi} - x_{\sigma s,\sigma\Phi}$ maps to zero in $X(G^{AH})$, all $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, $s \in S$, and $\Phi \in \mathcal{S}$. As $\mathrm{Gal}(E/\mathbb{Q})$ acts simply transitively on S, this implies that, for a fixed $s_0 \in S$, $X(G^{AH})$ is generated by the image of $\{x_{s_0,\Phi}, x_0 \mid \Phi \in \mathcal{S}\}$.

Let $d(\Phi) \geq 0$ be integers such that $\Sigma\, d(\Phi)\Phi = d/2$ (constant function on S) where $d = \Sigma d(\Phi)$. Then (c) shows that

$$W = \otimes_E H_1(A_\Phi,\mathbb{Q})^{\otimes_E d(\Phi)}(-d/2) = \Lambda^d_E\, H_1(\oplus A_\Phi^{d(\Phi)},\mathbb{Q})(-d/2)$$
$$\subset H_d(\oplus A_\Phi^{d(\Phi)},\mathbb{Q})(-d/2)\ .$$

consists of absolute Hodge cycles. The remark then shows that $\sum d(\Phi)x_{s,\Phi} - d/2$ maps to zero in $X(G^{AH})$ for all s.

Let $X = X(\Pi E^\times \times \mathbb{G}_m)/\sum \mathbb{Z}\,(x_{\sigma s,\sigma\Phi} - x_{s,\Phi})$, and regard $\{x_{s_0,\Phi}, x_0 \mid \Phi \in \mathcal{S}\}$ as a basis for X. We know that $X(\Pi E^\times \times \mathbb{G}_m) \twoheadrightarrow X(G^{AH})$ factors through X, and that therefore $Y \supset Y(G^{AH})$ $(\supset Y(G^H))$ where Y is the submodule of $Y(\Pi E^\times \times \mathbb{G}_m)$ dual to X.

<u>Lemma</u> 5.2. The submodule $Y(G^H)^\perp$ of X orthogonal to $Y(G^H)$ is equal to $\{\sum d(\Phi)x_{s_0,\Phi} - \frac{d}{2}x_0 \mid \sum d(\Phi)\Phi = \frac{d}{2}, \sum d(\Phi) = d\}$; it is generated by elements $\sum d(\Phi)x_{s_0,\Phi} - (d/2)x_0$, $\sum d(\Phi)\Phi = d/2$, $d(\Phi) \geq 0$ all Φ.

Proof: As $Y(G^H)$ is the $\mathrm{Gal}(E/\mathbb{Q})$-submodule of Y generated by μ, we see that $x = \sum d(\Phi)\, x_{s_0,\Phi} - d/2\, x_0 \in Y(G^H)^{\perp}$ if and only if $\langle \sigma\mu, x\rangle = 0$ all $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$. But $\mu = \sum \Phi(s) y_{s,\Phi} + y_0$ and $\sigma\mu = \sum \Phi(s) y_{\sigma s,\Phi} + x_0$, and so $\langle \sigma\mu, x\rangle = \sum d(\Phi)\Phi(\sigma^{-1}s_0) - d/2$. The first assertion is now obvious.

As $\Phi + \iota\Phi = 1$, $x_{s_0,\Phi} + x_{s_0,\iota\Phi} - x_0 \in Y(G^H)^{\perp}$ and has positive coefficients $d(\Phi)$. By adding enough elements of this form to an arbitrary element $x \in Y(G^H)^{\perp}$ we obtain an element with coefficients $d(\Phi) \geq 0$, which completes the proof of the lemma.

The lemma shows that $Y(G^H)^{\perp} \subset \mathrm{Ker}(X \twoheadrightarrow X(G^{AH})) = Y(G^{AH})^{\perp}$. Hence $Y(G^H) \subset Y(G^{AH})$ and it follows that $G^H = G^{AH}$; the proof is complete.

6. Completion of the proof; consequences.

Let A be an abelian variety over $\mathbb{C}$ and let t_α, $\alpha \in I$, be Hodge cycles on A (relative to $\mathbb{C} \xrightarrow{id} \mathbb{C}$). To prove the Main Theorem 2.11 we have to show the t_α are absolute Hodge cycles. Since we know the result for abelian varieties of CM-type, (2.15) shows that it remains only to prove the following proposition.

Proposition 6.1. There exists a connected, smooth (not necessarily complete) algebraic variety S over $\mathbb{C}$ and an abelian scheme $\pi\colon\ Y \to S$ such that

(a) for some $s_0 \in S$, $Y_{s_0} = A$;

(b) for some $s_1 \in S$, Y_{s_1} is of CM-type;

(c) the t_α extend to elements that are rational and of bidegree (0,0) everywhere in the family.

The last condition means the following. Suppose that t_α belongs to the tensor space $T_\alpha = H^1_B(A)^{\otimes m(\alpha)} \otimes \ldots$; then there is a section t of $R^1\pi_*\mathbb{Q}^{\otimes m(\alpha)}\otimes\ldots$ over the universal covering $\tilde{S}$ of S (equivalently, over a finite covering of S) such that for $\tilde{s}_0$ mapping to s_0, $t_{\tilde{s}_0} = t_\alpha$, and for all $\tilde{s} \in \tilde{S}$, $t_{\tilde{s}} \in H^1_B(Y_{\tilde{s}})^{\otimes m(\alpha)} \otimes \ldots$ is a Hodge cycle.

We sketch a proof of (6.1). (See also V.2). The parameter variety S will be a Shimura variety and (b) will hold for a dense set of points s_1.

We can assume that one of the t_α is a polarization θ for A. Let $H = H_1(A,\mathbb{Q})$ and let G be the subgroup of $GL(H) \times \mathbb{G}_m$ fixing the t_α. The Hodge structure on H defines a homomorphism $h_0\colon \mathbb{C}^\times \to G(\mathbb{R})$. Let $G^0 = \operatorname{Ker}(G \to \mathbb{G}_m)$; then $\operatorname{ad}(h_0(i))$ is a Cartan involution on $G^0_{\mathbb{C}}$ because the real form of $G^0_{\mathbb{C}}$ corresponding to it fixes the positive definite form $\psi(x,h(i)y)$ on $H \otimes \mathbb{R}$ where ψ is a Riemann form for θ. In particular, G is reductive (see 3.6).

Let $X = \{h\colon \mathbb{C}^\times \to G(\mathbb{R}) \mid h \text{ conjugate to } h_0 \text{ under } G(\mathbb{R})\}$. Each $h \in X$ defines a Hodge structure on H of type $\{(-1,0),(0,-1)\}$ relative to which each t_α is of bidegree

(0,0). Let $F^{0}(h) = H^{0,-1} \subset H \otimes \mathbb{C}$. Since $G(\mathbb{R})/K_\infty \xrightarrow{\approx} X$, where K_∞ is the centralizer of h_0, there is an obvious real differentiable structure on X, and the tangent space to X at h_0, $tgt_{h_0}(X) \approx Lie(G_{\mathbb{R}})/Lie(K_\infty)$. In fact X is a Hermitian symmetric domain. The Grassmanian, $Grass_d(H\otimes\mathbb{C}) \overset{df}{=} \{W \subset H \otimes \mathbb{C} \mid W \text{ of dimension } d\}$, $d = \dim(A)$, is a complex analytic manifold (even an algebraic variety). The map $\phi\colon X \to Grass_d(H \otimes \mathbb{C})$, $h \longmapsto F^{0}(h)$, is a real differentiable map, and is injective (because the Hodge filtration determines the Hodge decomposition). The map on tangent spaces factors into

$$\begin{array}{ccc} tgt_{h_0}(X) = Lie(G_{\mathbb{R}})/Lie(K_\infty) & \hookrightarrow & End(H\otimes\mathbb{C})/F^{0}End(H\otimes\mathbb{C}) = tgt_{\phi(h_0)}(Grass) \\ \downarrow \approx & \nearrow & \\ Lie(G_{\mathbb{C}})/F^{0}(Lie(G_{\mathbb{C}})), & & \end{array}$$

the maps being induced by $G(\mathbb{R}) \hookrightarrow G(\mathbb{C}) \hookrightarrow GL(H\otimes\mathbb{C})$. (The filtrations on $Lie(G_{\mathbb{C}})$ and $End(H\otimes\mathbb{C})$ are those corresponding to the Hodge structures defined by h_0). Thus $d\phi$ identifies $tgt_{h_0}(X)$ with a complex subspace of $tgt_{\phi(h_0)}(Grass)$, and so X is an almost-complex submanifold of $Grass_d(H\otimes\mathbb{C})$. It follows that it is a complex manifold (see Deligne [6,1.1] for more details). (There is an alternative, more group-theoretic description of the complex structure; see Knapp [1 , 2.4, 2.5]).

To each point h of X we can associate a complex torus $F^{0}(h)\backslash H\otimes\mathbb{C}/H(\mathbb{Z})$, where $H(\mathbb{Z})$ is some fixed lattice in H. For example, to h_0 is associated $F^{0}(h_0)\backslash H\otimes\mathbb{C}/H(\mathbb{Z}) = tgt_0(A)/H(\mathbb{Z})$, which is an abelian variety representing A.

From the definition of the complex structure on X it is clear that these tori form an analytic family B over X.

Let $\Gamma = \{g \in G(\mathbb{Q}) \mid (g-1)H(\mathbb{Z}) \subset nH(\mathbb{Z})\}$ some fixed integer n. For a suitably large $n \geq 3$, Γ will act freely on X, and so $\Gamma \backslash X$ will again be a complex manifold. The theorem of Baily and Borel [1] shows that $S = \Gamma \backslash X$ is an algebraic variety.

Γ acts compatibly on B, and on forming the quotients we obtain a complex analytic map $\pi\colon Y \to S$ with $Y = \Gamma \backslash B$. For $s \in S$, Y_s is a polarized complex torus (hence an abelian variety) with level n structure (induced by $H_1(B_h, \mathbb{Z}) \xrightarrow{\approx} H(\mathbb{Z})$ where h maps to s). The solution M_n of the moduli problem for polarized abelian varieties with level n-structure in the category of algebraic varieties is also a solution in the category of complex analytic manifolds. There is therefore an analytic map $\psi\colon S \longrightarrow M_n$ such that Y is the pull-back of the universal family on M_n. A theorem of Borel [2,3.10] shows that ψ is automatically algebraic, from which it follows that Y/S is an algebraic family.

For some connected component S^o of S, $\pi^{-1}(S^o) \to S^o$ will satisfying (a) and (c) of the proposition. To prove (b) we shall show that, for some $h \in X$ close to h_0, B_h is of CM-type (cf. Deligne [3,5.2]).

Recall (§5) that an abelian variety is of CM-type if and only if its Mumford-Tate group is a torus. From this it follows that B_h, $h \in X$, is of CM-type if and only if h factors through a subtorus of G defined over $\mathbb{Q}$.

Let T be a maximal torus, defined over $\mathbb{R}$, of the algebraic group K_∞. (See Borel-Springer [1] for a proof that T exists.) Since $h_0(\mathbb{C}^\times)$ is contained in the centre of K_∞, $h_0(\mathbb{C}^\times) \subset T(\mathbb{R})$. If T' is any torus in $G_{\mathbb{R}}$ containing T then T' will centralize h_0 and so $T' \subset K_\infty$; T is therefore maximal in $G_{\mathbb{R}}$. For a general (regular) element λ of Lie(T), T is the centralizer of λ. Choose a $\lambda' \in \mathrm{Lie}(G)$ that is close to λ in $\mathrm{Lie}(G_{\mathbb{R}})$ and let T' be the centralizer of λ' in G. Then T' is a maximal torus of G that is defined over $\mathbb{Q}$ and $T' = gTg^{-1}$ where g is an element of $G(\mathbb{R})$ that is close to 1. Thus $h = \mathrm{ad}(g)\circ h_0$ is close to h_0 and B_h is of CM-type.

This completes the proof of the main theorem. We end this section by giving two immediate consequences.

Let X be a variety over a field k and let $\gamma \in H^{2p}(X_{et},\mathbb{Q}_\ell)(p)$, $\ell \neq \mathrm{char}(k)$; then Tate's conjecture asserts that γ is algebraic if and only if there exists a subfield k_0 of k finitely generated over the prime field, a model X_0 of X over k_0, and a $\gamma_0 \in H^{2p}(X_0\otimes\bar{k}_0,\mathbb{Q}_\ell)(p)$ mapping to γ that is fixed by $\mathrm{Gal}(\bar{k}_0/k_0)$. (Only the last condition is not automatic.)

Corollary 6.2. Let A be an abelian variety over $\mathbb{C}$. If Tate's conjecture is true for A then so also is the Hodge conjecture.

Proof: We first remark that, for any variety X over $\mathbb{C}$, Tate's conjecture implies that all absolute Hodge cycles on X are algebraic. For (2.9) shows that there exists a subfield k_0 of $\mathbb{C}$ finitely generated over $\mathbb{Q}$ and a model X_0 of

X over k_0 such that $\mathrm{Gal}(\bar{k}_0/k_0)$ acts trivially on $C^p_{AH}(X_0 \otimes \bar{k}_0)$. If we let $C^p_{alg}(X_0 \otimes \bar{k}_0)$ be the $\mathbb{Q}$-linear subspace of $C^p_{AH}(X_0 \otimes \bar{k}_0)$ of algebraic cycles, then Tate's conjecture shows that the images of C^p_{AH} and C^p_{alg} in $H^{2p}(X_{et}, \mathbb{Q}_\ell)(p)$ generate the same $\mathbb{Q}_\ell$-linear subspaces. Thus $C^p_{alg} \otimes \mathbb{Q}_\ell = C^p_{AH} \otimes \mathbb{Q}_\ell$, and $C^p_{alg} = C^p_{AH}$.

Now let A be an abelian variety over $\mathbb{C}$ and let $t \in H^{2p}(A, \mathbb{C})$ be rational of bidegree (p,p). If $t_0 \in H^{2p}(A, \mathbb{Q})$ maps to t, then the image t' of t_0 in $H^{2p}_{\mathbb{A}}(A)(p)$ is a Hodge cycle relative to $\mathbb{C} \xrightarrow{id} \mathbb{C}$. The main theorem shows that t' is an absolute Hodge cycle, and the remark shows that it is algebraic.

<u>Remark</u> 6.3. The above result was first proved independently by Piatetski-Shapiro [1] and Deligne (unpublished) by an argument similar to that which concluded the proof of the main theorem. ((6.2) is easy to prove for varieties of CM-type; in fact, Pohlmann [1] shows that the two conjectures are equivalent in that case.) We mention also that Borovoĭ [1] shows that, for an abelian variety X over a field k, the $\mathbb{Q}_\ell$-subspace of $H^{2p}(X_{et}, \mathbb{Q}_\ell)(p)$ generated by cycles that are Hodge relative to an embedding $\sigma: k \hookrightarrow \mathbb{C}$ is independent of the embedding.

<u>Corollary</u> 6.4. Let A be an abelian variety over $\mathbb{C}$ and let G^H be the Mumford-Tate group of A. Then $\dim(G^H) \geq \mathrm{tr.deg}_k k(p_{ij})$ where the p_{ij} are the periods of A.

<u>Proof</u>: Same as that of (1.6).

7. Algebraicity of values of the Γ-function

The following result generalizes (1.5)

Proposition 7.1. Let $\bar{k}$ be an algebraically closed subfield of $\mathbb{C}$, and let V be a variety of dimension n over $\bar{k}$. If $\sigma \in H^B_{2r}(V)$ maps to an absolute Hodge cycle γ under

$$H^B_{2r}(V) \xrightarrow{(2\pi i)^{-r}} H^B_{2r}(V)(-r) \xrightarrow{\approx} H_B^{2n-2r}(V)(n-r) \hookrightarrow H_{\mathbb{A}}^{2n-2r}(V_{\mathbb{C}})(n-r)$$

then, for any C^∞ differential r-form ω on $V_{\mathbb{C}}$ whose class $[\omega]$ in $H^{2r}_{DR}(V/\mathbb{C})$ lies in $H^{2r}_{DR}(V/\bar{k})$,

$$\int_\sigma \omega \in (2\pi i)^r \bar{k}.$$

Proof: Proposition 2.9 shows that γ arises from an absolute Hodge cycle γ_o on $V/\bar{k}$. Let $(\gamma_o)_{DR}$ be the component of γ in $H^{2n-2r}_{DR}(V/\bar{k})$. Then, as in the proof of (1.5),

$$\int_\sigma \omega = (2\pi i)^r \, \mathrm{Tr}_{DR}((\gamma_o)_{DR} \cup [\omega]) \in (2\pi i)^r H^{2n}_{DR}(V/\bar{k}) = (2\pi i)^r \bar{k}.$$

In the most important case of the proposition, k will be the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, and it will then be important to know not only that the period

$$P(\sigma,\omega) \overset{df}{=} (2\pi i)^{-r} \int_\sigma \omega$$

is algebraic, but also which field it lies in. We begin by describing a general procedure for finding this field and then illustrate it by an example in which V is a Fermat hyperspace and the period is a product of values of the Γ-function.

Let V now be a variety over a number field $k \subset \mathbb{C}$ and let S be a finite abelian group acting on V over k.

If $\alpha: S \to \mathbb{C}^\times$ is a character of S taking values in $k^\times$ and H is a k vector space on which S acts k-linearly, then we write

$$H_\alpha = \{v \in H \mid sv = \alpha(s)v \ , \ \text{all} \ \ s \in S\} \ .$$

Assume that all Hodge cycles on $V_{\mathbb{C}}$ are absolutely Hodge and that $H^{2r}(V(\mathbb{C}),\mathbb{C})_\alpha$ has dimension 1 and is of bidegree (r,r). Then $(C^r_{AH}(\bar{V}) \otimes k)_\alpha$, where $\bar{V} = V \otimes_k \bar{\mathbb{Q}}$, has dimension one over k. The actions of S and $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ on $H^{2r}_{DR}(\bar{V}/\bar{\mathbb{Q}}) = H^{2r}_{DR}(V/k) \otimes_k \bar{\mathbb{Q}}$ commute because the latter acts through its action on $\bar{\mathbb{Q}}$; they therefore also commute on $C^r_{AH}(\bar{V}) \otimes k$, which embeds into $H^{2r}_{DR}(\bar{V}/\bar{\mathbb{Q}})$. It follows that $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ stabilizes $(C^r_{AH}(\bar{V}) \otimes k)_\alpha$ and, as this has dimension 1, there is a character $\chi: \mathrm{Gal}(\bar{\mathbb{Q}}/k) \to k^\times$ such that

$$\tau\gamma = \chi(\tau)^{-1}\gamma \ , \ \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/k) \ , \ \gamma \in (C^r_{AH}(\bar{V}) \otimes k)_\alpha .$$

<u>Proposition</u> 7.2. With the above assumptions, let $\sigma \in H^B_{2r}(V)$ and let ω be a C^∞ differential 2r-form on $V(\mathbb{C})$ whose class $[\omega]$ in $H^{2r}_{DR}(V/\mathbb{C})$ lies in $H^{2r}_{DR}(V/k)_\alpha$; then $P(\sigma,\omega)$ lies in an abelian algebraic extension of k, and

$$\tau(P(\sigma,\omega)) = \chi(\tau) \ P(\sigma,\omega) \ , \quad \text{all} \quad \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/k) \ .$$

<u>Proof</u>: Regard $[\omega] \in H^{2r}_{DR}(V/\mathbb{C})_\alpha = (C^r_{AH}(\bar{V}) \otimes \mathbb{C})_\alpha$; then $[\omega] = z\gamma$ for some $z \in \mathbb{C}$, $\gamma \in (C^r_{AH}(\bar{V}) \otimes k)_\alpha$. Moreover $P(\sigma,\omega) \overset{df}{=} (\frac{1}{2\pi i})^r \int_\sigma \omega = z\gamma(\sigma \otimes (2\pi i)^{-r}) \in zk$, where we are regarding γ as an element of $H^{2r}_B(V)(r) \otimes k = H^B_{2r}(V)(-r)^\vee \otimes k$. Thus $P(\sigma,\omega)^{-1}[\omega] \in (C^r_{AH}(\bar{V}) \otimes k)$. As $[\omega] \in H^{2r}_{DR}(\bar{V}/\bar{\mathbb{Q}}) = C^r_{AH}(\bar{V}) \otimes \bar{\mathbb{Q}}$,

this shows that $P(\sigma,\omega) \in \bar{\mathbb{Q}}$. Moreover, $\tau(P(\sigma,\omega)^{-1}[\omega]) = \chi(\tau)^{-1}(P(\sigma,\omega)^{-1}[\omega])$. On using that $\tau[\omega] = [\omega]$, we deduce that $\tau(P(\sigma,\omega)) = \chi(\tau)\ P(\sigma,\omega)$.

Remark 7.3 (a) Because $C^r_{AH}(\bar{V})$ injects into $H^{2r}(\bar{V}_{et},\mathbb{Q}_\ell)(r)$, χ can be calculated from the action of $Gal(\bar{\mathbb{Q}}/k)$ on $H^{2r}(\bar{V}_{et},\mathbb{Q}_\ell)_\alpha(r)$.

(b) The argument in the proof of the proposition shows that $\sigma \otimes (2\pi i)^{-r} \in H^B_{2r}(V)(-r)$ and $P(\sigma,\omega)^{-1}[\omega] \in H^{2r}_{DR}(\bar{V}/\bar{\mathbb{Q}})$ are different manifestations of the same absolute Hodge cycle.

The Fermat hypersurface

We shall apply (7.2) to the Fermat hypersurface

$$V:\ X_0^d + X_1^d + \dots + X_{n+1}^d = 0$$

of degree d and dimension n , which will be regarded as a variety over $k \overset{df}{=} \mathbb{Q}(e^{2\pi i/d})$. As above we write $\bar{V} = V \otimes_k \bar{\mathbb{Q}}$, and we shall often drop the subscript on $V_{\mathbb{C}}$.

It is known that the motive of V is contained in the category of motives generated by motives of abelian varieties (see (II 6.26)), and therefore (2.11) shows that every Hodge cycle on V is absolutely Hodge (cf. (II 6.27)) .

Let μ_d be the group of d^{th} roots of 1 in $\mathbb{C}$, and let $S = \bigoplus_{i=o}^{n+1} \mu_d/(\text{diagonal})$. Then S acts on V/k according to the formula:

$$(\zeta_o: \dots)(x_o: \dots) = (\zeta_o x_o: \dots)\ ,\quad \text{all}\ (x_o: \dots) \in V(\mathbb{C})\ .$$

The character group of S will be identified with

$$X(S) = \{\underline{a} \in (\mathbb{Z}/d\,\mathbb{Z})^{n+2} \mid \underline{a} = (a_o, \ldots, a_{n+1}),\ \Sigma a_i = o\};$$

$\underline{a} \in X(S)$ corresponds to the character

$$\underline{\zeta} = (\zeta_o : \ldots) \longmapsto \underline{\zeta}^{\underline{a}} \overset{df}{=} \Pi\ \zeta_i^{a_i}\ .$$

For $a \in \mathbb{Z}/d\,\mathbb{Z}$ we let $\langle a \rangle$ denote the representative of a in $\mathbb{Z}$ with $1 \leq \langle a \rangle \leq d$, and for $\underline{a} \in X(S)$ we let $\langle \underline{a} \rangle = d^{-1}\ \Sigma \langle a_i \rangle \in \mathbb{N}$.

If $H(V)$ is a cohomology group on which there is a natural action of k, we have a decomposition

$$H(V) = \oplus\ H(V)_{\underline{a}}\ ,\ H(V)_{\underline{a}} = \{v \mid \underline{\zeta} v = \underline{\zeta}^{\underline{a}} v\ ,\ \underline{\zeta} \in S\}$$

Let $(\mathbb{Z}/d\,\mathbb{Z})^{\times}$ act on $X(S)$ in the obvious way, $u(a_o, \ldots) = (ua_o, \ldots)$, and let $[\underline{a}]$ be the orbit of $\underline{a}$. The irreducible representations of S over $\mathbb{Q}$ (and hence the idempotents of $\mathbb{Q}[S]$) are classified by these orbits, and so $\mathbb{Q}[S] = \Pi\ \mathbb{Q}[\underline{a}]$ where $\mathbb{Q}[\underline{a}]$ is a field whose degree over $\mathbb{Q}$ is equal to the order of $[\underline{a}]$. The map $\underline{\zeta} \longmapsto \underline{\zeta}^{\underline{a}} : S \to \mathbb{C}$ induces an embedding $\mathbb{Q}[\underline{a}] \hookrightarrow k$. Any cohomology group decomposes as $H(V) = \oplus\ H(V)_{[\underline{a}]}$ where $H(V)_{[\underline{a}]} \otimes \mathbb{C} = \bigoplus_{\underline{a}' \in [\underline{a}]} (H(V) \otimes \mathbb{C})_{\underline{a}'}$.

Calculation of the cohomology

Proposition 7.4 The dimension of $H^n(V,\mathbb{C})_{\underline{a}}$ is 1 if no $a_i = o$ or if all $a_i = 0$; otherwise $H^n(V,\mathbb{C})_{\underline{a}} = 0$.

Proof: The map

$$(x_o : x_1 : \ldots) \longmapsto (x_o^d : x_1^d : \ldots) : \mathbb{P}^{n+1} \longrightarrow \mathbb{P}^{n+1}$$

defines a finite surjective map $\pi\colon V \to P^n$ where $P^n(\approx \mathbb{P}^n)$ is the hyperplane $\Sigma X_i = 0$. There is an action of S on $\pi_*\mathbb{C}$ which induces a decomposition $\pi_*\mathbb{C} \approx \oplus(\pi_*\mathbb{C})_{\underline{a}}$, and $H^r(V,\mathbb{C}) \xrightarrow{\approx} H^r(P^n,\pi_*\mathbb{C})$, being compatible with the actions of S, gives rise to isomorphisms $H^r(V,\mathbb{C})_{\underline{a}} \xrightarrow{\approx} H^r(P^n,(\pi_*\mathbb{C})_{\underline{a}})$. The sheaf $(\pi_*\mathbb{C})_{\underline{a}}$ is locally constant of dimension 1 except over the hyperplanes $H_i\colon X_i = o$ corresponding to i for which $a_i \neq o$, where it is ramified. Clearly $(\pi_*\mathbb{C})_{\underline{o}} = \mathbb{C}$, and so $H^r(P^n,(\pi_*\mathbb{C})_{\underline{o}}) \approx H^r(\mathbb{P}^n,\mathbb{C})$ for all r. It follows that $H^r(P^n,(\pi_*\mathbb{C})_{\underline{a}}) = o$, $r \neq n$, $\underline{a} \neq \underline{o} = (o,\ldots,o)$, and so $(-1)^n \dim H^n(P^n,(\pi_*\mathbb{C})_{\underline{a}})$, $\underline{a} \neq \underline{o}$, is equal to the Euler-Poincaré characteristic of $(\pi_*\mathbb{C})_{\underline{a}}$. We have

$$EP(P^n,(\pi_*\mathbb{C})_{\underline{a}}) = EP(P^n - \bigcup_{a_i \neq o} H_i,\mathbb{C}) .$$

Suppose first that no a_i is zero. Then

$$(x_o\colon \ldots\colon x_n\colon -\Sigma x_i) \leftrightarrow (x_o\colon \ldots\colon x_n)\colon P^n \xleftrightarrow{\approx} \mathbb{P}^n$$

induces

$$P^n - \bigcup_{i=o}^{n+1} H_i \xleftrightarrow{\approx} \mathbb{P}^n - \bigcup_{i=o}^{n} H_i \cup P^{n-1} ,$$

where H_i denotes the coordinate hyperplane in $\mathbb{P}^{n+1}$ or $\mathbb{P}^n$. As

$$(\mathbb{P}^n - \cup H_i \cup P^{n-1}) \amalg (P^{n-1} - \cup H_i) = \mathbb{P}^n - \cup H_i ,$$

and $\mathbb{P}^n - \cup H_i$, being topologically isomorphic to $(\mathbb{C}^\times)^n$, has Euler-Poincaré characteristic zero, we have

$EP(P^n - \bigcup^{n+1} H_i) = -EP(P^{n-1} - \bigcup^{n} H_i) = \ldots = (-1)^n EP(P^\circ) = (-1)^n$.

If some, but not all, a_i are zero, then $P^n - \cup H_i \approx (\mathbb{C}^\times)^r \times \mathbb{C}^{n-r}$

with $r \geq 1$, and so $EP(P^n - \bigcup H_i) = 0^r \times 1^{n-r} = 0$.

Remark 7.5. The above proof shows also that the primitive cohomology of V ,

$$H^n(V,\mathbb{C})_{prim} = \bigoplus_{\underline{a}\neq \underline{o}} H^n(V,\mathbb{C})_{\underline{a}} .$$

The action of S on $H^n(V,\mathbb{C})$ respects the Hodge decomposition, and so $H^n(V,\mathbb{C})_{\underline{a}}$ is pure of bidegree (p,q) for some p , q with $p + q = n$.

Proposition 7.6. If no $a_i = o$, then $H^n(V,\mathbb{C})_{\underline{a}}$ is of bidegree (p,q) with $p = \langle a\rangle - 1$.

Proof: We apply the method of Griffiths [1,§8]. When V is a smooth hypersurface in $\mathbb{P}^{n+1}$, Griffiths shows that the maps in

$$\begin{array}{ccccccc} H^{n+1}(\mathbb{P}^{n+1},\mathbb{C}) & \xrightarrow{o} & H^{n+1}(\mathbb{P}^{n+1} - V,\mathbb{C}) & \longrightarrow & H_V^{n+2}(\mathbb{P}^{n+1},\mathbb{C}) & \longrightarrow & H^{n+2}(\mathbb{P}^{n+1},\mathbb{C}) \\ & & & & \downarrow \simeq & & \\ & & & & H^n(V)(-1) & & \end{array}$$

induce an isomorphism

$$H^{n+1}(\mathbb{P}^{n+1} - V,\mathbb{C}) \xrightarrow{\approx} H^n(V)(-1)_{prim}$$

and that the Hodge filtration on $H^n(V)(-1)$ has the following explicit interpretation: identify $H^{n+1}(\mathbb{P}^{n+1} - V,\mathbb{C})$ with $\Gamma(\mathbb{P}^{n+1} - V, \Omega^{n+1})/d\Gamma(\mathbb{P}^{n+1} - V,\Omega^n)$ and let

$$\Omega_p^{n+1}(V) = \{\omega \in \Gamma(\mathbb{P}^{n+1} - V,\Omega^{n+1}) \mid \omega \text{ has a pole of order } \leq p \text{ on } V\} ;$$

then the map $R: \Omega_p^{n+1}(V) \to H^n(V,\mathbb{C})$ determined by

$$\langle\sigma, R(\omega)\rangle = \frac{1}{2\pi i}\int_\sigma \omega \ , \ \text{all} \quad \sigma \in H_n(V,\mathbb{C})$$

induces an isomorphism

$$\Omega_p^{n+1}(V)/d\Omega_{p-1}^n \xrightarrow{\approx} F^{n-p}\, H^n(V)(-1)_{prim} = F^{n-p+1}\, H^n(V)_{prim} \ .$$

(For example, if $p = 1$, R is the residue map $\Omega_1^{n+1}(V) \to F^n H^n(V) = H^o(V,\Omega^n)$).

Let f be the irreducible polynomial defining V. As $\Omega^{n+1}_{\mathbb{P}^{n+1}}(n + 2) \approx O_{\mathbb{P}^{n+1}}$ has basis

$$\omega_o = \Sigma(-1)^i\, X_i\, dX_o \wedge \ldots \wedge \hat{dX_i} \wedge \ldots \wedge dX_n \ ,$$

any differential form $\omega = P\omega_o/f^p$ with P a homogeneous polynomial of degree $p \deg(f) - (n + 2)$ lies in $\Omega_p^{n+1}(V)$. In particular, when V is our Fermat hypersurface,

$$\omega = \frac{X_o^{\langle a_o\rangle -1} \cdots X_{n+1}^{\langle a_{n+1}\rangle -1}}{(X_o^d + \ldots + X_{n+1}^d)^{\langle \underline{a}\rangle}}\,\omega_o = \frac{X_o^{\langle a_o\rangle} \cdots X_{n+1}^{\langle a_{n+1}\rangle}}{(X_o^d + \ldots + X_{n+1}^d)^{\langle \underline{a}\rangle}}\,\Sigma(-1)^i \frac{dX_o}{X_o}\wedge \ldots \wedge \frac{\hat{dX_i}}{X_i} \wedge \ldots$$

lies in $\Omega^{n+1}_{\langle \underline{a}\rangle}(V)$. For $\underline{\zeta} \in S$, $\underline{\zeta} X_i = \zeta_i^{-1} X_i$, and so $\underline{\zeta}\omega = \underline{\zeta}^{-\underline{a}}\,\omega$. This shows that $H^n(V,\mathbb{C})_{-\underline{a}} \subset F^{n-\langle \underline{a}\rangle +1}\, H^n(V,\mathbb{C})$. Since $\langle -\underline{a}\rangle - 1 = n + 1 - \langle \underline{a}\rangle$, we can rewrite this inclusion as $H^n(V,\mathbb{C})_{\underline{a}} \subset F^{\langle \underline{a}\rangle -1}\, H^n(V,\mathbb{C})$. Thus $H^n(V,\mathbb{C})_{\underline{a}}$ is of bidegree (p,q) with $p \geq \langle \underline{a}\rangle - 1$. The complex conjugate of $H^n(V,\mathbb{C})_{\underline{a}}$ is $H^n(V,\mathbb{C})_{-\underline{a}}$, and is of bidegree (q,p). Hence

$$n - p = q \geq \langle -\underline{a}\rangle - 1 = n + 1 - \langle \underline{a}\rangle$$

and so $p \leq \langle \underline{a}\rangle - 1$.

Recall that $H_B^n(V)_{[\underline{a}]} = \bigoplus_{\underline{a}'\in[\underline{a}]} H_B^n(V)_{\underline{a}'}$; thus (7.4) shows that $H_B^n(V)_{[\underline{a}]}$ has dimension 1 over $\mathbb{Q}[\underline{a}]$ when no a_i is zero and otherwise $H_B^n(V)_{[\underline{a}]} \cap H_B^n(V)_{\mathrm{prim}} = 0$.

Corollary 7.7. Assume no $a_i = 0$; $H_B^n(V)_{[\underline{a}]}$ is purely of bidegree $(\frac{n}{2},\frac{n}{2})$ if and only if $\langle u\underline{a}\rangle$ is independent of u .

Proof: As $\langle\underline{a}\rangle + \langle-\underline{a}\rangle = n + 2$, $\langle u\underline{a}\rangle$ is constant if and only if $\langle u\underline{a}\rangle = \frac{n}{2} + 1$ for all $u \in (\mathbb{Z}/d\,\mathbb{Z})^\times$, i.e. if and only if $\langle\underline{a}'\rangle = n/2 + 1$ for all $\underline{a}' \in [\underline{a}]$. Thus the corollary follows from the proposition.

Corollary 7.8. If no $a_i = 0$ and $\langle u\underline{a}\rangle$ is constant, then $C_{AH}^n(\bar{V})_{[\underline{a}]}$ has dimension one over $\mathbb{Q}[\underline{a}]$.

Proof: This follows immediately from (7.7) since, as we have remarked, all Hodge cycles on V are absolutely Hodge.

The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ on the étale cohomology.

Let $\mathfrak{p}$ be a prime ideal of k not dividing d , and let $\mathbb{F}_q$ be the residue field of $\mathfrak{p}$. Then $d|q-1$ and reduction modulo $\mathfrak{p}$ defines an isomorphism $\mu_d \xrightarrow{\approx} \mathbb{F}_d^\times$ whose inverse we denote t . Fix an $\underline{a} = (a_0,\ldots,a_{n+1}) \in X(S)$ with all a_i nonzero, and define a character $\varepsilon_i\colon \mathbb{F}_q^\times \to \mu_d$ by

$$\varepsilon_i(x) = t(x^{(1-q)/d})^{a_i} , \quad x \neq 0 .$$

As $\Pi\,\varepsilon_i = 1$, $\Pi\,\varepsilon_i(x_i)$ is well-defined for $\underline{x} = (x_0:\ldots:x_{n+1}) \in \mathbb{P}^{n+1}(\mathbb{F}_q)$, and we define a Jacobi sum

$$J(\varepsilon_o,\ldots,\varepsilon_{n+1}) = (-1)^n \sum_{\underline{x}\in P^n(\mathbb{F}_q)} \prod_{i=o}^{n+1} \varepsilon_i(x_i)$$

where P^n is the hyperplane $\Sigma X_i = o$ in $\mathbb{P}^{n+1}$. (As usual, we set $\varepsilon_i(o) = o$.) Let ψ be a nontrivial additive character $\psi: \mathbb{F}_q \to \mathbb{C}^\times$ and define Gauss sums

$$g(\mathcal{Y}, a_i, \psi) = -\sum_{x\in\mathbb{F}_q} \varepsilon_i(x)\ \psi(x)$$

$$g(\mathcal{Y}, \underline{a}) = q^{-\langle\underline{a}\rangle} \prod_{i=o}^{n+1} g(\mathcal{Y}, a_i, \psi) .$$

<u>Lemma</u> 7.9. The Jacobi sum $J(\varepsilon_o,\ldots,\varepsilon_{n+1}) = q^{\langle\underline{a}\rangle -1}\ g(\mathcal{Y},\underline{a})$.

<u>Proof</u>: $q^{\langle\underline{a}\rangle} g(\mathcal{Y},\underline{a}) = \prod_{i=o}^{n+1} (-\sum_{x\in\mathbb{F}_q} \varepsilon_i(x)\ \psi(x))$

$$= (-1)^n \sum_{\underline{x}\in\mathbb{F}_q^{n+1}} ((\prod_{i=o}^{n+1} \varepsilon_i(x_i))\ \psi(\Sigma x_i)),\ \underline{x} = (x_o,\ldots),$$

$$= (-1)^n \sum_{\underline{x}\in\mathbb{P}^n(\mathbb{F}_q)} \sum_{\lambda\in\mathbb{F}_q^\times} (\prod_{i=o}^{n+1} \varepsilon_i(\lambda x_i)\ \psi(\lambda\Sigma x_i)) .$$

We can omit the λ from $\Pi\,\varepsilon_i(\lambda x_i)$, and so obtain

$$q^{\langle\underline{a}\rangle} g(\mathcal{Y},\underline{a}) = (-1)^n \sum_{\underline{x}} (\prod_{i=o}^{n+1} \varepsilon_i(x_i)) \sum_{\lambda\in\mathbb{F}_q^\times} \psi(\lambda\Sigma x_i) .$$

Since $\sum_{\underline{x}} \prod_{i=o}^{n+1} \varepsilon_i(x_i) = \prod_{i=o}^{n+1} (\Sigma_{x\in\mathbb{F}_q} \varepsilon_i(x)) = o$, we can replace the sum over $\lambda\in\mathbb{F}_q^\times$ by a sum over $\lambda\in\mathbb{F}_q$. From

$$\sum_{\lambda \in \mathbb{F}_q} \psi(\lambda \Sigma x_i) = \begin{cases} q & \text{if } \Sigma x_i = 0 \\ 0 & \text{if } \Sigma x_i \neq 0 \end{cases}$$

we deduce finally that

$$q^{\langle \underline{a} \rangle} g(\wp, \underline{a}) = (-1)^n q \sum_{\underline{x} \in P^n(\mathbb{F}_q)} \left(\prod_{i=0}^{n+1} \varepsilon_i(x_i) \right)$$

$$= q\, J(\varepsilon_0, \ldots, \varepsilon_n) \ .$$

Note that this shows that $g(\wp, \underline{a})$ is independent of ψ and lies in k.

Let ℓ be a prime such that $\ell \nmid d$, $\wp \nmid \ell$, and $d | \ell - 1$. Then $\mathbb{Q}_\ell$ contains a primitive d^{th} root of 1 and so, after choosing an embedding $k \hookrightarrow \mathbb{Q}_\ell$, we can assume $g(\wp, \underline{a}) \in \mathbb{Q}_\ell$.

<u>Proposition</u> 7.10. Let $F_\wp \in \mathrm{Gal}(\bar{\mathbb{Q}}/k)^{ab}$ be a geometric Frobenius element of $\wp \nmid d$; for any $v \in H^n(\bar{V}_{et}, \mathbb{Q}_\ell)_{\underline{a}}$,

$$F_\wp v = q^{\langle \underline{a} \rangle - 1} g(\wp, \underline{a})\, v$$

<u>Proof</u>: As $\wp \nmid d$, V reduces to a smooth variety $V_\wp$ over $\mathbb{F}_q$ and the proper-smooth base change theorem shows that there is an isomorphism $H^n(\bar{V}, \mathbb{Q}_\ell) = H^n(\bar{V}_\wp, \mathbb{Q}_\ell)$ compatible with the action of S and carrying the action of $F_\wp$ on $H^n(\bar{V}, \mathbb{Q}_\ell)$ into the action of the Frobenius endomorphism Frob on $H^n(\bar{V}_\wp, \mathbb{Q}_\ell)$. The comparison theorem shows that $H^n(\bar{V}, \mathbb{Q}_\ell)_{\underline{a}}$ has dimension 1, and so it remains to compute

$$\mathrm{Tr}(F_\wp | H^n(\bar{V}, \mathbb{Q}_\ell)_{\underline{a}}) = \mathrm{Tr}(\mathrm{Frob} | H^n(\bar{V}_\wp, \mathbb{Q}_\ell)_{\underline{a}}) \ .$$

Let $\pi : V_{\mathfrak{X}} \to P^n$ be as before. Then $H^n(\bar{V}_{\mathfrak{X}}, \mathbb{Q}_\ell)_{\underline{a}} = H^n(P^n, (\pi_* \mathbb{Q}_\ell)_{\underline{a}})$, and the Lefschetz trace formula shows that

$$(-1)^n \operatorname{Tr}(\mathrm{Frob} \mid H^n(P^n, (\pi_* \mathbb{Q}_\ell)_{\underline{a}}) = \sum_{\underline{x} \in P^n(\mathbb{F}_q)} \operatorname{Tr}(\mathrm{Frob} \mid ((\pi_* \mathbb{Q}_\ell)_{\underline{a}})_{\underline{x}}) \qquad (7.10.1)$$

where $((\pi_* \mathbb{Q}_\ell)_{\underline{a}})_{\underline{x}}$ is the stalk of $(\pi_* \mathbb{Q}_\ell)_{\underline{a}}$ at $\underline{x}$.

Fix an $\underline{x} \in P^n(\mathbb{F}_q)$ with no x_i zero, and let $\underline{y} \in V_{\mathfrak{X}}(\overline{\mathbb{F}}_q)$ map to $\underline{x}$; thus $y_i^d = x_i$ all i. Then $\pi^{-1}(\underline{x}) = \{\underline{\zeta y} \mid \underline{\zeta} \in S\}$, and $(\pi_* \mathbb{Q}_\ell)_{\underline{x}}$ is the vector space $\mathbb{Q}_\ell^{\pi^{-1}(x)}$.

If ϕ denotes the arithmetic Frobenius automorphism (i.e., the generator $z \mapsto z^q$ of $\mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)$) then

$$\phi(y_i) = y_i^q = x_i^{\frac{q-1}{d}} y_i = t(x_i^{\frac{q-1}{d}}) y_i, \qquad 0 \le i \le n+1$$

and so

$$\mathrm{Frob}(\underline{y}) = \underline{\eta}\, \underline{y} \quad \text{where} \quad \underline{\eta} = (\ldots : t(x_i^{\frac{1-q}{d}}) : \ldots) \in S .$$

Thus Frob acts on $(\pi_* \mathbb{Q}_\ell)_{\underline{x}}$ as $\underline{\eta}$, and for $v \in ((\pi_* \mathbb{Q}_\ell)_{\underline{a}})_{\underline{x}}$ we have

$$\mathrm{Frob}(v) = \underline{\eta}\, v = \underline{\eta}^{\underline{a}} v, \quad \underline{\eta}^{\underline{a}} = \prod_{i=o}^{n+1} \varepsilon_i(x_i) \in k \subset \mathbb{Q}_\ell .$$

Consequently

$$\operatorname{Tr}(\mathrm{Frob} \mid ((\pi_* \mathbb{Q}_\ell)_{\underline{a}})_{\underline{x}}) = \prod_{i=o}^{n+1} \varepsilon_i(x_i) .$$

If some $x_i = o$ then both sides are zero ($(\pi_* \mathbb{Q}_\ell)_{\underline{a}}$ is ramified over the coordinate hyperplanes), and so on summing over $\underline{x}$ and applying (7.10.1) and (7.9), we obtain the proposition.

Corollary 7.11. Assume that no a_i is zero and that $\langle u\underline{a}\rangle$ is constant. Then, for any $v \in H^n(\bar{V}_{et}, \mathbb{Q}_\ell)_{\underline{a}}(\frac{n}{2})$,

$$F_{\wp} v = g(\wp, \underline{a})\, v .$$

Proof: The hypotheses on $\underline{a}$ imply that $\langle\underline{a}\rangle = n/2 + 1$; therefore, if we write $v = v_o \otimes 1$ with $v_o \in H^n(\bar{V}_{et}, \mathbb{Q}_\ell)_{\underline{a}}$, then

$$F_{\wp} v = F_{\wp} v_o \otimes F_{\wp} 1 = q^{n/2} g(\wp, \underline{a})\, v_o \otimes q^{-n/2} = g(\wp, \underline{a})\, v .$$

Calculation of the periods

Recall that the Γ-function is defined by

$$\Gamma(s) = \int_o^\infty e^{-t}\, t^s \frac{dt}{t} , \quad s > o ,$$

and satisfies the following equations

$$\Gamma(s)\,\Gamma(1-s) = \pi(\sin \pi s)^{-1}$$

$$\Gamma(1+s) = s\,\Gamma(s) .$$

The last equation shows that, for $s \in \mathbb{Q}^\times$, the class of $\Gamma(s)$ in $\mathbb{C}/\mathbb{Q}^\times$ depends only on the class of s in $\mathbb{Q}/\mathbb{Z}$. Thus, for $\underline{a} \in X(S)$, we can define

$$\tilde{\Gamma}(\underline{a}) = (2\pi i)^{-\langle\underline{a}\rangle} \prod_{i=o}^{n+1} \Gamma(a_i/d) \in \mathbb{C}/\mathbb{Q}^\times .$$

Let V^o denote the open affine subvariety of V with equation

$$Y_1^d + \ldots + Y_{n+1}^d = -1 \qquad (\text{so } Y_i = X_i/X_o) .$$

Denote by Δ the n-simplex $\{(t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_i \geq o ,\ \Sigma\, t_i = 1\}$ and define $\sigma_o \colon \Delta \to V^o(\mathbb{C})$ to be

$$(t_1,\dots,t_{n+1}) \mapsto (\varepsilon\, t_1^{1/d},\dots,\varepsilon\, t_{n+1}^{1/d}) , \quad \varepsilon = e^{2\pi i/2d} = \sqrt[d]{-1} , \quad t_i^{1/d} > o .$$

Lemma 7.12, Let a_o , ..., a_{n+1} be positive integers such that $\Sigma\, a_i = o$. Then

$$\int_{\sigma_o(\Delta)} Y_1^{a_1} \dots Y_{n+1}^{a_{n+1}} \frac{dY_1}{Y_1}\wedge\dots\wedge\frac{dY_n}{Y_n} = \frac{1}{2\pi i}(1-\xi^{-a_o}) \prod_{i=o}^{n+1} \Gamma(\tfrac{a_i}{d})$$

where $\xi = e^{2\pi i/d}$.

Proof: Write ω_o for the integrand. Then

$$\int_{\sigma_o(\Delta)} \omega_o = \int_\Delta \sigma_o^*(\omega_o) = \int_\Delta (\varepsilon\, t_1^{1/d})^{a_1} \dots (\varepsilon\, t_{n+1}^{1/d})^{a_{n+1}} d^{-n} \frac{dt_1}{t_1}\wedge\dots\wedge\frac{dt_n}{t_n}$$

$$= c \int_\Delta t_1^{b_1} \dots t_{n+1}^{b_{n+1}} \frac{dt_1}{t_1}\wedge\dots\wedge\frac{dt_n}{t_n}$$

where $b_i = a_i/d$ and $c = \varepsilon^{a_1+\dots+a_{n+1}} (\frac{1}{d})^n$. On multiplying by $\Gamma(1-b_o) = \Gamma(1+b_1+\dots+b_{n+1}) = \int_o^\infty e^{-t}\, t^{b_1+\dots+b_{n+1}}\, dt$ we obtain

$$\Gamma(1-b_o) \int_{\sigma_o(\Delta)} \omega_o = c \int_o^\infty \int_\Delta e^{-t}\, t^{b_1+\dots+b_{n+1}}\, t_1^{b_1}\dots t_{n+1}^{b_{n+1}} \frac{dt_1}{t_1}\wedge\dots\wedge\frac{dt_n}{t_n}\wedge dt .$$

If, on the inner integral, we make the change of variables $s_i = tt_i$, the integral becomes

$$c \int_o^\infty \int_{\Delta(t)} e^{-t}\, s_1^{b_1}\dots s_{n+1}^{b_{n+1}} \frac{ds_1}{s_1}\wedge\dots\wedge\frac{ds_n}{s_n}\wedge dt$$

where $\Delta(t) = \{(s_1,\dots,s_{n+1}) \mid s_i \geq o , \Sigma\, s_i = t\}$. We now let $t = \Sigma\, s_i$, and obtain

$$\Gamma(1-b_o) \int_{\sigma_o(\Delta)} \omega = c \int_o^\infty \dots \int_o^\infty e^{-s_1-s_2\dots-s_{n+1}}\, s_1^{b_1}\dots s_{n+1}^{1+b_{n+1}} \frac{ds_1}{s_1}\wedge\dots\wedge\frac{ds_{n+1}}{s_{n+1}}$$

$$= c\,\Gamma(b_1)\,\Gamma(b_2)\dots\Gamma(b_n)\,\Gamma(1+b_{n+1})$$

$$= c\, b_{n+1}\,\Gamma(b_1)\dots\Gamma(b_{n+1}) .$$

The formula recalled above shows that $\Gamma(1-b_o) = \pi/(\sin\pi b_o)\,\Gamma(b_o)$, and so $c\,\Gamma(1-b_o)^{-1} = \varepsilon^{-a_o}\dfrac{\sin\pi b_o}{\pi}\,\Gamma(b_o)$ $\pmod{\mathbb{Q}^\times}$

$$= \frac{1}{\pi}\, e^{-2\pi i b_o/2}\left(\frac{e^{\pi i b_o} - e^{-\pi i b_o}}{2i}\right)\Gamma(b_o)$$

$$= \frac{1}{2\pi i}\,(1-\varepsilon^{-2a_o})\,\Gamma(b_o)\ .$$

The lemma is now obvious.

The group algebra $\mathbb{Q}[S]$ acts on the $\mathbb{Q}$-space of differentiable n-simplices in $V(\mathbb{C})$. For $\underline{a} \in X(S)$ and $\underline{\xi}_i = (1,\ldots,\xi,\ldots)$ ($\xi = e^{2\pi i/d}$ in i^{th} position), define

$$\sigma = \prod_{i=1}^{n+1} (1-\underline{\xi}_i)^{-1}\sigma_o(\Delta) \subset V^o(\mathbb{C})$$

where σ_o and Δ are as above.

<u>Proposition</u> 7.13. Let $\underline{a} \in X(S)$ be such that no a_i is zero, and let ω^o be the differential

$$Y_1^{a'_1} \cdots Y_{n+1}^{a'_{n+1}}\ \frac{dY_1}{Y_1}\wedge\cdots\wedge\frac{dY_n}{Y_n}$$

on V^o, where a'_i represents $-a_i$, and $a'_i \gg o$. Then

(a) $\underline{\xi}\omega^o = \underline{\xi}^{\underline{a}}\,\omega^o$;

(b) $\int_\sigma \omega^o = \dfrac{1}{2\pi i}\displaystyle\prod_{i=o}^{n+1}(1-\xi^{a_i})\,\Gamma\left(\frac{-a_i}{d}\right)$.

<u>Proof</u>: (a) This is obvious since $\underline{\zeta}\,Y_i = (\zeta_i/\zeta_o)^{-1}\,Y_i$.

(b) $\int_\sigma \omega^o = \int_{\sigma_o(\Delta)} \displaystyle\prod_{i=1}^{n+1}(1-\xi_i)\omega^o = \prod_{i=1}^{n+1}(1-\xi^{a_i})\int_{\sigma_o(\Delta)}\omega^o$

$$= \frac{1}{2\pi i}\prod_{i=o}^{n+1}(1-\xi^{a_i})\,\Gamma\left(-\frac{a_i}{d}\right)$$

Remark 7.14. From the Gysin sequence

$$(\mathbb{C}\approx)\ H^{n-2}(V-V^{o},\mathbb{C}) \to H^{n}(V,\mathbb{C}) \to H^{n}(V^{o},\mathbb{C}) \to o$$

we obtain an isomorphism $H^{n}(V,\mathbb{C})_{prim} \xrightarrow{\approx} H^{n}(V^{o},\mathbb{C})$, which shows that there is an isomorphism

$$H^{n}_{DR}(V/k)_{prim} \xrightarrow{\approx} H^{n}_{DR}(V^{o}/k) = \Gamma(V^{o},\Omega^{n})/d\Gamma(V^{o},\Omega^{n+1}) \ .$$

The class $[\omega^{o}]$ of the differential ω^{o} lies in $H^{n}_{DR}(V^{o}/k)_{\underline{a}}$. Correspondingly we get a C^{∞} differential n-form ω on $V(\mathbb{C})$ such that

(a) the class $[\omega]$ of ω in $H^{n}_{DR}(V/\mathbb{C})$ lies in $H^{n}_{DR}(V/k)_{\underline{a}}$,

and

(b) $\int_{\sigma}\omega = \frac{1}{2\pi i}\prod_{i=o}^{n+1}(1-\xi^{a_i})\ \Gamma\left(-\frac{a_i}{d}\right)$, where $\sigma = \prod_{i=1}^{n+1}(1-\underline{\xi}_i)^{-1}\sigma_o(\Delta)$.

Note that, if we regard V as a variety over $\mathbb{Q}$, then $[\omega]$ even lies in $H^{n}_{DR}(V/\mathbb{Q})$.

The theorem

Recall that for $\underline{a} \in X(S)$, we set

$$\tilde{\Gamma}(\underline{a}) = (2\pi i)^{-\langle\underline{a}\rangle}\prod_{i=o}^{n+1}\Gamma(a_i/d) \qquad (\in \mathbb{C}/\mathbb{Q}^{\times})$$

and for $\mathfrak{p}$ a prime of k not dividing d , we set

$$g(\mathfrak{p},\underline{a}) = q^{-\langle\underline{a}\rangle}\prod_{i} g(\mathfrak{p},a_i,\psi)\ ,\ g(\mathfrak{p},a_i,\psi) = -\sum_{x\in\mathbb{F}_q} t\left(x^{\frac{1-q}{d}}\right)^{a_i}\psi(x)$$

where q is the order of the residue field of $\mathfrak{p}$.

Theorem 7.15. Let $\underline{a} \in X(S)$ have no $a_i = o$ and be such that $\langle u\underline{a}\rangle = \langle\underline{a}\rangle$ $(= n/2 + 1)$ for all $u \in (\mathbb{Z}/d\,\mathbb{Z})^{\times}$.

(a) Then $\tilde{\Gamma}(\underline{a}) \in \bar{\mathbb{Q}}$ and generates an abelian extension of $k = \mathbb{Q}(e^{2\pi i/d})$.

(b) If $F_{\mathfrak{P}} \in \mathrm{Gal}(\bar{\mathbb{Q}}/k)^{ab}$ is the geometric Frobenius element at $\mathfrak{P}$, then

$$F_{\mathfrak{P}}(\tilde{\Gamma}(\underline{a})) = g(\mathfrak{P},\underline{a})\,\tilde{\Gamma}(\underline{a}) .$$

(c) For any $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\lambda_{\underline{a}}(\tau) \stackrel{\mathrm{df}}{=} \tilde{\Gamma}(\underline{a})/\tau\tilde{\Gamma}(\underline{a})$ lies in k ; moreover, for any $u \in (\mathbb{Z}/d\,\mathbb{Z})^{\times}$,

$$\tau_u(\lambda_{\underline{a}}(\tau)) = \lambda_{u\underline{a}}(\tau)$$

where τ_u is the element of $\mathrm{Gal}(k/\mathbb{Q})$ defined by u .

<u>Proof</u>: Choose $\sigma \in H_n^B(V)$ and ω as in (7.14). Then all the conditions of (7.2) are fulfilled with α the character $\underline{a}$. Moreover, (7.14) and (7.11) show respectively that

$$P(\sigma,\omega) = \xi(\underline{a})\,\tilde{\Gamma}(-\underline{a}) , \text{ where } \xi(\underline{a}) = \prod_{i=o}^{n+1}(1-\xi^{a_i})$$

and

$$\chi(F_{\mathfrak{P}}) = g(\mathfrak{P},\underline{a})^{-1} .$$

As $\xi(\underline{a}) \in k$, (7.2) shows that $\tilde{\Gamma}(-\underline{a})$ generates an abelian algebraic extension of k and that $F_{\mathfrak{P}}\tilde{\Gamma}(-\underline{a}) = g(\mathfrak{P},\underline{a})^{-1}\tilde{\Gamma}(-\underline{a})$. It is clear from this equation that $g(\mathfrak{P},\underline{a})$ has absolute value 1 (in fact, it is a root of 1); thus $g(\mathfrak{P},\underline{a})^{-1} = \overline{g(\mathfrak{P},\underline{a})} = g(\mathfrak{P},-\underline{a})$. This proves (a) and (b) for $-\underline{a}$ and hence for $\underline{a}$.

To prove (c) we have to regard V as a variety over $\mathbb{Q}$. If S is interpreted as an algebraic group, then its action on V is rational over $\mathbb{Q}$. This means that

$$\tau(\underline{\zeta}\,\underline{x}) = \tau(\underline{\zeta})\ \tau(\underline{x})\ ,\ \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\ ,\ \underline{\zeta} \in S(\bar{\mathbb{Q}})\ ,\ \underline{x} \in V(\bar{\mathbb{Q}})$$

and implies that

$$\tau(\underline{\zeta}\,\gamma) = \tau(\underline{\zeta})\ \tau(\gamma)\ ,\ \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\ ,\ \underline{\zeta} \in S(\bar{\mathbb{Q}}),\ \gamma \in C^n_{AH}(\bar{V})\ .$$

Therefore $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ stabilizes $C^n_{AH}(\bar{V})_{[\underline{a}]}$ and, as this is a one-dimensional vector space over $\mathbb{Q}[\underline{a}]$, there exists for any $\gamma \in C^n_{AH}(\bar{V})_{[\underline{a}]}$ a crossed homomorphism $\lambda\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}[\underline{a}]^\times$ such that $\tau(\gamma) = \lambda(\tau)\gamma$ for all τ. On applying the canonical map $C^n_{AH}(\bar{V})_{[\underline{a}]} \to (C^n_{AH}(\bar{V}) \otimes k)_{\underline{a}}$ to this equality, we obtain $\tau(\gamma \otimes 1) = \lambda(\tau)^{\underline{a}}\ (\gamma \otimes 1)$.

We take γ to be the image of $\sigma \otimes (2\pi i)^{-n/2} \in H^B_n(V)(-\frac{n}{2})$ in $C^n_{AH}(\bar{V})_{[\underline{a}]}$. Then (cf. 7.3), $(\gamma \otimes 1)_{DR} = P(\sigma,\omega)^{-1}[\omega]$, if $[\omega]$ is as in (7.14). Hence

$$\lambda(\tau)^{\underline{a}} = P(\sigma,\omega)/\tau\ P(\sigma,\omega) = \lambda_{-\underline{a}}(\tau)\,(\xi(\underline{a})/\tau\ \xi(\underline{a}))\ .$$

On comparing

$$\lambda_{\underline{a}}(\tau) = \lambda(\tau)^{-\underline{a}}\ (\tau\ \xi(-\underline{a})/\xi(-\underline{a}))\quad \text{and}$$

$$\lambda_{u\underline{a}}(\tau) = \lambda(\tau)^{-u\underline{a}}(\tau\ \xi(-u\underline{a})/\xi(-u\underline{a}))\ ,$$

and using that $\tau(\xi(-u\underline{a})) = \tau(\tau_u(\xi(-\underline{a})) = \tau_u(\tau\ \xi(-\underline{a}))$, one obtains (c) of the theorem.

Remark 7.16 (a) The first statement of the theorem, that $\tilde{\Gamma}(\underline{a})$ is algebraic, has an elementary proof; see the appendix by Koblitz and Ogus to Deligne [7].

(b) Part (b) of the theorem has been proved up to sign by Gross and Koblitz [1, 4.5] using p-adic methods.

Remark 7.17 Let I_d be the group of ideals of k prime to d , and consider the character

$$\mathfrak{a} = \prod \mathfrak{p}_i^{r_i} \longmapsto g(\mathfrak{a},\underline{a}) \stackrel{\text{df}}{=} \prod g(\mathfrak{p}_i,\underline{a})^{r_i} : I_d \to k^\times .$$

When $\underline{a}$ satisfies the conditions in the theorem, then this is an algebraic Hecke character (Weil [1], [3]; see also Deligne [5, §6]). This means that there exists an ideal m of k (dividing a power of d) and a homomorphism $\chi_{alg}: k^\times \to k^\times$ that is algebraic (i.e., defined by a map of tori) and such that, for all $x \in k^\times$ totally positive and $\equiv 1 (\mathrm{mod}\ m)$, $g((x),\underline{a}) = \chi_{alg}(x)$. There is then a unique character $\chi_{\underline{a}}: \mathrm{Gal}(\bar{\mathbb{Q}}/k)^{ab} \to k^\times$ such that $\chi_{\underline{a}}(F_{\mathfrak{p}}) = g(\mathfrak{p},\underline{a})$ for all $\mathfrak{p}$ prime to d . Part (b) of the theorem can be stated as

$$\sigma(\tilde{\Gamma}(\underline{a})) = \chi_{\underline{a}}(\sigma)\, \tilde{\Gamma}(\underline{a}) , \text{ all } \sigma \in \mathrm{Gal}(\bar{k}/k) .$$

(There is an elegant treatment of algebraic Hecke characters in Serre [2, II]. Such a character with conductor dividing a modulus m corresponds to a character χ of the torus S_m(loc. cit. p II-17). The map χ_{alg} is $k^\times \xrightarrow{\pi} T_m \hookrightarrow S_m \xrightarrow{\chi} k^\times$. One defines from χ a character χ_∞ of the idèle class group as in (loc. cit., II 2.7). Weil's determination of χ_{alg} shows that χ_∞ is of finite order; in particular it is trivial on the connected component of the idèle class group, and so gives rise to a character $\chi_{\underline{a}}: \mathrm{Gal}(\bar{\mathbb{Q}}/k)^{ab} \to k^\times$.)

Restatement of the theorem

For $b \in d^{-1}\mathbb{Z}/\mathbb{Z}$, we write $\langle b\rangle$ for the representative of b in $d^{-1}\mathbb{Z}$ with $1/d \leq \langle b\rangle \leq 1$. Let $\underline{b} = \Sigma\, n(b)\,\delta_b$ be an element of the free abelian group generated by the set $d^{-1}\mathbb{Z}/\mathbb{Z} - \{o\}$, and assume that $\Sigma\, n(b) \langle ub\rangle = c$ is an integer independent of $u \in \mathbb{Z}/d\,\mathbb{Z}$. Define

$$\tilde{\Gamma}(\underline{b}) = \frac{1}{(2\pi i)^c} \prod_b \Gamma(\langle b\rangle)^{n(b)} .$$

Let $\mathfrak{p}$ be a prime of k, not dividing d, and let $\mathbb{F}_q$ be the residue field at $\mathfrak{p}$. For ψ a non-trivial additive character of $\mathbb{F}_q$, define

$$g(\mathfrak{p},\underline{b}) = \frac{1}{q^c} \prod_b g(\mathfrak{p},b,\psi)^{n(b)} , \text{ where } g(\mathfrak{p},b,\psi) = -\sum_{x\in\mathbb{F}_q} t(x^{b(1-q)})\psi(x) .$$

As in (7.17), $\mathfrak{p} \mapsto g(\mathfrak{p},\underline{b})$ defines an algebraic Hecke character of k and a character $\chi_{\underline{b}}\colon \mathrm{Gal}(\bar{\mathbb{Q}}/k) \to \mathbb{C}^\times$ such that $\chi_{\underline{b}}(F_{\mathfrak{p}}) = g(\mathfrak{p},\underline{b})$ for all $\mathfrak{p} \nmid b$.

Theorem 7.18. Assume that $\underline{b} = \Sigma\, n(b)\,\delta_b$ satisfies the condition above.

(a) Then $\tilde{\Gamma}(\underline{b}) \in k^{ab}$, and for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/k)^{ab}$,

$$\sigma\tilde{\Gamma}(\underline{b}) = \chi_{\underline{b}}(\sigma)\,\tilde{\Gamma}(\underline{b}) .$$

(b) For $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, let $\lambda_{\underline{b}}(\tau) = \tilde{\Gamma}(\underline{b})/\tau\tilde{\Gamma}(\underline{b})$; then $\lambda_{\underline{b}}(\tau) \in k$, and, for any $u \in (\mathbb{Z}/d\,\mathbb{Z})^\times$,

$$\tau_u(\lambda_{\underline{b}}(\tau)) = \lambda_{u\underline{b}}(\tau) .$$

Proof: Suppose first that $n(b) \geq o$ for all b. Let $n + 2 = \Sigma\, n(b)$, and let $\underline{a}$ be an $(n+2)$-triple in which each $a \in \mathbb{Z}/d\,\mathbb{Z}$ occurs exactly $n(a/d)$ times. Write

$\underline{a} = (a_o, \ldots, a_{n+1})$. Then $\Sigma a_i = d(\Sigma n(b)b) = dc \pmod{d} = o$, and so $\underline{a} \in X(S)$. Moreover,

$$\langle u\underline{a}\rangle \overset{\text{df}}{=} \frac{1}{d} \Sigma \langle ua_i\rangle = \Sigma\, n(b) \langle ub\rangle = c$$

for all $u \in \mathbb{Z}/d\,\mathbb{Z}$. Thus $\langle u\underline{a}\rangle$ is constant, and $c = \langle \underline{a}\rangle$. We deduce that $\tilde{\Gamma}(\underline{a}) = \tilde{\Gamma}(\underline{b})$, $g(\psi, \underline{a}) = g(\psi, \underline{b})$, and $\chi_{\underline{a}} = \chi_{\underline{b}}$. Thus in this case, (7.18) follows immediately from (7.15) and (7.17).

Let $\underline{b}$ be arbitrary. For some N , $\underline{b} + N\underline{b}_o$ has positive coefficients, where $\underline{b}_o = \Sigma\, \delta_b$. Thus (7.18) is true for $\underline{b} + N\underline{b}_o$. Since $\tilde{\Gamma}(\underline{b}_1 + \underline{b}_2) = \tilde{\Gamma}(\underline{b}_1)\, \tilde{\Gamma}(\underline{b}_2) \pmod{\mathbb{Q}^\times}$ and $g(\psi, \underline{b}_1 + \underline{b}_2) = g(\psi, \underline{b}_1)\, g(\psi, \underline{b}_2)$ this completes the proof.

Remark 7.19. (a) Part (b) of the theorem determines $\Gamma(u\underline{b})$ (up to multiplication by a rational number) starting from $\Gamma(\underline{b})$.

(b) Conjecture 8.11 of Deligne [7] is a special case of part (a) of the above theorem. The more precise form of the conjecture, Deligne [7, 8.13], can be proved by a modification of the above methods.

Final Note. The original seminar of Deligne comprised fifteen lectures, given between 29/10/78 and 15/5/79. The first six sections of these notes are based on the first eight lectures of the seminar, and the final section on the last two lectures. The remaining five lectures (which the writer of these notes was unable to attend) were on the following topics:

(6/3/79) review of the proof that Hodge cycles on abelian varieties are absolutely Hodge; discussion of the expected action of the Frobenius endomorphism on the image of an absolute Hodge cycle in crystalline cohomology;

(13/3/79) definition of the category of motives using absolute Hodge cycles; semisimplicity of the category; existence of the motivic Galois group G ;

(20/3/79) fibre functors in terms of torsors; the motives of Fermat hypersurfaces and K 3-surfaces are contained in the category generated by abelian varieties;

(27/3/79) Artin motives; the exact sequence

$$1 \to G^{o} \to G \xrightarrow{\pi} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1 \ ;$$

identification of G^{o} with the Serre group, and description of the G^{o}-torsor $\pi^{-1}(\tau)$;

(3/4/79) action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on G^{o} ; study of $G \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$; Hasse principle for $H^{1}(\mathbb{Q}, G^{o})$.

Most of the material in these five lectures is contained in the remaining articles of this volume (especially IV).

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