Letter from Langlands to Saint-Aubin, 2020-02-21.

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Abstract

Translation of https://publications.ias.edu/sites/default/files/letter-to-Saint-Aubin-rpl.pdf

Dear Yvan,

Apparently I have had since the beginning of my career some bad habits, not a refusal to publish my writings but a hesitation. For example, I have never published my thesis, only a summary and it was only decades after its writing in 1959 that it was made available to anyone who wanted to consult it in the collection https://publications.ias.edu/rpl. In the meantime it had been consulted by others. In particular it was integrated into the book *Elliptic Operators and Lie Groups* of Derek Robinson and that perhaps saved it from being forgotten.

Despite this laziness I have rarely been accused of errors, which allows me to believe that I made very few and even perhaps none. Recently thanks to your message and the curiosity of Tony, I have learned, as you know, that other mathematicians or physicists believe that they have discovered and corrected an error in another text,

On-unitary-representations-of-the-Virasoro-algebra-rpl.pdf

I am not indifferent to the truth of this text but it concerns things that occupied me a long time ago and I am ready to leave to others, if there are any, for whom they are more important to judge their value and their correctness.

On the other hand, for what is related to my work on the automorphic forms attached to the function field on an elliptic curve defined on a finite field, I find it very important that it be understood and judged correctly. Why? I will try to explain.

But I have had difficulties even before with the theory of Eisenstein series. I would like to explain them because the source of the difficulties was similar, a reluctance or rather indifference to publication. Others published a book, a plagiarism of what I had already written and put into circulation without submitting it to a publisher. I had been invited in all innocence to write a review of this book for a journal of the American Mathematical Society. I accepted and, fortunately for me, I discovered that those others had not simply copied my article, but that they had partly modified it in introducing an important error. So luckily I was spared from some problems, although their book is, I believe, still available.

Despite that I have the impression that some mathematicians, even renowned mathematicians, continued to express their doubts. But not all. In particular Harish-Chandra was convinced by what I wrote and he even gave lectures at the Institute for Advanced Study in

Available at www.jmilne.org/math/.

which he explained my arguments for groups of rank one. My debt to him in this respect and in other ways too is huge.

But these problems happened a long time ago in the fifties. Much later another difficulty has presented itself and Harish-Chandra is no longer there. Too bad!

It is a question now of the theory of automorphic forms, where there are apparently two theories, the geometric theory and the arithmetic theory, of which the second is the most important and the first most approachable. The second is, as you know, essential to the theory of numbers. The first is much less important but, at the same time, it serves as a model for the second. For this reason it is important enough to understand the first before we attack the major difficulties of the second. It nevertheless remains that the second is much older than the first.

I wrote only one text on the geometric theory and I wrote it in Russian. Why? Because it was a way to amuse myself. When I started my studies in Vancouver, I was told that — do not forget we were then in the year 1953, a period when we found, albeit for a short time, a European culture — for a mathematician it was very important to know French, German, Russian, and even perhaps Italian. We now live another world, but I followed, in my own way, this advice and to my profit. But it was French and German — the works of Hecke and Siegel above all, as well as Bourbaki — which were important to me at the beginning of my career. Nevertheless, it was the Russian language that fascinated me. I studied it for two years at the university, my second and fourth. The teacher was the same twice, a White Russian, hardly older than me, an enthusiast of the Russian opera. I was a diligent student, but already married. Subsequently, not having an opportunity to use it I neglected the language.

But during my years at UBC, I did not come across the writings of Bourbaki or the French mathematicians. What I found, and that before going to the university, was a book *The Story of the World's Great Thinkers* by Ernest Trautner. I like to tell the story of me and this book. I have told it several times. This book was very popular in the thirties especially on the left. I had my father-in-law, or rather the one who would a few years later be my father-in-law. It may have been this book and the girl who would later be my wife who most influenced the young man without any serious ambition that I was in high school.

Much later in 1967, having decided to abandon mathematics and go with my family to Turkey I returned to Russian and started Turkish. I had this time, but now in Princeton, a teacher much older than me, but I was always very diligent. It was again a White Russian, the sister of a Russian engineering teacher, Gregorie Chebotarev, indeed a Cossack, who was at the time of the First World War a Russian soldier. After or during the revolution he left the country. Then he learnt how to bring his sister, who was then very young, to America. All this is told in his book, *Russia, My Native Land*, published first in English and then in Russian. In Princeton, at the time I attended her class, she had an assistant whose reputation I did not recognized, another Russian woman, the writer Nina Berberova.

I do not remember the married name of my teacher, but she appreciated me because I worked very hard. I was serious and I really wanted to master the language. But, what happened is that having abandoned mathematics I had the leisure to make frivolous calculations, which led to my letter to Weil. So I gave up Turkish and Russian to go back to mathematics. I tried to explain to her what it was, why I did not have time to come to her classes but she was angry with me — with reason — and refused to accept my excuses.

The decision to go to Turkey with my family, my wife and four children, was nevertheless still there. We stayed for a year, a difficult year for the family, especially for Charlotte, but a year that gave me a lot. I believe — but I'm not sure — that despite all Charlotte at least does not regret the visit. Me, I learned first that in the contemporary world it is not easy for an anglophone to learn foreign languages, that it has to be in a place where we speak the native language even with foreigners. In the case of Québec it seems rather that what is important is that it is francophones — or foreigners — who are not convinced that the French language is not just a minority language but also sometimes a useful secondary language.

But my path went through Turkey, which was not a failure but was not a success either.

A few years later, better informed, I spent a second year abroad, this time in Germany, which was more profitable. I learned German, which remained with regard to the literature my favorite language, although I do not find, for the moment, the time to read it. Then I started spending more time in Québec, of which you know the consequences. With these two experiences I have returned to Turkey, where I found like in Québec many friends among former students. In the meantime I was able several times to spend a few weeks in Turkey, which allowed me to return to the language, that I have never really mastered, but I've been able to rewrite a few French texts for popular mathematics magazines.

But the advisor in Vancouver advised me to learn two other languages, Italian and Russian. Although Italian is unfortunately no longer important to mathematicians, I had started a few years ago, long after a holiday in Italy, a hike that ended in Florence, reading not-too-difficult Italian novels, for example, *Pane e Vino* by the writer Ignazio Silone a rather left novel, which I had already read in English because of its political message. Unfortunately, attracted probably by a mathematical question, I stopped reading Italian and I never returned. Too bad! Unfortunately I do not have much hope of returning.

Russian was very important for me, although it too has never seriously appeared in my mathematical reading. I nevertheless admire the work of Kolmogorov, or rather his style and his mathematical achievements. I have in fact his complete work in three volumes, acquired thanks to a German friend and a friend of his in Russia, and one of my activities in Gananoque will I hope be to read some pages from time to time. I do not know exactly when I started reading Russian seriously. I have read, however, very irregularly many books by the important authors of the nineteenth century and also the first half of the twentieth century, books of all kinds: the famous trip through Europe by the historian Nikolai Karamzin, the writings of Tolstoi, Gogol, and others, and among the most recent the story of Mikhail Sholokhov and also others.

Nevertheless it is not a language with which I am at my ease. I have only visited Russia once and I am very grateful to Dmitry Lebedev, who arranged my invitation to Moscow, as well as his family, his wife, their son and his sister-in-law for a pleasant afternoon at their at home in the Moscow countryside. Anyway it is this brief stay in Moscow which led to a return to the language, although I was somewhat disappointed by the audience. Indeed, I had prepared in Russian a coherent sequence of lectures. To my surprise the listeners insisted that I speak English. As far as I understand, it was not because they could all understand me more easily in English, let us say rather in American, but because they are so indulged in this language. Finally I accepted a compromise — half American half Russian, half an hour in Russian followed by half an hour in English. In principle, it was not so bad. However, I have a profound aversion to those people, especially Europeans, who believe that the American language is the key to success, glory, and happiness. They remind me of a restauranteur I met in a Parisian restaurant while visiting with some German friends. We were orderering our meal from a waitress, when the maître de restaurant realized that I, who was speaking to her, had an accent he believed American. He intervened in English. I explained to him that it was useless and rude to speak English, because the lady, our friend, spoke only German. He abruptly abandoned us to the waitress without apologizing. As you know there are many academics like this, so at his level, all over the world and the Russians seem to be no better than others, perhaps even worse.

Anyway the visit to Russia reminded me of my old linguistic ambitions. At the same time, I had already looked at the Russian theory of automorphic forms on a function field over a finite field. The absence in it of a Hecke theory disturbed me, especially the non-existence of eigenvalues and eigenfunctions. So, and here I must insist, in my opinion this is not an error neither of the Russians nor of me, but a fact that, at least in my opinion, the Russian definitions lead to a theory fundamentally different from the classical theories of Hecke, Siegel, and their successors. I am not even convinced of the value of their theory in its present form. I can, however, believe that their theory may be of some use as a theory with other goals than a theory in the style of Hecke.

It is true, however, in my opinion that geometric theory is necessarily other than the arithmetic theory. At the same time, it seems to me that it is necessary for the two to be similar, the first being easier than the second. This is the case with my test. What concerns me here is not the possibility of a theory other than the one I proposed for the group GL(2) and a geometric curve. It is that remarks of a mathematician perhaps ignorant and certainly lazy could prevent the creation of a geometric theory which is, if not deeper, at least comparable to the arithmetic theory, still largely conjectural, begun in the twentieth century.

To be more precise, Edward Frenkel was invited, in part thanks to a suggestion from me, who believed that he had the necessary skill in the theory of automorphic forms, to give an exposition during a conference in Minneapolis in which he was supposed to briefly explain the content of my Russian article.

Unfortunately he was not able, despite my exhortations, to understand that to create a geometric theory in the style of Hecke, that is to say with eigenvalues and eigenfunctions, we need a theory that is not that of the Russians. That is what I underlined in the sentence at the end of section VI of my article[1] [2] [3]

> возможно что это самостоятельность $L^2(\mu,D)$ и $L^2(\mu,A)$ то, что отличает L^2-теорию от пучковой теории в [G]

To be more precise, the theory I described in my Russian article for the group GL(2) and elliptic curves, begins with an article by Atiyah, an article that is in my opinion one of his best and it remains, I believe, to do the same for any curve and any reductive group. This would be a pleasant and instructive problem for vigorous young mathematicians. It is a shame — even worse, a sin — to discourage them by not explaining its importance for the theory of automorphic forms, at least for the geometric theory.

Having done so, one could, if one is convinced of its value, try to create a theory similar to that of my article for every curve and suitable group. Unfortunately, this is not a job for me. I will not have the time or the strength to do that. I also confess that in the short time I have left, I prefer to think about things outside of mathematics, including those that I find fascinating but for which I unfortunately did not have time when I was younger and devoted almost entirely to mathematics.

What I hope of young people, at least a few, is that they take advantage of the possibilities offered by the geometric theory even if they are not of the level of arithmetic theory. As I

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[1] It is possible that it is the independence of $L^2(\mu,D)$ and $L^2(\mu,A)$ that distinguishes L^2-theory from the bundle theory in [G]. (ReversoContext.)

[2] It is possible that this independence is $L^2(\mu,D)$ and $L^2(\mu,A)$ what distinguishes the L^2-theory from the bundle theory in [G] (Google translate).

[3] It's possible that this independence $L^2(\mu,D)$ and $L^2(\mu,A)$ is what distinguishes L^2-theory from the bundle theory in [G]. (Bing translator.)

have just written it will be necessary to start by pursuing the ideas of Atiyah, but generalizing his article, quoted above. But this will not, in my opinion, be easy. I haven't tried it myself, especially I don't know how to guess the structure of Bun_G in general. I would also point out that Atiyah's article, which corresponds to the group $\mathrm{GL}(n)$ group on an elliptic curve, is not easy, and as far as I know it has only been read by a small number of mathematicians.

The structure of Bun_G being known, it will require an understanding of all the eigenfunctions attached to a given curve and a given reductive group. In general, this last question will be much more difficult than for elliptic curves, the case dealt with by Atiyah. You see that this is not a question for the old!

That done, we can say that we have created a general geometric theory. More specifically, we want to prove that the unramified automorphic forms attached to a given group G are described by homomorphisms of a group, the Galois group, into another group, ${}^LG$, called the L-group. With ramification the structures will be more complicated. In addition there are other problems related to what is called in English L-indistinguishability, which is for the moment simply a distraction. All this is to say that in my opinion there is in the geometric theory a huge number of serious problems accessible to young people and that we do not need people who by frivolity, laziness, and ignorance, to divert them from these problems.

In addition, the so-called geometric theory is much simpler than arithmetic theory. I admit myself, and I'm not alone, that I do not know exactly what to expect from this last theory. We have seen that with Fermat's last theorem but, if you will, even with the theorem of quadratic reciprocity, that it has important arithmetic constituents, but the recent works of Sarnak suggest that there is an important influence from transcendental numbers. So we are very far from the end. I'll never make it myself.

So we see that there are a lot of problems more difficult than those that I have raised in the Russian text: arithmetic theory; ramified theory; the unramified theory for groups other than GL(2); the unramified theory on curves other than those of genus 1. It is a shame (disgrace) then not to have discovered the theory in the simplest case, and it is even worse, not to have recognized its discovery by another. Believing in a false theory is also embarrassing. I hope this does not turn out to be my fate.

I add, because it's easy to forget that what we are looking for is a theory like that created by Hecke!

I also admit that I did not devote myself sufficiently to the arithmetic theory to create it but simply to understand what its form will be.

Lastly I confess that I do not think that this essay is too harsh.

I will send it to Tony Pulido so that he can put it on the site when I am safe from all misunderstanding and all malice. I was afraid that this would not be for tomorrow! But I've become more optimistic.

À bientôt, Robert