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Shimura Varieties: Conjugates and the Action of Complex Conjugation

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A basic problem in the theory of Shimura varieties is to describe how an automorphism of the complex numbers acts on a variety and on its special points. For example Shimura's conjecture, concerning the existence of a canonical model, is the special case of this problem in which the automorphism fixes the reflex field of the variety or of the special point. In [1, p 417-18] Langlands states two conjectures concerning this problem, and the main purpose of this paper is to prove these conjectures for all Shimura varieties of abelian type. (This class of Shimura varieties, defined in §1 of this paper, contains all those for which a canonical model is constructed in Deligne [3]).

The first conjecture describes the action of complex conjugation on the complex points of a Shimura variety that has a real canonical model. Recall that a Shimura variety $Sh(G, X)$ is defined by a $\mathbb{Q}$-rational reductive group G and a family X of homomorphisms $\mathbb{C}^\times \to G(\mathbb{R})$ satisfying certain conditions. Initially $Sh(G,X)$ is defined as a complex variety but is expected to have a model over a certain number field $E(G,X)$ called the reflex field. A canonical model for $Sh(G,X)$ is a variety $M(G,X)$ over $E(G,X)$

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satisfying certain conditions sufficient to determine it uniquely. In the case that $E(G,X)$ is real and the canonical model exists, complex conjugation defines an involution θ of $Sh(G,X)$. It is important to have an explicit description of θ in order, for example, to compute the factor at infinity of the zeta function of $Sh(G,X)$. The conjecture of Langlands suggests such a description. When the canonical model is a moduli variety, and so has a direct description, the proof of the conjecture is straightforward. This is not usually the case, and just as the construction of the canonical model is intricate in general, so must be the proof of the conjecture. In particular it must involve an analogous assertion for connected Shimura varieties. Such an assertion is proved in Shih [1] for connected Shimura varieties that are of primitive abelian type C, and this result is the starting point of our proof of the conjecture for all Shimura varieties of abelian type. (Since θ does not preserve the connected component the analogous assertion takes on quite a different form from the original; it becomes rather a statement about the action of a "negative" element of $G(\mathbb{Q})$.)

We can apply an automorphism τ of $\mathbb{C}$ to the polynomial equations defining $Sh(G,X)$ and so obtain a conjugate variety $\tau Sh(G,X)$; the second conjecture concerns the identification of $\tau Sh(G,X)$. Since $\tau M(G,X) = M(G,X)$ when τ fixes $E(G,X)$, this is only a problem when τ does not fix the reflex field (or the canonical model is not known to exist). Kazhdan [1] shows that, when $Sh(G,X)$ is compact, $\tau Sh(G,X)$ is isomorphic to the Shimura variety defined by a second pair $({}^{\tau}G, {}^{\tau}X)$ but unfortunately his method provides very little information about ${}^{\tau}G$ and ${}^{\tau}X$. For Shimura curves Doi and Naganuma [1] show $\tau Sh(G,X)$ is isomorphic to $Sh({}^{\tau}G, {}^{\tau}X)$ for an explicit pair $({}^{\tau}G, {}^{\tau}X)$ and Shih [2] proves the same result for a Shimura variety of primitive abelian type A or C. The conjecture of Langlands suggests a description of $({}^{\tau}G, {}^{\tau}X)$ in the general case. This we verify for all Shimura varieties of abelian type.

In [3] Langlands announced a conjecture, which we shall refer to as conjecture C, that describes in a very precise way how an automorphism of $\mathbb{C}$ acts on any Shimura variety and its special points. The statement is based on the construction of a remarkable extension of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by the Serre group, which Langlands calls the Taniyama group. Conjecture C implies the previous two conjectures. Although expected, because of its greater precision, to be more amenable to proof then the earlier two conjectures, conjecture C seems to lie much deeper. We do, however, show that (at least for Shimura varieties of abelian type) it is equivalent to a statement (conjecture CM) involving only abelian varieties of CM-type. This conjecture CM can be regarded as a strengthening of the main theorem of complex multiplication in that it describes how any automorphism of $\mathbb{C}$ acts on an abelian variety of CM-type and its points of finite order. As a consequence we find that in order to prove conjecture C for all Shimura varieties of abelian type it suffices to prove it for those defined by a group of symplectic similitudes. Moreover we show that if one replaces the Taniyama group by the reductive group associated with the category of motives over $\mathbb{Q}$ generated by abelian varieties of potential CM-type, then conjecture C becomes true. Thus, if the Taniyama group is isomorphic to this motivic Galois group (with all its structure) then conjecture C is true; we also prove the converse of this assertion.

We would like to thank P. Deligne and R. Langlands for making available to us pre-prints of their work and D. Shelstad for a letter on which we have based Proposition 7.2 and preceding discussion. One of us was fortunate to be able to spend seven months during 1978-79 at I.H.E.S. and have numerous discussions with P. Deligne, which have profoundly influenced this paper.

Notations and conventions.

For Shimura varieties and algebraic groups we generally follow the notations of Deligne [3]. Thus a reductive algebraic group G is always connected, with derived group G^{der}, adjoint group G^{ad}, and centre $Z = Z(G)$. (We assume also that G^{ad} has no factors of type E_8.) A central extension is an epimorphism $G \to G'$ whose kernel is contained in $Z(G)$, and a covering is a central extension such that G is connected and the kernel is finite. If G is reductive, then $\rho\colon \tilde{G} \to G^{der}$ is the universal covering of G^{der}.

A superscript $+$ refers to a topological connected component; for example $G(\mathbb{R})^+$ is the identity connected component of $G(\mathbb{R})$ relative to the real topology, and $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. For G reductive, $G(\mathbb{R})_+$ is the inverse image of $G^{ad}(\mathbb{R})^+$ in $G(\mathbb{R})$ and $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. In contrast to Deligne [3], we use the superscript ^ to denote both completions and closures since we wish to reserve the superscript $-$ for certain negative components.

We write $Sh(G,X)$ for the Shimura variety defined by a pair (G,X) and $Sh^\circ(G, G', X^+)$ for the connected Shimura variety defined by a triple (G,G',X^+). The canonical model of $Sh(G,X)$ is denoted by $M(G,X)$.

Vector spaces are finite-dimensional, number fields are of finite degree over $\mathbb{Q}$ (and usually contained in $\mathbb{C}$), and $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. If V is a vector space over $\mathbb{Q}$ and R is a $\mathbb{Q}$-algebra, we often write $V(R)$ for $V \otimes R$.

We write $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$, $\mathbb{A}^f = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ for the ring of finite adèles of $\mathbb{Q}$, and $\mathbb{A} = \mathbb{R} \times \mathbb{A}^f$ for the ring of adèles of $\mathbb{Q}$. For E a number field, $\mathbb{A}_E^f$ and $\mathbb{A}_E$ denote $E \otimes_{\mathbb{Q}} \mathbb{A}^f$ and $E \otimes \mathbb{A}$. The groups of idèles of E is $\mathbb{A}_E^\times$ and the idèle class group is $C_E^\times = \mathbb{A}_E^\times / E^\times$.

If A is an abelian variety, $A_n = \ker(n: A \to A)$, $TA = \varprojlim A_n$, and $V^f(A) = \mathbb{Q} \otimes TA$. Throughout the paper an abelian variety A will be systematically confused with its isogeny class; thus only $V^f(A)$ (not TA), $H_r(A, \mathbb{Q})$ (not $H_r(A, \mathbb{Z})$), and $H^r(A_{et}, \mathbb{Q}_\ell)$ (not $H^r(A_{et}, \mathbb{Z}_\ell)$) are defined, and $\mathrm{Hom}(A,B)$ means $\mathrm{Hom}(A,B) \otimes \mathbb{Q}$.

Complex conjugation is denoted by $z \longmapsto \iota z$.

We use [*] to denote an equivalence class containing *; for example, if $x \in X$ and $g \in G(\mathbb{A}^f)$ then $[x, g]$ denotes the element of $Sh(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^f) / Z(\mathbb{Q})^\wedge$ containing (x,g). The Hecke operator $[x,g] \longmapsto [x,gg']$ is denoted by $T(g')$. The symbol $A \overset{df}{=} B$ means A is defined to be B or that A equals B by definition.

We normalize the reciprocity isomorphism of class field theory so that a uniformizing parameter corresponds to the reciprocal of the (arithmetic) Frobenius element; we thus agree with Deligne [3] and Tate [1].

For the Weil group, we follow the notations of Tate [1]. In particular, for a topological group Γ, Γ^c denotes the closure of the commutator subgroup of Γ and $\Gamma^{ab} = \Gamma/\Gamma^c$.

For Galois cohomology and torsors (= principal homogeneous spaces) we follow the notations of Serre [1].

The notations concerning Hodge structures are reviewed in Appendix A.

There are many differences of sign between this paper and Langlands [3]; the reason is that this author, despite his statement on p. 224, appears to be using the opposite of the above sign convention for the reciprocity map and hence a different notion of the Weil group from that in Tate [1].

Contents

I. Shimura varieties.

In §1 we review the basic definitions concerning Shimura varieties, introduce the notion of a Shimura variety of abelian type, and discuss the relation between Shimura varieties and connected Shimura varieties. In §2 we describe how some Shimura varieties can be realized as moduli varieties for abelian varieties carrying Hodge cycles, and in §3 we prove a result that will enable us to handle reductive groups whose derived groups are not simply connected.

1. Shimura varieties of abelian type.

A Shimura variety $Sh(G,X)$ is defined by a pair (G,X), comprising a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$, that satisfies the following axioms:

(1.1a) the Hodge structure defined on $Lie(G_{\mathbb{R}})$ by any $h \in X$ is of type $\{(-1, 1), (0,0), (1, -1)\}$ (cf. Appendix A);

(1.1b) for any $h \in X$, ad $h(i)$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$;

(1.1c) the group G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group.

Then $Sh(G,X)$ has complex points $G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)/Z(\mathbb{Q})^\wedge$, where Z is the centre of G and $Z(\mathbb{Q})^\wedge$ the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}^f)$.

A connected Shimura variety $Sh^o(G,G',X^+)$ is defined by a triple (G,G',X^+) comprising an adjoint group G over $\mathbb{Q}$, a covering G' of G, and a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ such that G and the $G(\mathbb{R})$-conjugacy class of X containing X^+ satisfy (1.1). The topology $\tau(G')$ on $G(\mathbb{Q})$ is that for which the images of the congruence subgroups of $G'(\mathbb{Q})$ form a fundamental system of neighbourhoods of the identity, and $Sh^o(G,G',X^+)$ has complex points $\varprojlim \Gamma\backslash X^+$ where Γ runs over the arithmetic subgroups of $G(\mathbb{Q})^+$ that are open relative to the topology $\tau(G')$ (Deligne [3, 2.1.8]).

The relation between the two notions of Shimura variety is as follows: let (G,X) be as in the first paragraph and let X^+ be some connected component of X; then X^+ can be regarded as a $G^{ad}(\mathbb{R})^+$-conjugacy class of maps $\mathbb{S} \to G^{ad}_{\mathbb{R}}$ and $Sh^o(G^{ad}, G^{der}, X^+)$ can be identified with the connected component of $Sh(G,X)$ that contains the image of $X^+ \times \{1\}$.

We recall that the reflex field $E(G,X)$ of (G,X) is the subfield of $\mathbb{C}$ that is the field of definition of the $G(\mathbb{C})$-conjugacy class of μ_h, any $h \in X$, (see Appendix A for this notation) and that $E(G,X^+)$ is defined to equal $E(G,X)$ if X^+ is a connected component of X (Deligne [3, 2.2.1]).

The following easy lemma will be needed in comparing the Shimura varieties defined by (G,X) and (G^{ad}, G^{der}, X^+).

<u>Lemma 1.2.</u> Let $G_1 \to G$ be a central extension of reductive groups over $\mathbb{C}$; let M be a $G(\mathbb{C})$-conjugacy class of homomorphisms $\mathbb{G}_m \to G$ and let M_1 be a $G_1(\mathbb{C})$-conjugacy class lifting M. Then $M_1 \to M$ is bijective.

<u>Proof</u>. The map is clearly surjective and so it suffices to show that, for $\mu_1 \in M_1$ lifting $\mu \in M$, the centralizer of μ_1 is the inverse image of the centralizer of μ. Since the centralizer of μ_1 contains the center of G_1, we only have to show the map on centralizers is surjective. We can construct a diagram

$$C \times G_2 \to G_1 \to G$$

in which the first map, and the composite $G_2 \to G$ are coverings. After replacing μ_1 and μ by multiples, we can assume μ_1 lifts to a homomorphism $(\mu',\mu''): \mathbb{G}_m \to C \times G_2$. Then the centralizer of (μ',μ'') maps into the centralizer of μ_1, and onto the centralizer of μ.

Let (G,X) be as in (1.1) with G adjoint and $\mathbb{Q}$-simple; if every $\mathbb{R}$-simple factor of $G_{\mathbb{R}}$ is of one of the types A, B, C, $D^{\mathbb{R}}$, $D^{\mathbb{H}}$, or E (in the sense of Deligne [3, 2.3.8]; see also Appendix B) then G will be said to be of that type. When G' is a covering of G, we say that (G,G') (or (G,G',X)) is of <u>primitive abelian type</u> if G is of type A, B, C, or $D^{\mathbb{R}}$ and G' is the universal covering of G, or if G is of type $D^{\mathbb{H}}$ and G' is the double covering described in Deligne [3, 2.3.8] (see also Appendix B).

<u>Notations 1.3</u>. If (G,X) satisfies (1.1) and G is adjoint and $\mathbb{Q}$-simple, then there is a totally real number field F_o and and absolutely simple group G^s over F_o such that $G = \mathrm{Res}_{F_o/\mathbb{Q}} G^s$. For any embedding $v\colon F_o \hookrightarrow \mathbb{R}$, let $G_v = G^s \otimes_{F_{o,v}} \mathbb{R}$, and write I_c and I_{nc} for the sets of embeddings for which $G_v(\mathbb{R})$ is compact and noncompact. Let F be a quadratic totally imaginary extension of F_o and let $\Sigma = (\sigma_v)_{v \in I_c}$ be a set of embeddings $\sigma_v\colon F \hookrightarrow \mathbb{C}$ such that $\sigma_v | F_o = v$; we define h_{Σ} to be the Hodge structure on F (regarded as a vector space over $\mathbb{Q}$) such that $(F \otimes_{\mathbb{Q}} \mathbb{C})^{-1,0}$, $(F \otimes_{\mathbb{Q}} \mathbb{C})^{0,-1}$, and $(F \otimes_{\mathbb{Q}} \mathbb{C})^{0,0}$ are

the direct summands of $F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^{\mathrm{Hom}(F,\mathbb{C})}$ corresponding to Σ, $\iota\Sigma$, and $\{\sigma : F \hookrightarrow \mathbb{C} \mid \sigma|F_o \in I_{nc}\}$.

Proposition 1.4. Let G be a $\mathbb{Q}$-simple adjoint group and assume that (G,G',X) is of primitive abelian type. For any pair (F,Σ) as above there exists a diagram

$$(G,X) \longleftarrow (G_1,X_1) \hookrightarrow (CSp(V), S^{\pm})$$

such that $G_1^{ad} = G$, $G_1^{der} = G'$, and $E(G_1,X_1) = E(G,X)\, E(F^{\times},h_{\Sigma})$.

Proof. This is Deligne [3, 2.3.10].

Remark 1.5(a). We shall need a supplement to the proposition. Consider an h in X that is special, and so factors through $T_{\mathbb{R}}$ where T is a $\mathbb{Q}$-rational maximal torus in G. The inverse image of T in G_1 is a $\mathbb{Q}$-rational maximal torus $T_1 \subset G_1$, and h lifts to an h_1 in X_1 factoring through T_1. We claim that $E(T_1,h_1) = E(T,h)\, E(F^{\times},h_{\Sigma})$.

To see this, first note that it is obvious that $E(T_1,h_1) \supset E(T,h)$ and $E(T_1,h_1) \supset E(G_1,X_1)$. As the proposition shows that $E(G_1,X_1) \supset E(F^{\times},h_{\Sigma})$, we see that $E(T_1,h_1) \supset E(T,h)\, E(F^{\times},h_{\Sigma})$. For the reverse inclusion, let σ be an automorphism of $\overline{\mathbb{Q}}$ fixing $E(T,h)$ and $E(F^{\times},h_{\Sigma})$, and let μ_1, μ, and μ_{Σ} be the cocharacters of T_1, T, and $F^{\times}$ corresponding to h_1, h, and h_{Σ}. Since σ fixes $E(T,h)$, it fixes $E(G,X)$, and the proposition shows that it fixes $E(G_1,X_1)$; thus $\sigma\mu_1$ is $G_1(\mathbb{C})$-

2. Shimura varieties as moduli varieties.

We shall want to make use of the notion of an absolute Hodge cycle on a variety (Deligne [4,0.7]) and the important result (Deligne [5]) that any Hodge cycle on an abelian variety is an absolute Hodge cycle. Let A be an abelian variety over an algebraically closed field $k \subset \mathbb{C}$; we shall always identify a Hodge cycle on A with its Betti realization. By this we mean the following. Let $V = H_1(A_{\mathbb{C}}, \mathbb{Q})$ (usual Betti homology) and note that V has a natural Hodge structure and that its dual $\check{V} = H^1(A,\mathbb{Q})$. If $H^1_{dR}(A)$ denotes the de Rham cohomology of A over k then there is a canonical isomorphism $H^1_{dR}(A) \otimes_k \mathbb{C} \xrightarrow{\approx} \check{V}(\mathbb{C})$. There is also a canonical isomorphism $V^f(A) \xrightarrow{\approx} V(\mathbb{A}^f)$. A Hodge cycle s on A is to be an element of some space $V^{\otimes m} \otimes \check{V}^{\otimes n}(p)$ (see Appendix A) such that:

(2.1a) s is of type $(0,0)$ for the Hodge structure defined by that on V;

(2.1b) there is an $s_{dR} \in (H^1_{dR}(A)^{\vee})^{\otimes m} \otimes H^1_{dR}(A)^{\otimes n}$ that corresponds to s under the isomorphism induced by $H^1_{dR}(A) \otimes_k \mathbb{C} \approx V(\mathbb{C})$ and $\mathbb{C} \approx 2\pi i\mathbb{C}$;

(2.1c) there is an $s_{et} \in V^f(A)^{\otimes m} \otimes (V^f(A)^{\vee})^{\otimes n} \otimes (\varprojlim \mu_n(k))^{\otimes p}$ that corresponds to s under the isomorphism induced by $V(\mathbb{A}^f) \approx V^f(A)$ and $2\pi i \hat{\mathbb{Z}} \xrightarrow{\exp} \varprojlim \mu_n(\mathbb{C})$.

Let τ be an automorphism of $\mathbb{C}$; then τA is an abelian variety over $\tau k \subset \mathbb{C}$ and the above-mentioned result of Deligne shows that τs is a well-defined Hodge cycle on τA: it has $(\tau s)_{dR} = s_{dR} \otimes 1 \in H_{dR}(\tau A) = H_{dR}(A) \otimes_{k,\tau} k$ and $(\tau s)_{et} = \tau s_{et}$.

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Certain Shimura varieties can be described as parameter spaces for families of abelian varieties. Let (G,X) satisfy (1.1), and assume there is an embedding $(G,X) \hookrightarrow (CSp(V), S^{\pm})$ where V is a vector space over $\mathbb{Q}$, $CSp(V)$ is the group of symplectic similitudes corresponding to some non-degenerate skew-symmetric form ψ on V, and $S^{\pm}$ is the Siegel double space (in the sense of Deligne [3, 1.3.1]). There will be some family of tensors $(s_\alpha)_{\alpha \in J}$ in spaces of the form $V^{\otimes m} \otimes \check{V}^{\otimes n}(p)$ such that $G = \mathrm{Aut}(V, (s_\alpha)) \subset GL(V) \times \mathbb{G}_m$ (see Appendix A). We shall always take ψ to be one of the s_α; then the projection $G \to \mathbb{G}_m$ is defined by the action of G on ψ.

Consider triples $(A, (t_\alpha)_{\alpha \in J}, k)$ with A an abelian variety over $\mathbb{C}$, (t_α) a family of Hodge cycles on A, and k is an isomorphism $k: V^f(A) \xrightarrow{\approx} (V(\mathbb{A}^f)$ under which t_α corresponds to s_α for each $\alpha \in J$. We define Ⓐ(G,X,V) to be the set of isomorphism classes of triples of this form that satisfy the following conditions:

(2.2a) there exists an isomorphism $H_1(A, \mathbb{Q}) \xrightarrow{\approx} V$ under which s_α corresponds to t_α for each $\alpha \in J$;

(2.2b) the map $\mathbb{S} \xrightarrow{h_A} GL(H_1(A, \mathbb{R}))$ defined by the Hodge structure on $H_1(A, \mathbb{R})$, when composed with the map $GL(H_1(A, \mathbb{R})) \to GL(V(\mathbb{R}))$ induced by an isomorphism as in (a), lies in X.

We let $g \in G(\mathbb{A}^f)$ act on a class $[A, (s_\alpha), k] \in$ Ⓐ(G,X,V) as follows: $[A, (t_\alpha), k]g = [A, (t_\alpha), g^{-1}k]$.

Proposition 2.3. There is a bijection $Sh(G,X) \xrightarrow{\approx} (A)(G,X,V)$ commuting with the actions of $G(\mathbb{A}^f)$.

Proof: Corresponding to $[h,g] \in Sh(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)$, we choose A to be the abelian variety associated with the Hodge structure (V,h). Thus $H_1(A, \mathbb{Q}) = V$ and the s_α can be regarded as Hodge cycles on A. As $V^f(A) = V(\mathbb{A}^f)$ we can define k to be $V^f(A) = V(\mathbb{A}^f) \xrightarrow{g^{-1}} V(\mathbb{A}^f)$. It is easily checked that the class $[A, (t_\alpha), h] \in (A)(G,X,V)$ depends only on the class $[h,g] \in Sh(G,X)$.

Conversely, let $(A, (t_\alpha), k)$ represent a class in $(A)(G,X,V)$. We choose an isomorphism $f: H_1(A, \mathbb{Q}) \to V$ as in (2.2a) and define h to be $f\, h_A f^{-1}$ (cf. 2.2b) and g to be $V(\mathbb{A}^f) \xrightarrow{k^{-1}} V^f(A) \xrightarrow{f\otimes 1} V(\mathbb{A}^f)$. If f is replaced by qf, then (h,g) is replaced by $(ad(q) \circ h, qg)$, and $q \in G(\mathbb{Q})$.

Remark 2.4. The above proposition can be strengthened to show that $Sh(G,X)$ is the solution of a moduli problem over $\mathbb{C}$. Since the moduli problem is defined over $E(G,X)$, $Sh(G,X)$ therefore has model over $E(G,X)$ which, because of the main theorem of complex multiplication, is canonical. This is the proof of Deligne [3, 2.3.1] hinted at in the last paragraph of the introduction to that paper.

3. A result on reductive groups; applications.

The following proposition will usually be applied to replace a given reduction group by one whose derived group is simply connected.

Proposition 3.1. (cf. Langlands [3, p 228-29]). Let G be a reductive group over a field k of characteristic zero and let L be a finite Galois extension of k that is sufficiently large to split some maximal torus in G. Let $G' \to G^{der}$ be a covering of the derived group of G. Then there exists a central extension defined over k

$$1 \longrightarrow N \longrightarrow G_1 \longrightarrow G \longrightarrow 1$$

such that G_1 is a reductive group, N is a torus whose group of characters $X^*(N)$ is a free module over the group ring $\mathbb{Z}[\mathrm{Gal}(L/k)]$, and $(G_1^{der} \longrightarrow G^{der}) = (G' \longrightarrow G^{der})$.

Proof: The construction of G_1 will use the following result about modules.

Lemma 3.2. Let $\underline{G}$ be a finite group and M a finitely generated G-module. Then there exists an exact sequence of $\underline{G}$-modules $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ in which P_0 is free and finitely generated as a $\mathbb{Z}$-module and P_1 is a free $\mathbb{Z}[\underline{G}]$-module.

Proof: Write M_0 for M regarded as an abelian group, and choose an exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M_0 \longrightarrow 0$$

of abelian groups with F_0 (and hence F_1) finitely generated and free. On tensoring this sequence with $\mathbb{Z}[\underline{G}]$ we obtain an exact sequence of $\underline{G}$-modules

$$0 \longrightarrow \mathbb{Z}[\underline{G}] \otimes F_1 \longrightarrow \mathbb{Z}[G] \otimes F_0 \longrightarrow \mathbb{Z}[G] \otimes M_0 \longrightarrow 0$$

whose pull-back relative to the injection

$$(m \mapsto \Sigma g \otimes g^{-1}m): \; M \hookrightarrow \mathbb{Z}[\underline{G}] \otimes M_0$$

has the required properties.

We now prove (3.1). Let T be a maximal torus in G that splits over L and let T' be the inverse image of T under $G' \to G^{der} \subset G$; it is a maximal torus in G'. An application of (3.2) to the $\underline{G} = \mathrm{Gal}(L/k)$-module $M = X_*(T)/X_*(T')$ provides us with the bottom row of the following diagram, and we define Q to be the fibred product of P_0 and $X_*(T)$ over M :

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_*(T') & = & X_*(T') & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & Q & \longrightarrow & X_*(T) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the terms of the middle row of the diagram are torsion-free, the $\mathbb{Z}$-linear dual of the sequence is also exact, and hence corresponds, via the functor X^*, to an exact sequence

$$1 \to N \to T_1 \to T \to 1$$

of tori. The map $X_*(T') \to Q = X_*(T_1)$ corresponds to a map $T' \to T_1$ lifting $T' \to T$. Since the kernel of $T' \to T_1$ is finite, the torsion-freeness of $P_0 = \operatorname{coker}(X_*(T') \to X_*(T_1))$ thus implies that $T' \to T_1$ is injective. On forming the pull-back of the above sequence of tori relative to $Z \hookrightarrow T$, where $Z = Z(G)$, we obtain an exact sequence

$$1 \to N \to Z_1 \to Z \to 1$$

As T' contains $Z' = Z(G')$, $T' \hookrightarrow T_1$ induces an inclusion $Z' \hookrightarrow Z_1$. The group G can be written as a fibred sum, $G = \tilde{G} *_{\tilde{Z}} Z$, where $\tilde{G}$ is the universal covering group of G^{der} and $\tilde{Z} = Z(\tilde{G})$ (Deligne [3.2.0.1]). We can identify G' with a quotient of $\tilde{G}$. Define $G_1 = \tilde{G} *_{\tilde{Z}} Z_1$. It is easy to check that $Z_1 \to Z$ induces a surjection $G_1 \to G$ with kernel $N \subset Z_1 = Z(G_1)$ and that $\tilde{G} \to G_1$ induces an isomorphism $G' \xrightarrow{\approx} G_1^{der}$. Finally, we note that $X_*(N)$ is a free $\mathbb{Z}[\underline{G}]$-module and $X^*(N)$ is the $\mathbb{Z}$-linear dual of $X_*(N)$.

Remark 3.3 (a) The torus N in (3.1) is a product of copies of $\operatorname{Res}_{L/k}\mathbb{G}_m$. Thus $H^1(k', N_{k'}) = 0$ for any field $k' \supset k$, and the sequence $1 \to N(k') \to G_1(k') \to G(k') \to 1$ is exact.

(b) Let $\tilde{T}$ be the inverse image of T (or T') in $\tilde{G}$. Then the maps $\tilde{T} \to T' \hookrightarrow T_1$ and $Z_1 \hookrightarrow T_1$ induce an isomorphism $\tilde{T} *_{\tilde{Z}} Z_1 \xrightarrow{\approx} T_1$. Thus T_1 can be identified with a subgroup of G_1, and the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & T_1 & \longrightarrow & T & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

commutes. Obviously T_1 is a maximal torus in G_1.

<u>Application 3.4</u>. Let (G,X) satisfy (1.1), let $h \in X$ be special, and let T be a maximal torus such that h factors through $T_{\mathbb{R}}$. Let $G' \to G^{der}$ be some covering. Take k to be $\mathbb{Q}$ and L to split T, and construct $T_1 \subset G_1 \to G$ as above. Choose some $\mu_1 \in X_*(T_1)$ mapping to $\mu_h \in X_*(T)$. Then μ_1 obviously commutes with $\iota\mu_1$ and so defines a homomorphism $h_1\colon \mathbb{S} \to T_{\mathbb{R}} \subset G_{\mathbb{R}}$. We let X_1 be the $G(\mathbb{R})$- conjugacy class of maps containing h_1. The pair (G_1,X_1) satisfies (1.1) because, modulo centres, (G_1,X_1) and (G,X) are equal.

It is possible to choose μ_1 so that $E(G_1,X_1) = E(G,X)$. To prove this we first show that the image $\bar{\mu}_h$ of μ_h in M is fixed by $\mathrm{Aut}(\mathbb{C}/E(G,X))$, where M is as in the proof of (3.1):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_*(T') & = & X_*(T') & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_*(N) & \longrightarrow & X_*(T_1) & \longrightarrow & X_*(T) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_*(N) & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We have to show $\tau\mu_h - \mu_h$ lifts to an element of $X_*(T')$ or, equivalently, an element of $X_*(G')$, for any $\tau \in \mathrm{Aut}(\mathbb{C}/E(G,X))$. By proceeding as in the proof of (Deligne [3, 2.5.5]) one can construct a central extension $G_2 \to G_E$ of G_E with $G_2^{der} = \tilde{G}_E$ and a homomorphism $\mu_2\colon\ \mathbb{G}_m \to G_{2\bar{\mathbb{Q}}}$, lifing μ, whose conjugacy class is defined over $E = E(G,X)$. Then, for any $\tau \in \mathrm{Aut}(\mathbb{C}/E)$, $\tau\mu_2 - \mu_2 \in X_*(G_2^{der}) = X_*(\tilde{G}_E)$ and maps to $\tau\mu - \mu \in X_*(T)$.

We now use the fact that $X_*(N)$ is a free $\mathrm{Gal}(LE/E)$-module to deduce the existence of a $\mu_1 \in X_*(T_1)$ mapping to $\mu \in X_*(T)$ and whose image $\bar{\mu}_1$ in P_0 is fixed by $\mathrm{Aut}(\mathbb{C}/E)$. The map $G_1 \to G$ induces an isomorphism $W(G_1,T_1) \xrightarrow{\approx} W(G,T)$ of Weyl groups. Let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ and suppose $\tau\mu = \omega \circ \mu$ with $\omega \in W(G,T)$. If $\omega_1 \in W(G_1,T_1)$ maps to ω, then $\omega_1 \circ \mu_1$ maps to $\tau\mu$ in $X_*(T)$ and $\bar{\mu}_1 = \tau\bar{\mu}_1$ in P_0; thus $\omega_1 \circ \mu_1 = \tau\mu_1$. It follows that τ fixes $E(G_1,X_1)$, and so $E(G,X) \supset E(G_1,X_1)$. The reverse inclusion is automatic.

We can apply this to a triple (G,G',X^+) defining a connected Shimura variety. Thus there exists a pair (G_1,X_1) satisfying (1.1) and such that $(G_1^{ad},G_1^{der},X_1^+) \approx (G,G',X^+)$, $E(G_1,X_1) = E(G,X^+)$, and $X^*(Z(G_1))$ is a free $Gal(L/\mathbb{Q})$-module for some finite Galois extension L of $\mathbb{Q}$ (cf. Deligne [3, 2.7.16]). The last condition implies $G_1(k) \to G_1^{ad}(k) = G(k)$ is surjective for any field $k \supset \mathbb{Q}$.

Application 3.5. Let G be a reductive group over a field k of characteristic zero, and let $\rho\colon \tilde{G} \to G^{der} \subset G$ be the universal covering of G^{der}. When k is a local or global field and k' is a finite extension of k, there is a canonical norm map $N_{k'/k}\colon G(k')/\rho\tilde{G}(k') \to G(k)/\rho\tilde{G}(k)$ (Deligne [3, 2.4]). We shall use (3.1) to give a more elementary construction of this map.

If G is commutative, $N_{k'/k}$ is just the usual norm map $G(k') \to G(k)$.

Next assume G^{der} is simply connected and let $T = G/G^{der}$. If in the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & G(k)/\tilde{G}(k') & \longrightarrow & T(k') & \longrightarrow & H^1(k',\tilde{G}) \\ & & & & \Big\downarrow N_{k'/k} & & \\ 1 & \longrightarrow & G(k)/\tilde{G}(k) & \longrightarrow & T(k) & \longrightarrow & H^1(k,\tilde{G}) \end{array}$$

the map $G(k')/\tilde{G}(k') \to H^1(k,\tilde{G})$ is a zero, we can define $N_{L'/k}$ for G to be the restriction of $N_{k'/k}$ for T.

When k is local and nonarchimedean then $H^1(k,\tilde{G}) = 0$, and so the map is zero. When k is local and archimedean we can suppose $k = \mathbb{R}$ and $k' = \mathbb{C}$; then $N_{\mathbb{C}/\mathbb{R}}: T(\mathbb{C}) \to T(\mathbb{R})$ maps into $T(\mathbb{R})^+$, and any element of $T(\mathbb{R})^+$ lifts to an element of $G(\mathbb{R})$ (even to an element of $Z(G)(\mathbb{R})$). When k is global, we can apply the Hasse principle.

In the general case we choose an exact sequence

$$1 \longrightarrow N \longrightarrow G_1 \longrightarrow G \longrightarrow 1$$

as in (3.1). From the diagram

$$\begin{array}{ccccccc} N(k') & \longrightarrow & G_1(k')/\tilde{G}(k') & \longrightarrow & G(k')/\rho\tilde{G}(k') & \longrightarrow & 1 \\ \downarrow N_{k'/k} & & \downarrow N_{k'/k} & & & & \\ N(k) & \longrightarrow & G_1(k)/\tilde{G}(k) & \longrightarrow & G(k)/\rho\tilde{G}(k) & \longrightarrow & 1 \end{array}$$

we can deduce a norm map for G.

Let k be a number field. If we take the restricted product of the norm maps for the completions of k, and form the quotient by the norm map for k, we obtain the map

$$N_{k'/k}: \pi(G_{k'}) \longrightarrow \pi(G_k)$$

of Deligne [3,2.4.0.1], where $\pi(G_k) = G(\mathbb{A}_k)/(G(k).\rho\tilde{G}(\mathbb{A}_k))$.

Application 3.6. Let G be a reductive group over $\mathbb{Q}$ such that G^{ad} has no factor over $\mathbb{Q}$ whose real points form a compact group, and let G' be an inner twist of G. Thus for some Galois extension L of $\mathbb{Q}$ there is an isomorphism

$f\colon G_L \xrightarrow{\approx} G'_L$ such that, for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$, $(\sigma f)^{-1} \circ f = \mathrm{ad}\,\alpha_\sigma$ with $\alpha_\sigma \in G^{ad}(L)$. We shall show that f induces a canonical isomorphism $\pi(f)\colon \pi(G) \xrightarrow{\approx} \pi(G')$. (Recall $\pi(G) = G(\mathbb{A})/(G(\mathbb{Q})\cdot\rho\tilde{G}(\mathbb{A}))$.)

If f is defined over $\mathbb{Q}$, for example if G is commutative, then $\pi(f)$ exists because π is a functor.

Next assume that G^{der} is simply connected, and let $\bar{f}$ be the isomorphism from $T = G/G^{der}$ to $T' = G'/G'^{der}$ induced by f. A theorem of Deligne [1,2.4] shows that the vertical arrows in the following diagram are isomorphisms

$$\begin{array}{ccc} \pi(G) & \overset{\pi(f)}{\dashrightarrow} & \pi(G') \\ \downarrow\approx & & \downarrow\approx \\ \pi_0(\pi(T)) & \xrightarrow[\approx]{\pi_0(\pi(\bar{f}))} & \pi_0(\pi(T')) \end{array} .$$

We define $\pi(f)$ to make the diagram commute.

In the general case we choose an exact sequence

$$1 \to N \to G_1 \to G \to 1$$

as in (3.1) with G_1^{der} simply connected. Note that $G_1^{ad} = G^{ad}$ so that we can use the same cocycle to define an inner twist $f_1\colon G_{1L} \to G'_{1L}$. The first case considered above allows us to assume f_1 lifts f. Remark (3.3a) shows that $\pi(G_1) \to \pi(G)$ is surjective, and we define $\pi(f)$ to make the following diagram commute:

$$\begin{array}{ccccccc}
\pi(N) & \longrightarrow & \pi(G_1) & \longrightarrow & \pi(G) & \longrightarrow & 1 \\
\downarrow{\scriptstyle \pi(f|N)} & & \downarrow{\scriptstyle \pi(f_1)} & & \downarrow{\scriptstyle \pi(f)} & & \\
\pi(N') & \longrightarrow & \pi(G_1') & \longrightarrow & \pi(G') & \longrightarrow & 1
\end{array}$$

Note that, if $f: G_L \to G_L'$ and $f': G_L' \to G_L''$ define G' and G'' as inner twists of G and G', then $\pi(f')\circ\pi(f) = \pi(f'\circ f)$. Also, that if $f: G \to G$ is of the form $f = \underset{\sim}{ad}\, q$ for $q \in G^{ad}(\mathbb{Q})$, then $\pi(f)$ is induced by $\underset{\sim}{ad}\, q: G(\mathbb{A}) \to G(\mathbb{A})$ and hence is the identity map (Deligne [3,2.0.15]). On combining these two remarks we find that $\pi(f): \pi(G) \to \pi(G')$ depends only on G' and the cocycle (α_σ), for f can only be replaced by $f\circ\underset{\sim}{ad}(q)$ with $q \in G^{ad}(\mathbb{Q})$, and $\pi(f\circ\underset{\sim}{ad}\, q) = \pi(f)\circ\pi(\underset{\sim}{ad}\, q) = \pi(f)$.

II The Conjectures of Langlands

We begin, in §4, by reviewing the basic properties of the Serre group. In §5 we discuss abstractly the notion of an extension of the Serre group by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In §6 and §7 we review, with some complements, Langlands's construction of the Taniyama group and his conjectures concerning Shimura varieties.

§4. The Serre group.

Let $L \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}$, let Γ be the set of embeddings of L into $\mathbb{C}$, and write $L^{\times}$ for $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m$. Any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ defines an element $[\rho]$ of Γ, which may be regarded as a character of $L^{\times}$. Then Γ is a basis for $X^*(L^{\times})$. An element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $X^*(L^{\times})$ by $\sigma(\Sigma b_\rho[\rho]) = \Sigma b_\rho[\sigma\rho] = \Sigma b_{\sigma^{-1}\rho}[\rho]$. The quotient of $L^{\times}$ by the Zariski closure of any sufficiently small arithmetic subgroup has character group $X^*(L^{\times}) \cap (Y^0 \oplus Y^-)$ where

$$Y^0 = \{\chi \in X^*(L^{\times}) \otimes \mathbb{Q} \mid \sigma\chi = \chi \text{, all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$$

$$Y^- = \{\chi \in X^*(L^{\times}) \otimes \mathbb{Q} \mid c\chi = -\chi, \text{ all } c \text{ of the form } c = \sigma\iota\sigma^{-1}\}$$

(Serre [3, II-31, Cor.1]). Thus this quotient is independent of the arithmetic subgroup; it is called the <u>Serre group</u> S^L of L (or, sometimes, the <u>connected</u> Serre group). One checks easily that $X^*(S^L)$ is the subgroup of $X^*(L^{\times})$ of χ satisfying

$$(4.1) \qquad (\sigma-1)(\iota+1)\chi = 0 = (\iota+1)(\sigma-1)\chi \text{, all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

There is a canonical homomorphism $h = h^L\colon \mathbb{S} \to S^L_{\mathbb{R}}$ and hence (see Appendix A) corresponding homomorphisms $w_h\colon \mathbb{G}_m \to S^L_{\mathbb{R}}$ and $\mu = \mu^L\colon \mathbb{G}_m \to S^L_{\mathbb{C}}$. They determine the following maps on the character groups:

$$X^*(h) = (\Sigma b_\rho[\rho] \mapsto (b_1, b_\iota)\colon X^*(S^L) \longrightarrow X^*(\mathbb{S}) = \mathbb{Z} \oplus \mathbb{Z})$$

$$X^*(w_h) = (\Sigma b_\rho[\rho] \mapsto -b_1 - b_\iota)$$

$$X^*(\mu) = (\Sigma b_\rho[\rho] \mapsto b_1)$$

Note that w_h is defined over $\mathbb{Q}$. The pair (S^L,μ^L) is universal: for any $\mathbb{Q}$-rational torus T that is split over L and cocharacter μ of T satisfying (4.1) there is a unique $\mathbb{Q}$-rational homomorphism $S^L \xrightarrow{\rho_\mu} T$ such that $\rho_\mu \circ \mu^L = \mu$. In particular there are no nontrivial automorphisms of (S^L,μ^L) .

For $\mathbb{C} \supset L' \supset L \supset \mathbb{Q}$ and L' of finite degree over $\mathbb{Q}$, the norm map induces a homomorphism $S^{L'} \to S^L$ sending $h^{L'}$ to h^L . The (connected) <u>Serre group</u> S is defined to be the pro-algebraic group $\varprojlim S^L$. There is a canonical homomorphism $h = h_{can} = \varprojlim h^L\colon \mathbb{S} \to S_{\mathbb{R}}$ and corresponding cocharacter $\mu = \mu_{can}\colon \mathbb{G}_m \to S_{\mathbb{C}}$. For any L , S^L is the largest quotient of S that splits over L .

We review the properties of S that we shall need to use.

(4.2). The topology induced on $S^L(\mathbb{Q})$ by the embedding $S^L(\mathbb{Q}) \hookrightarrow S^L(\mathbb{A}^f)$ is the discrete topology; thus $S^L(\mathbb{Q})$ is closed in $S^L(\mathbb{A}^f)$. This is a consequence of Chevalley's theorem, which says that any arithmetic subgroup of the $\mathbb{Q}$-rational points of a torus is open relative to the adelic topology, because the subgroup $\{1\}$ of $S^L(\mathbb{Q})$ is arithmetic.

(4.3). Make $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ act on the group Λ of locally constant functions $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}$ by transport of structure: thus $(\sigma\lambda)(\rho) = \lambda(\sigma^{-1}\rho)$. The map $X^*(S^L) \to \Lambda$ that sends $\chi = \Sigma b_\rho[\rho]$ to the function $\rho \mapsto b_\rho$ identifies $X^*(S^L)$ with the subset Λ^L of Λ comprising those functions that are constant on left cosets of $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$ in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and satisfy (4.1). On passing to the limit over L, we find that $X^*(S)$ becomes identified with the subgroup of Λ of functions satisfying (4.1).

(4.4). Let $\mathbb{Q}^{cm}$ be the union of all subfields of $\overline{\mathbb{Q}}$ of CM-type; it is the largest subfield on which ι and σ commute for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The condition (4.1) is equivalent to the following condition:

(4.1') λ is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{cm})$ and $\lambda(\iota\sigma) + \lambda(\sigma)$ is independent of σ.

In particular, for a given L, $\Lambda^L \subset \Lambda^F$ where $F = L \cap \mathbb{Q}^{cm}$ is the maximal CM-subfield of L (or $\mathbb{Q}$). Since obviously $\Lambda^L \supset \Lambda^F$, they must be equal: $S^L \xrightarrow{\approx} S^F$.

(4.5). (Deligne) Let F be a CM-field with maximal real subfield F_0. There is an exact commutative diagram (of algebraic groups):

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & F^\times/F_0^\times & \xrightarrow{\approx} & S^F/hw(\mathbb{Q}^\times) & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F^\times & \longrightarrow & S^F & \longrightarrow & 1 \\
 & & \uparrow \wr & & \uparrow & & \uparrow hw & & \\
1 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F_0^\times & \xrightarrow{\text{norm}} & \mathbb{Q}^\times & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

To prove this it suffices to show that the square at bottom-right commutes, and the top horizontal arrow is injective, but both of these are easily seen on the character groups. Thus there is an exact sequence

$$1 \longrightarrow F_0^\times \longrightarrow F^\times \times \mathbb{Q}^\times \longrightarrow S^F \longrightarrow 1 .$$

We can deduce that, for any field $k \supset \mathbb{Q}$, there is an injection $H^1(k,S^F) \hookrightarrow Br(F_0 \otimes k)$ where Br denotes the Brauer group. It follows that, when k is a number field, the Hasse principle holds for $H^1(k,S^F)$: the map $H^1(k,S^F) \to \oplus H^1(k_v,S^F)$ is injective. The remark (4.4) shows that this is also true without assuming F to be a CM-field.

(4.6) Let $\lambda \in X^*(S)$ and let T_λ be the $\mathbb{Q}$-rational torus such that $X^*(T_\lambda)$ is the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-submodule of $X^*(S)$ generated by λ. Thus T_λ is a quotient of S and h_{can} defines a homomorphism $h\colon \mathbb{S} \to T_\lambda$. For any $\mathbb{Q}$-rational representation of T_λ, $T_\lambda \hookrightarrow GL(V)$, (V,h) is a $\mathbb{Q}$-rational Hodge structure with weight $n = -(\lambda(1)+\lambda(\iota))$ and Mumford-Tate group $MT(V,h) = T_\lambda$ (Appendix A). The condition (4.1') shows that ι acts as -1 on $Ker(\lambda' \mapsto \lambda'(1)+\lambda'(\iota)\colon X^*(T_\lambda) \to \mathbb{Z})$; thus $(T_\lambda/w_h(\mathbb{G}_m))(\mathbb{R})$ is compact, and (V,h) is polarizable (Deligne [2,2.8]). It follows easily that $S = \varprojlim MT(V,h)$ where the limit is over the $\mathbb{Q}$-rational polarizable Hodge structures (V,h) of CM-type. In other words, S is the group associated with the Tannakian category of Hodge structures of this type.

(4.7) (Serre). It is an easy combinatorial exercise to show that $X^*(S)$ is generated by functions λ such that $\lambda(\sigma)$ is 0 or 1 and $\lambda(\sigma) + \lambda(\iota\sigma) = 1$. If λ is of this type then, for any representation $T_\lambda \hookrightarrow GL(V)$ of T_λ, (V,h) is a $\mathbb{Q}$-rational polarizable Hodge structure of CM-type and weight -1; it therefore corresponds to an abelian variety. Thus $S = \varprojlim MT(A)$ where the limit is over abelian varieties (over $\mathbb{C}$) of CM-type. In other words, the Tannakian category of $\mathbb{Q}$-rational polarizable Hodge structures of CM-type is generated by those arising from abelian varieties.

(4.8) If L is Galois over $\mathbb{Q}$, then $Gal(L/\mathbb{Q})$ acts on $L^\times = Res_{L/\mathbb{Q}}\mathbb{G}_m$ and this action induces an action on the quotient S^L. Thus there is an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $\mathbb{Q}$-rational pro-algebraic group S. It is important to distinguish carefully between the two natural actions of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S(\overline{\mathbb{Q}})$, the first of which arises from the (algebraic) action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on S and the second from the (Galois) action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}$.

5. <u>Extensions of</u> $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ <u>by</u> S.

By an extension of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by S we shall mean a projective system

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^{L'} & \longrightarrow & \underset{\sim}{T}^{L'} & \longrightarrow & Gal(L'^{ab}/\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N_{L'/L}} & & \downarrow & & \downarrow{\scriptstyle can} & & \\ 1 & \longrightarrow & S^{L} & \longrightarrow & \underset{\sim}{T}^{L} & \longrightarrow & Gal(L^{ab}/\mathbb{Q}) & \longrightarrow & 1 \end{array} \qquad (L \subset L')$$

There is no page 34

of extensions of $\mathbb{Q}$-rational pro-algebraic groups; the indexing set is all finite Galois extensions of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$. The group $Gal(L^{ab}/\mathbb{Q})$ is to be regarded as a pro-system of finite constant algebraic groups in the obvious way, and the action of $Gal(L^{ab}/\mathbb{Q})$ on S^L determined by the extension is to be the algebraic action described in (4.8). On passing to the limit we obtain an extension

$$1 \longrightarrow S \longrightarrow \underset{\sim}{T} \longrightarrow Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 .$$

We shall always assume there to be a splitting of the extension over $\mathbb{A}^f$, i.e., a compatible family of continuous homomorphic sections sp^L: $Gal(L^{ab}/\mathbb{Q}) \to \underset{\sim}{T}^L(\mathbb{A}^f)$. In the limit this defines a continuous homomorphism $sp: Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \underset{\sim}{T}(\mathbb{A}^f)$.

Fix an L . The general theory of affine group schemes (Demazure-Gabriel [1,V.2]) shows that, for some finite quotient G' of $G = Gal(L^{ab}/\mathbb{Q})$, $\underset{\sim}{T}^L$ will be the pull-back of an extension of $\underset{-}{G}'$ by S^L:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^L & \longrightarrow & \underset{\sim}{T}^L & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & S^L & \longrightarrow & \underset{\sim}{T}' & \longrightarrow & G' & \longrightarrow & 1 . \end{array}$$

Since S^L splits over L , Hilbert's theorem 90 shows that $H^1(L,S^L) = 0$, and so $\underset{\sim}{T}'(L) \longrightarrow G'$ is surjective. Thus we can choose a section a': $\underset{-}{G}' \to \underset{\sim}{T}'_L$, which will automatically be a morphism of algebraic varieties. On pulling back to G , we get a section $a = a^L: \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to T^L_L$ which is a morphism of pro-algebraic varieties. The choice of such an a gives us the following data.

(5.1). A 2-cocycle (d_{τ_1,τ_2}) for $\mathrm{Gal}(L^{ab}/\mathbb{Q})$ with values in the algebraic group S^L_L, defined by $d_{\tau_1,\tau_2} = a(\tau_1)a(\tau_2)a(\tau_1\tau_2)^{-1}$.

(5.2). A family of 1-cocycles $c(\tau) \in Z^1(L/\mathbb{Q}, S^L(L))$, one for each $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$, defined by $c_\sigma(\tau)a(\tau) = \sigma a(\tau)$. ($\mathrm{Gal}(L/\mathbb{Q})$ acts on $S^L(L)$ through its action on the field L.)

(5.3). A continuous map $b: \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}^f_L)$ defined by $b(\tau)sp^L(\tau) = a(\tau)$.

These satisfy the following relations:

(5.4). $d_{\tau_1,\tau_2} \cdot c_\sigma(\tau_1) \cdot \tau_1(c_\sigma(\tau_2)) = \sigma d_{\tau_1,\tau_2} \cdot c_\sigma(\tau_1\tau_2)$,

(5.5). $d_{\tau_1,\tau_2} = b(\tau_1) \cdot \tau_1 b(\tau_2) \cdot b(\tau_1\tau_2)^{-1}$,

(5.6). $c_\sigma(\tau) = b(\tau)^{-1} \cdot \sigma(b(\tau))$

for $\tau_1, \tau_2, \tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ and $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. (We have used the convention that $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ acts on $S^L(L)$ through its action on S^L, and $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ acts on $S^L(L)$ through its action on the field L.) In fact, the first relation is a consequence of the other two.

Note that b determines (d_{τ_1,τ_2}) and the $(c_\sigma(\tau))$, and that the image $\bar{b}(\tau)$ of $b(\tau)$ in $S^L(\mathbb{A}^f_L)/S^L(L)$ is uniquely determined by the extension and sp^L (independently of the choice of a) .

Proposition 5.7. A mapping $\bar{b}$: $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)/S^L(L)$ arises (as above) from an extension of S^L by $\mathrm{Gal}(L^{ab}/\mathbb{Q})$ and a splitting if and only if it satisfies the following conditions:

(a) $\sigma(\bar{b}(\tau)) = \bar{b}(\tau)$, all $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$, $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$;

(b) $\bar{b}(\tau_1\tau_2) = \bar{b}(\tau_1)\cdot\tau_1\bar{b}(\tau_2)$, all $\tau_1, \tau_2 \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$;

(c) $\bar{b}$ lifts to a continuous map b: $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)$ such that the map $(\tau_1,\tau_2) \mapsto d_{\tau_1,\tau_2} \overset{df}{=} b(\tau_1)\cdot\tau_1 b(\tau_2)\cdot b(\tau_1\tau_2)^{-1}$ is locally constant. Moreover, the extension (together with the splitting) is determined by $\bar{b}$ up to isomorphism.

Proof: We shall only show how to construct the extension from $\bar{b}$, the rest being easy. Choose a lifting b of $\bar{b}$ as in (c) . The family d_{τ_1,τ_2} is a 2-cocycle which takes values in the algebraic group S_L^L . It therefore defines an extension

$$1 \longrightarrow S_L^L \longrightarrow T_L^L \longrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow 1$$

of pro-algebraic groups over L together with a section a: $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \to T_L^L$ that is a morphism of pro-varieties. Define T^L to be the pro-algebraic group scheme over $\mathbb{Q}$ such that $T^L(\bar{\mathbb{Q}}) = T_L^L(\bar{\mathbb{Q}})$ with $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acting by the formula:

$\sigma(s\cdot a(\tau)) = c_\sigma(\tau)\cdot \sigma s \,.\, a(\tau)$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $s \in S^L(\overline{\mathbb{Q}})$, $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$, $c_\sigma(\tau) \overset{\mathrm{df}}{=} b(\tau)^{-1}\cdot \sigma b(\tau) \in S^L(L)$. There is an exact sequence

$$1 \longrightarrow S^L \longrightarrow \underset{\sim}{T}^L \longrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow 1\,.$$

For each $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$, $b(\tau)^{-1}a(\tau) \in S^L(\mathbb{A}^f_L)^{\mathrm{Gal}(L/\mathbb{Q})} = S^L(\mathbb{A}^f)$, and $\tau \mapsto sp(\tau) \overset{\mathrm{df}}{=} b(\tau)^{-1}a(\tau)$ is a homomorphism. As b is continuous, so also is sp.

<u>Corollary 5.8</u>. To define an extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by S (together with a splitting over $\mathbb{A}^f$) it suffices to give maps $\bar{b}^L\colon\ \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}^f_L)/S^L(L)$ satisfying the conditions of (5.7) and such that, whenever $L \subset L'$,

$$\begin{array}{ccc}
\mathrm{Gal}(L'^{ab}/\mathbb{Q}) & \xrightarrow{\ \bar{b}^{L'}\ } & S^{L'}(\mathbb{A}^f_{L'})/S^{L'}(L') \\
\Big\downarrow \text{can} & & \Big\downarrow N_{L'/L} \\
\mathrm{Gal}(L/\mathbb{Q}) & \xrightarrow{\ \bar{b}^{L}\ } & S^{L}(\mathbb{A}^f_{L})/S^{L}(L))
\end{array}$$

commutes.

<u>Remark 5.9</u>. Let $\underset{\sim}{T}$ be an extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by S. For any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, multiplication in $\underset{\sim}{T}$ makes $\pi^{-1}(\tau)$ into a torsor for S, and $sp(\tau)$ is a point of the torsor with values in $\mathbb{A}^f$ (i.e. a trivialization of the torsor over $\mathbb{A}^f$). In the above we have implicitly regarded $\pi^{-1}(\tau)$ as a left torsor, because that is the convention of Langlands [3]. It is however

both more convenient and more conventional to regard $\pi^{-1}(\tau)$ as a right S-torsor. With this point of view it is natural to associate with $\underset{\sim}{T}$ cocycles $(\gamma_\sigma(\tau))$ and a map β defined as follows: let L be a finite Galois extension of $\mathbb{Q}$ and choose a section $\tau \mapsto a(\tau)$ to $\underset{\sim}{T}^L \to \mathrm{Gal}(L^{ab}/\mathbb{Q})$ that is a morphism of pro-algebraic varieties; then

$$\sigma a(\tau) = a(\tau)\gamma_\sigma(\tau), \quad \text{for} \quad \sigma \in \mathrm{Gal}(L/\mathbb{Q}),\ \tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q}) \ , \text{ and}$$

$$sp(\tau)\beta(\tau) = a(\tau) \quad \text{for} \quad \tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q}) \ .$$

The following relations hold:

$$\gamma_\sigma(\tau) = \beta(\tau)^{-1} \cdot \sigma(\beta(\tau))$$

$$\bar{\beta}(\tau_1\tau_2) = \tau_2^{-1}\bar{\beta}(\tau_1) \cdot \bar{\beta}(\tau_2) \ .$$

The new objects are related to the old as follows:

$$\gamma_\sigma(\tau) = \tau^{-1}c_\sigma(\tau)$$

$$\beta(\tau) = \tau^{-1}b(\tau) \ .$$

Define $c'(\tau)$ and $b'(\tau)$ by the formulas (5.2) and (5.3) but with $a(\tau)$ replaced by the section $\tau \mapsto a'(\tau) = a(\tau^{-1})^{-1}$. Then

$$\gamma_\sigma(\tau) = c'_\sigma(\tau^{-1})^{-1}$$

$$\beta(\tau) = b'(\tau^{-1})^{-1}$$

In particular, we see that $\gamma(\tau)$ and $c(\tau^{-1})^{-1}$ are cohomologous and $\bar{\beta}(\tau) = \bar{b}(\tau^{-1})^{-1}$.

<u>Example 5.10</u>. In the preceding discussion there is no need to take the base field to be $\mathbb{Q}$. We shall use this method to construct for any number field $L \subset \overline{\mathbb{Q}}$, a canonical extension

$$1 \longrightarrow S^L \longrightarrow (T^L)^{ab} \xrightarrow{\pi} \mathrm{Gal}(L^{ab}/L) \longrightarrow 1$$

of pro-algebraic groups over $\mathbb{Q}$, together with a splitting over $\mathbb{A}^f$. According to (5.7), such an extension corresponds to a map $\bar{b}\colon \mathrm{Gal}(L^{ab}/L) \to S^L(\mathbb{A}^f_L)/S^L(L)$ satisfying conditions similar to (a), (b), and (c) of that proposition. In fact we shall define a map $\bar{b}\colon \mathrm{Gal}(L^{ab}/L) \to S^L(\mathbb{A}^f)/S^L(\mathbb{Q}) \subset S^L(\mathbb{A}^f_L)/S^L(L)$ and so (a) will be obvious (and the cocycles $c(\tau)$ trivial). Note that $\mathrm{Gal}(L^{ab}/L)$ acts trivially on S^L and so (b) requires that $\bar{b}$ be a homomorphism.

The canonical element $\mu^L \in X_*(S^L)$ is defined over L, and so gives rise to a homomorphism of algebraic groups,

$$\mathrm{NR}\colon \quad L^\times \xrightarrow{\mathrm{Res}_{L/\mathbb{Q}}(\mu^L)} \mathrm{Res}_{L/\mathbb{Q}}\, S^L_L \xrightarrow{N_{L/\mathbb{Q}}} S^L .$$

Consider

$$\begin{array}{llcl}
\mathrm{NR}(\mathbb{A})\colon & \mathbb{A}^\times_L & \longrightarrow & S^L(\mathbb{A}) \\
 & \cup & & \cup \\
\mathrm{NR}(L)\colon & L^\times & \longrightarrow & S^L(\mathbb{Q})
\end{array}$$

The reciprocity morphism (Deligne [3,2.2.3])

$$r_L = r_L(S^L, h^L): \quad \mathrm{Gal}(L^{ab}/L) \longrightarrow S^L(\mathbb{A}^f)/S^L(\mathbb{Q})$$

is defined to be the reciprocal of the composite of the following maps: the reciprocity law isomorphism $\mathrm{Gal}(L^{ab}/L) \overset{\approx}{\longleftrightarrow} \pi_0(\mathbb{A}_L^\times/L^\times)$, the map $\pi_0(\mathbb{A}_L^\times/L^\times) \to \pi_0(S^L(\mathbb{A})/S^L(\mathbb{Q}))$ defined by NR , and the projection $\pi_0(S^L(\mathbb{A})/S^L(\mathbb{Q})) \to S^L(\mathbb{A}^f)/S^L(\mathbb{Q}))$. We define $\bar{b}(\tau) = r_L(\tau)^{-1}$. It satisfies (a) and (b) of (5.7).

According to (4.2), $S^L(\mathbb{Q})$ is a discrete subgroup of $S^L(\mathbb{A}^f)$, and hence of $S^L(\mathbb{A})$. Thus there is an open subgroup U of $\mathbb{A}_L^\times$ such that NR: $\mathbb{A}_L^\times \to S^L(\mathbb{A})$ is 1 on $U \cap L^\times$. If $F \supset L$ corresponds to $U \subset \mathbb{A}_L^\times$, then there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(L^{ab}/F) & \longrightarrow & \mathrm{Gal}(L^{ab}/L) & \longrightarrow & \mathrm{Gal}(F/L) & \longrightarrow & 1 \\
 & & \downarrow b & \swarrow b & \downarrow \bar{b} & & & & \\
 & & S^L(\mathbb{A}^f) & \longrightarrow & S^L(\mathbb{A}^f)/S(L) & & & &
\end{array}$$

in which b^{-1}: $\mathrm{Gal}(L^{ab}/F) \to S^L(\mathbb{A})$ is induced by NR: $U/U \cap L^\times \to S^L(\mathbb{A})$. It is easy to extend b to a continuous map $\mathrm{Gal}(L^{ab}/L) \to S^L(\mathbb{A}^f)$ lifting $\bar{b}$: choose a set S' of representatives for $\mathrm{Gal}(F/L)$ in $\mathrm{Gal}(L^{ab}/L)$, choose an element $b(s) \in S^L(\mathbb{A}^f)$ mapping to $\bar{b}(s)$ for each $s \in S'$, and define $b(sg) = b(s)b(g)$ for $s \in S'$, $g \in \mathrm{Gal}(L^{ab}/F)$. This map b satisfies (c) of (5.7) because, when restricted to $\mathrm{Gal}(L^{ab}/F)$, it is a homomorphism.

Remark 5.11. The extension constructed in (5.10) is, up to sign, that defined by Serre [3]. For a sufficiently large modulus $\underline{m}$ the group $T_{\underline{m}} = T/\bar{E}_{\underline{m}}$ of (ib.,p II-8) is the Serre group S^L, and $C_{\underline{m}} = \mathrm{Gal}(L_{\underline{m}}/L)$ for some $L_{\underline{m}} \subset L^{ab}$. Thus the sequence (ib.,p II-9) can be written

$$1 \longrightarrow S^L \longrightarrow S_{\underline{m}} \longrightarrow \mathrm{Gal}(L_{\underline{m}}/L) \longrightarrow 1 .$$

On passing to the limit over increasing $\underline{m}$, this becomes

$$1 \longrightarrow S^L \longrightarrow (\underset{\sim}{T}^L)^{ab} \longrightarrow \mathrm{Gal}(L^{ab}/L) \longrightarrow 1 .$$

The splitting (over $\mathbb{Q}_\ell$) is defined in (ib., 2.3).

6. The Taniyama group.

We denote the Weil group of a local or global field L by W_L. Let v denote the prime induced on $\overline{\mathbb{Q}}$, or a subfield L of $\overline{\mathbb{Q}}$, by the fixed inclusion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and let L_v denote the closure of L in $\overline{\mathbb{Q}}_v = \mathbb{C}$. According to Tate [1] there is a homomorphism $i_v\colon W_{\mathbb{Q}_v} \to W_{\mathbb{Q}}$ such that the diagrams

$$\begin{array}{ccc} L_v & \xrightarrow[\approx]{rec_v} & W^{ab}_{L_v} \\ \downarrow{\scriptstyle can} & & \downarrow{\scriptstyle i^{ab}_v} \\ C_L & \xrightarrow[\approx]{rec_L} & W^{ab}_L \end{array} \qquad \begin{array}{ccc} W_{\mathbb{Q}_v} & \xrightarrow{\phi_v} & \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v) \\ \downarrow{\scriptstyle i_v} & & \cap \\ W_{\mathbb{Q}} & \xrightarrow{\phi} & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \end{array}$$

commute for all number fields L contained in $\overline{\mathbb{Q}}$. The constructions that follow will be independent of the choice of i_v, but we shall ignore this question by fixing an i_v. If $L \subset \overline{\mathbb{Q}}$ is a finite Galois extension of $\mathbb{Q}$ then i_v induces a map from $W_{L_v/\mathbb{Q}_v} \overset{df}{=} W_{\mathbb{Q}_v}/W^c_{L_v}$ to $W_{L/\mathbb{Q}} \overset{df}{=} W_{\mathbb{Q}}/W^c_L$ which makes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_v^{\times} & \longrightarrow & W_{L_v/\mathbb{Q}_v} & \longrightarrow & \mathrm{Gal}(L_v/\mathbb{Q}_v) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle i_v} & & \downarrow & & \\ 1 & \longrightarrow & C_L & \longrightarrow & W_{L/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & 1 \end{array} \qquad (6.1)$$

commute.

We note that there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_L & \longrightarrow & W_{L/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \rightarrow & \mathrm{Gal}(L^{ab}/L) & \rightarrow & \mathrm{Gal}(L^{ab}/\mathbb{Q}) & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & 1 \end{array}$$

in which the vertical arrows are surjective.

Let T be a torus over $\mathbb{Q}$; by analogy with $T(L) = X_*(T) \otimes L^\times$, $T(\mathbb{A}^f) = X_*(T) \otimes \mathbb{A}^f$ etc., we shall write $T(C_L)$ for $X_*(T) \otimes C_L$. If $\mu \in X_*(T)$ and a belongs to a $\mathbb{Q}$-algebra R (or C_L) then we write a^μ for $\mu \otimes a \in T(R)$.

Fix such a torus T and an element $\mu \in X_*(T)$, and let $L \subset \overline{\mathbb{Q}}$ be a number field splitting T. For each $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ that satisfies

$$(1 + \iota)(\tau^{-1} - 1)\mu = 0 \tag{6.3}$$

and lifting $\tilde{\tau}$ of τ to $W_{L/\mathbb{Q}}$ (using the map in (6.2)) we shall define an element $b_0(\tilde{\tau},\mu) \in T(C_L)/T(L_\infty^\times)$, where $L_\infty = L \otimes_{\mathbb{Q}} \mathbb{R}$.

Choose a section $\sigma \longmapsto w_\sigma$ to $W_{L/\mathbb{Q}} \longrightarrow \mathrm{Gal}(L/\mathbb{Q})$ such that:

(6.4a) $w_1 = 1$;

(6.4b) $w_\iota \in W_{L_v/\mathbb{Q}_v} \subset W_{L/\mathbb{Q}}$;

(6.4c) for some choice of H containing 1 and such that $\mathrm{Gal}(L/\mathbb{Q}) = H \cup H\iota$ (disjoint union), $w_{\sigma\iota} = w_\sigma w_\iota$ for all $\sigma \in H$.

Of course, the last two conditions are trivial if $L \subset \mathbb{R}$. Corresponding to w there is a 2-cocycle $(a_{\sigma,\tau})$, defined by $w_\sigma w_\tau = a_{\sigma,\tau} w_{\sigma\tau}$. Let $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ satisfy (6.3) and let $\tilde{\tau} \in W_{L/\mathbb{Q}}$ map to it. Choose $c_{\sigma,\tilde{\tau}} \in C_L$ to satisfy $w_\sigma \tilde{\tau} = c_{\sigma,\tilde{\tau}} w_{\sigma,\tau}$, and define

$$b_0(\tilde{\tau},\mu) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} c_{\sigma,\tilde{\tau}}^{\sigma\mu} \in T(C_L)/T(L_\infty).$$

Lemma 6.5. The element $b_0(\tilde{\tau},\mu)$ is independent of the choice of the section w; it is fixed by $\mathrm{Gal}(L/\mathbb{Q})$.

Proof. (Langlands [3, p. 221; p. 223]).

On tensoring

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L^{\times} & \longrightarrow & \mathbb{A}_L^{f\,\times} & \longrightarrow & \mathbb{A}_L^{f\,\times}/L^{\times} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle -1} & & \downarrow & & \downarrow{\scriptstyle \approx} & & \\ 1 & \longrightarrow & L_{\infty}^{\times} & \longrightarrow & C_L & \longrightarrow & C_L/L_{\infty}^{\times} & \longrightarrow & 1 \end{array} \qquad (6.6a)$$

with $X_*(T)$ we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T(L) & \longrightarrow & T(\mathbb{A}_L^f) & \longrightarrow & T(\mathbb{A}_L^f)/T(L) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle -1} & & \downarrow & & \downarrow{\scriptstyle \approx} & & \\ 1 & \longrightarrow & T(L_{\infty}) & \longrightarrow & T(C_L) & \longrightarrow & T(C_L)/\,T(L_{\infty}) & \longrightarrow & 1\,. \end{array} \qquad (6.6b)$$

(The -1 reminds us that the map is the reciprocal of the obvious inclusion.) We define $\bar{b}(\tilde{\tau},\mu)$ to be the element of $T(\mathbb{A}_L^f)/T(L)$ corresponding to $b_0(\tilde{\tau},\mu)$. Lemma 6.5 shows that it lies in $(T(\mathbb{A}_L^f)/T(L))^{\mathrm{Gal}(L/\mathbb{Q})}$ and hence gives rise to an element $c(\tilde{\tau},\mu) \in H^1(L/\mathbb{Q},\, T(L))$ through the boundary map in the exact sequence

$$1 \to T(\mathbb{Q}) \longrightarrow T(\mathbb{A}^f) \longrightarrow (T(\mathbb{A}_L^f)/T(L))^{\mathrm{Gal}(L/\mathbb{Q})} \longrightarrow H^1(L/\mathbb{Q}, T(L)).$$

LEMMA 6.7. The cohomology class $c(\tilde{\tau},\mu)$ depends only on the image of $\tilde{\tau}$ in $\mathrm{Gal}(L/\mathbb{Q})$.

Proof. Suppose $\tilde{\tau}'$ and $\tilde{\tau}$ have the same image in $\mathrm{Gal}(L/\mathbb{Q})$; then $\tilde{\tau}' = u\tilde{\tau}$ with $u \in C_L$, and $c_{\sigma,\tilde{\tau}'} = \sigma(u)c_{\sigma,\tilde{\tau}}$. Thus $b_0(\tilde{\tau},\mu)$ is multiplied by $\Pi\sigma(u)^{\sigma\mu} = NR(u)$, where NR is the map of algebraic groups $L^{\times} \xrightarrow{\mathrm{Res}(\mu)} \mathrm{Res}_{L/\mathbb{Q}} T_L \xrightarrow{N_{L/\mathbb{Q}}} T$. Choose an element $\tilde{u} \in \mathbb{A}_L^f$ such that $\tilde{u}$ and u represent the same element in $C_L/L_\infty^{\times}$. Then $NR(\tilde{u}) \in T(\mathbb{A}^f)$ has the same image as $NR(u)$ in $T(C_L)/T(L_\infty^{\times})$, and we see that $b(\tilde{\tau}',\mu) = \overline{NR}(\tilde{u})\, b(\tilde{\tau},\mu)$ where $\overline{NR}(\tilde{u})$ denotes the image of $NR(\tilde{u})$ in $T(\mathbb{A}^f)/T(\mathbb{Q}) \subset T(\mathbb{A}_L^f)/T(L)$. Hence $c(\tilde{\tau},\mu) = c(\tilde{\tau}',\mu)$.

Thus we can write $c(\tau,\mu)$ for $c(\tilde{\tau},\mu)$ where $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ (or even $\mathrm{Gal}(L/\mathbb{Q})$).

Lemma 6.8. Up to multiplication by an element of the closure $T(\mathbb{Q})^{\wedge}$ of $T(\mathbb{Q})$ in $T(\mathbb{A}^f)$, $b(\tilde{\tau},\mu)$ depends only on τ (and not $\tilde{\tau}$).

Proof. From (6.2) we see that $\tilde{\tau}$ can be multiplied only by an element u of the identity component of C_L. An argument as in the proof of (6.7) shows that multiplying $\tilde{\tau}$ by u corresponds to multiplying $b(\tilde{\tau},\mu)$ by $\overline{NR}(\tilde{u})$, where $\tilde{u}$ is a lifting of u to $\mathbb{A}_L^f$. But $\tilde{u}$ is in the closure of $L^{\times} \subset (\mathbb{A}_L^f)^{\times}$, and so $\overline{NR}(\tilde{u})$ is in the closure of $T(\mathbb{Q})$.

Thus, for any $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ satisfying (6.3), there is a well-defined element $\overline{b}(\tau,\mu) \in T(\mathbb{A}_L^f)/T(L)\,T(\mathbb{Q})^{\wedge}$.

Example 6.9. For any T and μ, $\bar{b}(\iota,\mu)$ is defined; we show that it is 1. We can take $\tilde{\iota} = w_\iota$. If $\sigma \in H$ (see 6.4), then $w_\sigma \tilde{\iota} = w_\sigma w_\iota = w_{\sigma\iota}$, and $c_{\sigma,\tilde{\iota}} = 1$; moreover $w_{\sigma\iota}\tilde{\iota} = w_\sigma w_\iota w_\iota = w_\sigma a_{\iota,\iota} = \sigma(a_{\iota,\iota})w_\sigma$ and $c_{\sigma\iota,\tilde{\iota}} = \sigma(a_{\iota,\iota}) \in L_\infty^\times$. Clearly $b_0(\tilde{\iota},\mu) = 1$.

Proposition 6.10. Assume that μ is defined over $E \subset L$. Then $\bar{b}(\tau,\mu)$ is defined for all $\tau \in \mathrm{Gal}(L^{ab}/E)$ and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(L^{ab}/E) & \xrightarrow{\bar{b}(-,\mu)} & T(\mathbb{A}_L^f)/T(L)\ T(\mathbb{Q})^\wedge \\ \downarrow \text{rest} & & \uparrow \\ \mathrm{Gal}(E^{ab}/E) & \xrightarrow{r_E(T,h)^{-1}} & T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge \end{array}$$

in which $r_E(T,h)$ is the reciprocity morphism (Deligne [3,2.2.3]). In particular, $c(\tau,\mu)$ is trivial.

Proof. Let $\tau \in \mathrm{Gal}(L^{ab}/E)$. Then τ fixes μ, and so (6.3) is satisfied and $\bar{b}(\tau,\mu)$ is defined. We may choose the section w to $W_{L/\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q})$ in such a way that $w_\tau = \tilde{\tau}$ maps to τ in $\mathrm{Gal}(L^{ab}/\mathbb{Q})$. Then $c_{\sigma,\tilde{\tau}} = a_{\sigma,\tau}$. Let R be a set of representatives for $\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/E)$. We have

$$b_0(\tilde{\tau},\mu) = \prod_{\rho \in R} \prod_{\sigma \in \mathrm{Gal}(L/E)} a_{\rho\sigma,\tau}^{\rho\mu} \quad (\text{since } \sigma\mu = \mu)$$

$$= \prod_{\rho \in R} \Big(\prod_\sigma (\rho a_{\sigma,\tau} \cdot a_{\rho,\sigma\tau} \cdot a_{\rho,\sigma}^{-1})\Big)^{\rho\mu}$$

$$= \prod_{\rho \in R} (\rho a)^{\rho\mu}, \quad \text{where } a = \prod_\sigma a_{\sigma,\tau}.$$

To evaluate a, we use the commutative diagram (Tate[3, W_3])

$$\begin{array}{ccc} C_E & \xrightarrow{r_E} & W_E^{ab} \\ \downarrow & & \downarrow t \\ C_L & \xrightarrow{r_L} & W_L^{ab} \end{array}$$

where t is the transfer map arising from the inclusion $W_L \hookrightarrow W_E$. Clearly $r_L(a) = \prod r_L(a_{\sigma,\tau}) = t(\tilde{\tau} W_E^c)$. Thus a is an element of C_E that maps to $\tau | E^{ab}$ in $\mathrm{Gal}(E^{ab}/E)$. Let $\tilde{a} \in \mathbb{A}_E^f$ represent the same element in $C_E/E_\infty^\times$ as a. Then $\bar{b}(\tau,\mu)$ is the image of $\tilde{a}$ under $\mathbb{A}_E^{f\times} \xrightarrow{NR} T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge$, and this equals $r_E(T,h)(\tau|E^{ab})^{-1}$.

We now apply the above theory to construct the Taniyama group of a finite Galois extension L of $\mathbb{Q}$, $L \subset \bar{\mathbb{Q}}$. To do so, we take the torus T to be S^L and μ to be the canonical co-character of S^L (see §4). Since $S^L(\mathbb{Q})$ is closed in $S^L(\mathbb{A}^f)$ the above constructions give a map $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow (S^L(\mathbb{A}_L^f)/S^L(L))^{\mathrm{Gal}(L/\mathbb{Q})}$ which we denote by $\bar{b}$(or $\bar{b}^L$).

<u>Proposition 6.11</u>. The map $\bar{b}$ satisfies the conditions of (5.7) and so defines an extension

$$1 \longrightarrow S^L \longrightarrow T^L \longrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow 1$$

together with a continuous splitting over $\mathbb{A}^f$.

<u>Proof</u>. It is proved in Langlands [3, p. 223] that $\bar{b}(\tau_1\tau_2) = \bar{b}(\tau_1) \cdot \tau_1\bar{b}(\tau_2)$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Gal}(L^{ab}/\mathbb{Q}) & \xrightarrow{\bar{b}} & S^L(\mathbb{A}_L^f)/S^L(L) \\ \uparrow & & \uparrow \\ \mathrm{Gal}(L^{ab}/L) & \xrightarrow{b} & S^L(\mathbb{A}^f) \end{array}$$

where b is the map defined in (5.9). The diagram commutes because of (6.10). It is easy to extend b to a continuous map $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}^f)$ lifting $\bar{b}$ (see the proof of 5.9). Then b satisfies (5.7c) because its restriction to $\mathrm{Gal}(L^{ab}/F)$ is a homomorphism, where F is the finite extension of L defined in the proof of (5.9).

The extension, together with the splitting, is the Taniyama group of L. The next lemma implies that the Taniyama groups for varying L form a projective system: we have an extension of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S in the sense of §5.

Lemma 6.12. If $L' \supset L$ then

$$\begin{array}{ccc} \mathrm{Gal}(L'^{ab}/\mathbb{Q}) & \xrightarrow{\bar{b}^{L'}} & S^{L'}(\mathbb{A}_{L'}^f)/S^{L'}(L') \\ \downarrow \text{rest.} & & \downarrow N_{L'/L} \\ \mathrm{Gal}(L^{ab}/\mathbb{Q}) & \xrightarrow{\bar{b}^{L}} & S^L(\mathbb{A}_L^f)/S^L(L) \end{array}$$

commutes.

Proof. This follows from the corresponding statement for b_0; see Langlands [3, p. 222-23].

Proposition 6.13. Let T be a torus over $\mathbb{Q}$, let $\mu \in X_*(T)$, and let ι be an automorphism of $\mathbb{C}$. Assume (6.3) holds, so that

$c(\tau,\mu) \in H^1(L/\mathbb{Q}, T(L))$ is defined for L a sufficiently large number field. The image of $c(\tau,\mu)$ in $H^1(L_v/\mathbb{Q}_v, T(L_v))$ is represented by $\mu(-1)/\tau^{-1}\mu(-1) \in \mathrm{Ker}(1+\iota : T(\mathbb{C}) \to T(\mathbb{C}))$.

Proof. The image of $c(\tau,\mu)$ in $H^1(\mathbb{C}/\mathbb{R}, T(\mathbb{C}))$ is the cup-product of the local fundamental class in $H^2(\mathbb{C}/\mathbb{R}, \mathbb{C}^\times)$ with the element of $H^{-1}(\mathbb{C}/\mathbb{R}, X_*(T))$ represented by $(1-\tau^{-1})\mu$. Thus the proposition is a consequence of the following easy lemma.

Lemma 6.14. For any torus T over $\mathbb{R}$, the map $H^{-1}(\mathbb{C}/\mathbb{R}, X_*(T)) \longrightarrow H^1(\mathbb{C}/\mathbb{R}, T)$ induced by cupping with the fundamental class in $H^2(\mathbb{C}/\mathbb{R}, \mathbb{C}^\times)$ sends the class represented by $\chi \in X_*(T)$ to the class represented by $\chi(-1)$.

Remark 6.15. For any $\mathbb{Q}$-rational torus T, split by L, and cocharacter μ satisfying (6.3) relative to $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ we have defined an element $\bar{b}(\tau,\mu) \in T(\mathbb{A}_L^f)/T(L)\, T(\mathbb{Q})^\wedge$. It is natural also to define $\bar{\beta}(\tau,\mu) = \bar{b}(\tau^{-1},\mu)^{-1}$ and $\gamma(\tau,\mu) = c(\tau^{-1},\mu)^{-1}$ (cf. 5.9). If μ satisfies the stronger condition (4.1) then there is a unique homomorphism $\rho_\mu : S^L \to T$ such that $\rho_\mu \circ \mu^L = \mu$, and we have $\bar{\beta}(\tau,\mu) = \rho_\mu(\bar{\beta}(\tau))$ and $\gamma(\tau,\mu) = \rho_\mu(\gamma(\tau))$.

§7. The conjectures of Langlands.

Let (G,X) satisfy (1.1). Before discussing the conjectures of Langlands concerning $Sh(G,X)$ we review some of the properties of (G,X) over $\mathbb{R}$.

Let $h \in X$ be special (in the sense of Deligne [3,2.2.4]), and let T be a $\mathbb{Q}$-rational maximal torus such that h factors through $T_{\mathbb{R}}$. Let $\mu = \mu_h$ be the cocharacter corresponding to h. According to (1.1b) $\mathrm{ad}\, h(i)$ is a Cartan involution on $G_{\mathbb{R}}^{ad}$, and hence on $G_{\mathbb{R}}^{der}$. Thus $\mathfrak{g}^{der} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{g}^{der} = \mathrm{Lie}(G_{\mathbb{R}}^{der}) = \mathrm{Lie}(G_{\mathbb{R}})^{der}$ and $\mathrm{Ad}\, h(i)$ acts as 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. According to (1.1a) there is a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{c}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$$

where $\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{R}})$, $\mathfrak{c} = \mathrm{Lie}(Z(G)_{\mathbb{R}})$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$, and $\mathrm{Ad}\, \mu(z)$ acts as z on $\mathfrak{p}^{+}$ and ιz on $\mathfrak{p}^{-}$. (Thus $\mathfrak{g}^{0,0} = \mathfrak{c}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$, $\mathfrak{g}^{-1,1} = \mathfrak{p}^{+}$ and $\mathfrak{g}^{1,-1} = \mathfrak{p}^{-}$.) As $T_{\mathbb{C}}$ is a maximal torus in $G_{\mathbb{C}}$, we also have a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha \in R} \mathfrak{g}_{\alpha}$$

where $\mathfrak{t} = \mathrm{Lie}(T_{\mathbb{R}})$ and $R \subset \mathfrak{t}_{\mathbb{C}}^{\vee}$ is the set of roots of (G,T). A root α is said to be <u>compact</u> or <u>noncompact</u> according as $\mathfrak{g}_{\alpha} \subset \mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{g}_{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$.

<u>Remark 7.1</u>. If $Y \in \mathfrak{g}_{\alpha}$ then $\mathrm{Ad}(\mu(-1))Y = \alpha(\mu(-1))Y - (-1)^{\langle\alpha,\mu\rangle}Y$. Since $\mathrm{Ad}\mu(-1)$ acts on $\mathfrak{k}_{\mathbb{C}}$ as $+1$ and on $\mathfrak{p}_{\mathbb{C}}$ as -1, this shows that α is compact or noncompact according as $\langle\alpha,\mu\rangle$ is

even or odd.

Note that $T^{der} \overset{df}{=} T \cap G^{der}$ is anisotropic because $\underline{t}^{der} \subset \underline{k}$. Let N be the normalizer of T in G and let $W = N(\mathbb{C})/T(\mathbb{C})$ be the Weyl group. As $\underline{\iota}$ acts as -1 on $R \subset \underline{t}^{der}_{\mathbb{C}}$, it commutes with the action of any reflection s_α. Hence $\underline{\iota}$ acts trivially on W and there is an exact cohomology sequence.

$$1 \longrightarrow T(\mathbb{R}) \longrightarrow N(\mathbb{R}) \xrightarrow{\text{ad}} W \xrightarrow{\delta} H^1(\mathbb{R},T)$$

where, for $\underline{\omega} \in W$ lifting to $w \in N(\mathbb{C})$, $\delta(\underline{\omega})$ is represented by $w^{-1} \cdot {}_\iota w \in \operatorname{Ker}(1 + \iota : T(\mathbb{C}) \to T(\mathbb{C}))$.

Proposition 7.2. The class $\delta(\omega)$ is represented by $(\omega^{-1}\mu)(-1)/\mu(-1) \in T(\mathbb{C})$.

Proof: Note that $\delta(\omega_1\omega_2) = \omega_2^{-1}\ \delta(\omega_1) \cdot \delta(\omega_2)$ while $(\omega_1\omega_2)^{-1}\mu(-1)/\mu(-1) = \omega_2^{-1}(\omega_1^{-1}\mu(-1)/\mu(-1)) \cdot (\omega_2^{-1}\mu(-1)/\mu(-1))$ and so it suffices to prove the proposition for a generator s_α of W.

We make the identifications $T(\mathbb{C}) = X_*(T) \otimes \mathbb{C}^\times$, $\underline{t}_{\mathbb{C}} = X_*(T) \otimes \mathbb{C}$, and $\underline{t}^\vee_{\mathbb{C}} = X^*(T) \otimes \mathbb{C}$. If $\alpha^\vee$ is a coroot and H_α is the element of $\underline{t}_{\mathbb{C}}$ corresponding to α, then $\exp \pi i H_\alpha = \alpha^\vee(-1)$. Let $X_\alpha \in \underline{g}_\alpha$ and $X_{-\alpha} \in \underline{g}_{-\alpha}$ be such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. As ${}_\iota\alpha = -\alpha$, we have that ${}_\iota H_\alpha = -H_\alpha$ and that ${}_\iota X_\alpha = cX_{-\alpha}$ and ${}_\iota X_{-\alpha} = dX_\alpha$ with $c, d \in \mathbb{C}$. The conditions $[X_\alpha, X_{-\alpha}] = H_\alpha$ and $\iota^2 = 1$ imply that $cd = 1$ and ${}_\iota c.d = 1$, and so c is real and $d = c^{-1}$. If we replace X_α by aX_α then we must replace $X_{-\alpha}$ by $\frac{1}{a}X_{-\alpha}$ and c by a^2c. Thus, for a given α, there are two possiblities:

either X_α can be chosen so that $\iota X_\alpha = -X_{-\alpha}$ or X_α can be chosen so that $\iota X_\alpha = X_{-\alpha}$. In the first case α is compact and in the second is is noncompact.

Assume that α is compact; then the map $\underline{su}_2 \to \underline{g}$ such that $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \mapsto H_\alpha$, $\begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix} \mapsto X_\alpha$, $\begin{pmatrix}0 & 0\\ -i & 0\end{pmatrix} \mapsto X_{-\alpha}$ lifts to a homomorphism $SU_2 \to G_{\mathbb{R}}$ (defined over $\mathbb{R}$). The image w of $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ in $G(\mathbb{R})$ represents s_α. Thus $\delta(s_\alpha) = 1$ in this case. On the other hand, $s_\alpha(\mu)-\mu = -\langle\alpha,\mu\rangle\alpha^\vee$, and so $s_\alpha\mu(-1)/\mu(-1) = \alpha^\vee(-1)^{-\langle\alpha,\mu\rangle} = 1$ (by 7.1).

If α is noncompact, then the map $\underline{sl}_2 \to \underline{g}$ such that $\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \mapsto H_\alpha$, $\frac{1}{2}\begin{pmatrix}-i & 1\\ 1 & -i\end{pmatrix} \mapsto X_\alpha$, $\frac{1}{2}\begin{pmatrix}i & 1\\ 1 & -i\end{pmatrix} \mapsto X_{-\alpha}$ lifts to a homomorphism $SU_2 \to G_{\mathbb{R}}$. The image w of $\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ in $G(\mathbb{C})$ represents s_α. Then $w^{-1}.\iota w$ is the image of $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, which is $\exp \pi i H_\alpha = \alpha^\vee(-1)$. On the other hand
$s_\alpha\mu(-1)/\mu(-1) = \alpha^\vee(-1)^{-\langle\alpha,\mu\rangle} = \alpha^\vee(-1)$ (by 7.1).

<u>Corollary 7.3</u>. If the reflex field $E(G,X)$ of (G,X) is real then there exists an $n \in N(\mathbb{R})$ such that $\underline{ad}(n)\circ\mu = \iota\mu$.

<u>Proof</u>: Since ι fixes $E(G,X)$ there is an element w in $G(\mathbb{C})$, which we can choose to lie in $N(\mathbb{C})$, such that $\iota\mu = \underline{ad}(w)\circ\mu$. The proposition shows that the image of $\underline{ad}\, w$ in $H^1(\mathbb{C}/\mathbb{R}, T(\mathbb{C}))$ is represented by $(\iota-1)\mu(-1)$, and therefore is zero. Thus there is an $n \in N(\mathbb{R})$ representing $\underline{ad}\, w$.

When the reflex field $E(G,X)$ is real and $Sh(G,X)$ has a canonical model over $E(G,X)$ then ι defines an antiholomorphic involution of $Sh(G,X)$. One of the conjectures of Langlands gives an explicit description of this involution.

Let h, as before, be special and let ${}^{\iota}h$ be the element of X corresponding to $\iota\mu$. If n is as in the corollary, then $\underset{\sim}{ad}(n)\circ h = {}^{\iota}h$. Since K_∞ is the centralizer of $h(i)$, and of ${}^{\iota}h(i)$, we see that n normalizes K_∞. Thus $g \mapsto gn : G(\mathbb{R}) \to G(\mathbb{R})$ induces a map on the quotient $G(\mathbb{R})/K_\infty$, which we can transfer to X by means of the isomorphism $g \mapsto \underset{\sim}{ad}g\circ h : G(\mathbb{R})/K_\infty \xrightarrow{\approx} X$. Thus we obtain an antiholomorphic isomorphism $\eta = (\underset{\sim}{ad}g\circ h \mapsto \underset{\sim}{ad}(gn)\circ h : X \to X)$

Conjecture B. (Langlands [1, p. 418], [2, p. 2.7, Conjecture B], [3, p. 234]). The involution of $Sh(G,X)$ defined by ι is $[x,g] \mapsto [\eta(x),g]$.

Remark 7.4. (a) If $h' = \underset{\sim}{ad}(g)\circ h$ with $g \in G(\mathbb{R})$ then $\mu_{h'} = \underset{\sim}{ad}(g)\circ\mu_h$ and $\iota\mu_{h'} = \iota(\underset{\sim}{ad}\, g\circ\mu_h) = \underset{\sim}{ad}(g)\circ\iota\mu_h = \underset{\sim}{ad}(gn)\circ\mu_h$. Thus $\eta(h') = {}^{\iota}h'$. In particular the validity of the conjecture is independent of the choice of the special point h.

(b) Since the two automorphisms of $Sh(G,X)$, $[x,g] \longmapsto \iota[x,g]$ and $[x,g] \longrightarrow [\eta(x),g]$, are continuous and commute with the Hecke operators they will be equal if they agree at one point (Deligne [1, 5.2]). Thus, to prove conjecture B, it suffices to show that $\iota[h,1] = [\eta(h),1]$ $(=[{}^{\iota}h,1])$ for a single special h.

(c) In the case that the canonical model of $Sh(G,X)$ is a moduli variety over $E(G,X)$, it is easy to verify conjecture B. Suppose, for example, that $(G,X) = (CSp(V), S^{\pm})$, so that $G = Aut(V,s)$ with s a non-degenerate skew-symmetric form on V. Then (2.3) shows that there is a bijection $Sh(G,X) \xrightarrow{\approx} \mathcal{A}(G,X,V)$ where $\mathcal{A}(G,X,V)$ consists of isomorphism classes of triples $[A,t,k]$

satisfying certain conditions. More precisely, $Sh(G,X)$ is the solution of a moduli problem over $\mathbb{C}$. The moduli problem is defined over $\mathbb{Q}$ $(=E(G,X))$ and so $Sh(G,X)$ has a model $M(G,X)$ over $\mathbb{Q}$, which the main theorem of complex multiplication shows to be a canonical model. If we set $\tau[A,t,k] = [\tau A, \tau t, k\tau^{-1}]$ for $\tau \in Aut(\mathbb{C})$ and $[A,t,k] \in \mathbb{A}(G,X,V)$, then $Sh(G,X) \xrightarrow{\approx} \mathbb{A}(G,X,V)$ commutes with actions of $Aut(\mathbb{C})$.

Let $h \in X$. Then $[h,1] \in Sh(G,X)$ corresponds to $[A,t,k]$ where A is the abelian variety defined by (V,h) (see Appendix A), $t = s$, and k is $V^f(A) = V(\mathbb{A}^f) \xrightarrow{1} V(\mathbb{A}^f)$. Since $\iota : ({}_\iota A)(\mathbb{C}) \to A(\mathbb{C})$ is a homeomorphism, it defines an isomorphism $f : H_1({}_\iota A, \mathbb{Q}) \xrightarrow{\approx} H_1(A,\mathbb{Q}) = V$. The canonical isomorphism $H^1_{dR}(A) \otimes_{\mathbb{C},\iota} \mathbb{C} \xrightarrow{\approx} H^1_{dR}({}_\iota A)$ preserves the Hodge filtrations, from which it follows easily that ${}^\iota h = f \circ h' \circ f^{-1}$ where h' defines the Hodge structure on $H_1({}_\iota A, \mathbb{Q})$. Since t corresponds to ιt under f, and the map $f \otimes 1 : V^f({}_\iota A) \longrightarrow V(\mathbb{A}^f)$ is ι^{-1}, we see that $[{}_\iota A, {}_\iota t, k\iota^{-1}]$ corresponds to $[{}^\iota h, 1] \in Sh(G,X)$. Thus ${}_\iota[h,1] = [{}^\iota h, 1]$ which, according to the above remark, proves conjecture B.

A similar argument suffices to prove the conjecture in the case that there is a embedding $(G,X) \hookrightarrow (CSp(V), S^{\pm})$ (see 2.4). It is however clear from (b) above and Deligne [1, 1.15] that if conjecture B is true for $Sh(G',X')$ and (G,X) embeds in (G',X'), then conjecture B is true for $Sh(G,X)$.

The conjecture of Langlands concerning conjugates of Shimura varieties is expressed in terms of the Taniyama group; thus let

$$1 \longrightarrow S \rightarrow \underset{\sim}{T} \xrightarrow{\pi} Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1$$

be the extension, and $sp : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \underset{\sim}{T}(\mathbb{A}^f)$ the splitting, defined in §6. For any $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, ${}^\tau S \overset{df}{=} \pi^{-1}(\tau)$ is a right S-torsor, and $sp(\tau) \in {}^\tau S(\mathbb{A}^f)$ defines a trivialization of ${}^\tau S$ over $\mathbb{A}^f$. (For any finite Galois extension L of $\mathbb{Q}$ and $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ we can also define an S^L-torsor ${}^\tau S^L$; it corresponds to the cohomology class $\gamma(\tau) \in H^1(L/\mathbb{Q}, S^L)$; (see 5.9)).

Let G, X, h, μ, T be as at the start of this section. As $T^{ad}_{\mathbb{R}}$ is anisotropic, $\mu^{ad} \overset{df}{=} (\mathbb{G}_m \xrightarrow{\mu} T \longrightarrow T^{ad})$ satisfies (4.1) and so factors into $\mathbb{G}_m \xrightarrow{\mu_{can}} S \xrightarrow{\rho_\mu} T^{ad} \subset G^{ad}$. Thus S acts on G, and we can use ${}^\tau S$ to twist G: we define ${}^\tau G$ (or ${}^{\tau,\mu}G$) to be ${}^\tau S \times^S G$. (If $L \supset \mathbb{Q}$ splits T then there is an isomorphism $f : G_L \xrightarrow{\approx} {}^\tau G_L$ such that $\sigma f = f \circ \underset{\sim}{ad}\, \gamma_\sigma(\tau, \mu^{ad})$.) Note that the action of S on T is trivial, and so $T = {}^\tau S \times^S T \subset {}^\tau G$. Define ${}^\tau h$ to be the homomorphism $\mathbb{S} \rightarrow {}^\tau G_{\mathbb{R}}$ associated with $\mathbb{G}_m \xrightarrow{\tau\mu} T_{\mathbb{C}} \subset {}^\tau G_{\mathbb{C}}$, and ${}^\tau X$ (or ${}^{\tau,\mu}X$) to be the $G(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \rightarrow {}^\tau G_{\mathbb{R}}$ containing ${}^\tau h$. The element $sp(\tau) \in {}^\tau S(\mathbb{A}^f)$ provides a canonical isomorphism $g \mapsto sp(\tau).g : G(\mathbb{A}^f) \longrightarrow {}^\tau G(\mathbb{A}^f)$, which we write as $g \longmapsto {}^\tau g$ (or $g \longmapsto {}^{\tau,\mu}g$). Langlands has shown [3, p. 231] that $({}^\tau G, {}^\tau X)$ satisfies (1.1); moreover [3, p. 233] that if h' is a second special point of X and $\mu' = \mu_{h'}$, then there is an isomorphism

$$\phi(\tau;\mu',\mu) : Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) \longrightarrow Sh({}^{\tau,\mu'}G, {}^{\tau,\mu'}X)$$

such that

$$\phi(\tau;\mu',\mu) \circ \mathcal{T}({}^{\tau,\mu}g) = \mathcal{T}({}^{\tau,\mu'}g) \circ \phi(\tau;\mu',\mu) .$$

<u>Conjecture</u> C. (Langlands [3, p. 232-33]) (a) For any special $h \in X$ there is an isomorphism

$$\phi_{\tau} = \phi_{\tau,\mu_h} : \tau Sh(G,X) \xrightarrow{\approx} Sh({}^{\tau}G, {}^{\tau}X)$$

such that

$$\phi_{\tau}(\tau[h,1]) = [{}^{\tau}h,1]$$
$$\phi_{\tau} \circ \tau(T(g)) = T({}^{\tau}g) \circ \phi_{\tau}, \text{ all } g \in G(\mathbb{A}^f).$$

(b) If h is a second special element of X and $\mu = \mu_h$, $\mu' = \mu_h$, then

$$\begin{array}{ccc} \tau Sh(G,X) & \xrightarrow{\phi_{\tau,\mu}} & Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X) \\ & \searrow^{\phi_{\tau,\mu'}} & \downarrow \phi(\tau;\mu',\mu) \\ & & Sh({}^{\tau,\mu'}G, {}^{\tau,\mu}X) \end{array}$$

commutes.

<u>Remark 7.5</u>. For a given h there is at most one map $\phi_{\tau,\mu}$ satisfying the conditions in part (a) of the conjecture (this follows from Deligne [1, 5.2]).

We note one consequence of conjecture C. Assume that $Sh(G,X)$ has a canonical model $(M(G,X), f : M(G,X)_{\mathbb{C}} \xrightarrow{\approx} Sh(G,X))$, and let $h \in X$ be special with associated cocharacter μ. Then for any automorphism τ of $\mathbb{C}$, $\tau M(G,X)$ is defined over $\tau E(G,X)$, and obviously $\tau E(G,X) = E({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. Moreover, if we make

${}^{\tau}g \in {}^{\tau}G(\mathbb{A}^f)$ act on $\tau M(G,X)$ as $\tau(\hat{T}(g))$, then $(\tau M(G,X),\ \tau M(G,X)_{\mathbb{C}} \xrightarrow{\phi_{\tau,\mu} \circ \tau f} Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X))$ satisfies the condition, relative to h , to be a canonical model. Part (b) of conjecture C shows that everything is essentially independent of h , and so $\tau M(G,X)$ is a canonical model for $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. For the sake of reference, and because it is the original form of conjecture C, we state another conjecture which is a weak form of this consequence.

Conjecture A. (Langlands [1, p. 417], [2, p. 2.5]) Assume that $Sh(G,X)$ has a canonical model $(M(G,X),f)$, and let h be some special point of X with associated cocharacter μ . Then there exists an isomorphism $g \mapsto g' : G(\mathbb{A}^f) \to {}^{\tau,\mu}G(\mathbb{A}^f)$ such that, if $g' \in {}^{\tau}G(\mathbb{A}^f)$ is made to act on $\tau M(G,X)$ as $\tau(\hat{T}(g))$, then $\tau M(G,X)$ is a canonical model for $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$.

Remark 7.6. Conjecture A appears to depend on the choice of h . One can, however, use the maps $\phi(\tau;\mu',\mu)$ to show that if the conjecture is true with one special point h then it is true with any special h .

We shall need to use several properties of the maps $\phi(\tau;\mu',\mu)$. Thus we prove them for the Shimura varieties of interest to us, namely those of abelian type. We begin by defining the maps in an easy case.

Let (G,X) satisfy (1.1). Assume:

(7.7a) for all special $h \in X$ and all $\tau \in Aut(\mathbb{C})$,

$$(\tau-1)\ (\iota+1)\mu_h = o = (\iota+1)\ (\tau-1)\mu_h\ ;$$

(7.7b) if h is special and $\rho_h : S \to G$ is the map defined by μ_h (see §4) then the element $\gamma(\tau,\mu) \overset{df}{=} \rho_h(\gamma(\tau))$ of $H^1(\mathbb{Q},G)$ is independent of h.

Now fix two special points h and h' of X and let $\mu = \mu_h$ and $\mu' = \mu_{h'}$. We write $'G$ for ${}^{\tau,\mu}G$, $"G$ for ${}^{\tau,\mu'}G$, etc.. Let L be some large finite Galois extension of $\mathbb{Q}$ and let $a(\tau)$ be a section to $\underset{\sim}{T}^L \to \mathrm{Gal}(L^{ab}/\mathbb{Q})$. Then there are defined $\beta(\tau) \in S(\mathbb{A}_L^f)$, $\beta(\tau,\mu) \overset{df}{=} \rho_h(\beta(\tau)) \in G(\mathbb{A}_L^f)$, and $\beta(\tau,\mu') \overset{df}{=} \rho_{h'}(\beta(\tau)) \in G(\mathbb{A}_L^f)$, and cocycles $\gamma_\sigma(\tau)$, $\gamma_\sigma(\tau,\mu) \overset{df}{=} \rho_h(\gamma_\sigma(\tau))$ and $\gamma_\sigma(\tau,\mu')$. Moreover there are maps $f' = (g \mapsto a(\tau).g) : G_L \xrightarrow{\approx} 'G_L$, $f" = (g \mapsto a(\tau).g) : G_L \xrightarrow{\approx} "G_L$, and $f = f" \circ f'^{-1} : 'G_L \to "G_L$. According to (7.7b) there is a $v \in G(L)$ such that $\gamma_\sigma(\tau,\mu') = v^{-1}.\gamma_\sigma(\tau,\mu).\sigma v$. The map $f_1 = f \circ \mathrm{ad}\, f'(v^{-1}) : 'G_L \xrightarrow{\approx} "G_L$ is defined over $\mathbb{Q}$ and sends $'X$ into $"X$. It therefore defines an isomorphism $Sh(f_1) : Sh('G,'X) \xrightarrow{\approx} Sh("G,"X)$.

As $B \overset{df}{=} \beta(\tau,\mu')\, v^{-1}\, \beta(\tau,\mu)^{-1}$ is fixed by $\mathrm{Gal}(L/\mathbb{Q})$ it lies in $G(\mathbb{A}^f)$, and hence $'B \overset{df}{=} {}^{\tau,\mu}\beta = f'(\beta(\tau,\mu)^{-1}\, \beta(\tau,\mu')\, v^{-1})$ lies in $'G(A^f)$. We define $\phi(\tau;\mu',\mu)$ to be the composite $Sh(f_1) \circ \mathcal{T}('B)$. Thus

$$\phi(\tau;\mu'.\mu)\ [x,'g] = [f_1 \circ x,\ "(Bg)]$$

Evidently,

$$\phi(\tau;\mu',\mu) \circ \mathcal{T}('g) = \mathcal{T}("g) \circ \phi(\tau;\mu',\mu)\ .$$

Replace $a(\tau)$ by $a(\tau)u$ with $u \in S^L(L)$, and let $u_1 = \rho_h(u)$ and $u_2 = \rho_{h'}(u)$. This forces the following changes:

f'	f	$\gamma_\sigma(\tau,\mu)$	v^{-1}	$\beta(\tau)$
$f'\circ \underset{\sim}{ad}\, u_1$	$f\circ \underset{\sim}{ad}(f'(u_2u_1^{-1}))$	$u_1^{-1}\,\gamma_\sigma(\tau,\mu)\sigma u_1$	$u_2^{-1}v^{-1}u_1$	$\beta(\tau)u$.

Thus f_1 and B are unchanged, and so also is $\phi(\tau;\mu',\mu)$. If v^{-1} is replaced by $v^{-1}\,u^{-1}$ where $u \in G(L)$ satisfies $u = \gamma_\sigma(\tau,\mu).\sigma u$, then $[\underset{\sim}{ad}\, f'(u^{-1})\circ x\, ,\, f'(u^{-1})g] = [\, x,g]$ for any $[x,g] \in Sh('G,'X)$ because $f'(u) \in {'G}(\mathbb{Q})$. Again $\phi(\tau;\mu',\mu)$ is unchanged, and is therefore well-defined.

Example 7.8. Let $(G,X) = (CSp(V),\, S^{\pm})$. For $h \in X$ special, we can use $\rho_h : S \to CSp(V)$ to define an action of S on V . Let ${}^{\tau,\mu}V = {}^{\tau}S \times^{S} V$; clearly ${}^{\tau,\mu}G = CSp({}^{\tau,\mu}V)$. The element $sp(\tau) \in S(\mathbb{A}^f)$ defines an isomorphism $sp(\tau,\mu) : V(\mathbb{A}^f) \to {}^{\tau,\mu}V(\mathbb{A}^f)$. Under the bijections $Sh(G,X) \xrightarrow{\approx} (A)(G,X,V)$ defined in (2.3), $\phi(\tau;\mu',\mu)$ corresponds to the map $[A,t,\, sp(\tau,\mu)\circ k] \to [A,t,sp(\tau,\mu')\circ k]$

Example 7.9. Suppose $h' = \underset{\sim}{ad}\, q\circ h$ with $q \in G(\mathbb{Q})$. Then $B = q$ and $v^{-1} = q$. Thus $\phi(\tau;\mu',\mu)$ is the map

$$[x,'q] \mapsto [f\circ \underset{\sim}{ad}\, f''(q)\circ x\ ,\ ''(qg)]\ .$$

Note that, even without the assumption (7.7), this expression gives a well-defined map.

To be able to apply the above discussion, we need to know when (7.7) holds. Clearly (7.7a) is valid if G is an adjoint group or if there is a map $(G,X) \to (CSp(V),\, S^{\pm})$ such that the kernel of $G \to CSp(V)$ is finite.

Lemma 7.10. The pair (G,X) satisfies (7.7) if it is of abelian type and G is adjoint.

Proof. We can assume G to $\mathbb{Q}$-simple. There is a diagram

$$(G,X) \longleftarrow (G_1,X_1) \longrightarrow (CSp(V),\ S^{\pm})$$

such that $G_1^{ad} = G$, $G_1^{der} = \tilde{G}$, and $G \longrightarrow CSp(V)$ has finite kernel (cf. 1.4). We shall prove (7.7b) holds for (G_1,X_1). To show that the two classes $\gamma(\tau,\mu)$ and $\gamma(\tau,\mu')$ are equal in $H^1(\mathbb{Q},G_1)$ it suffices to show they have the same images in $H^1(\mathbb{Q},G_1/G_1^{der})$ and in $H^1(\mathbb{R},G_1)$ (see 10.3). The first is obvious since μ and μ' map to the same element of $X_*(G_1/G_1^{der})$. For the second we use (6.13). Thus $\gamma = ((\tau-1)\mu)(-1)$ and $\gamma' = ((\tau-1)\mu')(-1)$ represent the images of $\gamma(\tau,\mu)$ and $\gamma(\tau,\mu')$ in $H^1(\mathbb{R},G_1)$. For any $z \in G(\mathbb{C})$ we write $z(\mu)$ for $\underset{\sim}{ad}\ z \circ \mu$. A direct calculation shows that if $\mu' = x(\mu)$, $x \in G(\mathbb{R})$, then

$$x^{-1}\gamma'\ x\ \gamma^{-1} = (x^{-1}\ .\ \tau x - \tau)\mu\ (-1)\ .$$

Let T be a maximal $\mathbb{Q}$-rational torus in G such that μ factors through $T(\mathbb{C})$, and let N be the normalizer of T. If $w \in N(\mathbb{C})$ then

$$\iota w.\ \gamma.\ w^{-1}\ \gamma^{-1} = (\iota w.\ w^{-1})\ [(w-1)(\tau-1)\mu(-1)]$$

According to (7.2), ${}_{\iota}w.\ w^{-1} = ({}_{\iota}c.\ c^{-1})\ [(w-1)\mu(-1)]$ for some $c \in T(\mathbb{C})$.

Thus

$$ {}_{\iota}w.\ \gamma.\ w^{-1}\gamma^{-1} = ({}_{\iota}c.\ c^{-1})\ [(w-1)\tau\mu(-1)]\ . $$

If we choose w to act on the roots of (G,T) as $x^{-1}.\ \tau x.\ \tau^{-1}$, then $(w-1)\tau\mu(-1) = (x^{-1}\tau x-\tau)\ \mu(-1)$, and it follows that $x\ \gamma'\ x^{-1} = {}_{\iota}(c\ w).\ \gamma.\ (c\ w)^{-1}$, which completes the proof.

<u>Lemma 7.11</u>. Let (G,G',X^{+}) define a connected Shimura variety and assume (G,X) is of abelian type. Then there exists a map $(G_0,X_0) \longrightarrow (G,X)$ such that $G_0^{ad} = G$, $G_0^{der} = G'$, $G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ is surjective, and (G_0,X_0) satisfies (7.7).

<u>Proof</u>. Clearly the lemma is true for a product if it is true for each factor, and is true for (G,G',X^{+}) if it is true for $(G,\tilde{G},X^{+})$. Thus we can assume G is $\mathbb{Q}$-simple and $G' = \tilde{G}$. Choose (G_1,X_1) as in the proof of (7.10). Let L be a finite Galois extension of $\mathbb{Q}$ that splits $Z(G_1)$. There exists a subjective map $M \longrightarrow X^{*}(Z(G_1))$ with M a finitely-generated free $\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})]$-module. Let $Z(G_1) \hookrightarrow Z$ be the corresponding map of tori, and define $G_0 = \tilde{G} *_{Z(\tilde{G})} Z$ (see Deligne [3,2.0.1]). The map $Z(G_1) \hookrightarrow Z$ induces an inclusion $G_1 \hookrightarrow G_0$, and we define X_0 to be the composite of X_1 with this inclusion. Then (G_0,X_0) satisfies (7.7) because (G_1,X_1) does, and $G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ is surjective because $Z(G_0) = Z$ and $H^1(\mathbb{Q},Z) = 0$.

Let (G,G',X^+) and (G_0,X_0) be as in (7.11), and let h and h' be special elements of X^+. Write $\mu = \mu_h$ and $\mu' = \mu_{h'}$. The map

$$\phi(\tau;\mu',\mu) : Sh('G_0,'X_0) \xrightarrow{\approx} Sh(''G_0,''X_0)$$

induces an isomorphism

$$\phi^0(\tau;\mu',\mu) : Sh^0('G,'G','X) \longrightarrow Sh^0(''G,''G',''X) .$$

(As before, we have substituted ' and " for the superscripts τ,μ and τ,μ'.) The usual argument shows that ϕ^0 is independent of (G_0,X_0). Moreover, the surjectivity of $G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ shows that

$$\phi^0(\tau;\mu',\mu) \circ '\gamma. = ''\gamma. \circ \phi^0(\tau;\mu',\mu)$$

for all $\gamma \in G(\mathbb{Q})^{+\wedge}$ (rel G') where $\gamma.$ denotes the canonical left action on Sh^0. (For the fact that $\gamma \mapsto \gamma' = {}^{\tau,\mu}\gamma$ maps $G(\mathbb{Q})^{+\wedge}$ into $'G(\mathbb{Q})^{+\wedge}$, see 16.1.)

<u>Proposition 7.12</u>. Let (G,X) be such that (G^{ad},X) is of abelian type. Then there is a unique family of isomorphisms

$$\phi(\tau;\mu',\mu) : Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) \longrightarrow Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X) ,$$

$\tau \in Aut(\mathbb{C})$, $\mu = \mu_h$, $\mu' = \mu_{h'}$ with h and h' special, such that:

(a) $\phi(\tau;\mu',\mu) \circ (T)('g) = (T)(''g) \circ \phi(\tau;\mu',\mu)$, $g \in G(\mathbb{A}^f)$;

(b) $\phi(\tau;\mu'',\mu') \circ \phi(\tau;\mu',\mu) = \phi(\tau;\mu'',\mu)$;

(c) if h and h' belong to the same connected component X^+ of X , then $\phi(\tau;\mu',\mu)$ restricted to the connected component of $Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$ is the map $\phi^0(\tau;\mu',\mu)$ defined above;

(d) if $h' = \underset{\sim}{ad}(q) \circ h$ with $q \in G(\mathbb{Q})$, then $\phi(\tau;\mu',\mu)$ is the map defined in (7.9).

<u>Proof</u>. There is clearly at most one family of maps with these properties. To show the existence one uses the standard technique for extending a map from the connected component of a variety to the whole variety (see Deligne [3, 2.7], or §17).

<u>Remark 7.13</u>. In the case that τ fixes $E(G,X)$, we define in (10.8) below a map $\phi(\tau;\mu) : Sh(G,X) \longrightarrow Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X)$. On comparing the two definitions one finds that

$$\phi(\tau;\mu',\mu) = \phi(\tau;\mu') \circ \phi(\tau;\mu)^{-1} .$$

<u>Remark 7.14</u>. Let $h' = \underset{\sim}{ad}(q) \circ h$ with $q \in G(\mathbb{Q})$, and assume part (a) of conjecture C holds. One checks directly that $\phi = \phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu}$ has the following properties:

$$\phi(\tau[h',1]) = \phi(\tau[h,q^{-1}]) = [{}^{\tau,\mu'}h',1]$$
$$\phi \circ \tau(T)(g) = (T)({}^{\tau,\mu'}g) \cdot \phi .$$

Thus $\phi = \phi_{\tau,\mu'}$, and part (b) of the conjecture holds (for μ and μ')

III. Abelian varieties of CM-type

In §8 we make a calculation showing that a certain torsor over $\mathbb{R}$, which arises naturally from the study of abelian varieties of CM-type, is isomorphic to a torsor ${}^{\tau}S_{\mathbb{R}}$ occurring in the Taniyama group. This result enables us, in §7, to state a conjecture generalizing the main theorem of complex multiplication. The final section relates conjecture C, which concerns Shimura varieties, to this new conjecture, which concerns only abelian varieties of CM-type.

§8. A cocycle calculation.

Let A be an abelian variety over $\mathbb{C}$ of CM-type, so that there is a product F of CM-fields acting on $\mathbb{A}$ in such a way that $H_1(A,\mathbb{Q})$ is a free F-module of rank 1. Assume that there is a homogeneous polarization $[\psi]$ on A whose Rosati involution stabilizes $F \subset \mathrm{End}(A)$ and induces ι on it. Let $F_0 = \{f \in F \mid \iota f = f\}$; thus F_0 is a product of totally real fields. Note that the Hodge structure h on $V = H_1(A,\mathbb{Q})$ is compatible with the action of F. Let $\psi \in [\psi]$ be a polarization of A (or (V,h)) ; for any choice of an element $f \in F^{\times}$ with $\iota f = -f$ there exists a unique F-Hermitian form ϕ on V such that $\psi(x,y) = \mathrm{Tr}_{F/\mathbb{Q}}(f\phi(x,y))$.

Let Σ be the set of embeddings $F_0 \hookrightarrow \mathbb{C}$; then

$$H_1(A,\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\sigma \in \Sigma} V_\sigma \ , \text{ where } \ V_\sigma = V \otimes_{F_0,\sigma} \mathbb{C} \ .$$

Moreover:

V_σ is a free $F \otimes_{F_{0,\sigma}} \mathbb{C}$-module of rank 1;

$V_\sigma = V_\sigma^+ \oplus V_\sigma^-$, where $h(z)$ acts as z on V_σ^+ and ιz on V_σ^- ;

ϕ defines a Hermitian form ϕ_σ on V_σ such that $\phi_\sigma > 0$ on V_σ^+ and $\phi_\sigma < 0$ on V_σ^- .

Let τ be an automorphism of $\mathbb{C}$ and let $V' = H_1(\tau A,\mathbb{Q})$. The action of F on A induces an action of F on τA , and $[\psi]$ gives rise to a homogeneous polarization $[\tau\psi]$ on τA . Thus there is a decomposition $H_1(\tau A,\mathbb{C}) = \oplus_{\sigma \in \Sigma} V'_\sigma$ where the V'_σ have similar structures to the V_σ .

Our purpose is to construct an isomorphism $\theta : H_1(A,\mathbb{C}) \longrightarrow H_1(\tau A,\mathbb{C})$ that is $F \otimes \mathbb{C}$-linear and takes $[\psi]$ to $[\tau\psi]$. It will suffice to define θ on each component V_σ of $H_1(A,\mathbb{C})$.

As $H_1(A,\mathbb{C})$ is canonically dual to the de Rham cohomology group $H^1_{dR}(A)$, and $H^1_{dR}(\tau A) = H^1_{dR}(A) \otimes_{\mathbb{C},\tau}\mathbb{C}$, we see that $H_1(\tau A,\mathbb{C}) = H_1(A,\mathbb{C}) \otimes_{\mathbb{C},\tau}\mathbb{C}$. Under this identification, the two actions of F correspond, and $\tau\psi$ corresponds to ψ .

Fix a $\sigma \in \Sigma$, and consider V_σ and V'_σ . Since F_0 acts on $V_\sigma \otimes_{\mathbb{C},\tau}\mathbb{C}$ through $\tau\sigma$, we see that we must have $V'_\sigma = V_{\tau^{-1}\sigma} \otimes_{\mathbb{C},\tau}\mathbb{C}$. There is an $F_0 \otimes \mathbb{C}$-linear isomorphism $\theta_1 : V_\sigma \xrightarrow{\approx} V'_\sigma$ and, since F acts on V^+_σ and V^-_σ through distinct enbeddings $F \hookrightarrow \mathbb{C}$, exactly one of the following must hold:

$$(+) \quad \theta_1 : V^+_\sigma \xrightarrow{\approx} V'^+_\sigma \ , \quad \theta_1 : V^-_\sigma \xrightarrow{\approx} V'^-_\sigma \ ;$$

$$(-) \quad \theta_1 : V^+_\sigma \xrightarrow{\approx} V'^-_\sigma \ , \quad \theta_1 : V^-_\sigma \xrightarrow{\approx} V'^+_\sigma \ .$$

Choose a basis for V_σ compatible with the decomposition $V_\sigma = V^+_\sigma \oplus V^-_\sigma$ and define θ_σ to be θ_1 in case $(+)$ and to be the composite of θ_1 with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ in case $(-)$. Then θ_σ is an $F \otimes_{F_0,\sigma} \mathbb{C}$-linear isomorphism $V_\sigma \longrightarrow V'_\sigma$ taking ϕ_σ to a multiple of ϕ'_σ .

<u>Lemma 8.1</u>. With the above notations, there exists an isomorphism $\theta : H_1(A,\mathbb{C}) \longrightarrow H_1(\tau A,\mathbb{C})$ such that:

(a) $\theta \circ f = f \circ \theta$ for all $f \in F$;

(b) $\theta(\tau[\psi]) = [\psi]$;

(c) ${}_{\iota}\theta = \theta.(\tau\mu(-1)/\mu(-1))$

Proof. Define $\theta = \oplus_{\sigma}\theta_{\sigma}$ and note that ${}_{\iota}\theta_{\sigma} = \theta_{\sigma}$ in case (+) while ${}_{\iota}\theta_{\sigma} = -\theta_{\sigma}$ in case (-) . On the other hand, $\mu(-1)$ acts as $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ on $V_{\sigma} = V_{\sigma}^{+} \oplus V_{\sigma}^{-}$ and $\tau\mu(-1) = \mu(-1)$ in case (+) while $\tau\mu(-1) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ in case (-) .

We shall need a slightly more precise result.

Proposition 8.2. Let A be an abelian variety over $\mathbb{C}$ that is of CM-type, and let τ be an automorphism of $\mathbb{C}$. There exists an isomorphism $\theta : H_1(A,\mathbb{C}) \longrightarrow H_1(\tau A,\mathbb{C})$ such that:

(a) $\theta(s) = \tau s$ for all Hodge cycles s on A ,

(b) ${}_{\iota}\theta = \theta.\gamma$ where γ is the class in $H^1(\mathbb{R},MT(A))$ represented by $\tau\mu(-1)/\mu(-1)$. (MT(A) = Mumford-Tate group of

Proof. Note that, if we let

$$P(R) = \{\theta : H_1(A,R) \xrightarrow{\approx} H_1(\tau A,R) \mid \theta \text{ satisfies } (8.2a)\}$$

for any $\mathbb{Q}$-algebra R , then P is a right MT(A)-torsor. Proposition (8.2) describes the class of $P_{\mathbb{R}}$ in $H^1(\mathbb{R},MT(A))$. The lemma shows that image of the class in $H^1(\mathbb{R},T)$ is correct, where T is the subtorus of $F^{\times}$ of elements whose norm to F_0 lies in $\mathbb{Q}^{\times}$.

We shall complete the proof of the proposition by showing that $H^1(\mathbb{R}, MT(A)) \to H^1(\mathbb{R},T)$ is injective.

The norm map N_{F/F_0} defines a surjection $T \to \mathbb{G}_m$, and we define ST and SMT to make the rows in the following diagram exact:

$$\begin{array}{ccccccccc} 1 & \to & SMT & \to & MT & \to & \mathbb{G}_m & \to & 1 \\ & & \cap & & \cap & & \| & & \\ 1 & \to & ST & \to & T & \to & \mathbb{G}_m & \to & 1 \end{array}$$

This diagram gives rise to an exact commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}^\times & \to & H^1(\mathbb{R},SMT) & \to & H^1(\mathbb{R},MT) & \to & 0 \\ \| & & \downarrow & & \downarrow & & \\ \mathbb{R}^\times & \to & H^1(\mathbb{R},ST) & \to & H^1(\mathbb{R},T) & \to & 0 \end{array}$$

Note that ST (and hence SMT) is anisotropic over $\mathbb{R}$, and that for an anisotropic torus S', $H^1(\mathbb{R},S') = \mathrm{Ker}(S'(\mathbb{C}) \xrightarrow{2} S'(\mathbb{C}))$. Thus $H^1(\mathbb{R},SMT) \to H^1(\mathbb{R},ST)$ is injective, and the five-lemma shows that $H^1(\mathbb{R},MT) \to H^1(\mathbb{R},T)$ is injective.

Remark 8.3. Let A, F, and $V = H_1(A,\mathbb{Q})$ be as in the first paragraph. Then h can be regarded as a map $h : \mathbb{S} \to F^\times(\mathbb{R})$ (thinking of $F^\times$ as a $\mathbb{Q}$-rational torus). It is clear from the discussion preceeding (8.1) that τA is the abelian variety corresponding to $(V, {}^\tau h)$, whose ${}^\tau h$ is the map $\mathbb{S} \to F^\times(\mathbb{R})$ with associated cocharacter $\tau\mu_h \in X_*(F^\times)$.

§9. Conjugates of abelian varieties of CM-type.

Let A be an abelian variety of CM-type over $\mathbb{C}$, let $V = H_1(A,\mathbb{Q})$, and let h be the (natural) Hodge structure on V. Fix some family $(s_\alpha)_{\alpha \in J}$ of tensors such that the Mumford-Tate group $MT(A)$ of A is $\mathrm{Aut}(V,(s_\alpha))$ (see Appendix A). The canonical map $S \xrightarrow{\rho} MT(A)$ induces an action of S on $(V,(s_\alpha))$ and, for any automorphism τ of $\mathbb{C}$, we define $({}^\tau V,({}^\tau s_\alpha)) = {}^\tau S \times^S (V,(s_\alpha))$. The element $sp(\tau) \in {}^\tau S(\mathbb{A}^f)$ defines an isomorphism

$$v \mapsto sp(\tau)v : (V(\mathbb{A}^f),(s_\alpha)) \xrightarrow{\approx} ({}^\tau V(\mathbb{A}^f),({}^\tau s_\alpha)) ,$$

which we shall again denote by $sp(\tau)$.

Lemma 9.1. There is an isomorphism $f : (H_1(\tau A,\mathbb{Q}),(\tau s_\alpha)) \xrightarrow{\approx} ({}^\tau V,({}^\tau s_\alpha))$

Proof. Let P_A be the functor such that, for any $\mathbb{Q}$-algebra R, $P_A(R)$ is the set of isomorphisms $(H_1(A,R),(s_\alpha)) \xrightarrow{\approx} (H_1(\tau A,R),(\tau s_\alpha))$. Clearly P_A is representable, and is a right $MT(A)$-torsor. Since $P_A \times^{MT(A)} (H_1(A,\mathbb{Q}),(s_\alpha)) = (H_1(\tau A,\mathbb{Q}),(\tau s_\alpha))$, to prove the lemma it suffices to show that P_A is isomorphic to the $MT(A)$-torsor $\rho_*({}^\tau S)$. We shall show this simultaneously for all abelian varieties (over $\mathbb{C}$, of CM-type) whose Mumford-Tate groups are split by a fixed finite Galois extension L of $\mathbb{Q}$.

According to (4.7), $S^L = \varprojlim MT(A)$, and it will suffice to show that the two S^L-torsors $P = \varprojlim P_A$ and ${}^\tau S^L$ are isomorphic. As $H^1(\mathbb{Q},S^L)$ satisfies the Hasse principle (see 4.5) this only has to be shown locally. The isomorphisms

$H_1(A,\mathbb{Q}_\ell) = V^f(A) \underset{\approx}{\xrightarrow{\tau}} V^f(\tau A) = H_1(\tau A,\mathbb{Q}_\ell)$ show P to be trivial over $\mathbb{Q}_\ell$, while $sp(\tau) \in {}^\tau S^L(\mathbb{Q}_\ell)$ shows ${}^\tau S^L_{\mathbb{Q}_\ell}$ to be trivial. Finally (6.12) and (8.2) show ${}^\tau S^L_{\mathbb{R}}$ and $P_{\mathbb{R}}$ define the same cohomology class in $H^1(\mathbb{R}, S^L)$ and are therefore isomorphic.

Note that f is uniquely determined up to right multiplication by an element of $MT(A)(\mathbb{Q})$.

Conjecture CM (first form). The isomorphism f of (9.1) can be chosen to make the following diagram commute:

$$\begin{array}{ccc} V^f(A) & \xrightarrow{\tau} & V^f(\tau A) \\ \| & & \downarrow f\otimes 1 \\ V(\mathbb{A}^f) & \xrightarrow{sp(\tau)} & {}^\tau V(\mathbb{A}^f) \end{array}$$

We next restate the conjecture in a form that is closer to the usual statements of the main theorem of complex multiplication. Let $T = MT(A)$, and choose a polarization ψ for (V,h) which we shall assume to be one of the s_α . From the inclusion $(T,\{h\}) \hookrightarrow (CSp(V), S^{\pm})$ we obtain, as in §2, a bijection

$$Sh(T,\{h\}) \xrightarrow{\approx} \mathcal{A}(T,\{h\},V)$$

where $\mathcal{A}(T,\{h\},V)$ consists of certain isomorphism classes of triples $(A',(t_\alpha),k)$.

The torus T continues to act on ${}^\tau V$, and in fact $T = Aut({}^\tau V, ({}^\tau s_\alpha))$. One of the ${}^\tau s_\alpha$ is ${}^\tau\psi$, which is a polarization for $({}^\tau V, {}^\tau h)$, where ${}^\tau h$ is the homomorphism $\mathbb{S} \to T$ corresponding to $\tau\mu_h$. Thus we have an inclusion

$(T,\{{}^{\tau}h\}) \hookrightarrow (CSp({}^{\tau}V), S^{\pm})$ and, as before, a bijection

$$Sh(T,\{{}^{\tau}h\}) \xrightarrow{\approx} \mathbb{A}(T,\{{}^{\tau}h\},{}^{\tau}V) .$$

We define $\chi_{\tau} : \mathbb{A}(T,\{h\},V) \longrightarrow \mathbb{A}(T,\{{}^{\tau}h\},{}^{\tau}V)$ to be the mapping that sends $[A',(t_{\alpha}),k]$ to the class $[\tau A',(\tau t_{\alpha}),{}^{\tau}k]$ where ${}^{\tau}k$ is the composite $V^{f}(\tau A) \xrightarrow{\tau^{-1}} V^{f}(A) \xrightarrow{k} V(\mathbb{A}^{f}) \xrightarrow{sp(\tau)} {}^{\tau}V(\mathbb{A}^{f})$. Lemma (9.1) shows that $[\tau A',(\tau t_{\alpha}),{}^{\tau}k]$ satisfies condition (2.1a) to lie in $\mathbb{A}(T,\{{}^{\tau}h\},{}^{\tau}V)$ and (8.3) shows that it satisfies (2.1b).

Conjecture CM (second form). The following diagram commutes:

$$\begin{array}{ccc}
[h,g] & Sh(T,\{h\}) \xrightarrow{\approx} & \mathbb{A}(T,\{h\},V) \\
\downarrow & \downarrow\approx & \chi_{\tau}\downarrow\approx \\
[{}^{\tau}h,g] & Sh(T,\{{}^{\tau}h\}) \xrightarrow{\approx} & \mathbb{A}(T,\{{}^{\tau}h\},{}^{\tau}V)
\end{array}$$

It is easy to check that the two forms of the conjecture are equivalent.

Example 9.2(a). Suppose, in the above, that there exists an isomorphism $a(\tau) : (V,(s_{\alpha})) \xrightarrow{\approx} ({}^{\tau}V,({}^{\tau}s_{\alpha}))$, i.e. that the cocycle $\gamma(\tau,\mu) \in H^{1}(\mathbb{Q},T)$ is trivial. Then $\bar{\beta}(\tau,\mu)$ lies in $T(\mathbb{A}^{f})/T(\mathbb{Q})^{\wedge}$ and is represented by $\beta(\tau,\mu) \in T(\mathbb{A}^{f})$ where $sp(\tau)\ \beta(\tau,\mu) = a(\tau)$ (as maps $V(\mathbb{A}^{f}) \longrightarrow {}^{\tau}V(\mathbb{A}^{f})$) . If we use $a(\tau)$ to identify $\mathbb{A}(T,\{{}^{\tau}h\},V)$ with $\mathbb{A}(T,\{{}^{\tau}h\},{}^{\tau}V)$ then χ_{τ} becomes identified with the map $[A,(t_{\alpha}),k] \longmapsto [\tau A,(\tau t_{\alpha}),\beta(\tau,\mu)^{-1}\circ k\circ \tau^{-1}]$ i.e. with the composite $\mathbb{A}(T,\{h\},V) \xrightarrow{I} \mathbb{A}(T,\{{}^{\tau}h\},V) \xrightarrow{\beta(\tau,\mu)} \mathbb{A}(T,\{{}^{\tau}h\},V)$.

Thus, in this case, the second form of conjecture CM asserts that

$$\begin{array}{ccc} [h,g] & \qquad & \mathrm{Sh}(T,\{h\}) \xrightarrow{\approx} \mathcal{A}(T,\{h\},V) \\ \downarrow & & \downarrow \qquad\qquad\qquad \downarrow \tau \\ [{}^{\tau}h, g\beta(\tau,\mu)^{-1}] & & \mathrm{Sh}(T,\{{}^{\tau}h\}) \xrightarrow{\approx} \mathcal{A}(T,\{{}^{\tau}h\},V) \end{array}$$

commutes.

(b) Suppose that τ fixes the reflex field $E(T,\{h\})$; then (6.10) shows that $\bar{b}(\tau,\mu) = r_E(T,h)(\tau)^{-1}$, and so $\bar{\beta}(\tau,\mu) = r_E(T,h)(\tau^{-1}) = r_E(T,h)(\tau)^{-1} \in T(\mathbb{A}^f)/T(\mathbb{Q})^{\wedge}$. It follows that $\gamma(\tau,\mu)$ is trivial, and (a) shows that, once $({}^{\tau}V,({}^{\tau}s_\alpha))$ has been identified with $(V,(s_\alpha))$, conjecture CM becomes the statement that the action of τ on $\mathcal{A}(T,\{h\},V)$ corresponds to the action of $\mathcal{T}(\tilde{r}(\tau))$ on $\mathrm{Sh}(T,\{h\})$, where $\tilde{r}(\tau) \in T(\mathbb{A}^f)$ represents $r_E(T,h)(\tau)$. This is precisely the statement of the main theorem of complex multiplication to be found, for example, in Deligne [1, 4.19]. Thus the conjecture is a generalization of that theorem.

<u>Example 9.3</u>. Let F be a CM-field and Σ a CM-type for F . Let A be an abelian variety (an actual abelian variety - not an isogeny class of abelian varieties!) of type (F,Σ) . Then $H_1(A,\mathbb{Z})$ is a locally free module of rank one over the ring of integers O_F in F , and hence defines an element $\underline{I}(A)$ of $\mathrm{Pic}(O_F)$. Consider

$$\begin{array}{c} (S^L(\mathbb{A}_L^f)/S^L(L))^{Gal(L/\mathbb{Q})} \\ \downarrow \\ 1 \longrightarrow T(\mathbb{A}^f)/T(\mathbb{Q}) \longrightarrow (T(\mathbb{A}_L^f)/T(L))^{Gal(L/\mathbb{Q})} \longrightarrow H^1(L/\mathbb{Q},T) \end{array}$$

where $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$, L is a (sufficiently large) finite Galois extension of $\mathbb{Q}$, and the vertical map is induced by the canonical map $\rho : S^L \to T$. As $H^1(L/\mathbb{Q},T) = 0$, the image of $\bar{\beta}(\tau)$ in $(T(\mathbb{A}_L^f)/T(L))^{Gal(L/\mathbb{Q})}$ arises from an element $\beta'(\tau) \in T(\mathbb{A}^f)/T(\mathbb{Q})$. This defines an ideal class $\underline{I}(\tau) \in \mathrm{Pic}(O_F)$, and the conjecture predicts that $\underline{I}(\tau A) = \underline{I}(\tau)\, \underline{I}(A)$.

Example 9.4. Let F_0 be a totally real number field, let F_1 and F_2 be distinct, totally imaginary, quadratic extensions of F_0 , and let $F = F_1F_2$. For each $\sigma \in I \stackrel{df}{=} \mathrm{Hom}(F_0,\mathbb{C})$ choose an extension σ_1 of σ to F_1 and an extension σ_2 of σ to F_2 . Write σ' and σ'' for the elements of $\mathrm{Hom}(F,\mathbb{C})$ such that

$$\sigma' = \sigma_1 \text{ on } F_1 , \quad \sigma'' = \iota\sigma_1 \text{ on } F_1$$
$$\sigma' = \sigma_2 \text{ on } F_2 , \quad \sigma'' = \sigma_2 \text{ on } F_2 .$$

Let Σ_0 be a subset of $\mathrm{Hom}(F_0,\mathbb{C})$ and define

$$\Sigma = \{\sigma' | \sigma \in I\} \cup \{\sigma'' | \sigma \in \Sigma_0\} \cup \{\iota\sigma'' | \sigma \notin \Sigma_0\} .$$

Then Σ is a CM-type for F . The sets of complex embeddings $\Sigma_0, \Sigma_1 = \{\sigma_1 | \sigma \in \Sigma_0\}, \Sigma_2 = \{\sigma_2 | \sigma \notin \Sigma_0\}$, and Σ define $\mathbb{Q}$-rational Hodge structures on the vector spaces F_0, F_1, F_2, and F , and hence homomorphisms $h_j : \mathbb{S} \longrightarrow (F_j \otimes \mathbb{R})^\times$ for $j = 0, \ldots$. Let μ_0, μ_1, μ_2, and $\mu = \mu_1\mu_2$ be the corresponding cocharacters, and let E_0, E_1, and E_2 be the reflex fields $E(F_0, h_0)$, $E(F_1, h_1)$, and $E(F_2, h_2)$. In the following we assume that E_1 and E_2 are linearly disjoint over E_0. As E_0 is totally real, this assumption allows us to consider an automorphism τ of $\mathbb{C}$ over E_0 such that

$$\begin{aligned} \tau &= \text{id} \quad \text{on } E_1 \\ &= \iota \quad \text{on } E_2 \end{aligned}$$

Note that $(\iota+1)(\tau-1)\mu_1 = 0 = (\iota+1)(\tau-1)\mu_2$, and so $(\iota+1)(\tau-1)\mu = 0$. If we let $T = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ then μ_0, μ_1, μ_2, and μ can be regarded as elements of $X_*(T)$. Thus, if L is large enough to split T, there are defined elements $\bar{\beta}(\tau, \mu_j) \in (T(\mathbb{A}_L^f)/T(L))^{\mathrm{Gal}(L/\mathbb{Q})}$ for $j = 0, \ldots$. As $\mu = \mu_1\mu_2$, $\bar{\beta}(\tau, \mu) = \bar{\beta}(\tau, \mu_1)\bar{\beta}(\tau, \mu_2)$. Since $\tau = \text{id}$ on E_1 and $\iota\tau = \text{id}$ on E_2, (6.10) shows that

$$\begin{aligned} \bar{\beta}(\tau, \mu_1) &= r_{E_1}(\tau | E_1^{ab})^{-1} \in T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge \\ \bar{\beta}(\iota\tau, \mu_2) &= r_{E_2}(\iota\tau | E_2^{ab})^{-1} \in T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge . \end{aligned}$$

From (6.11) and (5.9) we know $\bar{\beta}(\tau, \mu_2) = (\iota\tau)^{-1}\bar{\beta}(\iota, \mu_2) \cdot \bar{\beta}(\iota\tau, \mu_2)$, and (6.9) shows $\bar{\beta}(\iota, \mu_2) = 1$. Thus

$$\bar{\beta}(\tau, \mu) = r_{E_1}(\tau | E_1^{ab})^{-1}\, r_{E_2}(\iota\tau | E_2^{ab})^{-1} \in T(\mathbb{A}^f)/T(\mathbb{Q}) .$$

Let A be an abelian variety over $\mathbb{C}$ with complex multiplication by F and of CM-type Σ. Choose identifications of $H_1(A,\mathbb{Q})$ and $H_1(\tau A,\mathbb{Q})$ with F. Then $V^f(A)$ and $V^f(\tau A)$ are identified with $\mathbb{A}_F^f$ and $sp(\tau) : V^f(A) \to V^f(\tau A)$ is multiplication by an element $\beta(\tau,\mu)^{-1} \in (\mathbb{A}_F^f)^\times = T(\mathbb{A}^f)$. This $\beta(\tau,\mu)$ lifts $\bar{\beta}(\tau,\mu) = r_{E_1}(\tau)^{-1} r_{E_2}(\iota\tau)^{-1}$. Conjecture CM asserts in this case that the two maps

$$V^f(A) \xrightarrow{\ \tau\ } V^f(\tau A)$$

$$V^f(A) \xrightarrow{\beta(\tau,\mu)^{-1}} V^f(\tau A)$$

are equal.

This last statement is, apart from notation, Theorem 9 of Shih [1] (see also §15 below).

Remark 9.5. Let A be an abelian variety of potential CM-type defined over a number field k. Conjecture CM would imply that the zeta function of A is an alternating product of L-series associated to complex representations of the Weil group of k. Deligne has proved this result without, however, proving the conjecture (cf. Deligne [6]).

§10. Conjecture C, conjecture CM, and canonical models.

Let $(M(G,X),\ f: M(G,X)_{\mathbb{C}} \xrightarrow{\approx} Sh(G,X))$ be a canonical model for $Sh(G,X)$ (Deligne [3, 2.2.5]) and, for each automorphism τ of $\mathbb{C}$ fixing $E(G,X)$, set $\psi_\tau = f \circ (\tau f)^{-1}$. These isomorphisms $\psi_\tau: \tau Sh(G,X) \to Sh(G,X)$ satisfy the following conditions:

(10.1a) $\psi_{\tau_1\tau_2} = \psi_{\tau_1} \circ (\tau_1\psi_{\tau_2}),\ \tau_1,\tau_2 \in \mathrm{Aut}(\mathbb{C}/E(G,X))$;

(10.1b) $\psi_\tau \circ \tau(\mathcal{T}(g)) = \mathcal{T}(g) \circ \psi_\tau,\ \tau \in \mathrm{Aut}(\mathbb{C}/E(G,X)),\ g \in G(\mathbb{A}^f)$;

(10.1c) let $h \in X$ be special and assume that τ fixes the reflex field $E(h)$ of h; then $\psi_\tau(\tau[h,1]) = [h, \tilde{r}(\tau)]$. (Here $\tilde{r}(\tau) \in G(\mathbb{A}^f)$ represents $r_E(T,h)(\tau) \in T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge$ where T is some $\mathbb{Q}$-rational torus such that h factors through $T_{\mathbb{R}}$ and $r_E(T,h)$ is the reciprocity morphism (Deligne [3, 2.2.3]).) Note that the family (ψ_τ) is uniquely determined by (G,X): if $(M(G,X)', f')$ is a second canonical model, there is an isomorphism $q: M(G,X)' \to M(G,X)$ such that $f' = f \circ q_{\mathbb{C}}$, and so $f' \circ (\tau f')^{-1} = f \circ (\tau f)^{-1} = \psi_\tau$. Moreover, descent theory shows that every family (ψ_τ) satisfying (10.1) arises from a canonical model for $Sh(G,X)$.

If τ fixes $E(G,X)$ and $M(G,X)$ is a canonical model for $Sh(G,X)$, then $\tau M(G,X) = M(G,X)$ is again canonical model for $Sh(G,X)$, and so conjecture A suggests that we should have $Sh(G,X) \xrightarrow{\approx} Sh({}^\tau G, {}^\tau X)$. We shall prove this. Thus let (G,X) be any pair satisfying (1.1) and let $h \in X$ be special. Choose a

$\mathbb{Q}$-rational maximal torus T in G such that h factors through $T_{\mathbb{R}}$, and let $\mu = \mu_h$. If τ is an automorphism of $\mathbb{C}$ that fixes $E(G,X)$ then $\tau\mu$ and μ have the same weight; thus $(1 + \iota)\tau\mu = (1 + \iota)\mu$, and (see 6.15) there is a well-defined cohomology class $\gamma(\tau,\mu) \in H^1(\mathbb{Q}, T)$.

<u>Lemma</u> 10.2. The image of $\gamma(\tau,\mu)$ in $H^1(\mathbb{Q},G)$ is trivial.

<u>Proof</u>: After replacing (G,X) with the pair (G_1,X_1) constructed in (3.4), we can assume G^{der} is simply connected. Let $H = G/G^{der}$ and let μ' be the composite of μ with $G \to H$. As $\tau\mu$ is conjugate to μ, $\tau\mu' = \mu'$ and (6.10) shows that $\gamma(\tau,\mu')$ is trivial.

Let $w \in G(\mathbb{C})$ normalize $T(\mathbb{C})$ and be such that $\tau\mu = \underset{\sim}{ad}\, w \circ \mu$. According to (6.13), the image of $\gamma(\tau,\mu)$ in $H^1(\mathbb{R}, G)$ is represented by $\tau\mu(-1)/\mu(-1) = (\underset{\sim}{ad}\, w \circ \mu)(-1)/\mu(-1)$ which (see 7.2) is also represented by $w \cdot \iota w^{-1}$; it is therefore trivial. The lemma is now a consequence of the following easy result.

<u>Sublemma 10.3</u>. Let G be a reductive group over $\mathbb{Q}$ such that G^{der} is simply connected. An element γ of $H^1(\mathbb{Q},G)$ is trivial if its images in $H^1(\mathbb{Q}, G/G^{der})$ and $H^1(\mathbb{R}, G)$ are trivial.

We continue with the notations of the second paragraph of this section; thus $h \in X$ is special, $\mu = \mu_h$, and τ fixes $E(G,X)$. Choose an element $a(\tau) \in {}^{\tau}S(\overline{\mathbb{Q}})$ and let $f: G_{\overline{\mathbb{Q}}} \to {}^{\tau}G_{\overline{\mathbb{Q}}}$

be the isomorphism $g \longmapsto a(\tau).g$. It will often be convenient to regard f as being defined over L where L is some sufficiently large finite Galois extension of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$. Let $\beta(\tau) = sp(\tau)^{-1}\, a(\tau) \in S(\mathbb{A}_L^f)$ and let $\beta(\tau, \overline{\mu})$ be the image of $\beta(\tau)$ in $T^{ad}(\mathbb{A}_L^f)$ under the map $\rho_{\overline{\mu}} : S \to T^{ad}$ defined by $\mu^{ad} \underline{\underline{df}}\ \overline{\mu} \underline{\underline{df}} (\mathbb{G}_m \overset{\mu}{\to} T \to T^{ad})$. Recall (6.8) that we have also defined an element $\overline{\beta}(\tau,\mu) \in T(\mathbb{A}_L^f)/T(L)T(\mathbb{Q})^\wedge$. Since $\beta(\tau,\overline{\mu})$ and $\overline{\beta}(\tau,\mu)$ have the same image in $T(\mathbb{A}_L^f)/Z(\mathbb{A}_L^f)\,T(L)\,T(\mathbb{Q})^\wedge$ we can choose an element $\tilde{\beta}(\tau,\mu) \in T(\mathbb{A}_L^f)$ that lifts both $\beta(\tau,\overline{\mu})$ and $\overline{\beta}(\tau,\mu)$; it is determined up to multiplication by an element of $Z(\mathbb{A}_L^f) \cap T(L)\ T(\mathbb{Q})^\wedge = Z(L)\ Z(\mathbb{Q})^\wedge$. (Note that $T(\mathbb{Q})Z(\mathbb{Q})^\wedge = T(\mathbb{Q})^\wedge$ because $T^{ad}(\mathbb{Q})$ is a discrete subgroup of $T^{ad}(\mathbb{A}^f)$.) Let $\sigma\tilde{\beta}(\tau,\mu) = \tilde{\beta}(\tau,\mu)\gamma_\sigma$; then (γ_σ) is a 1-cocycle representing $\gamma(\tau,\mu) \in H^1(\mathbb{Q},T)$. We have $\sigma f = f \circ \underset{\sim}{ad}\gamma_\sigma$. The lemma shows that there is an element $v \in G(\overline{\mathbb{Q}})$ such that $\gamma_\sigma = v^{-1}.\sigma v$ for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. We define an isomorphism $f_1 : G \to {}^\tau G$ and an element $\beta_1(\tau,\mu) \in G(\mathbb{A}^f)$ by the formulas:

$$f_1 = f \circ \underset{\sim}{ad}\ v^{-1} \tag{10.4a}$$

$$\beta_1(\tau,\mu) = \tilde{\beta}(\tau,\mu)v^{-1} \tag{10.4b}$$

<u>Remark</u> 10.5. In the above we have had to choose an $a(\tau)$, $\tilde{\beta}(\tau,\mu)$, and v. For example, if $a(\tau)$ is replaced by $a(\tau)u$, $u \in S(L)$, then $\beta(\tau,\overline{\mu})$ is replaced by $\beta(\tau,\overline{\mu})\ \rho_{\overline{\mu}}(u)$. We show that the cosets defined by $\beta_1(\tau,\mu)$ and $\beta_1(\tau,\mu)^{-1}$ in $G(\mathbb{Q}) \backslash G(\mathbb{A}^f)/Z(\mathbb{Q})^\wedge$ are independent of all choices.

Consider the exact commutative diagram

$$\begin{array}{ccccc} 1 \to T(\mathbb{Q})\backslash T(\mathbb{A}^f) & \longrightarrow & (T(L)\backslash T(\mathbb{A}_L^f))^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},T(L)) \\ \downarrow & & \downarrow & & \downarrow \\ 1 \to G(\mathbb{Q})\backslash G(\mathbb{A}^f) & \longrightarrow & (G(L)\backslash G(\mathbb{A}_L^f))^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},G(L)) \end{array}$$

in which the vertical arrows are induced by the inclusion $T \hookrightarrow G$. On dividing by $Z(\mathbb{Q})^{\wedge}$ we obtain

$$\begin{array}{ccccc} 1 \longrightarrow T(\mathbb{A}^f)/T(\mathbb{Q})^{\wedge} & \longrightarrow & (T(L)\backslash T(\mathbb{A}_L^f)/T(\mathbb{Q})^{\wedge})^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},T(L)) \\ \downarrow & & \downarrow & & \downarrow \\ 1 \longrightarrow G(\mathbb{Q})\backslash G(\mathbb{A}^f)/Z(\mathbb{Q})^{\wedge} & \longrightarrow & (G(L)\backslash G(\mathbb{A}_L^f)/Z(\mathbb{Q})^{\wedge})^{\mathrm{Gal}(L/\mathbb{Q})} & \longrightarrow & H^1(L/\mathbb{Q},G(L)). \end{array}$$

Lemma 10.2 shows that the image of $\overline{\beta}(\tau,\mu)$ (or $\overline{\beta}(\tau,\mu)^{-1}$) under the middle vertical arrow lies in $G(\mathbb{Q})\backslash G(\mathbb{A}^f)/Z(\mathbb{Q})^{\wedge}$; it is represented by $\beta_1(\tau,\mu)$.

<u>Remark</u> 10.6. Everything is much simpler when μ satisfies (4.1). Then there is a map $\rho_\mu\colon S \to T$ and we can choose $\tilde{\beta}(\tau,\mu) = \beta(\tau,\mu) \overset{\mathrm{df}}{=} \rho_\mu(\beta(\tau))$. A change in the choices of $a(\tau)$ and v forces the following changes:

$a(\tau)$	$\beta(\tau,\mu)$	γ_σ	v	f	$\beta_1(\tau,\mu)$
$a(\tau)u_0$	$\beta(\tau,\mu)u_2$	$u_2^{-1}\gamma_\sigma\sigma u_2$	u_3vu_2	$f \circ \mathrm{ad}\,u_2$	$\beta_1(\tau,\mu)u_3^{-1}$

$u_0 \in S(L)$, $u_2 = \rho_\mu(u_0) \in T(L)$, $u_3 \in G(\mathbb{Q})$.

We shall abuse notation by writing ${}^\tau h$ also for the map $\mathbb{S} \to G_{\mathbb{R}}$ associated with $\tau\mu\colon \mathbb{C}_m \to G_{\mathbb{C}}$; thus ${}^\tau h$ (in the sense of §7) $= f \circ {}^\tau h$(this sense).

Lemma 10.7. Regard v as an element of $G(\mathbb{C})$; then $\underset{\sim}{ad}\, v \circ {}^{\tau}h \in X$.

Proof. Let $w \in G(\mathbb{C})$ normalize $T(\mathbb{C})$ and be such that $\tau\mu = \underset{\sim}{ad}\, w \circ \mu$. Then (see the proof of 10.2) v and w represent the same cocycle, and so $v^{-1}w \in G(\mathbb{R})$. Hence $\underset{\sim}{ad}\, v \circ {}^{\tau}h = \underset{\sim}{ad}\, v \circ \underset{\sim}{ad}\, w \circ h \in X$.

Since $\underset{\sim}{ad}\, v \circ {}^{\tau}h \in X$, and $f_1 \circ \underset{\sim}{ad}\, v \circ {}^{\tau}h = {}^{\tau}h \in {}^{\tau}X$, we see that $f_1 : G \xrightarrow{\approx} {}^{\tau}G$ defines an isomorphism $Sh(f_1):Sh(G,X) \xrightarrow{\approx} Sh({}^{\tau}G, {}^{\tau}X)$.

Proposition 10.8. Let $\phi(\tau;\mu)$ be the map

$$Sh(f_1) \circ \mathcal{T}(\beta_1(\tau,\mu)) : Sh(G,X) \xrightarrow{\approx} Sh({}^{\tau,\mu}G, {}^{\tau,\mu}X).$$

Then $\phi(\tau;\mu)$ is independent of the choices of $a(\tau)$, $\tilde{\beta}(\tau,\mu)$, and v; moreover

$$\phi(\tau;\mu)[\underset{\sim}{ad}\, v \circ {}^{\tau}h,\ \beta_1(\tau,\mu)^{-1}] = [{}^{\tau}h,1]$$

$$\phi(\tau;\mu) \circ \mathcal{T}(g) = \mathcal{T}({}^{\tau,\mu}g) \circ \phi(\tau;\mu).$$

Proof. The formula $\phi(\tau;\mu)[x,g] = [f_1 \circ x,\ f_1(g\beta_1(\tau,\mu))]$ shows immediately that $\phi(\tau;\mu)$ maps $[\underset{\sim}{ad}\, v \circ {}^{\tau}h,\ \beta_1(\tau,\mu)^{-1}]$ to $[{}^{\tau}h,1]$ and that $\phi(\tau;\mu)\, \mathcal{T}(g) = \mathcal{T}(g')\, \phi(\tau;\mu)$ with $g' = f_1(\beta_1^{-1}g\beta_1) = f \circ \underset{\sim}{ad}\, \beta(\tau,\bar{\mu})(g) = {}^{\tau}g$. The independence assertion is a consequence of this, the following lemma, and Deligne [1,5.2].

Lemma 10.9. The element $[\underset{\sim}{ad}v \circ {}^{\tau}h, \beta_1(\tau,\mu)^{-1}] \in Sh(G,X)$ is independent of the choices of $a(\tau)$, $\tilde{\beta}(\tau,\mu)$, and v.

Proof. Suppose that, after a change in the choices of $a(\tau)$, $\tilde{\beta}(\tau,\mu)$ and v, the elements β_1 and v are replaced by β_1' and v'. Remark (10.5) shows that $(\beta_1')^{-1} = u\beta_1^{-1}z$ with $u \in G(\mathbb{Q})$ and $z \in Z(\mathbb{Q})^\wedge$, and that $\underset{\sim}{ad}(\beta_1' v') = \underset{\sim}{ad}(\beta(\tau,\bar{\mu})u_1) = \underset{\sim}{ad}(\beta_1 v u_1)$ with $u_1 \in T^{ad}(L)$. Thus $\underset{\sim}{ad}(z^{-1}\beta_1 u^{-1} v') = \underset{\sim}{ad}(\beta_1 v u_1)$ and, on cancelling the β_1, we find $\underset{\sim}{ad}(u^{-1}v') = \underset{\sim}{ad}(vu_1)$. Hence $[\underset{\sim}{ad}v' \circ {}^{\tau}h, (\beta_1')^{-1}] = [\underset{\sim}{ad}v' \circ {}^{\tau}h, u\beta_1^{-1}z] = [\underset{\sim}{ad}u^{-1}v' \circ {}^{\tau}h, \beta_1^{-1}] = [\underset{\sim}{ad}v \circ \underset{\sim}{ad}u_1 \circ {}^{\tau}h, \beta_1^{-1}] = [\underset{\sim}{ad}v \circ {}^{\tau}h, \beta_1^{-1}]$ because ${}^{\tau}h$ maps into $T(\mathbb{R})$ and $u_1 \in T^{ad}(L)$.

Remark 10.10. Under the hypothesis of (10.6), the map $\phi(\tau;\mu)$ becomes $[x,g] \longmapsto [f \circ \underset{\sim}{ad}v^{-1} \circ x, \; f(v^{-1} g\beta(\tau,\mu))]$ and the element in (10.9) becomes $[\underset{\sim}{ad}v \circ {}^{\tau}h, \; v\beta(\tau,\mu)^{-1}]$. Both can be directly shown to be independent of all choices.

Proposition 10.11. Assume that $Sh(G,X)$ has a canonical model and let (ψ_τ), $\tau \in Aut(\mathbb{C}/E(G,X))$, be the corresponding family of maps as in (10.1) above. Conjecture C is true for $Sh(G,X)$ and a particular $\tau \in Aut(\mathbb{C}/E(G,X))$ if

$$\psi_\tau(\tau[h,1]) = [\underset{\sim}{ad}v \circ {}^{\tau}h, \; \beta_1(\tau,\mu)^{-1}] \qquad (10.12)$$

holds for all special $h \in X$.

Proof: Note that Lemma 10.9 shows (10.12) makes sense. Define $\phi_{\tau,\mu} = \phi(\tau;\mu) \circ \psi_\tau$. Then

$$\phi_{\tau,\mu}(\tau[h,1]) = \phi(\tau;\mu)[\underset{\sim}{\mathrm{ad}}v \circ {}^\tau h,\ \beta_1(\tau,\mu)^{-1}] \qquad \text{by (10.12)}$$

$$= [{}^\tau h, 1] \qquad \text{by (10.8)}$$

Moreover,

$$\phi_{\tau,\mu} \circ (\tau\mathcal{T}(g)) = \phi(\tau;\mu) \circ \mathcal{T}(g) \circ \psi_\tau$$

$$= \mathcal{T}({}^{\tau,\mu}g) \circ \phi_{\tau,\mu} \qquad \text{by (10.8)}$$

Thus $\phi_{\tau,\mu}$ satisfies condition (a) of conjecture C. Let h' be a second special point and let $\mu' = \mu_{h'}$. Then

$$\phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu} = \phi(\tau;\mu',\mu) \circ \phi(\tau;\mu) \circ \psi_\tau$$

$$= \phi_{\tau,\mu'}$$

because $\phi(\tau;\mu',\mu) = \phi(\tau;\mu') \circ \phi(\tau;\mu)^{-1}$ (7.13).

Remark 10.13. In certain situations, (10.12) simplifies. For example, under the hypothesis of (10.6) it becomes

$$\psi_\tau(\tau[h,1]) = [\underset{\sim}{\mathrm{ad}}v \circ {}^\tau h,\ v\,\beta(\tau,\mu)^{-1}] \qquad (10.13a)$$

(see 10.10). On the other hand, if we identify $Sh(G,X)$ with $M(G,X)_{\mathbb{C}}$, then (10.12) becomes

$$\tau[h,1] = [\underset{\sim}{\mathrm{ad}}v \circ {}^\tau h,\ \beta_1(\tau,\mu)^{-1}] \qquad (10.13b)$$

If $\gamma(\tau,\bar{\mu}) \overset{df}{=} \rho_{\bar{\mu}}(\gamma(\tau))$ is trivial in $H^1(\mathbb{Q}, \rho_{\bar{\mu}}(S))$ then there exists a $u \in S(L)$ such that $\rho_{\bar{\mu}}(u)^{-1}(\sigma\rho_{\bar{\mu}}(u)) = \gamma_\sigma$ (mod $Z(G)$).

After replacing $a(\tau)$ with $a(\tau)u$ one finds that f is defined over $\mathbb{Q}$, that $\tilde{\beta}(\tau,\mu)$ can be chosen to lie in $T(\mathbb{A}^f)$, and consequently that $v = 1$. Thus (10.12) becomes

$$\psi_\tau(\tau[h,1]) \quad = \quad [{}^\tau h,\ \tilde{\beta}(\tau,\mu)^{-1}] \tag{10.13c}$$

Finally, if τ fixes $E(h)$ then the hypothesis of (10.6) is satisfied, $\gamma(\tau)$ is trivial in $H^1(\mathbb{Q},\ S^{E(h)})$, and (10.12) can be written

$$\psi_\tau(\tau[h,1]) = [h,\ \beta(\tau,\mu)^{-1}] = [h,\ \tilde{r}(\tau)] \tag{10.13d}$$

(see 6.10), which is one of the defining conditions for $M(G,X)$ to be a canonical model (see 10.1c).

<u>Proposition 10.14</u>. Assume that conjecture C is true for $Sh(G,X)$ and all $\tau \in Aut(\mathbb{C}/E(G,X))$; then $Sh(G,X)$ has a canonical model and the maps ψ_τ (as in 10.1) satisfy $\psi_\tau = \phi(\tau;\mu_h)^{-1} \circ \phi_{\tau,\mu_h}$ for any special $h \in X$; equation (10.12) is true for all τ fixing $E(G,X)$ and all special $h \in X$.

<u>Proof:</u> Choose a special h and set $\psi_\tau = \phi(\tau;\mu)^{-1} \circ \phi_{\tau,\mu}$ with $\mu = \mu_h$. Arguments reverse to those in the proof of (10.11) show that ψ_τ is independent of h, that $\psi_\tau \circ \tau(T)(g) = (T)'(g) \circ \psi_\tau$, and that $\psi_\tau(\tau[h,1]) = [\underset{\sim}{ad}v \circ {}^\tau h,\ \beta_1(\tau,\mu)^{-1}]$. To complete the proof it must be shown that $\psi_{\tau_1\tau_2} = \psi_{\tau_1} \circ (\tau_1\psi_{\tau_2})$, but it can be checked directly that the two maps agree at the point $\tau_1\tau_2[h,1]$, and this implies they agree everywhere.

Corollary 10.15. In addition to the assumption of (10.14), suppose that $E(G,X) \subset \mathbb{R}$. Then conjecture B is true for $Sh(G,X)$.

Proof. If we identify $Sh(G,X)$ with $M(G,X)_{\mathbb{C}}$ then (10.14) and (10.13) show that $\iota[h,1] = [{}^{\iota}h, \tilde{\beta}(\iota,\mu)^{-1}]$ for any special h, where $\tilde{\beta}(\iota,\mu)$ has been chosen to be in $T(\mathbb{A}^f)$. But, according to (6.9), $\overline{\beta}(\iota,\mu) = 1$ and so $\beta(\iota,\mu) \in T(\mathbb{Q})$. Thus $\iota[h,1] = [{}^{\iota}h, \tilde{\beta}(\iota,\mu)^{-1}] = [{}^{\iota}h, 1]$, which implies conjecture B (7.4).

We come now to the relation between conjectures C and CM. Let A be an abelian variety of CM-type, let $V = H_1(A,\mathbb{Q})$, let h be the natural Hodge structure on V, and let ψ be a Riemann form for A. If T is the Mumford-Tate group of A then we have an embedding $(T,h) \hookrightarrow (CSp(V), S^{\pm})$.

Proposition 10.16. Conjecture CM is true for A and a given $\tau \in Aut(\mathbb{C})$ if and only if (10.12) holds for $Sh(CSp(V),S^{\pm})$, h, and τ.

Proof. Write (G,X) for $(CSp(V), S^{\pm})$. Recall (2.3) that there is a bijection $Sh(G,X) \xrightarrow{\approx} \mathcal{A}(G,X,V)$ where $\mathcal{A}(G,X,V)$ consists of certain isomorphism classes of triples (A',t,k). Let $\mu = \mu_h$; we define $\chi_{\tau,\mu} : \mathcal{A}(G,X,V) \longrightarrow \mathcal{A}({}^{\tau,\mu}G, {}^{\tau,\mu}X, {}^{\tau,\mu}V)$ to be the map $[A',t,k] \longmapsto [\tau A', \tau t, {}^{\tau}k]$ where ${}^{\tau}k$ is the composite $V^f(\tau A') \xrightarrow{\tau^{-1}} V^f(A') \xrightarrow{k} V(\mathbb{A}^f) \xrightarrow{sp(\tau)} {}^{\tau,\mu}V(\mathbb{A}^f)$.

Clearly there is a commutative diagram

$$\begin{array}{ccc} A(T,h,V) & \hookrightarrow & A(G,X,V). \\ \downarrow \chi_\tau & & \downarrow \chi_{\tau,\mu} \\ A(T,{}^\tau h,{}^{\tau,\mu}V) & \hookrightarrow & A({}^{\tau,\mu}G,{}^{\tau,\mu}X,{}^{\tau,\mu}V) \end{array}$$

where χ_τ is as defined in §9. On the other hand, as the canonical model for $Sh(G,X)$ is the moduli variety, $\tau: Sh(G,X) \to Sh(G,X)$ corresponds to the map $\tau : A(G,X,V) \to A(G,X,V)$ such that $[A',t,k] \longmapsto [{}^\tau A',\tau t, \tau k]$ (where $\tau k = k \circ \tau^{-1}$). It is easily verified that $\phi(\tau;\mu)$ corresponds to the map $[A',t,k] \longrightarrow [A',t,sp(\tau) \circ k]$; thus $\phi(\tau;\mu) \circ \tau$ corresponds to $\chi_{\tau,\mu}$. Since $\phi(\tau;\mu)$ is an isomorphism, (10.12) is equivalent to the equation $\phi(\tau;\mu)(\tau[h,1]) = [{}^\tau h,1]$, or, to the assertion that $\chi_{\tau,\mu}$ maps the triple corresponding to $[h,1]$ to the triple corresponding to $[{}^\tau h,1]$. But this is precisely the second form of conjecture CM.

Corollary 10.17. Conjecture CM is true if and only if conjecture C is true for all Shimura varieties of the form $Sh(CSp(V), S^\pm)$.

Proof. Combine (10.16) with (10.11) and (10.14).

Remark 10.18. The same arguments as above show that conjecture CM implies conjecture C for Shimura varieties of the form $Sh(G,X)$ when (G,X) embeds into $(CSp(V), S^\pm)$. We shall show, however, that conjecture C for Shimura varieties of the form $Sh(CSp(V), S^\pm)$ implies conjecture C for all Shimura varieties of abelian type. Thus, at least for these varieties, conjecture C is equivalent to a statement involving nothing more than abelian varieties of CM-type.

Remark 10.19. It is easy to verify conjecture CM in the case that $\tau = \iota$ (cf. 7.4c). On combining this remark with (9.2b) we find that conjecture CM is true whenever τ fixes the maximal real subfield of $E(G,X)$. In particular, conjecture CM is true for elliptic curves. Now (10.16) shows that conjecture C is true for $Sh(GL_2, S^{\pm})$. (Cf. Shimura [2, 6.9]).

IV. The action of complex conjugation.

Assuming $E(G,X)$ is totally real, we compute in §11 the action of complex conjugation on $\pi_0(Sh(G,X))$. This allows us (§12, §13) to formulate a conjecture (Conjecture B^0) for connected Shimura varieties, and show it to be equivalent to Conjecture B. In §14 we show that Conjecture B^0 is a special case of Conjecture CM. In §15 we prove Conjecture B^0 case by case for all connected Shimura varieties of primitive abelian type, and deduce Conjecture B for all Shimura varieties of abelian type.

§11. The action of ι on $\pi_0(Sh(G,X))$.

Let (G,X) satisfy (1.1) and assume $E(G,X) \subset \mathbb{R}$. The adjoint group G^{ad} is a product, $G^{ad} = \prod_{i=1}^{s} G_i$, of $\mathbb{Q}$-simple adjoint groups G_i. Each G_i can be written $G_i = \mathrm{Res}_{F_i/\mathbb{Q}} G^i$ where G^i is absolutely simple and the F_i are totally real (Deligne [3, 2.3.4]). For each embedding $v: F_i \hookrightarrow \mathbb{R}$ we obtain a group G^i_v over $\mathbb{R}$, and $G^i_v(\mathbb{R})$ is either compact or has exactly two connected components (Deligne [3, 1.2.8]). In the latter case we write $G^i_v(\mathbb{R})^+$ (or simply +) for the component containing 1 and $G^i_v(\mathbb{R})^-$ (or simply -) for the other component. Note that $G_i \otimes_{\mathbb{Q}} \mathbb{R} = \prod G^i_v$. Define:

$$G^{ad}(\mathbb{R})^+ = \{g \in G^{ad}(\mathbb{R}) \mid g \mapsto + \text{ for all } i \text{ and } v \text{ with } G^i_v(\mathbb{R}) \text{ non compact}\}$$

$$G^{ad}(\mathbb{R})^- = \{g \in G^{ad}(\mathbb{R}) \mid g \mapsto - \text{ for all } i \text{ and } v \text{ with } G^i_v(\mathbb{R}) \text{ non compact}\}$$

$$G^{ad}(\mathbb{R})^{\pm} = G^{ad}(\mathbb{R})^+ \cup G^{ad}(\mathbb{R})^- .$$

Clearly, $G^{ad}(\mathbb{R})^{\pm}$ is a normal subgroup of $G^{ad}(\mathbb{R})$, and there is an exact sequence

$$1 \to G^{ad}(\mathbb{R})^+ \longrightarrow G^{ad}(\mathbb{R})^{\pm} \longrightarrow \{\pm\} \to 1$$

For $* = +, -,$ or $\pm$, we define

$$G^{ad}(\mathbb{Q})^* = G^{ad}(\mathbb{R})^* \cap G(\mathbb{Q});$$

$$G(\mathbb{R})_* = \text{inverse image of } G^{ad}(\mathbb{R})^* \text{ in } G(\mathbb{R});$$

$$G(\mathbb{Q})_* = G(\mathbb{R})_* \cap G(\mathbb{Q}).$$

The real approximation theorem shows that there is an exact sequence

$$1 \longrightarrow G^{ad}(\mathbb{Q})^{+} \longrightarrow G^{ad}(\mathbb{Q}) \longrightarrow \{\pm\} \longrightarrow 1$$

<u>Remark 11.1</u>. Consider $\tilde{G} \xrightarrow{\rho} G \longrightarrow G^{ad}$. The group $\tilde{G}(\mathbb{R})$ is connected (Borel-Tits [1, 4]) and so the image of $\tilde{G}(\mathbb{R})$ in $G^{ad}(\mathbb{R})$ is $G^{ad}(\mathbb{R})^{+}$. Thus an element g of $G(\mathbb{R})$ is in $G(\mathbb{R})_{+}$ if and only if $g = \rho(\tilde{g})c$ for some $\tilde{g} \in \tilde{G}(\mathbb{R})$ and $c \in Z(\mathbb{R})$, where $Z = Z(G)$. Define T by the exact sequence

$$1 \longrightarrow G^{der} \longrightarrow G \xrightarrow{\nu} T \longrightarrow 1.$$

If $G^{der} = \tilde{G}$ then an element $g \in G(\mathbb{R})$ is in $G(\mathbb{R})_{+}$ if and only if $\nu(g) \in \nu(Z(\mathbb{R}))$.

Now let $h \in X$ be special and $\mu = \mu_h$. Choose a $\mathbb{Q}$-rational maximal torus T in G such that h factors through $T_{\mathbb{R}}$, and let N be the normalizer of T in G. We have seen (7.3) that there exists an $n \in N(\mathbb{R})$ such that $\underset{\sim}{ad}(n) \circ \mu = \iota\mu$. Since n takes h to ${}^{\iota}h$, n belongs to $G(\mathbb{R})_{-}$ (because ${}^{\iota}h$ and h^{-1} become equal when composed with $G_{\mathbb{R}} \to G^{ad}_{\mathbb{R}}$, and $h^{-1} \in X^{-}$, Deligne [3, 1.2.7]). In particular, we see that $G(\mathbb{R})_{\pm} \to \{\pm\}$ is surjective, and the real approximation theorem shows that there is an exact sequence

$$1 \longrightarrow G(\mathbb{Q})_{+} \longrightarrow G(\mathbb{Q}) \longrightarrow \{\pm\} \longrightarrow 1.$$

The element $n \in N(\mathbb{R})$ is unique modulo $N(\mathbb{R}) \cap K_\infty$, where K_∞ is the isotropy subgroup of h. However, on some occasions it is convenient to choose an n with the properties stated in the following Lemma.

Lemma 11.2. There exist $n \in N(\mathbb{R})$ and $w \in \rho(\tilde{G}(\mathbb{C}))$ such that $\underset{\sim}{\mathrm{ad}}(n) \circ \mu = \iota\mu$ and $\mu(-1) = wn$.

Proof.: Choose a maximal set of strongly orthogonal noncompact roots $\{\gamma_1, \ldots, \gamma_r\}$ of $\underset{\sim}{g}_{\mathbb{C}}^{der}$ with respect to $\underset{\sim}{t}_{\mathbb{C}}^{der}$ in the sense of Harish-Chandra, and use it to define a homomorphism of SL_2 to $\tilde{G}$ over $\mathbb{R}$ as usual (Ash et al. [1, III.2]). We can choose the γ_i's in such a way that $\langle \gamma_i, \mu \rangle = 1$ for all $i = 1, \ldots, r$. Put $w = (\rho \circ \phi)\left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right)$. Then $w \in N(\mathbb{C}) \cap \rho(\tilde{G}(\mathbb{C}))$. Furthermore,

$$w^{-1} \cdot \iota(w) = (\rho \circ \phi)\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (\Sigma_{i=1}^{r} \gamma_i^\vee)(-1) = (\mu - \iota\mu)(-1),$$

where $\gamma_i^\vee$ denotes the coroot of γ_i. Hence $n = w\mu(-1) \in N(\mathbb{R})$ and it has the required properties.

Recall (Deligne [3,2.1.14]) that the action of $G(\mathbb{A}^f)$ on $Sh(G,X)$ (on the right) induces an action of $G(\mathbb{A}^f)$ on $\pi_0(Sh(G,X))$ under which $\pi_0(Sh(G,X))$ becomes a principal homogeneous space for $\overline{\pi}_0\pi(G) = G(\mathbb{A}^f)/G(\mathbb{Q})_+^\wedge$. The image of $G(\mathbb{Q})_-$ in $G(\mathbb{A}^f)/G(\mathbb{Q})_+^\wedge \subset \mathrm{Aut}(\pi_0(Sh(G,X))$ is therefore an element of order 2. On the other hand, if $Sh(G,X)$ has a weakly canonical model over a real field then ι acts on $Sh(G,X)$ and hence on $\pi_0(Sh(G,X))$.

<u>Proposition 11.3</u>. Assume that $Sh(G,X)$ has a weakly canonical model over a real field E containing $E(G,X)$. Then for any $\alpha \in G(\mathbb{Q})_-$, the image of α in $G(\mathbb{A}^f)$ acts on $\pi_0(Sh(G,X))$ as ι.

<u>Proof</u>: According to Deligne [3,2.6.3], ι acts on $\pi_0(Sh(G,X))$ as $(\pi_0 N_{E/\mathbb{Q}} q_M)(\bar{e})$, where $\bar{e} \in \pi_0\pi(\mathbb{G}_{mE}) = \pi_0(\mathbb{A}_E^\times/E^\times)$ maps to $\iota \in Gal(E^{ab}/E)$, M denotes the $G(\mathbb{C})$-conjugacy class of maps $\mu\colon \mathbb{G}_m \longrightarrow G_{\mathbb{C}}$ corresponding to X, $q_M\colon \pi(\mathbb{G}_{mE}) \to \pi(G_E)$ and $N_{E/\mathbb{Q}}\colon \pi(G_E) \to \pi(G)$ are the maps defined in Deligne [3,2.4], and $\pi_0 N_{E/\mathbb{Q}} q_M$ is the composite

$$\pi_0\pi(\mathbb{G}_{mE}) \xrightarrow{\pi_0(N_{E/\mathbb{Q}} \circ q_M)} \pi_0\pi(G) \longrightarrow \bar{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0(G(\mathbb{R})_+).$$

The problem is to elucidate these maps.

Assume first that $G^{der} = \tilde{G}$. By definition M is defined over E. For any $k \supset E$, there is a map

$$\tilde{q}_M\colon \mathbb{G}_m(k) \longrightarrow (G/\tilde{G})(k)$$

with the following property (see Deligne [3,2.4]): let $\mu \in M$ be defined over $k' \supset k$, and denote the composite

$$\mathbb{G}_m(k') \xrightarrow{\mu} G(k') \longrightarrow (G/\tilde{G})(k')$$

by $\tilde{q}_\mu$; then $\tilde{q}_M = \tilde{q}_\mu$ over k'. Consider

$$\begin{array}{ccccccc} & & k^\times & \overset{\tilde{q}_M}{\searrow} & & & \\ & & \downarrow & & & & \\ 1 \longrightarrow & & G(k)/\tilde{G}(k) & \longrightarrow & (G/\tilde{G})(k) & \xrightarrow{\partial} & H^1(k,\tilde{G}). \end{array}$$

We see that the restriction of $\tilde{q}_M$ to the kernel $(k^{\times})_0$ of $\partial \circ \tilde{q}_M$ factors through $G(k)/\tilde{G}(k)$. Since $(E_v^{\times})_0 = E_v^{\times}$ for v a finite prime of E and $(E_v^{\times})_0 = (\mathbb{R}^{\times})^+$ for v a real prime, on forming a restricted product we obtain a map $q_M:(\mathbb{A}_E)^+ \longrightarrow G(\mathbb{A})/\tilde{G}(\mathbb{A})$. On passing to a quotient we obtain the map q_M from $\mathbb{A}_E^{\times}/E^{\times} = (\mathbb{A}_E^{\times})^+/(E^{\times})^+$ to $(\mathbb{A}_E^{\times})/\tilde{G}(\mathbb{A}_E)G(E) = \pi(G_E)$.

Let $e = (1,\ldots,1;\ 1,\ldots,1,\ -1) \in \mathbb{A}_E^{\times}$, where the final place corresponds to the real prime v_0 of E defined by the given embedding $E \hookrightarrow \mathbb{R}$. Then e represents $\bar{e} \in \pi_0(\mathbb{A}_E^{\times}/E^{\times})$. We compute $q_M([e]) \in \pi(G_E)$, where $[e]$ is the image of e in $\mathbb{A}_E^{\times}/E^{\times}$.

For $v \neq v_0$, $e_v = 1$, and $\tilde{q}_M(e_v)$ is represented by $1 \in G(E_v)$. For $v = v_0$, let $h \in X$ be special, let $\mu = \mu_h$ and choose n and w as in Lemma 11.9. Then $\tilde{q}_M = \tilde{q}$ over $\mathbb{C}$. Since $\mu(e_v) = \mu(-1) = wn$, $\mu(e_v)$ and n have the same image in $(G/\tilde{G})(\mathbb{R})$; thus $q_M(e_v)$ is defined and can be represented by $n \in G(E_v)$. We conclude that $(1,\ldots,1;\ 1,\ldots,1,n) \in G(\mathbb{A}_E)$ representes $q_M([e])$. It follows that $(N_{E/\mathbb{Q}}\, q_M)([e])$ is represented by $\xi = (1,\ldots,1;\ n) \in G(\mathbb{A})$, and $(\pi_0 N_{E/\mathbb{Q}}\, q_M)(\bar{e})$ is represented by the image $\bar{\xi}$ of ξ in $\bar{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0(G(\mathbb{R})_+)$:

$$\iota \text{ acts on } \pi_0(Sh(G,X)) \text{ as } \bar{\xi}.$$

Now for $\alpha \in G(\mathbb{Q})_-$, let $\alpha_0 = (\alpha,\ldots,\alpha;\ 1) \in G(\mathbb{A})$. Then $\alpha_0\xi^{-1} \in G(\mathbb{Q})G(\mathbb{R})_+$, and so the image $\bar{\alpha}_0$ of α_0 in $\bar{\pi}_0\pi(G)$ is $\bar{\xi}$.

Therefore $(\pi_0 N_{E/\mathbb{Q}} q_M)(\bar{e})$ is also represented by $\bar{\alpha}_0$. To complete the proof of this case, we observe that, when $\bar{\pi}_0\pi(G)$ is identified with $G(\mathbb{A}^f)/G(\mathbb{Q})^{\wedge}_{+}$, $\bar{\alpha}_0$ is the image of $\alpha \in G(\mathbb{Q}) \subset G(\mathbb{A}^f)$.

For the general case, one can repeat the argument with the group $(G/\tilde{G})(k)$ replaced by $\mathbb{H}^1(\tilde{G} \to G)$ (see Deligne [3,2.4]).

12. <u>Definition of</u> $\varepsilon\colon G(\mathbb{Q})^{\pm\wedge}(\text{rel } G') \longrightarrow \mathcal{E}_E(G,G',X^+)$.

Let (G,G',X^+) define a connected Shimura variety, as in §1. Recall that $E(G,X^+)$ is defined to be $E(G,X)$, where X is the $G(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to G_{\mathbb{R}}$ containing X^+. Assume that $E(G,X^+)$ is real. Then the discussion in §11 applies to $G = G^{ad}$ and we have groups $G(\mathbb{R})^+$, $G(\mathbb{R})^-$, ..., $G(\mathbb{Q})^{\pm}$, and exact sequences

$$1 \longrightarrow G(\mathbb{R})^+ \longrightarrow G(\mathbb{R})^{\pm} \longrightarrow \{\pm\} \longrightarrow 1,$$

$$1 \longrightarrow G(\mathbb{Q})^+ \longrightarrow G(\mathbb{Q})^{\pm} \longrightarrow \{\pm\} \longrightarrow 1.$$

Recall (Deligne [3, 2.5.7]) that for any $E \subset \overline{\mathbb{Q}}$ that is finite over $E(G,X)$, there is a canonical extension

$$1 \longrightarrow G(\mathbb{Q})^{+\wedge}(\text{rel } G') \longrightarrow \mathcal{E}_E(G,G',X^+) \longrightarrow \text{Gal}(\overline{\mathbb{Q}}/E) \longrightarrow 1.$$

In the following we assume $E \subset \mathbb{R}$.

<u>Proposition 12.1</u>. With the above assumptions and notations, there exists a canonical embedding

$$\varepsilon : G(\mathbb{Q})^{\pm\wedge}(\text{rel } G') \longrightarrow \mathcal{E}_E(G,G',X^+).$$

rendering

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \xrightarrow{\varepsilon} & \mathcal{E}_E(G,G',X^+) & \xrightarrow{\pi} & \text{Gal}(\overline{\mathbb{Q}}/E) & \longrightarrow & 1 \\ & & \big\downarrow & \nearrow_{\varepsilon} & & & & & \\ & & G(\mathbb{Q})^{\pm\wedge}(\text{rel } G') & & & & & & \end{array}$$

commutative and such that $(\pi\varepsilon)^{-1}(\iota) = G(\mathbb{Q})^{-\wedge}(\text{rel } G')$.

Proof: We first review Deligne's construction [3, 2.5] of the canonical extension. Choose a pair (G_1, X_1) satisfying (1.1) and such that $(G_1^{ad}, G_1^{der}, X_1^+) = (G, G', X^+)$ for some $X_1^+ \subset X_1$; it is possible to do this in such a way that $E(G_1, X_1) = E(G, X^+)$; see (3.4). The canonical extension is defined by the diagram

$$\begin{array}{ccccccccc} 1 \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \mathcal{E}_E(G,G',X^+) & \xrightarrow{\quad \pi \quad} & \text{Gal}(\bar{\mathbb{Q}}/E) & \longrightarrow 1 \\ & \| & & \downarrow f & & \downarrow r_{G_1,X_1} & \\ 1 \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \dfrac{G_1(\mathbb{A}^f)}{Z(\mathbb{Q})^\wedge} *_{G_1(\mathbb{Q})_+/Z(\mathbb{Q})} G(\mathbb{Q})^+ & \longrightarrow & \bar{\pi}_0\pi(G_1) & \longrightarrow 1 \end{array} \qquad (12.2)$$

in which r_{G_1,X_1} is the reciprocity law and Z is the center of G_1. The calculation made in the proof of (11.4) shows that

$$\pi^{-1}(\{1,\iota\}) \xrightarrow[\approx]{f} \frac{G_1(\mathbb{Q})^\wedge_\pm}{Z(\mathbb{Q})^\wedge} *_{G_1(\mathbb{Q})_+/Z(\mathbb{Q})} G(\mathbb{Q})^+$$

$$\approx \frac{G_1(\mathbb{Q})^\wedge_\pm}{Z(\mathbb{Q})^\wedge} *_{G_1(\mathbb{Q})_\pm/Z(\mathbb{Q})} G(\mathbb{Q})^\pm ,$$

which can be identified with $G(\mathbb{Q})^{\pm\wedge}(\text{rel } G')$, Deligne [3, 2.1.15.1] - We define ε to be the inverse isomorphism.

To see that ε is independent of the choice of (G_1, X_1), take another (G_2, X_2) with the same properties as (G_1, X_1). Let G_3 be the identity component of the fibre product $G_1 \times_G G_2$, and $X_3 = X_1 \times_X X_2$. Then (G_3, X_3) also has the same properties as (G_1, X_1). We see easily that, via the projections $G_3 \to G_1$ and $G_3 \to G_2$, (G_3, X_3), (G_1, X_1) and (G_2, X_2) all define the same ε.

Remark 12.3. Let the notations be as in the above proof. For simplicity, put $\mathcal{Y} = \frac{G_1(\mathbb{A}^f)}{Z(\mathbb{Q})^\wedge} *_{G_1(\mathbb{Q})_+/Z(\mathbb{Q})} G(\mathbb{Q})^+$.

Note that in the identification

$$G(\mathbb{Q})^{\pm\wedge}(\text{rel } G') = \frac{G_1(\mathbb{Q})^\wedge_\pm}{Z(\mathbb{Q})^\wedge} *_{G_1(\mathbb{Q})/Z(\mathbb{Q})_\pm} G(\mathbb{Q})^\pm,$$

$\alpha \in G(\mathbb{Q})^\pm$ is identified with $1 * \alpha$. Therefore, if $\alpha \in G(\mathbb{Q})^-$ lifts to $\alpha_1 \in G_1(\mathbb{Q})_-$, then $\varepsilon(\alpha)$ is the element of $\mathcal{E}_E(G, G', X^+)$ such that

$$f(\varepsilon(\alpha)) = \alpha_1 * 1 \in \mathcal{Y} \quad \text{and} \quad \pi(\varepsilon(\alpha)) = \iota \in \text{Gal}(\overline{\mathbb{Q}}/E).$$

In general, let γ_1 be an element of $G_1(\mathbb{Q})^-$, and let γ be its image in $G(\mathbb{Q})_-$. Then for any $\alpha \in G(\mathbb{Q})^-$, $\varepsilon(\alpha)$ is the element of $\mathcal{E}_E(G, G', X^+)$ such that

$$f(\varepsilon(\alpha)) = \gamma_1 * \gamma^{-1}\alpha \in \mathcal{Y} \quad \text{and} \quad \pi(\varepsilon(\alpha)) = \iota \in \text{Gal}(\overline{\mathbb{Q}}/E).$$

Assume (G, G', X) is of primitive abelian type. Then the pair (G_2, X_2) constructed in the proof of Deligne [3, 2.3.10] satisfies the conditions $G_2^{ad} = G$, $G_2^{der} = G'$, $(G_2, X_2) \to (G, X)$ and $E(G_2, X_2) = E(G, X)$, and so can be chosen as (G_1, X_1) in the proof above. However, this is not the most convenient one to use. We shall use a group G_3 that is larger than G_2.

Let the notations be as in (1.3) and (1.4). Recall that V is a vector space over F and $G_2 \subset GL(V)$. We take G_3 to be the $\mathbb{Q}$-algebraic group generated by G_2 and $F^\times$. Then (G_3, X_2) can be used instead of (G_2, X_2) as our (G_1, X_1). The extra properties (G_3, X_2) enjoys, which are established in the proof of [3, 2.3.10], are summarized in the following proposition, in which (G_3, X_2) is denoted by (G_2, X_2).

Proposition 12.4. Let the notations and assumptions be as in (1.4). Then there exists a diagram

$$(G_2, X_2) \longrightarrow (G, X) \longleftarrow (G_1, X_1) \hookrightarrow (CSp(V), S^\pm)$$

such that $G_1^{ad} = G_2^{ad} = G$, $G_1^{der} = G_2^{der} = G'$, $E(G_1, X_1) = E(G, X)\, E(F^\times, h_\Sigma)$, $E(G_2, X_2) = E(G, X)$, $G_1 \subset G_2$, $Z(G_2) \supset F^\times$ and $X_1 = \{h_2 h_\Sigma \mid h_2 \in X_2\}$.

§13. Statement of Conjecture B^0; equivalence with Conjecture B.

Let (G,G',X^+) define a connected Shimura variety as in §1. Recall (Deligne [3, 2.7.10]) that a weakly canonical model for $Sh^0(G,G',X^+)$ over $E \supset E(G,X^+)$ is a scheme $Sh^0(G,G',X^+)_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$ together with a left action of $\mathcal{E}_E(G,G',X^+)$ satisfying certain properties.

Let X be the conjugacy class of maps $\mathbb{S} \to G_{\mathbb{R}}$ containing X^+. Assume that $E(G,X^+) = E(G,X)$ is totally real. Fix a special $h_0 \in X$, and let $n \in N(\mathbb{R})$ and $\eta: X \to X$ be as in §7. Since $n \in G(\mathbb{R})^-$, as was remarked in §11, we see that $\underset{\sim}{ad}\, \alpha \circ \eta(x) \in X^+$ for all $\alpha \in X^+$ and $x \in X^+$.

Conjecture B^0. Assume that $Sh^0(G,G',X^+)$ has a weakly canonical model over a field $E \subset \mathbb{R}$; then for all $\alpha \in G(\mathbb{Q})^-$, the element $\varepsilon(\alpha) \in \mathcal{E}_E(G,G',X^+)$ acts on $Sh^0(G,G',X^+) = \varprojlim \Gamma\backslash X^+$ as follows: $[x] \longmapsto [\underset{\sim}{ad}\, \alpha \circ \eta(x)]$ for all $x \in X^+$.

Remark 13.1. Suppose α_1 and α_2 are both in $G(\mathbb{Q})^-$. Then $\alpha^+ = \alpha_1\alpha_2^{-1} \in G(\mathbb{Q})^+$. Hence $\varepsilon(\alpha_1) = \varepsilon(\alpha^+)\,\varepsilon(\alpha_2)$ and $[\underset{\sim}{ad}\, \alpha_1 \circ \eta(x)] = [\underset{\sim}{ad}\, \alpha^+ \circ \underset{\sim}{ad}\, \alpha_2 \circ \eta(x)] = \varepsilon(\alpha^+)[\underset{\sim}{ad}\, \alpha_2 \circ \eta(x)]$. Thus, Conjecture B^0 holds for all $\alpha \in G(\mathbb{Q})^-$ if and only if it does for one α.

Proposition 13.2. Let (G,X) satisfy (1.1), and assume $Sh(G,X)$ has a weakly canonical model over some field $E \subset \mathbb{R}$. Then Conjecture B holds for $Sh(G,X)$ if and only if Conjecture B^0 holds for $Sh^0(G^{ad}, G^{der}, X^+)$.

<u>Proof</u>: The proof is straightforward, but it is convenient first to review the various group actions on $Sh(G,X)$ and $Sh^0(G^{ad}, G^{der}, X^+)$.

The group $G^{ad}(\mathbb{Q})^{+\wedge}(\text{rel } G^{der})$ acts canonically on $Sh^0(G^{ad}, G^{der}, X^+)$ on the left. When $Sh^0(G^{ad}, G^{der}, X^+)$ is identified with the connected component $Sh^0(G,X)$ of $Sh(G,X)$ containing the image of $X^+ \times 1$, then the action of $\gamma \in G^{ad}(\mathbb{Q})^+$ is the restriction of

$$[x,g] \longmapsto \gamma[x,g] = [\gamma(x), \mathbf{ad}(\gamma)(g)], \; x \in X, \; g \in G(\mathbb{A}^f) .$$

By transport of structure, there is also a right action of $G^{ad}(\mathbb{Q})^+$ on $Sh(G,X)$:

$$[x,g]\gamma = \gamma^{-1}[x,g], \quad \gamma \in G^{ad}(\mathbb{Q})^+, \quad x \in X, \quad g \in G(\mathbb{A}^f).$$

The group $G(\mathbb{A}^f)$ acts on $Sh(G,X)$ on the right, via the Hecke operators. If $\gamma \in G^{ad}(\mathbb{Q})^+$ is the image of $\delta \in (\mathbb{Q})_+$, then the actions of γ and δ (considered as an element of $G(\mathbb{A}^f)$) agree. Thus there is a right action of

$$\mathcal{G} = \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^\wedge} *_{G(\mathbb{Q})_+/Z(\mathbb{Q})} G^{ad}(\mathbb{Q})^+$$

on $Sh(G,X)$: $[x,g](g' * \gamma) = [\gamma^{-1}(x), \mathbf{ad}\gamma^{-1}(gg')]$.

When $\mathcal{G}$ is made to act on $\pi_0(Sh(G,X))$, the stabilizer of the image of $Sh^0(G,X)$ is $G^{ad}(\mathbb{Q})^{+\wedge}$ (rel G^{der}), and $\pi_0(Sh(G,X))$ becomes a principal homogeneous space for the abelian quotient

$\bar{\pi}_0\pi(G) = G(\mathbb{A}^f)/G(\mathbb{Q})^{\wedge}_{+}$ of $\mathcal{G}$. These facts are summarized by an exact sequence:

$$1 \longrightarrow G^{ad}(\mathbb{Q})^{+\wedge} \ (\mathrm{rel}\ G^{der}) \longrightarrow \mathcal{G} \longrightarrow \bar{\pi}_0\pi(G) \longrightarrow 1.$$

Now assume $Sh(G,X)$ has a weakly canonical model over a finite extension E of $E(G,X)$. Then $Gal(\bar{\mathbb{Q}}/E)$ acts on $\pi_0(Sh(G,X))$ on the left. Since $\pi_0(Sh(G,X))$ is a principal homogeneous space for $\bar{\pi}_0\pi(G)$, the action of $Gal(\bar{\mathbb{Q}}/E)$ is described by a homomorphism $r\colon Gal(\bar{\mathbb{Q}}/E) \longrightarrow \bar{\pi}_0\pi(G)$ such that $\sigma\cdot r = x\cdot r(\sigma)$. The map r has an explicit description (Deligne [3,2.6]), and there is a commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G^{ad}(\mathbb{Q})^{+\wedge}(\mathrm{rel}\ G^{der}) & \longrightarrow & \mathcal{E}_E(G^{ad},G^{der},X^+) & \xrightarrow{\pi} & Gal(\bar{\mathbb{Q}}/E) & \longrightarrow & 1 \\
 & & \| & & \downarrow f & & \downarrow r & & \\
1 & \longrightarrow & G^{ad}(\mathbb{Q})^{+\wedge}(\mathrm{rel}\ G^{der}) & \longrightarrow & \mathcal{G} & \longrightarrow & \bar{\pi}_0\pi(G) & \longrightarrow & 1
\end{array}$$

Convert the right action of $\mathcal{G}$ on $Sh(G,X)$ to a left action, and consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G^{ad}(\mathbb{Q})^{+\wedge}(\mathrm{rel}\ G^{der}) & \longrightarrow & \mathcal{E}_E(G^{ad},G^{der},X^+) & \xrightarrow{\pi} & Gal(\bar{\mathbb{Q}}/E) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow f\times\pi & & \| & & \\
1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G}\times Gal(\bar{\mathbb{Q}}/E) & \longrightarrow & Gal(\bar{\mathbb{Q}}/E) & \longrightarrow & 1\ .
\end{array}$$

The action of $\mathcal{E}_E = \mathcal{E}_E(G^{ad},\ G^{der},\ X^+)$ on $Sh^0(G,X)$ arising, via this diagram, from the left actions of $\mathcal{G}$ and $Gal(\bar{\mathbb{Q}}/E)$ on $Sh(G,X)$, corresponds to the given action of $\mathcal{E}_E$ on the weakly canonical model of $Sh^0(G^{ad},\ G^{der},\ X^+)$ over E.

Now we prove the Proposition. Recall that E is assumed to be real. Fix an element $\alpha \in G^{ad}(\mathbb{Q})^-$ which lifts to an element $\alpha_1 \in$ of $G(\mathbb{Q})_-$. Then $\pi(\varepsilon(\alpha)) = \iota \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ and $f(\varepsilon(\alpha)) = \alpha_1 * 1 \in \mathcal{Y}$, see (12.3). Hence

$$\varepsilon(\alpha)[h,1] = (\iota[h,1])\,\textcircled{T}(\alpha_1)^{-1} \quad \text{for} \quad h \in X^+ .$$

Since Conjecture B holds if and only if

$$\iota[h,1] = [\eta(h),1] = [\underset{\sim}{\mathrm{ad}}\ \alpha_1 \circ \eta(h),\ \alpha_1] = [\underset{\sim}{\mathrm{ad}}\ \alpha \circ \eta(h),1]\textcircled{T}(\alpha_1),$$

this shows

$$\text{Conjecture B holds} \iff \varepsilon(\alpha)[h,1] = [\underset{\sim}{\mathrm{ad}}\ \alpha \circ \eta(h),\ 1]$$

$$\iff \text{Conjecture } B^0 \text{ holds for } \alpha .$$

This completes the proof, in view of (13.1).

14. A relation between Conjectures B and C.

We consider the situation of (1.4). Thus (G,G',X) defines a connected Shimura variety of primitive abelian type. Write $G = \mathrm{Res}_{F_0/\mathbb{Q}} G^s$ with F_0 totally real and G^s absolutely simple, and let I_c and I_{nc} be as in (1.3). Denote by F_0' the totally real number field corresponding to the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that stabilizes I_c. We have $F_0' \subset E(G,X)$.

Let $h \in X$ be special, and let $T \subset G$ be a $\mathbb{Q}$-rational torus such that h factors through $T_{\mathbb{R}}$. Let F be a quadratic totally imaginary extension of F_0 and let Σ be some family $(\sigma')_{\sigma \in I_c}$ of embeddings $\sigma'\colon F \to \mathbb{C}$ such that $\sigma'|F_0 = \sigma$. Denote by h_Σ the Hodge structure on F defined by Σ, see (1.3). We shall assume that (T,h) and (F,Σ) are such that there exists an automorphism τ of $\mathbb{C}$ with

$$\tau = \begin{cases} \mathrm{id} & \text{on } E(F^\times, h_\Sigma), \\ \iota & \text{on } E(T,h). \end{cases}$$

This is the case, for example, if $E(F^\times, h_\Sigma)$ and $E(T,h)$ are linearly disjoint over F_0'. Using Deligne [1,6.5] we know that for a given (T,h) there is always an (F,Σ) such that this holds. On the other hand, we can also start with an (F,Σ) and choose a (T,h) such that $E(T,h)$ is linearly disjoint from $E(F^\times, h_\Sigma)\,E(G,X)$ over $E(G,X)$, see Deligne [1,5.1]. Then (T,h) and (F,Σ) satisfy our assumption if $E(G,X)$ is totally real.

Remark 14.1. If (T,h) and (F,Σ) satisfy the assumption, then so do (T',h') and (F,Σ), where $T' = \mathrm{ad}\,\gamma(T)$ and $h' = \mathrm{ad}\,\gamma \circ h$ with $\gamma \in G(\mathbb{Q})$. This follows from the fact that $E(T', h') = E(T,h)$.

We assume now that $E(G,X)$ is totally real. Let (T,h), (F,Σ) and τ be as above. Consider the diagram

$$(G,X) \longleftarrow (G_1,X_1) \longrightarrow (CSp(V),\, S^{\pm})$$

constructed in Deligne [3, 2.3.10]. The information we need concerning this diagram is collected in Proposition 12.4. Lift (T,h) to $(T_1,h_1) \subset (G_1,X_1)$ as in (1.5); then $E(T_1,h_1) = E(T,h)\; E(F^{\times},h_{\Sigma})$. To simplify the notations, we put $E = E(G,X)$, $E_1 = E(G_1,X_1)$, $E(h) = E(T,h)$, $E_1(h) = E(T_1,h_1)$ and $F' = E(F^{\times}, h_{\Sigma})$. Thus we have $E_1 = EF'$ and $E_1(h) = E(h)F'$. Fix a component X^+ of X. We identify $Sh^0(G,G',X^+)$ (resp. $Sh(G_1,X_1)$) with its canonical model over E (resp. E_1).

Note that τ fixes E_1, because it fixes both F' and E, E being a totally real subfield of $E(h)$. Thus we are in the situation of §10. Let μ_1 be the cocharacter of T_1 associated to h_1, and define $v \in G_1(\overline{\mathbb{Q}})$ and $\beta_1(\tau,\mu_1) \in G_1(\mathbb{A}^f)$ as in (10.4).

Proposition 14.2. Conjecture B^0 holds for (G,G',X^+) if and only if $\tau[h_1,1] = [\mathrm{ad}\, v \circ {}^{\tau}h_1,\; \beta_1(\tau,\mu_1)^{-1}]$.

Recall that we have identified $Sh(G_1,X_1)$ with its canonical model. According to Proposition 10.11, the condition

$$\tau[h_1,1] = [\underset{\sim}{ad}\, v \circ {}^{\tau}h_1,\ \beta_1(\tau,\mu_1)^{-1}] \tag{14.3}$$

implies that Conjecture C holds for $(T_1,h_1) \hookrightarrow (G_1,X_1)$ and τ. As $(G_1,X_1) \hookrightarrow (CSp(V),\ S^{\pm})$, (14.3) is equivalent to Conjecture CM for A_1 and τ, where A_1 is the abelian variety of CM-type determined by $(T_1,h_1) \hookrightarrow (CSp(V),\ S^{\pm})$; see Proposition 10.16.

First we show that Proposition 14.2 is a consequence of the following assertion.

<u>Proposition 14.4</u>. Let the notations and assumptions be as above. Then $\varepsilon(\alpha)[h] = [\underset{\sim}{ad}\,\alpha \circ \eta(h)]$ for all $\alpha \in G(\mathbb{Q})^{-}$ (and for the given h) if and only if $\tau[h_1,1] = [\underset{\sim}{ad}\, v \circ {}^{\tau}h_1,\ \beta_1(\tau,\mu_1)^{-1}]$.

In fact, note that the $G(\mathbb{Q})^{+}$-orbit of $[h]$ is dense in $Sh^0(G,G',X^+)$. Therefore Conjecture B^0 holds for $Sh^0(G,G',X^+)$ if and only if $\varepsilon(\alpha)[h'] = [\underset{\sim}{ad}\,\alpha \circ \eta(h')]$ for all $\alpha \in G(\mathbb{Q})^{-}$ and all $[h']$ in the $G(\mathbb{Q})^{+}$- orbit of $[h]$. Let $\gamma \in G(\mathbb{Q})^{+}$, and consider $T' = \underset{\sim}{ad}\,\gamma(T)$ and $h' = \underset{\sim}{ad}\,\gamma \circ h$. By Remark 14.1, Proposition 14.4 also applies to (T',h'). Since (T,h) lifts to $(T_1,h_1) \subset (G_1,X_1)$, (T',h') lifts to (T_1',h_1'), where $T_1' = \underset{\sim}{ad}\,\gamma(T_1)$ and $h_1' = \underset{\sim}{ad}\,\gamma \circ h_1$. Moreover, ${}^{\tau}h_1' = \underset{\sim}{ad}\,\gamma \circ {}^{\tau}h_1$, and we can take $ad\,\gamma(\beta_1(\tau,\mu_1))$ as $\beta_1(\tau,\mu_1')$, where μ_1' is the cocharacter of T_1' associated to h_1', and take $\underset{\sim}{ad}\,\gamma(v)$ as the v for

(T_1', h_1'). Therefore, by Proposition 14.4, $\varepsilon(\alpha)[h'] = [\underset{\sim}{\mathrm{ad}}\ \alpha \circ \eta(h')]$ for all $\alpha \in G(\mathbb{Q})^-$ if and only if

$$\tau[h_1', 1] = [\underset{\sim}{\mathrm{ad}}\ \gamma(v) \circ {}^{\tau}h_1', \beta_1(\tau,\mu_1')^{-1}]. \qquad (14.5)$$

But we have

$$\tau[h_1', 1] = \tau[\underset{\sim}{\mathrm{ad}}\ \gamma \circ h_1, 1] = \gamma([h_1,1](\gamma * 1))$$
$$= (\tau[h_1,1])(\gamma * 1),$$

and
$$[\underset{\sim}{\mathrm{ad}}\ \gamma(v) \circ {}^{\tau}h_1', \beta_1(\tau,\mu_1')^{-1}]$$
$$= [\underset{\sim}{\mathrm{ad}}\ \gamma(v) \circ \underset{\sim}{\mathrm{ad}}\ \gamma \circ {}^{\tau}h_1, \underset{\sim}{\mathrm{ad}}\ \gamma(\beta_1\ (\gamma,\mu_1)^{-1})]$$
$$= [\underset{\sim}{\mathrm{ad}}\ \gamma \circ \underset{\sim}{\mathrm{ad}}\ v \circ {}^{\tau}h_1, \underset{\sim}{\mathrm{ad}}\ \gamma(\beta_1\ (\tau,\mu_1)^{-1})]$$
$$= [\underset{\sim}{\mathrm{ad}}\ v \circ {}^{\tau}h_1, \beta_1(\tau,\mu_1)^{-1}](\gamma * 1).$$

In other words, (14.5) holds for all h_1' in the $G(\mathbb{Q})^+$-orbit of h_1 if and only if it holds for h_1 (i.e. (14.3)). Putting these observations together, we obtain Proposition 14.2.

It remains to prove Proposition 14.4. Let $(G_2,X_2) \longrightarrow (G,X)$ be as in Proposition 12.4; thus $E(G_2,X_2) = E$, $G_2 \supset G_1$, $Z(G_2) \supset F^{\times}$ and $X_2 = \{x_1 h_{\Sigma}^{-1} \mid x_1 \in X_1\}$. Lift (T,h) to $(T_2,h_2) \subset (G_2,X_2)$. Then $T_2 \supset T_1F^{\times}$. Furthermore, using Lemma 1.2, one shows that $E(T_2,h_2) = E(T,h) = E(h)$ and $h_2 = h_1 h_{\Sigma}^{-1}$. Therefore h_2 factors through $T_2^* \overset{\mathrm{df}}{=} T_1F^{\times}$.

Let $\bar{r} = \bar{\beta}(\iota\tau,\mu_2)^{-1}$, where μ_2 is the cocharacter of T_2^* corresponding to h_2, and $\bar{s} = \bar{\beta}(\tau,\mu_{\Sigma})^{-1}$, where μ_{Σ} is the cocharacter of $F^\times$ corresponding to h_Σ. As $\iota\tau$ fixes $E(h)$, $\bar{r} = r_{E(h)}(T_2^\star, h_2)(\tau)$, and as τ fixes F', $\bar{s} = r_{F'}(F^\times, h_\Sigma)(\tau)$, see (6.10). Moreover, as $\mu_1 = \mu_2\mu_\Sigma$, the computation of (9.4) shows $\bar{\beta}(\tau,\mu_1)^{-1} = \bar{r}\,\bar{s}$.

Let L be a Galois extension of $\mathbb{Q}$ that splits $T_1, F^\times$ and $T_2^\star$. Since w_{h_1} is defined over $\mathbb{Q}$, we can define $\beta(\tau,\mu_1) \in T_1(\mathbb{A}^f)$, and choose v and $\beta_1(\tau,\mu_1)$ so that $\beta_1(\tau,\mu_1) = \beta(\tau,\mu_1)v^{-1}$, see (10.6). We have $\bar{r} \in T_2^\star(\mathbb{A}^f)/T_2^\star(\mathbb{Q})^\wedge$ and $\bar{s} \in F^\times(\mathbb{A}^f)/F^\times(\mathbb{Q})^\wedge$; let $r \in T_2^\star(\mathbb{A}^f)$ and $s \in F^\times(\mathbb{A}^f)$ be their respective representatives. Since $\bar{\beta}(\tau,\mu_1)^{-1} = \bar{r}\,\bar{s}$, we can choose r,s in such a way that $rs = z\beta(\tau,\mu_1)^{-1}$ with $z \in T_2^\star(L)$. Note that $zv^{-1} \in G_2(\mathbb{Q})$.

(a) Let $\tilde{G}_2 = \dfrac{G_2(\mathbb{A}^f)}{Z_2(\mathbb{Q})^\wedge} \; {}^{\star}{}_{G_2(\mathbb{Q})_+/Z_2(\mathbb{Q})} \; G(\mathbb{Q})^+$, where $Z_2 = Z(G_2)$, and consider the following diagram (Deligne [3, 2.5.3, 2.5.8, 2.5.10]).

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \tilde{G}_2 & \xrightarrow{\pi_2} & \bar{\pi}_0\pi(G_2) & \longrightarrow & 1 \\
 & & \| & & \uparrow f_2 & & \uparrow r_{G_2,X_2} & & \\
1 & \longrightarrow & G(\mathbb{Q})^{+}(\text{rel } G') & \longrightarrow & \mathcal{E}_E(G,G',X^+) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{Q}}/E) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & T(\mathbb{Q}) & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{Q}}/E(h)) & \longrightarrow & 1
\end{array}$$

Since $r \star 1 \in \tilde{G}_2$ and $\iota\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$ map to the same element in $\bar{\pi}_0\pi(G_2)$, they are both the image of an element

$\lambda = \lambda(h) \in \mathcal{E}_E(G,G',X^+)$. As $\iota\tau$ lies in $\mathrm{Gal}(\overline{\mathbb{Q}}/E(h))$ and $r * 1$ lies in

$$\frac{T_2(\mathbb{A}^f)}{Z_2'(\mathbb{Q})^\wedge} *_{T_2(\mathbb{Q})/Z_2'(\mathbb{Q})} T(\mathbb{Q}),$$

where $Z_2' = Z_2 \cap T_2$, the element $\lambda(h)$ lies in $\mathcal{E}'$. Therefore $\lambda(h)$ fixes the point $[h] \in Sh^0(G,G',X^+)$.

(b) Now consider $\varepsilon(\alpha) \in \mathcal{E}_E(G,G',X^+)$. As remarked in 12.3, we can use the diagram in (a) to define the map $\varepsilon\colon G(\mathbb{Q})^{\pm\wedge}(\text{rel } G') \longrightarrow \mathcal{E}_E$. Fix an element γ_1 of $G_1(\mathbb{Q})^-$ and let γ be its image in $G(\mathbb{Q})_-$. Since $G_1(\mathbb{Q}) \subset G_2(\mathbb{Q})$, the image of $\varepsilon(\alpha)$ in $\mathcal{G}_2$ is $\gamma_1 * \gamma^{-1}\alpha$, see 12.3. Therefore $\varepsilon(\alpha)\lambda(h) \in \mathcal{E}_E$ maps to $(\gamma_1 * \gamma^{-1}\alpha)(r * 1)$ in $\mathcal{G}_2$ and to $\iota(\iota\tau) = \tau$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

Consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \mathcal{E}_E & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/E) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \mathcal{E}_{E_1} & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/E_1) & \longrightarrow & 1
\end{array}$$

where $\mathcal{E}_{E_1} = \mathcal{E}_{E_1}(G,G',X^+)$. Since τ lies in $\mathrm{Gal}(\overline{\mathbb{Q}}/E_1)$, $\varepsilon(\alpha)\lambda(h) \in \mathcal{E}_E$ arises from an element $\varepsilon_1(\alpha,h) \in \mathcal{E}_{E_1}$. We have $\varepsilon_1(\alpha,h)\,[h] = \varepsilon(\alpha)\lambda(h)[h] = \varepsilon(\alpha)[h]$.

(c) Observe that (G_2, X_1) defines a Shimura variety, $(G_2, X_1) \longrightarrow (G, X)$, and $E(G_2, X_1) = E(G_1, X_1) = E_1$. Thus we have an exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \mathcal{G}_2 & \xrightarrow{\pi_2} & \bar{\pi}_0\pi(G_2) & \longrightarrow & 1 \\
 & & \| & & \uparrow \tilde{f}_2 & & \uparrow r_{G_2, X_1} & & \\
1 & \longrightarrow & G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & \mathcal{E}_{E_1} & \longrightarrow & \text{Gal}(\bar{\mathbb{Q}}/E_1) & \longrightarrow & 1
\end{array}$$

We show that $\tilde{f}_2(\varepsilon_1(\alpha, h)) = (\gamma_1 * \gamma^{-1}\alpha)(z\beta(\tau, \mu_1)^{-1} * 1)$.

We have a map $r_{F'}(F^\times, h_\Sigma)\colon \text{Gal}(\bar{\mathbb{Q}}/F') \longrightarrow \bar{\pi}_0\pi(F^\times) = F^\times(\mathbb{A}^f)/F^\times(\mathbb{Q})^\wedge$; composing this map with $\bar{\pi}_0\pi(F^\times) \longrightarrow \bar{\pi}_0\pi(G_2)$ (resp. $F^\times(\mathbb{A}^f)/F^\times(\mathbb{Q})^\wedge \longrightarrow \mathcal{G}_2$), we obtain a map

$$r_{F'}\colon \text{Gal}(\bar{\mathbb{Q}}/F') \longrightarrow \bar{\pi}_0\pi(G_2) \text{ (resp. } \tilde{r}_{F'}\colon \text{Gal}(\bar{\mathbb{Q}}) \longrightarrow \mathcal{G}_2).$$

Denote the product map of $\mathcal{E}_{E_1} \hookrightarrow \mathcal{E}_E$ and $\mathcal{E}_{E_1} \xrightarrow{\pi_1} \text{Gal}(\bar{\mathbb{Q}}/E_1) \hookrightarrow \text{Gal}(\bar{\mathbb{Q}}/F')$ by i, and the natural injection of $\text{Gal}(\bar{\mathbb{Q}}/E_1)$ into $\text{Gal}(\bar{\mathbb{Q}}/E) \times \text{Gal}(\bar{\mathbb{Q}}/F')$ by j. Then the diagram

$$\begin{array}{ccccc}
\mathcal{E}_{E_1} & \xrightarrow{i} & \mathcal{E}_E \times \text{Gal}(\bar{\mathbb{Q}}/F') & \xrightarrow{f_2 \times \tilde{r}_{F'}} & \mathcal{G}_2 \\
\downarrow & & \downarrow & & \downarrow \pi_2 \\
\text{Gal}(\bar{\mathbb{Q}}/E_1) & \xrightarrow{j} & \text{Gal}(\bar{\mathbb{Q}}/E) \times \text{Gal}(\bar{\mathbb{Q}}/F') & \xrightarrow{r_{G_2, X_2} \times r_{F'}} & \bar{\pi}_0\pi(G_2)
\end{array}$$

is commutative. Since $X_1 = \{x_2 h_\Sigma \mid x_2 \in X_2\}$, we have $(r_{G_2,X_2} \times r_{F'}) \circ j = r_{G_2,X_1}$ and $(f_2 \times \tilde{r}_{F'}) \circ i = \tilde{f}_2$. Thus $\tilde{f}_2(\varepsilon_1(\alpha,h)) = f_2(\varepsilon(\alpha)\lambda(h)) \cdot r_{F'}(\tau) = (\gamma_1 * \gamma^{-1}\alpha)(r * 1)(s * 1) = (\gamma_1 * \gamma^{-1}\alpha)(z\beta(\tau,\mu_1)^{-1} * 1)$.

(d) Next we show that on the canonical model of the Shimura variety $Sh(G_2,X_1)$,

$$[\underset{\sim}{ad}\,\alpha \circ {}^{\tau}h_1,\ 1]\ \tilde{f}_2(\varepsilon_1(\alpha,h)) = [\underset{\sim}{ad}\ v \circ {}^{\tau}h_1,\ \beta_1^{-1}],$$

where $\beta_1 = \beta_1(\tau,\mu_1) = \beta(\tau,\mu_1)v^{-1}$. In fact, for any $\delta_1 \in \dfrac{G_2(\mathbb{A}^f)}{Z_2(\mathbb{Q})^{\wedge}}$,

$$(\gamma_1 * \gamma^{-1}\alpha)(\delta_1 * 1) = (\gamma_1 \cdot (\underset{\sim}{ad}\ \gamma^{-1}\alpha)(\delta_1) * 1)(1 * \gamma^{-1}\alpha).$$

Therefore, for $x_1 \in X_1$,

$$\begin{aligned}
&[\underset{\sim}{ad}\,\alpha \circ x_1,\ 1](\gamma_1 * \gamma^{-1}\alpha)(\delta_1 * 1) \\
&= [\underset{\sim}{ad}\,\alpha \circ x_1,\ \gamma_1 \cdot (\underset{\sim}{ad}\ \gamma^{-1}\alpha)(\delta_1)](1 * \gamma^{-1}\alpha) \\
&= [\underset{\sim}{ad}(\alpha^{-1}\gamma) \circ \underset{\sim}{ad}\ \alpha \circ x_1,\ (\underset{\sim}{ad}\ \alpha^{-1}\gamma)(\gamma_1) \cdot \delta_1] \\
&= [\underset{\sim}{ad}((\underset{\sim}{ad}\ \alpha^{-1})(\gamma_1)) \circ x_1,\ (\underset{\sim}{ad}\ \alpha^{-1})(\gamma_1) \cdot \delta_1] \\
&= [x_1,\ \delta_1]
\end{aligned}$$

because $(\underset{\sim}{ad}\ \alpha^{-1})(\gamma_1) \in G_2(\mathbb{Q})$. Especially, in ~~veiw~~ view of (c),

$$\begin{aligned}
&[\underset{\sim}{ad}\ \alpha \circ {}^{\tau}h_1,\ 1]\tilde{f}_2\ (\varepsilon_1(\alpha,h)) \\
&= [\underset{\sim}{ad}\,\alpha \circ {}^{\tau}h_1,\ 1](\gamma_1 * \gamma^{-1}\alpha)(z\beta(\tau,\mu_1)^{-1} * 1)
\end{aligned}$$

$$= [\underset{\sim}{ad}\,\alpha \circ {}^{\tau}h_1,\ 1](\gamma_1 * \gamma^{-1}\alpha)(zv^{-1}\beta_1^{-1} * 1)$$

$$= [{}^{\tau}h_1,\ zv^{-1}\beta_1^{-1}]$$

$$= [\underset{\sim}{ad}(vz^{-1}) \circ {}^{\tau}h_1,\ \beta_1^{-1}] \qquad (\text{as } zv^{-1} \in G_2(\mathbb{Q}))$$

$$= [\underset{\sim}{ad}\, v \circ {}^{\tau}h_1,\ \beta_1^{-1}] \qquad (\text{as } z \in T_2(\mathbb{C})).$$

(e) The inclusion $(G_1, X_1) \hookrightarrow (G_2, X_1)$ induces maps $Sh(G_1, X_1) \hookrightarrow Sh(G_2, X_1)$, $\mathcal{G}_1 \hookrightarrow \mathcal{G}_2$ and $\overline{\pi}_0\pi(G_1) \hookrightarrow \overline{\pi}_0\pi(G_2)$ (Deligne [1, 1.15.3]). Note that the composite $\mathcal{E}_{E_1} \xrightarrow{f_1} \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ coincides with $\tilde{f}_2$. Since both $[\underset{\sim}{ad}\,\alpha \circ {}^{\tau}h_1,\ 1]$ and $[\underset{\sim}{ad}\, v \circ {}^{\tau}h_1,\ \beta_1^{-1}]$ are on the canonical model of the Shimura variety $Sh(G_1, X_1)$, the result (d) shows

$$[\underset{\sim}{ad}\,\alpha \circ {}^{\tau}h_1,\ 1]\, f_1(\varepsilon_1(\alpha,h)) = [\underset{\sim}{ad}\, v \circ {}^{\tau}h_1,\ \beta_1^{-1}].$$

(f) Finally we observe that $\tau\mu_1 = \tau\mu_2 \cdot \tau\mu_\Sigma = \iota\mu_2 \cdot \mu_\Sigma$ projects to $\iota\mu$ in $X_*(T)$. Thus $(T_1, {}^{\tau}h_1)$ is the lift of $(T, \eta(h))$ to (G_1, X_1). We also recall that $\pi_1(\varepsilon_1(\alpha,h)) = \tau$.

Therefore, for $\alpha \in G(\mathbb{Q})^-$,

$$\varepsilon(\alpha)[h] = [\underset{\sim}{ad}\,\alpha \circ \eta(h)]$$

$$\iff \varepsilon_1(\alpha,h)[h] = [\underset{\sim}{ad}\,\alpha \circ \eta(h)] \qquad (\text{by (b)})$$

$$\iff \tau[h_1, 1] = [\underset{\sim}{ad}\,\alpha \circ {}^{\tau}h_1,\ 1]\, f_1(\varepsilon_1(\alpha,h)) \qquad (\text{by (f)})$$

$$\iff \tau[h_1, 1] = [\underset{\sim}{ad}\, v \circ {}^{\tau}h_1,\ \beta_1^{-1}] \qquad (\text{by (e)})$$

This completes the proof of Proposition 14.4.

15. Proof of Conjecture B^0 .

In this section we prove Conjecture B^0 for (G,G',X^+) of primitive abelian type. For (G,G',X^+) of type C, this is done in Shih [1]. We shall use this result to prove Conjecture B^0 for all other cases. For completeness' sake, we start with a sketch of the proof for the type C case.

Every (G,G',X') of type C is obtained in the following fashion. Let F_0 be a totally real number field and B a quaternion algebra over F_0. We use σ to denote the main involution of B. Denote by I the set of embeddings of F_0 into $\mathbb{R}$, by I_{nc} the set of $\tau \in I$ at which B splits, and by I_c the complement of I_{nc}. Let Φ be a non-degenerate F_0-bilinear symmetric form on a free left B-module $\underline{\Lambda}$ of rank n such that

$$\Phi(bx,y) = \Phi(x,b^{\sigma}y) \quad \text{for} \quad x,y \in \Lambda \quad \text{and} \quad b \in B.$$

Let G_* be the similitude group of Φ , considered as an algebraic group over F_0 , and let $G_0 = \mathrm{Res}_{F_0/\mathbb{Q}} G_*$. There is a natural way of defining a $G_{0\mathbb{R}}$-conjugacy class X_0 of homomorphisms of $\mathbb{S}$ into $G_{0\mathbb{R}}$ such that (G_0,X_0) defines a Shimura variety, see Deligne [1, 6.3]. The reflex field $E(G_0,X_0)$ is totally real. Let $G = G_0^{ad}$ and $G' = G_0^{der}$. Let X_0^+ be a component of X_0. We can identify X_0^+ with a $G(\mathbb{R})^+$-conjugacy class X^+ of homomorphisms of $\mathbb{S}$ into $G_{\mathbb{R}}$. The triple (G,G',X^+) is of type C. The center Z_0 of G_0 is $\mathrm{Res}_{F_0/\mathbb{Q}}\mathbb{G}_m$. Thus

$$1 \longrightarrow F_0^{\times} \longrightarrow G_0(\mathbb{Q}) \longrightarrow G(\mathbb{Q}) \longrightarrow 1$$

is exact. In particular, $G(\mathbb{Q})^{+\wedge}(\text{rel } G') = G_0(\mathbb{Q})^{\wedge}_{+}/Z_0(\mathbb{Q})^{\wedge}$.

The first step towards proving Conjecture B^0 is to show that there is $t \in Z_0(\mathbb{A}^f) \cap G'(\mathbb{A}^f)$ such that

$$[\underset{\sim}{\mathrm{ad}}\ \alpha \circ \eta(h)] = \varepsilon(\alpha\lambda)[h] \quad \text{for all} \quad \alpha \in G(\mathbb{Q})^- \text{ and } h \in X^+, \qquad (15.1)$$

where λ denotes the image of t in $G_0(\mathbb{Q})^{\wedge}_{+}/Z_0(\mathbb{Q})^{\wedge} = G(\mathbb{Q})^{+\wedge}(\text{rel } G')$. (Note that an element t of $Z_0(\mathbb{A}^f)$ is in $G'(\mathbb{A}^f)$ if and only if $t^2 = 1$.) Two essential ingredients we need in proving the above claim are (i) uniqueness of canonical models and (ii) a concrete description of the automorphism group of $Sh^0(G,G',X^+)_{\mathbb{C}}$. For the former, we refer to Deligne [3, 2.7.19], and the latter, to T. Miyake [1], or to Milne-Shih [1]. The element t is unique modulo ± 1.

Let F be a quadratic totally imaginary extension of F_0, and consider the diagram

$$(G,X) \longleftarrow (G_1,X_1) \hookrightarrow (CSp(V), S^{\pm})$$

as in (1.4). Let $h \in X^+$ be special, and let $T \subset G$ be a $\mathbb{Q}$-rational torus such that h factors through $T_{\mathbb{R}}$. Lift (T,h) to $(T_1,h_1) \subset (G_1,X_1)$. Consider $(F^{\times},h_{\Sigma})$ and an automorphism τ of $\mathbb{C}$ as in §14. Since $(T_1,h_1) \hookrightarrow (CSp(V), S^{\pm})$, we have a diagram

$$\begin{array}{cccc}
[h_1, g_1] & Sh(T_1, h_1) & \xrightarrow{\approx} & \hat{A}(T_1, \{h_1\}, V) \\
\big\downarrow & \big\downarrow \approx & & \big\downarrow \approx \\
[{}^{\tau}h_1, g_1] & Sh(T_1, {}^{\tau}h_1) & \xrightarrow{\approx} & \hat{A}(T_1, \{{}^{\tau}h_1\}, V)
\end{array}$$

as in the second form of Conjecture CM(see §9). Using (15.1) and the argument of §14, we can show that the diagram is commutative if the left vertical map is replaced by $[h_1, g_1] \longmapsto [{}^{\tau}h_1, g_1]\lambda$.

Note that $\lambda = 1$ (i.e. $t = \pm 1$) if I_c is empty, because in this case $E(F^{\times}, h_{\Sigma}) = \mathbb{Q}$, so τ fixes the reflex field of (T_1, h_1) and Conjecture CM holds.

To get a more precise statement, we assume that (G_1, X_1) is constructed using Shimura's original method [1] (see also Deligne [1, §6]). Thus $V = \Lambda \otimes_{F_0} F$ and we have an exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z_0 & \longrightarrow & G_0 \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m & \longrightarrow & G_1 & \longrightarrow & 1 \\
 & & a & \longmapsto & (a, a^{-1}) & & & &
\end{array}$$

Note that t, when considered as an element of $G_1(\mathbb{A}^f)$, is in the center of $G(\mathbb{A}^f)$. We shall write $t = t(B,n)$ to emphasize its dependence on B and n. We choose (T,h) in the following way: Let P be a quadratic totally imaginary extension of F_0 that splits B. Then $T_0 = (\mathrm{Res}_{P/\mathbb{Q}} \mathbb{G}_m)^n$ can be embedded in G_0 and there is an $h_0 \in X_0^+$ that factors through $T_{0\mathbb{R}}$. We let (T,h) be the projection of (T_0, h_0) to (G, X^+).

With this choice of (G_1, X_1) and (T,h), T_1 is simply n copies of $\mathrm{Res}_{FP/\mathbb{Q}}\, \mathbb{G}_m$, and the abelian varieties (up to isogeny) that appear in the family Ⓐ$(T_1, \{h_1\}, V)$ are n-fold products of an abelian variety with FP as its field of complex multiplication. The conjugate of the family under τ is described by the map $[h_1, g_1] \longmapsto [h_1, g_1]\lambda = [{}^{\tau}h_1,\ t(B,n)g_1]$. From this we conclude that $t(B,n) \in (F_0 \otimes \mathbb{A}^f)^{\times}$ modulo ± 1 is independent of n. Actually it only depends on $\Sigma = I_{nc}$, the set of infinite places where B splits, and not on B, see Shih [1, Proposition 11].

Thus to a totally real number field k and a non-empty set Σ of embeddings of k into $\mathbb{R}$, we can associate a well-defined element $t(k,\Sigma)$ of $(k \otimes \mathbb{A}^f)^{\times}$ modulo ± 1. We remark that, the above considerations show that the statement at the end of Example 9.4 is correct if $\beta(\tau,\mu)$ is replaced by $t(F_0, \Sigma_0)\beta(\tau,\mu)$. Our goal is to prove that $t(k,\Sigma) = \pm 1$ for all k and Σ. This would complete the proof of Conjecture B^0 for (G, G', X^+) of type C, and also the proof of Example 9.4. We noted already that $t(k,\Sigma) = \pm 1$ if $\Sigma = I$, the set of all embeddings of k into $\mathbb{R}$.

By considering various families of the form Ⓐ$(T_1, \{h_1\}, V)$ and their conjugates, we obtain the following relations between $t(k,\Sigma)$'s. For simplicity, we shall use $t \equiv t'$ to mean that t is congruent to t' modulo ± 1. The fields k and k_1 are totally real.

(i) If (k_1, Σ_1) is an extension of (k, Σ), then $t(k,\Sigma) \equiv t(k_1,\Sigma_1)$ in $(k_1 \otimes \mathbb{A}^f)^\times$.

(ii) If $\gamma : k \longrightarrow k_1$ is an isomorphism, and Σ is the pull back of Σ_1 by γ, then $t(k_1,\Sigma_1) \equiv \gamma(t(k,\Sigma))$ in $(k_1 \otimes \mathbb{A}^f)^\times$.

(iii) Assume that k is normal over $\mathbb{Q}$, and Σ_1 and Σ_2 are two disjoint sets of embeddings of k into $\mathbb{R}$. Then $t(k,\Sigma_1)\, t(k,\Sigma_2) \equiv t(k, \Sigma_1 \cup \Sigma_2)$.

These functorial properties are all we need to conclude that $t(k,\Sigma) = \pm 1$ for any k and Σ. For details, see Shih [1, Theorem 16]. Thus we have shown that Conjecture B^0 holds for groups of type C, as well as the statement in Example 9.4.

Now turn to the proof of Conjecture B^0 in general. For each (G,G',X^+) of primitive abelian type, we shall take the corresponding (G_0,X_0) as given in Appendix B. We have $E(G_0,X_0) = E(G,X^+)$. In view of Proposition 13.2, we can either prove Conjecture B^0 for (G,G',X^+) or prove Conjecture B for (G_0,X_0). Recall that only those (G_0,X_0) with $E(G_0,X_0)$ totally real are under consideration.

(A) This is a trivial case, because (G_0,X_0) is embeddable in some $(CSp(V), S^{\pm})$, see K. Miyake [1].

$(B, D^{\mathbb{R}})$ According to Shih [3], in this case (G_0,X_0) can be embedded in some (G_1,X_1) such that $(G_1^{ad}, G_1^{der}, X_1^+)$ is of type C. Since Conjecture B holds for (G_1,X_1), it also holds for (G_0,X_0).

$(D^{\mathbb{H}})$ We use Proposition 14.2 here. Let the notations be as in Appendix B, case $(D^{\mathbb{H}})$. Let

$$q \sim \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} \quad (\varepsilon_i \in B)$$

be a diagonalization of q. Then for each i, $P_i = F_0(\varepsilon_i)$ is a CM-field, and $T_0 = \prod_{i=1}^{n} \mathrm{Res}_{P_i/\mathbb{Q}}\, \mathbb{G}_m$ can be embedded in G_0. Denote $\mathrm{Res}_{P_i/\mathbb{Q}}\, \mathbb{G}_m$ simply by $T_0^{(i)}$ so $T_0 = T_0^{(1)} \times \dots \times T_0^{(n)}$. Let X_0^+ be a connected component of X_0. We can embed T_0 in G_0 in such a way that some $h_0 \in X_0^+$ factors through $T_{0\mathbb{R}}$. Let $h_0^{(i)} : \mathbb{S} \longrightarrow T_{0\mathbb{R}}^{(i)}$ be the i-th factor of h_0, and let $E^{(i)} = E(T_0^{(i)}, h_0^{(i)})$. Then $E(T_0, h_0)$ is the composite of $E^{(1)}, \dots, E^{(n)}$.

Put $G = G_0^{ad}$, $G' = G_0^{der}$ and let X^+ be the $G(\mathbb{R})^+$-conjugacy class of homomorphisms of $\mathbb{S}$ into $G_{\mathbb{R}}$ induced by X_0^+. Let (T,h) be the image of (T_0, h_0) in (G, X^+). We have $E(T,h) = E(T_0, h_0)$, which is the composite of $E^{(1)}, \dots, E^{(n)}$.

Let F be a quadratic totally imaginary extension of F_0, and let Σ and h_Σ be as in (1.4). Consider the usual diagram

$$(G,X) \longleftarrow (G_1, X_1) \hookrightarrow (CSp(V), S^{\pm}).$$

As in the type C case, we can choose G_1 so that there is an exact sequence

$$1 \longrightarrow \mathrm{Res}_{F_0/\mathbb{Q}}\, \mathbb{G}_m \longrightarrow G_0 \times \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_m \longrightarrow G_1 \longrightarrow 1$$

$$a \longmapsto (a,\ a^{-1})$$

and take $\Lambda \otimes_{F_0} F$ as V. Let (T_1, h_1) be the lift of (T,h) to (G_1, X_1). Then $T_1 = \prod_{i=1}^{n} T_1^{(i)}$, where $T_1^{(i)} = \mathrm{Res}_{FP_i/\mathbb{Q}}\, \mathbb{G}_m$. We choose (T_0, h_0) and $(F^\times, h_\Sigma)$ in such a way that there exists an automorphism τ of $\mathbb{C}$ which induces the identity map on $E(F^\times, h_\Sigma)$, and the complex conjugation on $E(T, h)$, see §14.

The inclusion $(T_1, h_1) \hookrightarrow (CSp(V), S^{\pm})$ identifies the Shimura variety $Sh(T_1, h_1)$ with a family Ⓐ$(T_1, \{h_1\}, V)$ of abelian varieties. We show that Conjecture CM holds for (T_1, h_1) and ι. In view of Propositions 10.6 and 14.2, this would prove that Conjecture B^0 holds for (G, G', X^+).

Members of Ⓐ$(T_1, \{h_1\}, V)$ are (isogenous to) products $A_1 \times \ldots \times A_n$, where A_i is an abelian variety with complex multiplication by FP_i. Since $Sh(T_1, h_1)$ is the product of $Sh(T_1^{(i)}, h_1^{(i)})$, $i = 1, \ldots, n$, we only have to prove that Conjecture CM holds for τ and each individual $(T_1^{(i)}, h_1^{(i)})$. As $E^{(i)} = E(T_0^{(i)}, h_0^{(i)})$ is a CM-subfield of $E(T,h)$, τ acts as ι on $E^{(i)}$. Therefore Conjecture CM for $(T_1^{(i)}, h_1^{(i)})$ and τ is equivalent to the statement of Example 9.4. As we have established this statement while proving Conjecture B^0 for groups of type C, the proof of Conjecture B^0 for groups of type $D^{\mathbb{H}}$ is now completed.

Let (G,X) be of abelian type. By definition (see §1), there exist $(G_i, G_i', X_i^+)_i$ of primitive abelian type such that $G^{ad} = \Pi G_i$, G^{der} is a quotient of $\Pi G_i'$, and $X^+ \approx \Pi X_i^+$ for a suitable component X^+ of X. Assume $E(G,X)$ is totally real. Then $E(G^{ad}, X^+)$ and all $E(G_i, X_i^+)$ are totally real. As Conjecture B^0 holds for $Sh^0(G_i, G_i', X_i')$ for each i, it holds for $Sh^0(G^{ad}, G^{der}, X^+)$. Therefore Conjecture B holds for $Sh(G,X)$ in view of Proposition 13.2.

<u>Theorem 15.2</u>. Conjecture B holds for all $Sh(G,X)$ of abelian type (such that $E(G,X)$ is totally real).

<u>Remark 15.3</u>. Let V be a variety over a number field E. For a complex infinite prime $v: E \hookrightarrow \mathbb{C}$ of E the Hodge structure on $H^i(V \otimes_{E,v} \mathbb{C}, \mathbb{Q})$ defines a representation ρ^i of $\mathbb{C}^\times$, which we can regard as a representation of the Weil group $W_{\mathbb{C}}$. For a real prime v the involution of $H^i(V \otimes \mathbb{C}, \mathbb{Q})$ induced by ι enables one to define a representation of $W_{\mathbb{R}}$. In either case the factor $Z_v(V,s)$ of the zeta function corresponding to v is defined to be the alternating product of the L-series $L(s,\rho^i)$. Thus in order to compute the factors at infinity of the zeta function of a Shimura variety one must compute its cohomology and also the involution induced by ι (in the case of a real prime). The first of these is a problem in continuous cohomology and Theorem 15.2 reduces the second also to a problem in continuous cohomology. See Langlands [3, §7] where the assumption is made that conjecture B is true.

V. The conjugate of a Shimura variety

In §16 we state a version of conjecture C for connected Shimura varieties, and in the following section we prove that it is equivalent to the original version. This enables us to reduce the proof of conjecture C to the case of a Shimura variety defined by a group of symplectic similitudes. In §18 we twist the Taniyama group to obtain a group relative to which conjecture CM is automatically true. From this we can read off many consequences, including that conjecture A is true for Shimura varieties of abelian type. The final section, which is not used in the rest of the paper, contains a brief description of Deligne's theory of motives for absolute Hodge cycles; we include it in the hope that it may make the rest of the paper more comprehensible.

§16. Statement of conjecture C°.

Let (G,X) satisfy (1.1), let $h \in X$ be special, and let $\mu = \mu_h$. Recall that there is a unique homomorphism $\rho_{\bar{\mu}} : S \to G^{ad}$ such that $\rho_{\bar{\mu}} \circ \mu_{can} = \bar{\mu} \overset{df}{=} \mu^{ad}$; then $\rho_{\bar{\mu}}$ defines an action of S on G, and we write ${}^{\tau}G$ for ${}^{\tau}S \times^{S} G$ and $g \mapsto {}^{\tau}g : G(\mathbb{A}^f) \to {}^{\tau}G(\mathbb{A}^f)$ for $g \mapsto sp(\tau).g$.

<u>Lemma 16.1</u>. The isomorphism $g \mapsto {}^{\tau}g : G(\mathbb{A}^f) \to {}^{\tau}G(\mathbb{A}^f)$ maps the subgroup $G(\mathbb{Q})^{+\wedge}$ of $G(\mathbb{A}^f)$ into ${}^{\tau}G(\mathbb{Q})^{+\wedge}$ and $G(\mathbb{Q})^{\wedge}_{+}$ into ${}^{\tau}G(\mathbb{Q})^{\wedge}_{+}$.

<u>Proof</u>. Choose an element $a(\tau) \in {}^{\tau}S(L)$ for some finite Galois extension L of $\mathbb{Q}$, and let $f : G_L \to {}^{\tau}G_L$ be the isomorphism $g \to a(\tau).g$. In (3.6) we have defined an isomorphism $\pi(f) : \pi(G) \to \pi({}^{\tau}G)$ and it is easily checked that the following diagram commutes:

$$\begin{array}{ccc} g \mapsto {}^{\tau}g : & G(\mathbb{A}^f) & \longrightarrow & {}^{\tau}G(\mathbb{A}^f) \\ & \downarrow & & \downarrow \\ \pi(f) : & \pi(G) & \longrightarrow & \pi({}^{\tau}G) \end{array}.$$

Thus there is a commutative diagram

$$\begin{array}{ccc} g \mapsto {}^{\tau}g : & G(\mathbb{A}^f) & \longrightarrow & {}^{\tau}G(\mathbb{A}^f) \\ & \downarrow & & \downarrow \\ \pi_0\pi(f) : & \pi_0\pi(G) & \longrightarrow & \pi_0\pi({}^{\tau}G) \end{array}.$$

Since the kernels of the two vertical arrows in this diagram are $G(\mathbb{Q})^{+\wedge}$ and ${}^{\tau}G(\mathbb{Q})^{+\wedge}$ (Deligne [3, 2.5.1]), $g \mapsto {}^{\tau}g$ maps the first group into the second. Clearly $g \mapsto {}^{\tau}g$ maps $Z(G)(\mathbb{Q})$ into $Z({}^{\tau}G)(\mathbb{Q})$ and so it maps $G(\mathbb{Q})^{\wedge}_{+} = G(\mathbb{Q})^{+\wedge} Z(\mathbb{Q})$ into ${}^{\tau}G(\mathbb{Q})^{\wedge}_{+} = {}^{\tau}G(\mathbb{Q})^{+\wedge}({}^{\tau}Z(\mathbb{Q}))$.

Lemma 16.2. Let (G,G',X^{+}) define a connected Shimura variety, let $h \in X$ be special, and let $\mu = \mu_h$. Then there exists a unique isomorphism $g \mapsto {}^{\tau}g : G(\mathbb{Q})^{+\wedge}(\text{rel } G') \longrightarrow {}^{\tau}G(\mathbb{Q})^{+\wedge}(\text{rel } {}^{\tau}G')$ with the following property: for any map $(G_1,X_1) \longrightarrow (G,X)$ such that $G_1^{ad} = G$ and G_1^{der} is a covering of G' , the diagram

$$\begin{array}{ccc} g \mapsto {}^{\tau}g : G_1(\mathbb{Q})^{\wedge}_{+} & \longrightarrow & {}^{\tau}G_1(\mathbb{Q})^{\wedge}_{+} \\ \downarrow & & \downarrow \\ g \mapsto {}^{\tau}g : G(\mathbb{Q})^{+\wedge}(\text{rel } G') & \longrightarrow & {}^{\tau}G(\mathbb{Q})^{+\wedge}(\text{rel } G') \end{array}$$

commutes.

Proof. According to (3.4) we can choose a (G_1,X_1) , as in the statement of the lemma, such that $Z(G_1)$ is a torus having trivial cohomology. Then $G_1(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$ is surjective, and the equality

$$G(\mathbb{Q})^{+\wedge}(\text{rel } G') = G_1(\mathbb{Q})^{\wedge}_{+} *_{G_1(\mathbb{Q})_+} G(\mathbb{Q})^{+}$$

(Deligne [3, 2.1.6.2]) shows that $G_1(\mathbb{Q})^{\wedge}_{+} \longrightarrow G(\mathbb{Q})^{+\wedge}(\text{rel } G')$ is surjective. Thus we can define $g \mapsto {}^{\tau}g$ to be the map induced by its namesake on $G_1(\mathbb{Q})^{\wedge}_{+}$.

Let $(G_2,X_2) \longrightarrow (G,X)$ be a second map as in statement of the lemma and define G_3 to be the identity component of $G_2 \times_G G_1$. There is an X_3 for which there are maps $(G_3,X_3) \longrightarrow (G_1,X_1)$ and $(G_3,X_3) \longrightarrow (G_2,X_2)$. Since $\mathrm{Ker}(G_3 \longrightarrow G_2) = \mathrm{Ker}(G_1 \longrightarrow G)$, $G_3(\mathbb{Q}) \longrightarrow G_2(\mathbb{Q})$ is surjective and the image of $G_3(\mathbb{Q})^{\wedge}_{+}$ is dense in $G_2(\mathbb{Q})^{\wedge}_{+}$. Clearly the maps $g \mapsto {}^{\tau}g$ for G_3, G_1, and G are compatible, as are the same maps for G_3 and G_2. This forces the maps $g \mapsto {}^{\tau}g$ for G_2 and G to be compatible.

When necessary, we shall denote the map defined in the lemma by $\gamma \mapsto {}^{\tau,\mu}\gamma$.

Recall that any $\gamma \in G(\mathbb{Q})^{+\wedge}$ (rel G') defines an automorphism $\gamma.$ of $Sh^{\circ}(G,G',X^{+})$ which, when $\gamma \in G(\mathbb{Q})^{+}$, is equal to the family of maps $\underset{\sim}{\mathrm{ad}}\,\gamma : \Gamma\backslash X^{+} \longrightarrow \gamma\Gamma\gamma^{-1}\backslash X^{+}$.

<u>Conjecture C°</u>. Let (G,G',X^{+}) define a connected Shimura variety and let τ be an automorphism of $\mathbb{C}$.

a) For any special $h \in X^{+}$, with $\mu = \mu_h$, there is an isomorphism

$$\phi^{\circ}_{\tau} = \phi^{\circ}_{\tau,\mu} : \tau\, Sh^{\circ}(G,G',X^{+}) \longrightarrow Sh^{\circ}({}^{\tau}G, {}^{\tau}G', {}^{\tau}X^{+})$$

such that

$$\phi^{\circ}_{\tau}(\tau[h]) = [{}^{\tau}h]$$

$$\phi^{\circ}_{\tau} \circ \tau(\gamma.) = {}^{\tau}\gamma. \circ \phi^{\circ}_{\tau}, \quad \gamma \in G(\mathbb{Q})^{+\wedge} \text{ (rel } G').$$

b) If $h' \in X^{+}$ is a second special element and $\mu' = \mu_{h'}$, then

$$\begin{array}{ccc}
\tau\,\mathrm{Sh}^{\circ}(G,G',X^{+}) & \xrightarrow{\phi^{\circ}_{\tau,\mu}} & \mathrm{Sh}^{\circ}({}^{\tau,\mu}G,{}^{\tau,\mu}G',{}^{\tau,\mu}X^{+}) \\
& \searrow^{\phi^{\circ}_{\tau,\mu'}} & \downarrow{\scriptstyle \phi^{\circ}(\tau;\mu',\mu)} \\
& & \mathrm{Sh}^{\circ}({}^{\tau,\mu'}G,{}^{\tau,\mu'}G',{}^{\tau,\mu'}X^{+})
\end{array}$$

commutes.

(For $\phi^{\circ}(\tau;\mu',\mu)$, see §7.)

§17. Reduction of the proof of conjecture C to the case of the symplectic group.

Let (G,X) satisfy (1.1), let $\gamma \in G^{ad}(\mathbb{Q})$, and let $h \in X$ be special. If the image of γ in $G^{ad}(\mathbb{R})$ lifts to an element of $G(\mathbb{R})$, then $h' = ad\ \gamma \circ h$ is also a special point of X. Write $\mu = \mu_h$, $\mu' = \mu_{h'}$, and choose an $a(\tau) \in {}^{\tau}S^L(L)$ for some finite Galois extension of $\mathbb{Q}$. Then $f_1 = (a(\tau).g \mapsto a(\tau).\ qgq^{-1})$ is $\mathbb{Q}$-rational isomorphism ${}^{\tau,\mu}G \to {}^{\tau,\mu'}G$ which is independent of the choice of $a(\tau)$ and maps ${}^{\tau,\mu}X$ into ${}^{\tau,\mu'}X$.

<u>Lemma 17.1</u>. With the above notations, the composite

$$Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) \xrightarrow{\tau\gamma} Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) \xrightarrow{\phi(\tau;\mu',\mu)} Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X)$$

is equal to $Sh(f_1)$.

<u>Proof</u>. If γ lifts to an element of $G(\mathbb{Q})$, this is immediate from the definition of $\phi(\tau;\mu',\mu)$ (see 7.12d). Since we can always find a group with the same adjoint and derived groups as G, but with cohomologically trivial centre, this shows that the two maps agree on a connected component of $Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X)$. To complete the proof we only have to note that both maps transfer the action of $\mathbb{T}(g)$ on $Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X)$ into the action of $\mathbb{T}(f_1(g))$ on $Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X)$.

<u>Lemma 17.2</u>. Suppose conjecture C is true for (G,X) and let $h \in X$ be special with $\mu = \mu_h$. Then for any $\gamma \in G^{ad}(\mathbb{Q})^+$,

$$\phi_{\tau,\mu} \circ \tau(\gamma.) = {}^{\tau}\gamma. \circ \phi_{\tau,\mu} .$$

<u>Proof</u>. Consider the following diagram:

$$\begin{array}{ccccc}
\tau\ Sh(G,X) & \xrightarrow{\phi_{\tau,\mu}} & Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) & & \\
\downarrow{\scriptstyle \tau(\gamma.)} & & \downarrow{\scriptstyle {}^{\tau}\gamma.} & & \\
\tau\ Sh(G,X) & \xrightarrow{\phi_{\tau,\mu}} & Sh({}^{\tau,\mu}G,{}^{\tau,\mu}X) & & \Big\downarrow Sh(f_1) \\
& \searrow{\scriptstyle \phi_{\tau,\mu'}} & \downarrow{\scriptstyle \phi(\tau;\mu',\mu)} & & \\
& & Sh({}^{\tau,\mu'}G,{}^{\tau,\mu'}X) & &
\end{array}$$

Since we are assuming that the bottom triangle commutes, it suffices to show that the diagram commutes with the lower $\phi_{\tau,\mu}$ removed. But clearly

$$Sh(f_1) \circ \phi_{\tau,\mu}(\tau[h,1]) = [{}^{\tau}h',1] = \phi_{\tau,\mu'} \circ \tau(\gamma.)\ (\tau[h,1])$$

$$Sh(f_1) \circ \phi_{\tau,\mu} \circ \tau\mathbb{T}(g) = \mathbb{T}({}^{\tau,\mu'}g) \circ Sh(f_1) \circ \phi_{\tau,\mu}$$

$$\phi_{\tau,\mu'} \circ \tau(\gamma.) \circ \tau\mathbb{T}(g) = \mathbb{T}({}^{\tau,\mu'}g) \circ \phi_{\tau,\mu'} \circ \tau(\gamma.) ,$$

which completes the proof.

<u>Remark 17.3</u>. If, in (17.2), γ lifts to $\delta \in G(\mathbb{Q})$, then the statement of the lemma becomes $\phi_{\tau,\mu} \circ \mathbb{T}(\delta^{-1}) = \mathbb{T}({}^{\tau}\delta^{-1}) \circ \phi_{\tau,\mu}$, which is part of (a) of conjecture C.

<u>Proposition 17.4</u>. Let (G,X) satisfy (1.1) and let X^{+} be one connected component of X . Then conjecture C is true for $Sh(G,X)$ if and only if conjecture C° is true for $Sh^{\circ}(G^{ad},G^{der},X^{+})$.

Proof. Assume conjecture C and let $h \in X^+$ be special. Then $\phi_{\tau,\mu}$, with $\mu = \mu_h$, maps $\tau[h,1]$ to $[{}^\tau h,1]$ and therefore it maps $Sh^\circ(G,G',X^+)$ into $Sh^\circ({}^\tau G,{}^\tau G',{}^\tau X^+)$. We can therefore define $\phi^\circ_{\tau,\mu}$ to be the restriction of $\phi_{\tau,\mu}$ to $Sh^\circ(G,G',X^+)$. Part (a) of conjecture C° follows from part (a) of conjecture C and (17.2), while part (b) of conjecture C° follows from part (b) of conjecture C.

Next assume conjecture C holds for $Sh(G,X)$. Suppose that, for each special $h \in X^+$, we have extended $\phi^\circ_{\tau,\mu}$, $\mu = \mu_h$, to a map $\phi_{\tau,\mu} : \tau Sh(G,X) \longrightarrow Sh({}^\tau G,{}^\tau X)$ satisfying $\phi_{\tau,\mu} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^\tau g) \circ \phi_{\tau,\mu}$. Then $\phi_{\tau,\mu}(\tau[h,1]) = [{}^\tau h,1]$ and, for $\mu' = \mu_{h'}$, with $h' \in X^+$, $\phi_{\tau,\mu'} = \phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu}$, because the maps ϕ° have the corresponding properties. If h' is a special element of X, but $h' \notin X^+$, we write $h' = \text{ad}\, q \circ h$ with $h \in X^+$ and $q \in G(\mathbb{Q})$, and define $\phi_{\tau,\mu'}$ to be $\phi(\tau;\mu',\mu) \circ \phi_{\tau,\mu}$. We have already noted in (7.14) that this map automatically satisfies part (a) of conjecture C. That the entire family, (ϕ_{τ,μ_h}), $h \in X$ special, satisfies part b of conjecture C follows easily from the definitions and from (7.12b).

It remains to see how to extend $\phi^\circ_{\tau,\mu}$. For this we use Deligne [3, 2.7.3]. Write τSh for $\tau Sh(G,X)$ and ${}^\tau Sh$ for $Sh({}^\tau G,{}^\tau X)$. Recall (Deligne [3, 2.1.16]) that $G(\mathbb{A}^f)$ acts transitively on $\pi_0(\tau Sh)$ $(= \tau\,\pi_0(Sh))$ and that the stabilizer of $\tau e \overset{df}{=} \tau Sh^\circ(G^{ad},G^{der},X^+)$ is $G(\mathbb{Q})^\wedge_+$. Similarly ${}^\tau G(\mathbb{A}^f)$ acts transitively on $\pi_0({}^\tau Sh)$ and the stabilizer of ${}^\tau e$ is ${}^\tau G(\mathbb{Q})^\wedge_+$. We have compatible isomorphisms $G(\mathbb{A}^f) \longrightarrow {}^\tau G(\mathbb{A}^f)$ and $\pi_0(\tau Sh) \longrightarrow \pi_0({}^\tau Sh)$ (see the proof of 16.1). Thus giving a morphism $\tau Sh \longrightarrow {}^\tau Sh$ that is compatible with these two morphisms is equivalent to giving a morphism $\tau e \longrightarrow {}^{\tau}e$ that is equivariant for the actions of the stabilizers of τe and ${}^{\tau}e$. But $\phi^{\circ}_{\tau,\mu}$ is such a morphism.

Lemma 17.5. Suppose that (G,X) and (G',X') satisfy (1.1) and that there is a map $(G,X) \longrightarrow (G',X')$ with $G \rightarrow G'$ injective. If conjecture C is true for $Sh(G',X')$ then it is also true for $Sh(G,X)$.

Proof. According to Deligne [1, 1.15.1] the map $Sh(G,X) \longrightarrow Sh(G',X')$ is injective. A special point h of X maps to a special point h' of X' , and the map $\phi_{\tau,\mu_{h'}}$ sends $\tau[h,1]$ to $[{}^{\tau}h,1] \in Sh({}^{\tau}G,{}^{\tau}X) \subset Sh({}^{\tau}G',{}^{\tau}X')$. It therefore sends $\tau[h,g]$ to $[{}^{\tau}h,{}^{\tau}g] \in Sh({}^{\tau}G,{}^{\tau}X)$ for any $g \in G(\mathbb{A}^f)$, which implies that it maps $\tau Sh(G,X)$ into $Sh({}^{\tau}G,{}^{\tau}X)$. We define ϕ_{τ,μ_h} to be the restriction of $\phi_{\tau,\mu_{h'}}$ to $\tau Sh(G,X)$.

Lemma 17.6. If conjecture C° is true for $Sh^{\circ}(G,G',X^{+})$, and G'' is a quotient of G' , then conjecture C° is true for $Sh^{\circ}(G,G'',X^{+})$.

Proof. This follows immediately from the general fact that $Sh^{\circ}(G,G'',X^{+})$ is the quotient of $Sh^{\circ}(G,G',X^{+})$ by the kernel of the surjectuve map

$$G(\mathbb{Q})^{+\wedge}\ (\text{rel } G') \longrightarrow G(\mathbb{Q})^{+\wedge}\ (\text{rel } G'') .$$

Lemma 17.7. If conjecture C° is true for $Sh^\circ(G_i, G_i', X_i^+)$, $i = 1,\ldots,n$, then the conjecture is true for $Sh^\circ(\Pi G_i, \Pi G_i', \Pi X_i^+)$.

Proof. Easy.

Theorem 17.8. If conjecture C is true for all varieties of the form $Sh(CSp(V), S^{\pm})$ then it is true for all Shimura varieties of abelian type.

Proof. If conjecture C is true for varieties of the form $Sh(CSp(V), S^{\pm})$ then (1.4), (17.5), and (17.4) show that conjecture C° is true for all connected Shimura varieties of primitive abelian type. Then (17.6) and (17.7) show that conjecture C° is true for all connected Shimura varieties of abelian type. Finally (17.4) then implies that conjecture C is true for all Shimura varieties of abelian type.

Corollary 17.9. Conjecture CM implies that conjecture C is true for all Shimura varieties of abelian type.

Proof. Combine (10.17) with (17.8).

§18. Proof of conjecture A.

Throughout this section, all Shimura varieties will be of abelian type.

Let A be an abelian variety over $\mathbb{C}$ of CM-type, let $T = T_A$ be the Mumford-Tate group of A, and let $h = h_A : \mathbb{S} \longrightarrow T_A$ be the map defined by the Hodge structure on $V = H_1(A,\mathbb{Q})$. Choose an isomorphism $f : (H_1(\tau A,\mathbb{Q}),(\tau s_\alpha)) \xrightarrow{\approx} ({}^\tau V,({}^\tau s_\alpha))$ as in (9.1) and let $e_A(\tau)$ be the element of $T(\mathbb{A}^f)$ such that

$$V(\mathbb{A}^f) = V^f(A) \xrightarrow{\tau} V^f(\tau A) \xrightarrow{f\otimes 1} {}^\tau V(\mathbb{A}^f) \xrightarrow{sp(\tau)^{-1}} V(\mathbb{A}^f)$$

is multiplication by $e_A(\tau)^{-1}$. Note that the class $\bar{e}_A(\tau)$ of $e_A(\tau)$ in $T_A(\mathbb{A}^f)/T_A(\mathbb{Q})$ is well-defined, and that conjecture CM holds for A if and only if $\bar{e}_A(\tau) = 1$. In particular $\bar{e}_A(\tau) = 1$ if τ fixes $E(T,\{h\})$ (see 9.2b). We could also have defined $e_A(\tau)$ to be an element making

$$\begin{array}{ccc} [h,g] & Sh(T,\{h\}) \xrightarrow{\approx} & A(T,\{h\},V) \\ \updownarrow & \downarrow & \downarrow \chi_\tau \\ [{}^\tau h, e_A(\tau)g] & Sh(T,\{{}^\tau h\}) \xrightarrow{\approx} & A(T,\{{}^\tau h\},{}^\tau V) \end{array}$$

commute (see the second form of conjecture CM).

Fix a finite Galois extension L of $\mathbb{Q}$ and consider those abelian varieties A of CM-type such that T_A is split by L. For such an A, $\bar{e}_A(\tau)$ depends only on $\tau|L$. The maps $\bar{e}_A : Gal(L/\mathbb{Q}) \longrightarrow T_A(\mathbb{A}^f)/T_A(\mathbb{Q})$ for varying A are compatible and therefore define a map $\bar{e} = \bar{e}^L : Gal(L/\mathbb{Q}) \longrightarrow S^L(\mathbb{A}^f)/S^L(\mathbb{Q})$.

Lemma 18.1. For any $\tau_1, \tau_2 \in Gal(L/\mathbb{Q})$, $\bar{e}(\tau_1\tau_2) = \tau_2^{-1}\bar{e}(\tau_1).\bar{e}(\tau_2)$.

Proof. Let A have a Mumford-Tate group that is split by L , and let $\mu = \mu_A \in X_*(T_A)$ correspond to h_A . The homomorphism $\rho_\mu : S^L \to T_A$ defined by μ is induced by $L^\times \xrightarrow{\mu} T_A(L) \xrightarrow{N_{L/\mathbb{Q}}} T_A(\mathbb{Q})$. By definition $\bar{e}_A(\tau) = \rho_\mu(\bar{e}(\tau))$. If we choose an isomorphism $f : (H_1(\tau A,\mathbb{Q}), (\tau s_\alpha)) \xrightarrow{\approx} ({}^\tau V, ({}^\tau s_\alpha))$ then we can identify $MT(\tau A)$ with T_A , $h_{\tau A}$ with ${}^\tau h_A$, and $\mu_{\tau A}$ with $\tau\mu_A$. With these identifications, $\rho_{\tau\mu} = \rho_\mu \circ \tau^{-1}$.

The composite

$$\mathcal{A}(T_A,\{h\},V) \xrightarrow{\chi_{\tau_2}} \mathcal{A}(T_A,\{{}^{\tau_2}h\},{}^{\tau_2}V) \xrightarrow{\chi_{\tau_1}} \mathcal{A}(T_A,\{{}^{\tau_1\tau_2}h\},{}^{\tau_1\tau_2}V)$$

is equal to $\chi_{\tau_1\tau_2}$. Thus

$$\rho_\mu(\bar{e}(\tau_1\tau_2)) = \rho_\mu(\bar{e}(\tau_2))\ \rho_{\tau_2\mu}(\bar{e}(\tau_1)) = \rho_\mu(\bar{e}(\tau_2)\ .\ \tau_2^{-1}\ \bar{e}(\tau_1)) \ .$$

On passing to the inverse limit over A , we obtain the required formula.

For a given finite Galois extension L of $\mathbb{Q}$, the map

$$\tau \mapsto \bar{b}(\tau)\bar{e}(\tau^{-1})^{-1} : Gal(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)/S^L(L)$$

(with $\bar{b}$ as in §6) satisfies the conditions of (5.7), and for varying L the maps are compatible. Thus they define an extension

$$1 \to S \to \underset{\sim}{M} \xrightarrow{\tilde{\pi}} Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1$$

of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S (together with a splitting $\tilde{sp}$ over $\mathbb{A}^f$) in the sense of §5. We shall refer to this extension as the <u>motivic Galois group</u> (for an explanation of the name, and a much more natural definition, see §19). To distinguish objects associated with $\underset{\sim}{M}$ from the same objects for $\underset{\sim}{T}$, we shall use a tilde. Thus if $\tilde{a}(\tau)$ is a section to $\underset{\sim}{M}_L^L \to \mathrm{Gal}(L^{ab}/\mathbb{Q})$, then $\tilde{\beta}(\tau) \in S^L(\mathbb{A}_L^f)$ is defined by $\tilde{sp}(\tau)\ \tilde{\beta}\ (\tau) = \tilde{a}(\tau)$. For appropriate choices of a and $\tilde{a}$ we have $\tilde{\beta}(\tau) = \beta(\tau)\ e\ (\tau)$ with $e(\tau) \in S^L(\mathbb{A}^f)$ representing $\bar{e}(\tau)$. Then $\tilde{\gamma}_\sigma(\tau) = \gamma_\sigma(\tau)$ and so $\tau_{\tilde{S}} \overset{df}{=} \tilde{\pi}^{-1}(\tau) \approx \pi^{-1}(\tau) \overset{df}{=} \tau_S$. Thus Lemma 9.1 holds with $\underset{\sim}{T}$ replaced by $\underset{\sim}{M}$, and so it makes sense to ask whether conjecture CM is true for $\underset{\sim}{M}$.

<u>Remark 18.2</u>. Let extensions $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S correspond to families of maps $\bar{\beta}_1^L, \bar{\beta}_2^L : \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)/S^L(L)$. Then $\bar{\beta}_1^L = \bar{\beta}_2^L$ for all L if and only if $\underset{\sim}{M}_1 \approx \underset{\sim}{M}_2$ in the sense that there is an isomorphism $\psi : \underset{\sim}{M}_1 \to \underset{\sim}{M}_2$ of pro-algebraic groups such that:

(i)
$$\begin{array}{ccccccccc} 1 \to & S & \to & \underset{\sim}{M}_1 & \to & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \to 1 \\ & \| & & \downarrow\psi & & \| & \\ 1 \to & S & \to & \underset{\sim}{M}_2 & \to & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \to 1 \end{array}$$

commutes; and

(ii)
$$\begin{array}{ccc} \underset{\sim}{M}_{1\mathbb{A}^f} & \xleftarrow{sp_1} & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow\psi & & \| \\ \underset{\sim}{M}_{2\mathbb{A}^f} & \xleftarrow{sp_2} & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \end{array}$$

commutes.

There exists an isomorphism ψ satisfying (i), but not necessarily (ii), if and only if, for each L, $\bar{\beta}_1^L = \bar{\beta}_2^L \, \bar{e}^L$ where $\bar{e}^L$ lifts to a continuous map $e^L : \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow S^L(\mathbb{A}^f)$ satisfying $e(\tau_1\tau_2) = \tau_2^{-1}e(\tau_1).\, e(\tau_2)$.

<u>Proposition 18.3</u>. If, in the statement of conjecture CM, $\underset{\sim}{T}$ is replaced by $\underset{\sim}{M}$, then the conjecture becomes true. Conversely let $\underset{\sim}{M}_1$ be an extension of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S such that, for each $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the inverse image of τ in $\underset{\sim}{M}_1$ is isomorphic to ${}^{\tau}S$ as an S-torsor; if conjecture CM is true when T is replaced by $\underset{\sim}{M}_1$, then $\underset{\sim}{M}_1 \approx \underset{\sim}{M}$.

<u>Proof</u>. The first assertion is obvious; in fact our aim in twisting T was to define a group relative to which conjecture CM was true.

For the second assertion, note that conjecture CM is equivalent to the following statement: let L be a finite Galois extension of $\mathbb{Q}$ that splits T_A; then there exists an isomorphism $f : (H_1(\tau A, L), (\tau s_\alpha)) \xrightarrow{\approx} (V \otimes L, (s_\alpha))$ such that

$$\begin{array}{ccc} V^f(A) \otimes L & \xrightarrow{\tau} & V^f(\tau A) \otimes L \\ \| & & \downarrow 1\otimes f \\ V(\mathbb{A}_L^f) & \xrightarrow{\beta(\tau)} & V(\mathbb{A}_L^f) \end{array}$$

commutes. Thus if conjecture CM is true with $\underset{\sim}{M}_1$ then the map $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \xrightarrow{\bar{\beta}_1^L} S^L(\mathbb{A}_L^f)/S^L(L) \xrightarrow{\rho_\mu} T_A(\mathbb{A}_L^f)/T_A(L)$, where $\bar{\beta}_1^L$ corresponds to $\underset{\sim}{M}_1$, is equal to the same map defined by

$\tilde{\beta}^{-L}$. On passing to the inverse limit over A we find that $\beta_1^{-L} = \tilde{\beta}^{-L}$, and so $\underset{\sim}{M}_1 \approx \underset{\sim}{M}$.

Much of chapters II and III of this paper continues to hold if $\underset{\sim}{T}$ is replaced by $\underset{\sim}{M}$. In particular, maps $\tilde{\phi}(\tau;\mu',\mu)$ are defined (see 7.12) and it makes sense to ask whether conjecture C is true for $\underset{\sim}{M}$. The maps $\ddot{\phi}(\tau;\mu)$ of §10 are not defined in the same generality because, in their definition, we have used that $b(\tau,\mu)$ is defined whenever (6.3) holds. The alternative definition (see 10.6, 10.10) is, however, valid and provides a map $\varphi(\tau;\mu)$ whenever μ satisfies (4.1). (This condition holds when (G,X) satisfies (2.1.1.4) and (2.1.1.5) of Deligne [3]; that is, when the weight $w = w_h$ is defined over $\mathbb{Q}$ and $\underset{\sim}{ad}\, h(i)$ is a Cartan involution on $(G/w(\mathbb{G}_m))_{\mathbb{R}}$.)

<u>Theorem 18.4</u>. Conjecture C becomes true when $\underset{\sim}{T}$ is replaced by M. (Recall: we are only considering Shimura varieties of abelian type.)

<u>Proof</u>. As in (10.17) one proves that conjecture CM implies that conjecture C is true for Shimura varieties of the form $Sh(CSp(V),S^{\pm})$, and as in (17.8) that this implies that conjecture C is true for all Shimura varieties.

<u>Corollary 18.5</u>. Conjecture A is true (for Shimura varieties of abelian type).

Proof. This is an immediate consequence of (18.4) - see the discussion preceeding the statement of conjecture A in §7.

Corollary 18.6. Conjecture B is true for $Sh(G,X)$ provided the weight $w = w_h$ of any $h \in X$ is defined over $\mathbb{Q}$ and $\underset{\sim}{ad}\, h(i)$ is a Cartan involution on $(G/w(\mathbb{G}_m))_{\mathbb{R}}$ (and (G,X) is of abelian type).

Proof. We are, of course, assuming also that $E(G,X) \subset \mathbb{R}$. The hypotheses imply that, for any special $h \in X$, the cocharacter $\mu = \mu_h$ induces a map $\rho_\mu : S \to G$. We can substitute $\rho_\mu(\tilde{\beta}(\tau)$ for the element denoted by $\tilde{\beta}(\tau,\mu)$ in the proof of (10.15). In order to be able to apply the same argument as in that proof we have to show that $\tilde{\beta}(\iota) \in S^L(L)$. But conjecture CM is true for $\tau = \iota$ (cf. 10.19) and so $\bar{e}(\iota) = 1$. Thus $\bar{\bar{\beta}}(\iota) = \bar{\beta}(\iota)\ \bar{e}(\iota) = 1$ in $S^L(\mathbb{A}^f_L)/S^L(L)$.

Remark 18.7. There is a good reason why conjecture B is easy to prove under the hypotheses of (18.6): these hypotheses should imply that $Sh(G,X)$ is moduli variety for motives. (Cf. 7.4c)

Remark 18.8. Theorem (18.4) together with the proof of (10.14) show that $Sh(G,X)$ has a canonical model whenever (G,X) satisfies the conditions of (18.6). Presumably, if one defined maps $\Phi(\tau;\mu)$ (using $\underset{\sim}{M}$) for all Shimura varieties then one would recover the main theorem of Deligne [3], but there seems little point to this. (Except that, curiously, it would give a proof that does not involve $\mathbb{E}_E(G,G',X^+)$.)

Deligne has conjectured the following:

Conjecture D. The Taniyama group $\underset{\sim}{T}$ is isomorphic (in the sense of 18.2) to $\underset{\sim}{M}$.

He also suggested that this conjecture should be equivalent to Langland's conjecture C. We prove:

Proposition 18.9. Conjecture D is true if and only if conjecture C is true (for all Shimura varieties of abelian type).

Proof. If $\underset{\sim}{T} \approx \underset{\sim}{M}$ then (18.4) shows that conjecture C is true. Conversely assume that conjecture C is true for all Shimura varieties of the form $Sh(CSp(V), S^{\pm})$. Then (10.17) shows that conjecture CM is true, and (18.3) shows that $\underset{\sim}{T} \approx \underset{\sim}{M}$.

Remark 18.10. Deligne has shown [6] that if $\underset{\sim}{M}_1$ and $\underset{\sim}{M}_2$ are extensions of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ by S that give rise to the same S-torsors as $\underset{\sim}{T}$ for each $\tau \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, then there exists an isomorphism $\psi : \underset{\sim}{M}_1 \to \underset{\sim}{M}_2$ satisfying (i) of (18.2) (but not necessarily (ii)). Thus there exists a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S & \rightarrow & \underset{\sim}{T} & \longrightarrow & Gal(\bar{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 1 \\ & & \| & & \downarrow\approx & & \| & & \\ 1 & \longrightarrow & S & \longrightarrow & \underset{\sim}{M} & \longrightarrow & Gal(\bar{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 1 \end{array}$$

What remains to be shown is that the isomorphism can be chosen to carry sp into $\tilde{sp}$.

Let L be a finite Galois extension of $\mathbb{Q}$, and let K be a subfield of L. We shall write ${}_K\underset{\sim}{T}^L$ and ${}_K\underset{\sim}{M}^L$ for the pull-backs of $\underset{\sim}{T}^L$ and $\underset{\sim}{M}^L$ relative to $\mathrm{Gal}(L^{ab}/K) \hookrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q})$. Assume that L is a CM-field. If A is an abelian variety of CM-type whose Mumford-Tate group is split by L, then the reflex field of A is contained in L, and the main theorem of complex multiplication shows that comjecture CM is true for A and all τ fixing L (cf. 9.2b). Thus an obvious variant of (18.3) shows that ${}_L\underset{\sim}{T}^L \approx {}_L\underset{\sim}{M}^L$ (by an isomorphism preserving all structure including the splittings). Since conjecture CM is known to be true for $\tau = \iota$, it is also true for any τ fixing the maximal totally real subfield K of F; thus ${}_K\underset{\sim}{T}^L \approx {}_K\underset{\sim}{M}^L$. The results of §15 often allow one to replace K in this isomorphism by a subfield of L over which L has degree 4. In particular $\underset{\sim}{T}^L \approx \underset{\sim}{M}^L$ if L is the composite of two quadratic imaginary extensions of $\mathbb{Q}$.

§19. Motives.

Since the theory of motives has helped suggest a good part of the work described in the previous eighteen sections, we feel we should include a brief description of this theory.

Fix a field k. A k-linear tensor category is a k-linear category $\underset{\sim}{C}$ provided with a k-bilinear functor

$$(X,Y) \mapsto X \otimes Y : \underset{\sim}{C} \times \underset{\sim}{C} \longrightarrow \underset{\sim}{C}$$

and

(19.1a) functorial isomorphisms $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\approx} (X \otimes Y) \otimes Z$ for $X,Y,Z \in Ob(\underset{\sim}{C})$ (an associativity constraint),

(19.1b) functorial isomorphisms $\psi_{X,Y} : X \otimes Y \xrightarrow{\approx} Y \otimes X$ for $X,Y \in Ob(\underset{\sim}{C})$ (a commutativity constraint), and

(19.1c) an object 1 and functorial isomorphisms $\ell_X : X \xrightarrow{\approx} 1 \otimes X$, $r_X : X \xrightarrow{\approx} X \otimes 1$ for $X \in Ob(\underset{\sim}{C})$ (an identity constraint), all of which satisfy certain natural compatibility conditions (Saavedra [1, I2. 4.1]).

Let $\underset{\sim}{C}$ and $\underset{\sim}{C}'$ be k-linear tensor categories. A pair (F,c) comprising a k-linear functor $F : \underset{\sim}{C} \to \underset{\sim}{C}'$ and a functorial isomorphism $c_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\approx} F(X \otimes Y)$ is a tensor functor if it is compatible with the constraints on $\underset{\sim}{C}$ and $\underset{\sim}{C}'$ (Saavedra [1, I4.2]). If F and F' are two such functors then a morphism of k-linear functors $\lambda : F \longrightarrow F'$ is a tensor morphism if $\lambda_1 : F(1) \longrightarrow F'(1)$ is an isomorphism and if the diagrams

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{c_{X,Y}} & F(X\otimes Y) \\ \downarrow \lambda_X \otimes \lambda_Y & & \downarrow \lambda_{X\otimes Y} \\ F'(X) \otimes F'(Y) & \xrightarrow{c'_{X,Y}} & F'(X\otimes Y) \end{array}$$

commute. We write $\mathrm{Hom}_k^{\otimes}(F,F')$ for the set of such functors.

Examples 19.2. (a) The category $\underset{\sim}{\mathrm{Vec}}_k$ of finite-dimensional vector spaces over k is, in a natural way, a k-linear tensor category.

(b) Let G be an affine group scheme over k. The category $\underset{\sim}{\mathrm{Rep}}_k(G)$ of representations of G on finite dimensional vector spaces over k is a k-linear tensor category. The forgetful functor $\omega^G : \underset{\sim}{\mathrm{Rep}}_k(G) \longrightarrow \underset{\sim}{\mathrm{Vec}}_k$ is a tensor functor.

Let $(C,\otimes)$ be a k-linear tensor category and ω a tensor functor $\underset{\sim}{C} \to \underset{\sim}{\mathrm{Vec}}_k$. We say that $(C,\otimes)$ is Tannakian with fibre functor ω if there is an affine group scheme G over k and a tensor functor $F : \underset{\sim}{C} \to \underset{\sim}{\mathrm{Rep}}_k(G)$ that is an equivalence of categories and is such that ω is the composite $\underset{\sim}{C} \xrightarrow{F} \underset{\sim}{\mathrm{Rep}}_k(G) \xrightarrow{\omega^G} \underset{\sim}{\mathrm{Vec}}_k$. For conditions implying that a tensor category is Tannakian, see Saavedra [1].

Remark 19.3. For G an affine group scheme over k and A a k-algebra, let $\omega^G \otimes A$ be the functor $V \mapsto \omega^G(V) \otimes A$. Then G represents the functor of k-algebras $A \mapsto \mathrm{Aut}_A^{\otimes}(\omega^G \otimes A)$. Thus if $(\underset{\sim}{C},\otimes)$ is Tannakian with fibre functor ω, the group G corresponding to $(\underset{\sim}{C},\otimes,\omega)$ represents $A \mapsto \mathrm{Aut}_A^{\otimes}(\omega \otimes A)$.

We now assume that k is a subfield of $\mathbb{C}$ and let $\bar{k}$ be its algebraic closure in $\mathbb{C}$. The category of smooth projective (not necessarily connected) varieties over k, with the usual notion of morphism, will be denoted by $\underset{\sim}{V}(k)$. We write H_B for the functor $X \mapsto \oplus H^i(X(\mathbb{C}),\mathbb{Q})$ from $\underset{\sim}{V}(k)$ to graded vector spaces over $\mathbb{Q}$. We refer to Deligne [4, p. 317] for the notion of an absolute Hodge cycle of codimension p. Let $C^p_{AH}(X)$ be the $\mathbb{Q}$-vector space of all such cycles on X, and define $\mathrm{Mor}^p_{AH}(Y,X) = C^{p+\dim(Y)}_{AH}(X\times Y)$. There is a canonical map $\mathrm{Hom}(X,Y) \hookrightarrow \mathrm{Mor}^0_{AH}(Y,X)$ that sends a morphism $\phi: X \longrightarrow Y$ to the class of its graph Γ_ϕ.

The standard constructions (see, for example, Saavedra [1, VI.4]) show that there exists a triple $(\underset{\sim}{\mathrm{Mot}}_{AH}(k), \otimes, \omega)$, the <u>category of motives for absolute Hodge cycles</u>, and a contravariant functor $h : \underset{\sim}{V}(k) \longrightarrow \underset{\sim}{\mathrm{Mot}}_{AH}(k)$ with the following properties (which determine them uniquely).

(19.4) $(\underset{\sim}{\mathrm{Mot}}_{AH}(k), \otimes, \omega)$ is a Tannakian category; each object of $\underset{\sim}{\mathrm{Mot}}_{AH}(k)$ is graded by $\mathbb{Z}$, and this grading is compatible with tensor products; for any X in $V(k)$, $\omega(h(X)) = H_B(X)$ (as graded vector spaces); for any X and Y in $V(k)$, $h(X\times Y) = h(X)\otimes h(Y)$.

Let $L = h(\mathbb{P}^1)^2$, since $\omega(L) = H^2(\mathbb{P}^1,\mathbb{Q})$ is one-dimensional, there is an inverse object T in $\underset{\sim}{\mathrm{Mot}}_{AH}(k)$, which is called the Tate motive. For any $i \in \mathbb{Z}$ and $M \in \underset{\sim}{\mathrm{Mot}}_{AH}(k)$, write $M(i) = M \otimes T^{\otimes i}$.

(19.5) For any X,Y in $V(k)$, $\mathrm{Mor}^{j-i}_{AH}(X,Y) = \mathrm{Hom}(h(X)(i),h(Y)(j))$ in particular, $\mathrm{Mor}^0_{AH}(X,Y) = \mathrm{Hom}(h(X),h(Y))$.

If $\pi \in \mathrm{Mor}^0_{AH}(X,X)$ satisfies $\pi^2 = \pi$, we write $(X,\pi) = \mathrm{Ker}(\pi: h(X) \longrightarrow h(X))$. The category of <u>effective motives</u> $\underset{\sim}{\mathrm{Mot}}^+_{AH}(k)$ is defined to be the full subcategory of $\underset{\sim}{\mathrm{Mot}}_{AH}(k)$ whose objects are isomorphic to (X,π) for some X in $\underset{\sim}{V}(k)$.

(19.6) Any object in $\underset{\sim}{\mathrm{Mot}}_{AH}(k)$ is of the form $(X,\pi)(i)$ for some (X,π) in $\underset{\sim}{\mathrm{Mot}}^{+}_{AH}(k)$ and $i \in \mathbb{Z}$.

(19.7) For X in $\underset{\sim}{V}(k)$, let $id_X = \Sigma\, \pi^i_X \in C^{\dim(X)}_{AH}(X \times X)$; then $h(X)^i = (X,\pi^i_X)$.

(19.8) The constraints on $\underset{\sim}{\mathrm{Mot}}_{AH}(k)$ are determined by the following conditions:

for X,Y,Z in $\underset{\sim}{V}(k)$,

$\phi_{X,Y,Z} : h(X) \otimes (h(Y) \otimes h(Z)) \longrightarrow (h(X) \otimes h(Y)) \otimes h(Z)$ is obtained by applying h to the natural map $(X\times Y)\times Z \longrightarrow X\times(Y\times Z)$;

for X,Y in $\underset{\sim}{V}(k)$, let $\dot{\psi} = \oplus\, \dot{\psi}^{p,q}$ be the map obtained by applying h to the natural map $Y \times X \longrightarrow X \times Y$; then

$$\psi_{X,Y} : h(X) \otimes h(Y) \longrightarrow h(Y) \otimes h(X) \quad \text{is} \quad \oplus\, (-1)^{pq}\, \dot{\psi}^{p,q} ;$$

$1 = \mathrm{spec}\, k$, and $\ell_X : h(X) \longrightarrow 1 \otimes h(X)$ and $r_X : h(X) \longrightarrow h(X) \otimes 1$ are obtained by applying h to the natural maps $\mathrm{spec}(k) \times X \longrightarrow X$ and $X \times \mathrm{spec}(k) \longrightarrow X$.

(19.9) $\underset{\sim}{\mathrm{Mot}}_{AH}(k)$ is a semisimple category.

Example 19.10. Let $\underset{\sim}{V}^{0}(k)$ be the category of zero-dimensional smooth varieties over k. Define $(\underset{\sim}{\mathrm{Mot}}^{0}_{AH}(k), \otimes, \omega)$ to be the category $\underset{\sim}{\mathrm{Rep}}_{\mathbb{Q}}(\mathrm{Gal}(\bar{k}/k))$ of continuous finite-dimensional $\mathbb{Q}$-representations of $\mathrm{Gal}(\bar{k}/k)$, together with the obvious tensor product and fibre functor. This triple, with the functor $h = (X \mapsto \mathrm{Hom}(X(\bar{k}),\mathbb{Q}))$ satisfies (analogues of) the above conditions, and hence is the category of zero-dimensional motives.

Now let L be a finite Galois extension of $\mathbb{Q}$ and let A be an abelian variety over $\mathbb{Q}$ that is of potential CM-type and whose Mumford-Tate group is split by L. Write $\underset{\sim}{\mathrm{Mot}}^A(\mathbb{Q})$ for the Tannakian subcategory of $\underset{\sim}{\mathrm{Mot}}_{\mathrm{All}}(\mathbb{Q})$ generated by A, the Tate motive, and all zero-dimensional varieties over $\mathbb{Q}$ that are split by L^{ab}. We define $\underset{\sim}{M}^A$, the <u>motivic Galois group for A</u>, to be the $\mathbb{Q}$-rational affine group scheme associated with this category. The embedding of the zero-dimensional motives into $\underset{\sim}{\mathrm{Mot}}^A(\mathbb{Q})$ induces a surjection $\underset{\sim}{M}^A \to \mathrm{Gal}(L^{ab}/\mathbb{Q})$, and the map $\underset{\sim}{\mathrm{Mot}}^A(\mathbb{Q}) \to \underset{\sim}{\mathrm{Mot}}^A(\mathbb{C})$ induces an injection $MT(A) \hookrightarrow \underset{\sim}{M}^A$ (see Appendix A.3).

<u>Proposition 19.11</u>. The sequence

$$1 \to MT(A) \to \underset{\sim}{M}^A \to \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to 1$$

is exact; on passing to the inverse limit over A we obtain an exact sequence

$$1 \to S^L \to \underset{\sim}{M}^L \to \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to 1 \ ;$$

the action of $\mathrm{Gal}(L^{ab}/\mathbb{Q})$ on S^L arising from this sequence is the algebraic action (4.8).

<u>Proof</u>. See Deligne [5].

It is not difficult to obtain the following description of the points of $\underset{\sim}{M}^A$ in a $\mathbb{Q}$-algebra R : an element of $\underset{\sim}{M}^A(R)$ is given by a pair (f,τ) where $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ and f is an an isomorphism $(H_1(A,R),\ (s_\alpha)) \xrightarrow{\approx} (H_1(\tau A,R),\ (\tau s_\alpha))$, (s_α) being the family of all Hodge cycles on A. It follows that there is a canonical element $\tilde{sp}(\tau) \in \underset{\sim}{M}^A(\mathbb{A}^f)$, namely $H_1(A,\mathbb{A}^f) = V^f(A) \xrightarrow{\tau} V^f(\tau A) = H_1(\tau A,\mathbb{A}^f)$. Thus the extensions in (19.11) have a canonical splitting over $\mathbb{A}^f$. Since everything is compatible for varying L we obtain an extension $\underset{\sim}{M}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S, together with a splitting $\tilde{sp}$ over $\mathbb{A}^f$, in the sense of §5.

<u>Proposition 19.12</u>. The above definition of $\underset{\sim}{M}$ and $\tilde{sp}$ agrees with that in §18.

<u>Proof</u>. According to (18.3) we only have to show that conjecture CM becomes true when $\underset{\sim}{T}$ is replaced by the above $\underset{\sim}{M}$. It is clear from the above description of $\underset{\sim}{M}^A(R)$ that we can identify $({}^\tau V, ({}^\tau s_\alpha))$ with $(H_1(\tau A,\mathbb{Q}),\ (\tau s_\alpha))$. Thus the diagram in conjecture CM (first form) is

$$\begin{array}{ccc} V^f(A) & \xrightarrow{\tau} & V^f(\tau A) \\ \| & & \| \\ H_1(A,\mathbb{A}^f) & \xrightarrow{\tau} & H_1(\tau A,\mathbb{A}^f) \end{array}$$

Even the most weary reader will be able to observe that this commutes.

Appendix A. Hodge structures; Mumford-Tate groups.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. We usually identify $\mathbb{S}_{\mathbb{C}}$ with $\mathbb{G}_m \times \mathbb{G}_m$ through the isomorphism such that $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) \approx \mathbb{C}^\times \times \mathbb{C}^\times$ is $z \longmapsto (z, \iota z)$. Then ι acts on $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ by $\iota(z_1, z_2) = (\iota z_2, \iota z_1)$. There is a homomorphism $w : \mathbb{G}_m \to \mathbb{S}$ that is the inclusion $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on real points and the diagonal map on complex points. For any algebraic group G over $\mathbb{R}$ there is a one-one correspondence $h \leftrightarrow \mu_h$ between homomorphisms $h : \mathbb{S} \to G$ and homomorphisms $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$ for which μ commutes with $\iota\mu$; given h one defines $\mu(z) = h_{\mathbb{C}}(z,1)$, and given μ one defines $h_{\mathbb{C}}(z_1,z_2) = \mu(z_1).(\iota\mu)(z_2)$.

A <u>$\mathbb{Q}$-rational Hodge structure</u> on a vector space V over $\mathbb{Q}$ is a homomorphism $h : \mathbb{S} \to GL(V_{\mathbb{R}})$ whose weight, $w_h = (z \longmapsto h\ w(z^{-1})) : \mathbb{G}_m \longrightarrow GL(V_{\mathbb{R}})$, is defined over $\mathbb{Q}$. Then $V_{\mathbb{C}} = \oplus V^{p,q}$, where $V^{p,q} = \{v \in V_{\mathbb{C}} \mid h(z)v = z^{-p}(\iota z)^{-q}v\}$, and $V = \oplus V^n$, where $V^n = \{v \in V \mid w_h(r)v = r^n v\}$; note that $V^n_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$. An element $v \in V_{\mathbb{C}}$ is of <u>bidegree</u> (p,q) if it lies in $V^{p,q}$, and it is <u>rational</u> if it lies in $V \subset V_{\mathbb{C}}$. The vector space $(2\pi i)^n\mathbb{Q}$, together with the unique $\mathbb{Q}$-rational Hodge structure such that $((2\pi i)^n\mathbb{Q}) \otimes \mathbb{C} = ((2\pi i)^n\mathbb{Q})^{-n,-n}$, will be denoted by $\mathbb{Q}(n)$; thus $h(z)v = (N_{\mathbb{C}/\mathbb{R}}z)^n v$ for $v \in \mathbb{Q}(n) \otimes \mathbb{R}$. A $\mathbb{Q}$-rational Hodge structure on V induces a similar structure on the dual space $\check{V}$, and on any object $(V^{\otimes n_1} \otimes \check{V}^{\otimes n_2})(n_3) \overset{df}{=} V^{\otimes n_1} \otimes \check{V}^{\otimes n_2} \otimes \mathbb{Q}(1)^{\otimes n_3}$ of the tensor category generated by V, $\check{V}$, and $\mathbb{Q}(1)$. <u>A morphism of Hodge structures</u> $\phi : (V_1,h_1) \longrightarrow (V_2,h_2)$ is a $\mathbb{Q}$-linear map $\phi : V_1 \longrightarrow V_2$ such that $\phi_{\mathbb{R}}$ commutes with the actions of $\mathbb{S}$. The canonical map $\mathrm{Hom}(V_1,V_2) \xrightarrow{\approx} \check{V}_1 \otimes V_2$ identifies such morphisms with rational elements of $(\check{V}_1 \otimes V_2)_{\mathbb{C}}$ of bidegree $(0,0)$.

Let (V,h) be a $\mathbb{Q}$-rational Hodge structure of weight n, i.e. such that $V = V^n$. A polarization of (V,h) is a morphism of Hodge structures $\psi : V \otimes V \longrightarrow \mathbb{Q}(-n)$ such that $(x,y) \longmapsto (2\pi i)^n \psi(x,h(i)y) : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \longrightarrow \mathbb{R}$ is symmetric and positive definite. The category of polarizable $\mathbb{Q}$-rational Hodge structures is a semisimple Tannakian category (Saavedra [1, VI.2]).

For any abelian variety A over $\mathbb{C}$, there is a unique $\mathbb{Q}$-rational Hodge structure h on $V = H_1(A,\mathbb{Q})$ of weight -1 and such that $V^{0,-1}$ is the kernel of the exponential map $H_1(A,\mathbb{Q}) \otimes \mathbb{C} \longrightarrow \mathrm{Lie}(A)$. Any Riemann form for A defines a polarization of (V,h).

The map $A \longmapsto (H_1(A,\mathbb{Q}),h)$ defines an equivalence between the category of abelian varieties over $\mathbb{C}$ (up to isogeny) and the category of polarizable $\mathbb{Q}$-rational Hodge structures of weight -1.

Let (V,h) be a polarizable $\mathbb{Q}$-rational Hodge structure. The Mumford-Tate group $MT(V,h)$ of (V,h) can be described variously as follows:

(A.1) the smallest $\mathbb{Q}$-rational subgroup G of $GL(V) \times \mathbb{G}_m$ such that $G_{\mathbb{C}}$ contains the image of $\binom{\mu_h}{1} : \mathbb{G}_m \longrightarrow GL(V_{\mathbb{C}}) \times \mathbb{G}_m$;

(A.2) the smallest $\mathbb{Q}$-rational subgroup G' of $GL(V) \times \mathbb{G}_m$ such that $G'_{\mathbb{R}}$ contains the image of $\binom{h}{N_{\mathbb{C}/\mathbb{R}}} : \mathbb{S} \longrightarrow GL(V_{\mathbb{R}}) \times \mathbb{G}_m$;

(A.3) the affine group scheme G'' over $\mathbb{Q}$ associated with the Tannakian category of Hodge structures generated by V and $\mathbb{Q}(1)$ (cf. Saavedra [1]);

(A.4) the subgroup G''' of $GL(V) \times \mathbb{G}_m$ fixing all rational tensors of bidegree $(0,0)$ in spaces of the form $V^{\otimes n_1} \otimes \check{V}^{\otimes n_2}(n_3)$, $n_1, n_2 \in \mathbb{N}$, $n_3 \in \mathbb{Z}$. Indeed, it is clear that $G \subset G' \subset G'' \subset G'''$ and a standard result (Serre [2, Lemme 1]) shows that $G = G'''$.

The group $MT(V,h)$ is reductive. If it is commutative, and hence a torus, then (V,h) is said to be of CM-type.

Assume that (V,h) has weight -1, and let R be the centralizer of $h(\mathbb{S})$ in End (V), so that $R = \mathrm{End}(V,h)$. A polarization ψ of (V,h) induces a positive involution $*$ on R by the rule: $\psi(au,v) = \psi(u,a^*v)$; thus R is semi-simple. It is easily seen that (V,h) is of CM-type if and only if V is generated as an R-module by a single element or, equivalently, R contains a commutative étale subalgebra F such that $[F:\mathbb{Q}] = \dim V$; such an F will be a product of CM-fields. (Cf. Mumford [1, §2]). Let A be the abelian variety over $\mathbb{C}$ associated with (V,h). We often refer to $MT(V,h)$ as the Mumford-Tate group $MT(A)$ of A. Note that $R = \mathrm{End}(A)$. Clearly A is of CM-type (meaning that there is a product F of CM-fields acting on A in such a way that $H_1(A,\mathbb{Q})$ is a free F-module of rank 1) if and only if (V,h) is of CM-type. In this case $MT(A)$ is the image of

$$NR(\mu) : \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}}\mu} \mathrm{Res}_{E/\mathbb{Q}}\, T_E \xrightarrow{\mathrm{Norm}} T$$

where $T = \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_m$, $\mu = \mu_h : \mathbb{G}_m \to T_{\mathbb{C}} \subset GL(V_{\mathbb{C}})$, and E is the field of definition of μ (i.e. the reflex field for $Sh(T,\{h\})$) .

If A is an abelian variety defined over a subfield L of $\mathbb{C}$, then we write $MT(A)$ for $MT(A_{\mathbb{C}})$, but note that this depends on the embedding of L in $\mathbb{C}$. We say A is of <u>potential CM-type</u> if $A_{\mathbb{C}}$ is of CM-type.

Let (V,h) and (W,h) be $\mathbb{Q}$-rational Hodge structures and let $(s_\alpha)_{\alpha \in J}$ and $(t_\alpha)_{\alpha \in J}$ be tensors occurring in spaces of the form $V^{\otimes m} \otimes \check{V}^{\otimes n}(p)$ and $W^{\otimes m} \otimes \check{W}^{\otimes n}(p)$ respectively. Consider a $\mathbb{Q}$-algebra R and a pair (f,λ) where f is an isomorphism $V(R) \xrightarrow{\approx} W(R)$ and $\lambda \in R^\times$. If the maps

$$V(R)^{\otimes m} \otimes \check{V}(R)^{\otimes n}(p) \xrightarrow{\approx} W(R)^{\otimes m} \otimes \check{W}^{\otimes n}(R)\,(p)$$

defined by (f,λ) send s_α to t_α for each $\alpha \in J$, then we say that (f,λ) is an <u>isomorphism</u> $(V,(s_\alpha))\otimes R \xrightarrow{\approx} (W,(t_\alpha))\otimes R$. We also refer loosely to (f,λ) as an isomorphism $f : V(R) \longrightarrow W(R)$ making s_α correspond to t_α for each α .

Appendix B.

We give a list of classical reductive group G_o such that (G_o, X_o) defines a Shimura variety for a suitable X_o, and such that (G_o^{ad}, G_o^{der}) is of primitive abelian type. Every (G, G', X^+) of primitive abelian type is of the form $(G_o^{ad}, G_o^{der}, X_o^+)$ with some (G_o, X_o) from the following list. These (G_o, X_o) all have the property $E(G_o, X_o) = E(G_o^{ad}, X_o^+)$.

In the following, F_o is a totally real number field, and I is the set of all embeddings of F_o into $\mathbb{R}$. We use $\bar{z}$ to denote the complex conjugate of $z \in \mathbb{C}$.

(A) Let K be a quadratic totally imaginary extension of F_o, and A a central simple algebra over K , together with an involution σ of the second kind. Then $\{x \in A^\times | x x^\sigma \in F_o^\times\}$ defines a reductive group G_* over F_o . We put $G_o = \mathrm{Res}_{F_o/\mathbb{Q}} G_*$. The center of G_o is $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$.

For non-negative integers r and s , we put

$$I_{r,s} = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} ,$$

and

$$GU(r,s) = \{g \in GL_{r+s}(\mathbb{C}) \mid g I_{r,s} {}^t\bar{g} = \nu(g) I_{r,s}, \nu(g) \in \mathbb{R}^\times\} .$$

Then for each $\nu \in I$, there are non-negative integers r_ν and s_ν such that

$$G_o(\mathbb{R}) \cong \Pi_\nu GU(r_\nu, s_\nu) .$$

Let $I_{nc} = \{\nu \in I \mid r_\nu \cdot s_\nu \neq 0\}$ and let I_c be the complement of I_{nc}. Define $h_\nu : \mathbb{S} \cong \mathbb{C}^\times \longrightarrow GU(r_\nu, s_\nu)$ by

$$h_\nu(z) = \begin{cases} \begin{pmatrix} zI_{r_\nu} & 0 \\ 0 & \bar{z}I_{s_\nu} \end{pmatrix} & \text{if } \nu \in I_{nc} \\ 1 & \text{if } \nu \in I_c . \end{cases}$$

and define $h_o : \mathbb{S} \longrightarrow G_o(\mathbb{R})$ to be the product of h_ν's. Let X_o be the $G_o(\mathbb{R})$-conjugacy class of h_o. Then (G_o, X_o) defines a Shimura variety. For any connected component X_o^+ of X_o, $(G_o^{ad}, G_o^{der}, X_o^+)$ is of type A.

The reflex field $E(G_o, X_o)$ is either $\mathbb{Q}$ or a CM-field. The former case happens if and only if $r_\nu = s_\nu$ for all $\nu \in I$. In this case the map η defined in 7.3 takes h_o to $h_o' = \Pi_\nu h_\nu'$, where

$$h_\nu'(z) = \begin{cases} \begin{pmatrix} \bar{z}I_r & 0 \\ 0 & zI_r \end{pmatrix}, \; r = r_\nu = s_\nu, & \text{if } \nu \in I_{nc} \\ 1, & \text{if } I_c . \end{cases}$$

(B) Let $n \geq 3$ be an odd integer and q a quadratic form on an n-dimensional vector space over F_o such that the signature of q at a $\nu \in I$ is $(n,0)$, $(0,n)$, $(n-2,2)$ or $(2,n-2)$. The special Clifford group of q defines a reductive group G_* over F_o. We put $G_o = \mathrm{Res}_{F_o/\mathbb{Q}} G_*$. The center of G_o is $\mathrm{Res}_{F_o/\mathbb{Q}} G_m$.

We refer to Shih [3] for the description of X_o such that (G_o,X_o) defines a Shimura variety. The reflex field $E(G_o,X_o)$ is totally real. The derived group G_o^{der} is the spin group of q . $(G_o^{ad},G_o^{der},X_o^+)$ is of type B for any connected component X_o^+ of X_o .

(C) G_o is the similitude group of a hermitian form over a quaternion algebra whose center is F_o , see §15.

($D^{\mathbb{R}}$) There are two cases:

(1) Same as type B , except $n \geq 4$ is even.

(2) Let B be a totally indefinite quaternion algebra over F_o and denote by σ the main involution of B . Let q be a σ-antihermitian form on a left free B-module of rank $n \geq 2$. At each $\tau \in I$, q defines a quadratic form on a 2n-dimensional real vector space. We assume that its signature is (2n,0) , (0,2n) , (2n-2,2) or (2,2n-2) . Let G_* be the algebraic group over F_o defined by the special Clifford group of q , and let $G_o = Res_{F_o/\mathbb{Q}} G_*$. We define X_o as in Shih [3]. Then (G_o,X_o) defines a Shimura variety and (G_o^{ad},G_o^{der}) is of type $D^{\mathbb{R}}$.

In both cases $E(G_o,X_o)$ is totally real, and the center of G_o is $Res_{F_o/\mathbb{Q}} Z_*$, where Z_* is an extension of μ_2 by $\mathbb{G}_m$ over F_o .

($D^{\mathbb{H}}$) Let B be a quaternion algebra over F_o with main involution σ . Let q be a σ-anti-hermitian form on a free left B-module Λ of rank $n \geq 4$. Let I_{nc} be the set of $\tau \in I$ where B does not split, and let I_c be the complement of I_{nc} . As usual , we assume I_{nc} is non-empty; let r be its cardinality.

We assume also that at every $\tau \in I_c$, the real quadratic form defined by q is definite. Let G_o be the algebraic group over $\mathbb{Q}$ such that

$$G_o(\mathbb{Q}) = \{g \in GL_B(\Lambda) \mid gq^tg^\sigma = \nu(g)q, \nu(g) \in F_o^\times \text{ and } N(g) = \nu(g)^n\} ,$$

where N denotes the reduced norm from $\mathrm{End}_B(\Lambda)$ to F_o. Then $G_o(\mathbb{R})$ is isomorphic to the product of r copies of $GO^*(2n)$, where

$$GO^*(2n) = \left\{ g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in GL_{2n}(\mathbb{C}) \;\middle|\; g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} {}^t\bar{g} = \nu(g) \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \nu(g) \in \mathbb{R}^\times \text{ and } \det(g) = \nu(g)^n \right\}$$

Define $h_o : \mathbb{S} \cong \mathbb{C}^\times \longrightarrow G_o(\mathbb{R}) \cong (GO^*(2n))^r$ so that each component of h_o is given by

$$z \longmapsto \begin{pmatrix} zI_n & 0 \\ 0 & \bar{z}I_n \end{pmatrix} ,$$

and define X_o to be the $G_o(\mathbb{R})$-conjugacy class of h_o. Then (G_o, X_o) defines a Shimura variety. The center of G_o is $\mathrm{Res}_{F_o/\mathbb{Q}} G_m$.

The reflex field $E(G_o, X_o)$ is either a CM-field or a totally real field, depending on whether n is odd or even. Let $h_o' : \mathbb{S} \cong \mathbb{C}^\times \longrightarrow G_o(\mathbb{R}) \cong (GO^*(2n))^r$ be a homomorphism such that each component of h'_o is given by

$$z \mapsto \begin{pmatrix} \bar{z}I_n & 0 \\ & \\ 0 & zI_n \end{pmatrix}.$$

Then h'_o belongs to X_o if and only if n is even. In this case the map η defined in 7.3 takes h_o to h'_o.

When $n = 4$, we also allow G_o of the "mixed type". We let I_c be the set of $\tau \in I$ such that B splits at τ and the quadratic form over $\mathbb{R}$ determined by q at τ is definite. Denote the complement of I_c by I_{nc}. Let s (resp. r) be the number of $\tau \in I_{nc}$ at which B splits (resp. does not split). We assume $r > o$. If B splits at a $\tau \in I_{nc}$, we assume that the signature of the real quadratic form determined by q at τ is $(6,2)$ or $(2,6)$. Then

$$G_o(\mathbb{R}) \cong (GO^*(8))^r \times (GO(6,2)^+)^s ,$$

where

$$GO(6,2)^+ = \left\{ g \in GL_8(\mathbb{R}) \,|\, g\begin{pmatrix} I_6 & 0 \\ 0 & -I_2 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} I_6 & 0 \\ 0 & -I_2 \end{pmatrix}, \right.$$
$$\left. \nu(g) \in \mathbb{R}^\times \text{ and } \det g > o \right\} .$$

We define $h_o : \mathbb{S} \to G_o(\mathbb{R})$ componentwise: the homomorphism into the component $GO^*(8)$ is defined as before and the homomorphism into the component $GO(6,2)^+$ is given by

$$z \mapsto \begin{pmatrix} |z|^2 I_6 & & 0 & \\ & & & \\ & & \mathrm{Re}\, z^2 & \mathrm{Im}\, z^2 \\ 0 & & -\mathrm{Im}\, z^2 & \mathrm{Re}\, z^2 \end{pmatrix}$$

Let X_o be the $G_o(\mathbb{R})$-conjugacy class of h_o. Then (G_o, X_o) defines a Shimura variety.

The reflex field $E(G_o, X_o)$ is totally real. Let h_o' be the image of h_o under the map η of 7.3. Then the component of h_o' corresponding to the factor $GO^*(8)$ is

$$z \to \begin{pmatrix} \bar{z} I_4 & 0 \\ 0 & z I_4 \end{pmatrix},$$

and to the factor $GO(6,2)^+$, it is

$$z \mapsto \begin{pmatrix} |z|^2 I_6 & & 0 & \\ & & & \\ & & \mathrm{Re}\, z^2 & -\mathrm{Im}\, z^2 \\ 0 & & \mathrm{Im}\, z^2 & \mathrm{Re}\, z^2 \end{pmatrix}.$$

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