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III. LANGLANDS'S CONSTRUCTION OF THE TANIYAMA GROUP

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Introduction: In this article we give a detailed description of Langlands's construction of his Taniyama group. The first section reviews the definition and properties of the Serre group, and the following section discusses extensions of Galois groups by the Serre group. The construction itself is carried out in the third section, which also contains additional material required for V.

We mention that in [1] Langlands is using the opposite sign convention for the reciprocity law in class field theory from us and hence the opposite notion of the Weil group (although his statement at the bottom of p. 224 is misleading on this point). Thus, there are many sign differences between his article and ours.

Notation: Vector spaces are finite-dimensional, number fields are of finite degree over $\mathbb{Q}$ (and usually contained in $\mathbb{C}$), and $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For L a number field, $L^{ab} \subset \bar{\mathbb{Q}}$ denotes its abelian closure. For the Weil group, we follow the notations of Tate [2]. In particular, for a topological group Γ, Γ^c denotes the closure of the commutator subgroup of Γ and $\Gamma^{ab} = \Gamma/\Gamma^c$.

§1. The Serre group.

Let $L \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}$, let Γ be the set of embeddings of L into $\mathbb{C}$, and write $L^\times$ for $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m$. Any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ defines an element $[\rho]$ of Γ, which may be regarded as a character of $L^\times$. Then Γ is a basis for $X^*(L^\times)$. An element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $X^*(L^\times)$ by $\sigma(\Sigma b_\rho[\rho]) = \Sigma b_\rho[\sigma\rho] = \Sigma b_{\sigma^{-1}\rho}[\rho]$. The quotient of $L^\times$ by the Zariski closure of any sufficiently small arithmetic subgroup has character group $X^*(L^\times) \cap (Y^o \oplus Y^-)$ where

$$Y^o = \{\chi \in X^*(L^\times) \otimes \mathbb{Q} \mid \sigma\chi = \chi \text{ , all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$$

$$Y^- = \{\chi \in X^*(L^\times) \otimes \mathbb{Q} \mid c\chi = -\chi, \text{ all } c \text{ of the form } c = \sigma\iota\sigma^{-1}\}$$

(Serre [1, II-31, Cor.1]). Thus this quotient is independent of the arithmetic subgroup; it is called the Serre group S^L of L (or, sometimes, the connected Serre group). One checks easily that $X^*(S^L)$ is the subgroup of $X^*(L^\times)$ of χ satisfying

$$(1.1) \qquad (\sigma-1)(\iota+1)\chi = 0 = (\iota+1)(\sigma-1)\chi \text{ , all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) .$$

There is a canonical homomorphism $h = h^L\colon \mathbb{S} \to S^L_{\mathbb{R}}$ and hence corresponding homomorphisms $w_h\colon \mathbb{G}_m \to S^L_{\mathbb{R}}$ and $\mu = \mu^L\colon \mathbb{G}_m \to S^L_{\mathbb{C}}$. They determine the following maps on the character groups:

$$X^*(h) = (\Sigma b_\rho[\rho] \longmapsto (b_1, b_\iota)\colon X^*(S^L) \longrightarrow X^*(\mathbb{S}) = \mathbb{Z} \oplus \mathbb{Z})$$

$$X^*(w_h) = (\Sigma b_\rho[\rho] \longmapsto -b_1 - b_\iota)$$

$$X^*(\mu) = (\Sigma b_\rho[\rho] \longmapsto b_1)$$

Note that w_h is defined over $\mathbb{Q}$. The pair (S^L, μ^L) is universal: for any $\mathbb{Q}$-rational torus T that is split over L and cocharacter μ of T satisfying (1.1) there is a unique $\mathbb{Q}$-rational homomorphism $S^L \xrightarrow{\rho_\mu} T$ such that $\rho_\mu \circ \mu^L = \mu$. In particular there are no nontrivial automorphisms of (S^L, μ^L).

For $\mathbb{C} \supset L' \supset L \supset \mathbb{Q}$ and L' of finite degree over $\mathbb{Q}$, the norm map induces a homomorphism $S^{L'} \to S^L$ sending $h^{L'}$ to h^L. The (connected) <u>Serre group</u> S is defined to be the pro-algebraic group $\varprojlim S^L$. There is a canonical homomorphism $h = h_{can} = \varprojlim h^L : \mathbb{S} \to S_{\mathbb{R}}$ and corresponding cocharacter $\mu = \mu_{can}: \mathbb{G}_m \to S_{\mathbb{C}}$. For any L, S^L is the largest quotient of S that splits over L.

We review the properties of S that we shall need to use.

(1.2). The topology induced on $S^L(\mathbb{Q})$ by the embedding $S^L(\mathbb{Q}) \hookrightarrow S^L(\mathbb{A}^f)$ is the discrete topology; thus $S^L(\mathbb{Q})$ is closed in $S^L(\mathbb{A}^f)$. This is a consequence of Chevalley's theorem, which says that any arithmetic subgroup of the $\mathbb{Q}$-rational points of a torus is open relative to the adèlic topology, because the subgroup $\{1\}$ of $S^L(\mathbb{Q})$ is arithmetic.

(1.3). Make $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on the group Λ of locally constant functions $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}$ by transport of structure: thus $(\sigma\lambda)(\rho) = \lambda(\sigma^{-1}\rho)$. The map $X^*(S^L) \to \Lambda$ that sends $\chi = \Sigma b_\rho[\rho]$ to the function $\rho \mapsto b_\rho$ identifies $X^*(S^L)$ with the subset Λ^L of Λ comprising those functions that are constant on left cosets of $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and satisfy (1.1). On passing to the limit over L, we find that $X^*(S)$ becomes identified with the subgroup of Λ of functions satisfying (1.1).

(1.4). Let $\mathbb{Q}^{cm}$ be the union of all subfields of $\bar{\mathbb{Q}}$ of CM-type; it is the largest subfield on which ι and σ commute for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The condition (1.1) is equivalent to the following conditions:

(1.1') λ is fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}^{cm})$ and $\lambda(\iota\sigma) + (\sigma)$ is independent of σ.

In particular, for a given L, $\Lambda^L \quad \Lambda^F$ where $F = L \quad \mathbb{Q}^{cm}$ is the maximal CM-subfield of L (or is $\mathbb{Q}$). Since obviously $\Lambda^L \quad \Lambda^F$, they must be equal: $S^L \xrightarrow{\approx} S^F$.

(1.5). (Deligne) Let F be a CM-field with maximal real subfield F_0. There is an exact commutative diagram (of algebraic groups)

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F^\times/F_0^\times & \xrightarrow{\approx} & S^F/hw(\mathbb{Q}^\times) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \approx & & F^\times & & S^F & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle hw} & & \\
1 & \longrightarrow & \mathrm{Ker} & \longrightarrow & F_0^\times & \xrightarrow{\text{norm}} & \mathbb{Q}^\times & \longrightarrow & 1 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

To prove this it suffices to show that the square at bottom-right commutes, and the top horizontal arrow is injective, but both of these are easily seen on the character groups. Thus there is an exact sequence

$$1 \longrightarrow F_0^\times \longrightarrow F^\times \times \mathbb{Q}^\times \longrightarrow S^F \longrightarrow 1 .$$

We can deduce that, for any field $k \supset \mathbb{Q}$, there is an injection $H^1(k,S^F) \hookrightarrow Br(F_0 \otimes k)$ where Br denotes the Brauer group. It follows that, when k is a number field, the Hasse principle holds for $H^1(k,S^F)$: the map $H^1(k,S^F) \to \oplus H^1(k_v,S^F)$ is injective. The remark (1.4) shows that this is also true without assuming F to be a CM-field.

(1.6) Let $\lambda \in X^*(S)$ and let T_λ be the $\mathbb{Q}$-rational torus such that $X^*(T_\lambda)$ is the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-submodule of $X^*(S)$ generated by λ. Thus T_λ is a quotient of S and h_{can} defines a homomorphism $h\colon \mathbb{S} \to T_\lambda$. For any $\mathbb{Q}$-rational representation of T_λ, $T_\lambda \hookrightarrow GL(V)$, (V,h) is a $\mathbb{Q}$-rational Hodge structure with weight $n = -(\lambda(1)+\lambda(\iota))$ and Mumford-Tate group $MT(V,h) = T_\lambda$ (See II). The condition (1.1') shows that ι acts as -1 on $Ker(\lambda' \mapsto \lambda'(1)+\lambda'(\iota)\colon X^*(T_\lambda) \to \mathbb{Z})$; thus $(T_\lambda/w_h(\mathbb{G}_m))(\mathbb{R})$ is compact, and (V,h) is polarizable (Deligne [1, 2.8]). It follows easily that $S = \varprojlim MT(V,h)$ where the limit is over the $\mathbb{Q}$-rational polarizable Hodge structures (V,h) of CM-type. In other words, S is the group associated with the Tannakian category of Hodge structures of this type.

(1.7) (Serre). It is an easy combinatorial exercise to show that $X^*(S)$ is generated by functions λ such that $\lambda(\sigma)$ is 0 or 1 and $\lambda(\sigma) + \lambda(\iota\sigma) = 1$. If λ is of this type then, for any representation $T_\lambda \hookrightarrow GL(V)$ of T_λ, (V,h) is a $\mathbb{Q}$-rational polarizable Hodge structure of CM-type and weight -1; it therefore corresponds to an abelian variety. Thus $S = \varprojlim MT(A)$ where the limit is over abelian varieties (over $\mathbb{C}$) of CM-type. In other words, the Tannakian category of $\mathbb{Q}$-rational polarizable Hodge structures of CM-type is generated by those arising from abelian varieties.

(1.8) If L is Galois over $\mathbb{Q}$, then $Gal(L/\mathbb{Q})$ acts on $L^\times = Res_{L/\mathbb{Q}}\mathbb{G}_m$ and this action induces an action on the quotient S^L. Thus there is an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $\mathbb{Q}$-rational pro-algebraic group S. It is important to distinguish carefully between the two natural actions of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S(\overline{\mathbb{Q}})$, the first of which arises from the (algebraic) action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on S and the second from the (Galois) action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}$. See Langlands [1, p.220].

2. Extensions of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by S.

By an extension of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by S we shall mean a projective system

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^{L'} & \longrightarrow & \underset{\sim}{T}^{L'} & \longrightarrow & Gal(L'^{ab}/\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow N_{L'/L} & & \downarrow & & \downarrow \text{can} & & \\ 1 & \longrightarrow & S^{L} & \longrightarrow & \underset{\sim}{T}^{L} & \longrightarrow & Gal(L^{ab}/\mathbb{Q}) & \longrightarrow & 1 \end{array} \qquad (L \subset L')$$

of extensions of $\mathbb{Q}$-rational pro-algebraic groups; the indexing set is all finite Galois extensions of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$. The group $\mathrm{Gal}(L^{ab}/\mathbb{Q})$ is to be regarded as a pro-system of finite constant algebraic groups in the obvious way, and the action of $\mathrm{Gal}(L^{ab}/\mathbb{Q})$ on S^L determined by the extension is to be the algebraic action described in (1.8). On passing to the limit we obtain an extension

$$1 \longrightarrow S \longrightarrow \underset{\sim}{T} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 .$$

We shall always assume there to be a splitting of the extension over $\mathbb{A}^f$, i.e., a compatible family of continuous homomorphic sections sp^L: $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \rightarrow \underset{\sim}{T}^L(\mathbb{A}^f)$. In the limit this defines a continuous homomorphism sp: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \underset{\sim}{T}(\mathbb{A}^f)$.

Fix an L. The general theory of affine group schemes (Demazure-Gabriel [1,V.2]) shows that, for some finite quotient $\mathcal{G}'$ of $\mathcal{G} = \mathrm{Gal}(L^{ab}/\mathbb{Q})$, $\underset{\sim}{T}^L$ will be the pull-back of an extension of $\mathcal{G}'$ by S^L:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S^L & \longrightarrow & \underset{\sim}{T}^L & \longrightarrow & \mathcal{G} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & S^L & \longrightarrow & \underset{\sim}{T}' & \longrightarrow & \mathcal{G}' & \longrightarrow & 1 .
\end{array}$$

Since S^L splits over L, Hilbert's theorem 90 shows that $H^1(L,S^L) = 0$, and so $\underset{\sim}{T}'(L) \longrightarrow \mathcal{G}'$ is surjective. Thus we can choose a section a': $\mathcal{G}' \rightarrow \underset{\sim}{T}'_L$, which will automatically be a morphism of algebraic varieties. On pulling back to $\mathcal{G}$, we get a section $a = a^L\colon \operatorname{Gal}(L^{ab}/\mathbb{Q}) \to \underset{\sim}{T}^L_L$ which is a morphism of pro-algebraic varieties. The choice of such an a gives us the following data.

(2.1). A 2-cocycle (d_{τ_1,τ_2}) for $\operatorname{Gal}(L^{ab}/\mathbb{Q})$ with values in the algebraic group S^L_L, defined by $d_{\tau_1,\tau_2} = a(\tau_1)a(\tau_2)a(\tau_1\tau_2)^{-1}$.

(2.2). A family of 1-cocycles $c(\tau) \in Z^1(L/\mathbb{Q}, S^L(L))$, one for each $\tau \in \operatorname{Gal}(L^{ab}/\mathbb{Q})$, defined by $c_\sigma(\tau)a(\tau) = \sigma a(\tau)$. ($\operatorname{Gal}(L/\mathbb{Q})$ acts on $S^L(L)$ through its action on the field L.)

(2.3). A continuous map $b\colon \operatorname{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}^f_L)$ defined by $b(\tau)sp^L(\tau) = a(\tau)$.

These satisfy the following relations:

$$(2.4).\quad d_{\tau_1,\tau_2} \cdot c_\sigma(\tau_1) \cdot \tau_1(c_\sigma(\tau_2)) = \sigma d_{\tau_1,\tau_2} \cdot c_\sigma(\tau_1\tau_2)\ ,$$

$$(2.5).\quad d_{\tau_1,\tau_2} = b(\tau_1) \cdot \tau_1 b(\tau_2) \cdot b(\tau_1\tau_2)^{-1}\ ,$$

$$(2.6).\quad c_\sigma(\tau) = b(\tau)^{-1} \cdot \sigma(b(\tau))$$

for $\tau_1, \tau_2, \tau \in \operatorname{Gal}(L^{ab}/\mathbb{Q})$ and $\sigma \in \operatorname{Gal}(L/\mathbb{Q})$. (We have used the convention that $\tau \in \operatorname{Gal}(L^{ab}/\mathbb{Q})$ acts on $S^L(L)$ through its action on S^L, and $\sigma \in \operatorname{Gal}(L/\mathbb{Q})$ acts on $S^L(L)$ through its action on the field L.) In fact, the first relation is a consequence of the other two.

Note that b determines (d_{τ_1,τ_2}) and the $(c_\sigma(\tau))$, and that the image $\bar{b}(\tau)$ of $b(\tau)$ in $S^L(\mathbb{A}^f_L)/S^L(L)$ is uniquely determined by the extension and sp^L (independently of the choice of a) .

Proposition 2.7. A mapping $\bar{b}$: $Gal(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)/S^L(L)$ arises (as above) from an extension of S^L by $Gal(L^{ab}/\mathbb{Q})$ and a splitting if and only if it satisfies the following conditions:

(a) $\sigma(\bar{b}(\tau)) = \bar{b}(\tau)$, all $\tau \in Gal(L^{ab}/\mathbb{Q})$, $\sigma \in Gal(L/\mathbb{Q})$;

(b) $\bar{b}(\tau_1\tau_2) = \bar{b}(\tau_1) \cdot \tau_1\bar{b}(\tau_2)$, all $\tau_1, \tau_2 \in Gal(L^{ab}/\mathbb{Q})$;

(c) $\bar{b}$ lifts to a continuous map b: $Gal(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)$ such that the map $(\tau_1,\tau_2) \mapsto d_{\tau_1,\tau_2} \overset{df}{=} b(\tau_1)\cdot\tau_1 b(\tau_2)\cdot b(\tau_1\tau_2)^{-1}$ is locally constant. Moreover, the extension (together with the splitting) is determined by $\bar{b}$ up to isomorphism.

Proof. We shall only show how to construct the extension from $\bar{b}$, the rest being easy. Choose a lifting b of $\bar{b}$ as in (c) . The family d_{τ_1,τ_2} is a 2-cocycle which takes values in the algebraic group S_L^L . It therefore defines an extension

$$1 \longrightarrow S_L^L \longrightarrow \underset{\sim}{T}_L^L \longrightarrow Gal(L^{ab}/\mathbb{Q}) \longrightarrow 1$$

of pro-algebraic groups over L together with a section a: $Gal(L^{ab}/\mathbb{Q}) \to \underset{\sim}{T}_L^L$ that is a morphism of pro-varieties. Define $\underset{\sim}{T}^L$ to be the pro-algebraic group scheme over $\mathbb{Q}$ such that $\underset{\sim}{T}^L(\bar{\mathbb{Q}}) = \underset{\sim}{T}_L^L(\bar{\mathbb{Q}})$ with $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acting by the formula:

$\sigma(s\cdot a(\tau)) = c_\sigma(\tau)\cdot \sigma s \cdot a(\tau)$, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $s \in S^L(\bar{\mathbb{Q}})$
$\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$, $c_\sigma(\tau) \overset{df}{=} b(\tau)^{-1}\cdot \sigma b(\tau) \in S^L(L)$. There is an exact sequence

$$1 \longrightarrow S^L \longrightarrow \underset{\sim}{T}^L \longrightarrow \mathrm{Gal}(L^{ab}/\mathbb{Q}) \longrightarrow 1 .$$

For each $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q}$, $b(\tau)^{-1}a(\tau) \in S^L(\mathbb{A}^f_L)^{\mathrm{Gal}(L/\mathbb{Q})} = S^L(\mathbb{A}^f)$, and $\tau \longmapsto sp(\tau) \overset{df}{=} b(\tau)^{-1}a(\tau)$ is a homomorphism. As b is continuous, so also is sp .

<u>Corollary 2.8</u>. To define an extension of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S (together with a splitting over $\mathbb{A}^f$) it suffices to give maps Б^L: $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}^f_L)/S^L(L)$ satisfying the conditions of (2.7) and such that, whenever $L \subset L'$,

$$\begin{array}{ccccc}
\mathrm{Gal}(L'^{ab}/\mathbb{Q}) & & \xrightarrow{\quad \text{Б}^{L'} \quad} & & S^{L'}(\mathbb{A}^f_{L'})/S^{L'}(L') \\
\Big\downarrow \text{can} & & & & \Big\downarrow N_{L'/L} \\
\mathrm{Gal}(L^{ab}/\mathbb{Q}) & \xrightarrow{\text{Б}^L} & S^L(\mathbb{A}^f_L)/S^L(L) & \hookrightarrow & S^L(\mathbb{A}^f_{L'})/S^L(L')
\end{array}$$

commutes.

<u>Remark 2.9</u>. Let $\underset{\sim}{T}$ be an extension of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by S . For any $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, multiplication in $\underset{\sim}{T}$ makes $\pi^{-1}(\tau)$ into a torsor for S , and $sp(\tau)$ is a point of the torsor with values in $\mathbb{A}^f$ (i.e. a trivialization of the torsor over $\mathbb{A}^f$) . In the above we have implictly regarded $\pi^{-1}(\tau)$ as a left torsor, because that is the convention of Langlands [1]. It is however

both more convenient and more conventional to regard $\pi^{-1}(\tau)$ as a right S-torsor. With this point of view it is natural to associate with $\underset{\sim}{T}$ cocycles $(\gamma_\sigma(\tau))$ and a map β defined as follows: let L be a finite Galois extension of $\mathbb{Q}$ and choose a section $\tau \mapsto a(\tau)$ to $\underset{\sim}{T}^L \to \mathrm{Gal}(L^{ab}/\mathbb{Q})$ that is a morphism of pro-algebraic varieties; then

$$\sigma a(\tau) = a(\tau)\gamma_\sigma(\tau), \quad \text{for } \sigma \in \mathrm{Gal}(L/\mathbb{Q}),\ \tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q}), \text{ and}$$

$$sp(\tau)\beta(\tau) = a(\tau) \quad \text{for } \tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q}) .$$

The following relations hold:

$$\gamma_\sigma(\tau) = \beta(\tau)^{-1} \cdot \sigma(\beta(\tau)) ,$$

$$\overline{\beta}(\tau_1\tau_2) = \tau_2^{-1}\overline{\beta}(\tau_1) \cdot \overline{\beta}(\tau_2) .$$

The new objects are related to the old as follows:

$$\gamma_\sigma(\tau) = \tau^{-1}c_\sigma(\tau) ,$$

$$\beta(\tau) = \tau^{-1}b(\tau) .$$

Define $c'(\tau)$ and $b'(\tau)$ by the formulas (2.2) and (2.3) but with $a(\tau)$ replaced by the section $\tau \mapsto a'(\tau) = a(\tau^{-1})^{-1}$. Then

$$\gamma_\sigma(\tau) = c'_\sigma(\tau^{-1})^{-1} ,$$

$$\beta(\tau) = b'(\tau^{-1})^{-1} .$$

In particular, we see that $\gamma(\tau)$ and $c(\tau^{-1})^{-1}$ are cohomologous and $\overline{\beta}(\tau) = \overline{b}(\tau^{-1})^{-1}$.

Example 2.10. In the preceding discussion there is no need to take the base field to be $\mathbb{Q}$. We shall use this method to construct for any number field $L \subset \overline{\mathbb{Q}}$, a canonical extension

$$1 \longrightarrow S^L \longrightarrow (\underset{\sim}{T}^L)^{ab} \xrightarrow{\pi} \mathrm{Gal}(L^{ab}/L) \longrightarrow 1$$

of pro-algebraic groups over $\mathbb{Q}$, together with a splitting over $\mathbb{A}^f$. According to (2.7), such an extension corresponds to a map $\overline{b}$: $\mathrm{Gal}(L^{ab}/L) \to S^L(\mathbb{A}_L^f)/S^L(L)$ satisfying conditions similar to (a), (b), and (c) of that proposition. In fact we shall define a map $\overline{b}$: $\mathrm{Gal}(L^{ab}/L) \to S^L(\mathbb{A}^f)/S^L(\mathbb{Q}) \subset S^L(\mathbb{A}_L^f)/S^L(L)$ and so (a) will be obvious (and the cocycles $c(\tau)$ trivial). Note that $\mathrm{Gal}(L^{ab}/L)$ acts trivially on S^L and so (b) requires that $\overline{b}$ be a homomorphism.

The canonical element $\mu^L \in X_*(S^L)$ is defined over L , and so gives rise to a homomorphism of algebraic groups,

$$NR: \quad L^\times \xrightarrow{\mathrm{Res}_{L/\mathbb{Q}}(\mu^L)} \mathrm{Res}_{L/\mathbb{Q}}\, S_L^L \xrightarrow{N_{L/\mathbb{Q}}} S^L .$$

Consider

$$\begin{array}{llcl} NR(\mathbb{A}): & \mathbb{A}_L^\times & \longrightarrow & S^L(\mathbb{A}) \\ & \cup & & \cup \\ NR(L): & L^\times & \longrightarrow & S^L(\mathbb{Q}) \end{array}$$

The reciprocity morphism (Deligne [2,2.2.3])

$$r_L = r_L(S^L, h^L):\ \ \mathrm{Gal}(L^{ab}/L) \longrightarrow S^L(\mathbb{A}^f)/S^L(\mathbb{Q})$$

is defined to be the reciprocal of the composite of the following maps: the reciprocity law isomorphism $\mathrm{Gal}(L^{ab}/L) \overset{\approx}{\longleftrightarrow} \pi_0(\mathbb{A}_L^\times/L^\times)$, the map $\pi_0(\mathbb{A}_L^\times/L^\times) \to \pi_0(S^L(\mathbb{A})/S^L(\mathbb{Q}))$ defined by NR , and the projection $\pi_0(S^L(\mathbb{A})/S^L(\mathbb{Q})) \to S^L(\mathbb{A}^f)/S^L(\mathbb{Q}))$. We define $\bar{b}(\tau) = r_L(\tau)^{-1}$. It satisfies (a) and (b) of (2.7).

According to (1.2), $S^L(\mathbb{Q})$ is a discrete subgroup of $S^L(\mathbb{A}^f)$, and hence of $S^L(\mathbb{A})$. Thus there is an open subgroup U of $\mathbb{A}_L^\times$ such that NR: $\mathbb{A}_L^\times \to S^L(\mathbb{A})$ is 1 on $U \cap L^\times$. If $F \supset L$ corresponds to $U \subset \mathbb{A}_L^\times$, then there is a commutative diagram

$$\begin{array}{ccccccc}
1 \longrightarrow & \mathrm{Gal}(L^{ab}/F) & \longrightarrow & \mathrm{Gal}(L^{ab}/L) & \longrightarrow & \mathrm{Gal}(F/L) & \longrightarrow 1 \\
 & \downarrow b & \swarrow b & \downarrow \bar{b} & & & \\
 & S^L(\mathbb{A}^f) & \longrightarrow & S^L(\mathbb{A}^f)/S(L) & & &
\end{array}$$

in which b^{-1}: $\mathrm{Gal}(L^{ab}/F) \to S^L(\mathbb{A})$ is induced by NR: $U/U \cap L^\times \to S^L(\mathbb{A})$. It is easy to extend b to a continuous map $\mathrm{Gal}(L^{ab}/L) \to S^L(\mathbb{A}^f)$ lifting $\bar{b}$: choose a set S' of representatives for $\mathrm{Gal}(F/L)$ in $\mathrm{Gal}(L^{ab}/L)$, choose an element $b(s) \in S^L(\mathbb{A}^f)$ mapping to $\bar{b}(s)$ for each $s \in S'$, and define $b(sg) = b(s)b(g)$ for $s \in S'$, $g \in \mathrm{Gal}(L^{ab}/F)$. This map b satisfies (c) of (2.7) because, when restricted to $\mathrm{Gal}(L^{ab}/F)$, it is a homomorphism.

Remark 2.11. The extension constructed in (2.10) is, up to sign, that defined by Serre [1]. For a sufficiently large modulus $\mathfrak{m}$ the group $T_{\mathfrak{m}} = T/\bar{E}_{\mathfrak{m}}$ of (ib., p II-8) is the Serre group S^L, and $C_{\mathfrak{m}} = \mathrm{Gal}(L_{\mathfrak{m}}/L)$ for some $L_{\mathfrak{m}} \subset L^{ab}$. Thus the sequence (ib., p II-9) can be written

$$1 \longrightarrow S^L \longrightarrow S_{\mathfrak{m}} \longrightarrow \mathrm{Gal}(L_{\mathfrak{m}}/L) \longrightarrow 1 .$$

On passing to the limit over increasing $\mathfrak{m}$, this becomes

$$1 \longrightarrow S^L \longrightarrow (\underset{\sim}{T}^L)^{ab} \longrightarrow \mathrm{Gal}(L^{ab}/L) \longrightarrow 1 .$$

The splitting (over $\mathbb{Q}_\ell$) is defined in (ib., 2.3).

§3. The Taniyama group.

We denote the Weil group of a local or global field L by W_L. Let v denote the prime induced on $\bar{\mathbb{Q}}$, or a subfield L of $\bar{\mathbb{Q}}$, by the fixed inclusion $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and let L_v denote the closure of L in $\bar{\mathbb{Q}}_v = \mathbb{C}$. According to Tate [2] there is a homomorphism $i_v: W_{\mathbb{Q}_v} \to W_{\mathbb{Q}}$ such that the diagrams

$$\begin{array}{ccc} L_v & \xrightarrow[\approx]{r_v} & W^{ab}_{L_v} \\ \downarrow \text{can} & & \downarrow i^{ab}_v \\ C_L & \xrightarrow[\approx]{r} & W^{ab}_L \end{array} \qquad \begin{array}{ccc} W_{\mathbb{Q}_v} & \xrightarrow{\phi_v} & \mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v) \\ \downarrow i_v & & \downarrow \\ W_{\mathbb{Q}} & \xrightarrow{\phi} & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \end{array}$$

commute for all number fields L contained in $\bar{\mathbb{Q}}$. The constructions that follow will be independent of the choice of

i_v, but we shall ignore this question by fixing an i_v. If $L \subset \overline{\mathbb{Q}}$ is a finite Galois extension of $\mathbb{Q}$ then i_v induces a map from $W_{L_v/\mathbb{Q}_v} \overset{df}{=} W_{\mathbb{Q}_v}/W^c_{L_v}$ to $W_{L/\mathbb{Q}} \overset{df}{=} W_{\mathbb{Q}}/W^c_L$ which makes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_v^\times & \longrightarrow & W_{L_v/\mathbb{Q}_v} & \longrightarrow & \mathrm{Gal}(L_v/\mathbb{Q}_v) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow i_v & & \downarrow & & \\ 1 & \longrightarrow & C_L & \longrightarrow & W_{L/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & 1 \end{array} \tag{3.1}$$

commute.

We note that there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_L & \longrightarrow & W_{L/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \rightarrow & \mathrm{Gal}(L^{ab}/L) & \rightarrow & \mathrm{Gal}(L^{ab}/\mathbb{Q}) & \longrightarrow & \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & 1 \end{array} \tag{3.2}$$

in which the vertical arrows are surjective.

Let T be a torus over $\mathbb{Q}$; by analogy with $T(L) = X_*(T) \otimes L^\times$, $T(\mathbb{A}^f) = X_*(T) \otimes \mathbb{A}^f$ etc., we shall write $T(C_L)$ for $X_*(T) \otimes C_L$. If $\mu \in X_*(T)$ and a belongs to a $\mathbb{Q}$-algebra R (or C_L) then we write a^μ for $\mu \otimes a \in T(R)$.

Fix such a torus T and an element $\mu \in X_*(T)$, and let $L \subset \overline{\mathbb{Q}}$ be a number field splitting T. For each $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ that satisfies

$$(1 + \iota)(\tau^{-1} - 1)\mu = 0 \tag{3.3}$$

and lifting $\tilde{\tau}$ of τ to $W_{L/\mathbb{Q}}$ (using the map in (3.2)) we shall define an element $b_0(\tilde{\tau},\mu) \in T(C_L)/T(L_\infty^\times)$, where $L_\infty = L \otimes_{\mathbb{Q}} \mathbb{R}$.

Choose a section $\sigma \longmapsto w_\sigma$ to $W_{L/\mathbb{Q}} \longrightarrow \mathrm{Gal}(L/\mathbb{Q})$ such that:

(3.4a) $w_1 = 1$;

(3.4b) $w_\iota \in W_{L_v/\mathbb{Q}_v} \subset W_{L/\mathbb{Q}}$;

(3.4c) for some choice of H containing 1 and such that $\mathrm{Gal}(L/\mathbb{Q}) = H \cup H\iota$ (disjoint union), $w_{\sigma\iota} = w_\sigma w_\iota$ for all $\sigma \in H$.

Of course, the last two conditions are trivial if $L \subset \mathbb{R}$.

Corresponding to w there is a 2-cocycle $(a_{\sigma,\tau})$, defined by $w_\sigma w_\tau = a_{\sigma,\tau} w_{\sigma\tau}$. Let $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ satisfy (3.3) and let $\tilde{\tau} \in W_{L/\mathbb{Q}}$ map to it. Choose $c_{\sigma,\tilde{\tau}} \in C_L$ to satisfy $w_\sigma \tilde{\tau} = c_{\sigma,\tilde{\tau}} w_{\sigma\tau}$, and define

$$b_0(\tilde{\tau},\mu) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} c_{\sigma,\tilde{\tau}}^{\sigma\mu} \in T(C_L)/T(L_\infty).$$

<u>Lemma 3.5.</u> The element $b_0(\tilde{\tau},\mu)$ is independent of the choice of the section w; it is fixed by $\mathrm{Gal}(L/\mathbb{Q})$.

<u>Proof.</u> (Langlands [1. p. 221; p. 223].) Suppose $\sigma \mapsto w'_\sigma = e_\sigma w_\sigma$, $e_\sigma \in C_L$, is another section. We use $'$ to denote objects defined using this section. It is easy to see that

$$c'_{\sigma,\tilde{\tau}} = e_{\sigma\tau}^{-1} e_\sigma c_{\sigma,\tilde{\tau}} \quad \text{for all } \sigma \in \mathrm{Gal}(L/\mathbb{Q}).$$

Therefore

$$b_o(\tilde{\tau},\mu)' = \{\prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} (e_{\sigma\tau}^{\sigma\mu})^{-1} e_\sigma^{\sigma\mu}\} \; b_o(\tilde{\tau},\mu) .$$

We have to show that the product in $\{\ \}$ is congruent to 1 modulo $L_\infty^\times$. Consider $\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)$. Because of (3.4b), we have $e_\sigma \in L_v^\times$ and hence $\rho(e_\sigma) \in L_\infty^\times$ for all $\rho \in \mathrm{Gal}(L/\mathbb{Q})$. As

$$a'_{\rho,\sigma} = e_\rho \rho(e_\sigma)\, e_{\rho\sigma}^{-1}\, a_{\rho,\sigma} \quad ,$$

and both $a_{\rho,\sigma}$ and $a'_{\rho,\sigma}$ belong to $L_\infty^\times$, we have $e_\rho \equiv e_{\rho\sigma} \pmod{L_\infty^\times}$. Thus

$$\prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} (e_{\sigma\tau}^{\sigma\mu})^{-1}\, e_\sigma^{\sigma\mu} = \prod_\sigma (e_\sigma^{\sigma\tau^{-1}\mu})\, e_\sigma^{\sigma\mu}$$

$$= \prod_\sigma e_\sigma^{\sigma(1-\tau^{-1})\mu}$$

$$= \prod_{\eta \in H} \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} e_{\eta\sigma}^{\eta\sigma(1-\tau^{-1})\mu}$$

is congruent modulo $L_\infty^\times$ to

$$\prod_{\eta \in H} \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} e_\eta^{\eta\sigma(1-\tau^{-1})\mu} \quad ,$$

which is 1, because in view of (3.3),

$$\sum_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} \sigma(1-\tau^{-1})\mu = 0 \; .$$

Next we show that $b_0(\tilde{\tau},\mu)$ is fixed by $\mathrm{Gal}(L/\mathbb{Q})$. We have

$$\rho(c_{\sigma,\tilde{\tau}}) = a_{\rho,\sigma}\, a^{-1}_{\rho,\sigma\tau}\, c_{\rho\sigma,\tilde{\tau}} \quad \text{for all} \quad \rho,\sigma \in \mathrm{Gal}(L/\mathbb{Q}) ,$$

and hence

$$\rho(b_o(\tilde{\tau},\mu)) = \{\prod_{\sigma} (a_{\rho,\sigma}\, a^{-1}_{\rho,\sigma\tau})^{\rho\sigma\mu}\}\, b_o(\tilde{\tau},\mu) .$$

We can write the product in { } as

$$\prod_{\sigma} (a_{\rho,\sigma}{}^{\rho\sigma\mu})(a^{-1}_{\rho,\sigma}{}^{\rho\sigma\tau^{-1}\mu}) = \prod_{\sigma} a_{\rho,\sigma}{}^{\rho\sigma(1-\tau^{-1})\mu}$$

$$= \prod_{\eta \in H}\ \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} a_{\rho,\eta\sigma}{}^{\rho\eta\sigma(1-\tau^{-1})\mu} .$$

In view of (3.4c) we have $a_{\rho,\eta\sigma} \equiv a_{\rho,\eta} \pmod{L_\infty^\times}$ for all $\eta \in H$ and $\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)$. Hence the above product is congruent modulo $L_\infty^\times$ to

$$\prod_{\eta \in H}\ \prod_{\sigma \in \mathrm{Gal}(L_v/\mathbb{Q}_v)} a_{\rho,\eta}{}^{\rho\eta\sigma(1-\tau^{-1})\mu} ,$$

which is 1 because of (3.3) .

On tensoring

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L^{\times} & \longrightarrow & \mathbb{A}_L^{f\times} & \longrightarrow & \mathbb{A}_L^{f\times}/L^{\times} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle -1} & & \downarrow & & \downarrow{\scriptstyle \approx} & & \\ 1 & \longrightarrow & L_{\infty}^{\times} & \longrightarrow & C_L & \longrightarrow & C_L/L_{\infty}^{\times} & \longrightarrow & 1 \end{array} \qquad (3.6a)$$

with $X_*(T)$ we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T(L) & \longrightarrow & T(\mathbb{A}_L^f) & \longrightarrow & T(\mathbb{A}_L^f)/T(L) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle -1} & & \downarrow & & \downarrow{\scriptstyle \approx} & & \\ 1 & \longrightarrow & T(L_{\infty}) & \longrightarrow & T(C_L) & \longrightarrow & T(C_L)/T(L_{\infty}) & \longrightarrow & 1 \end{array} \qquad (3.6b)$$

(The -1 reminds us that the map is the reciprocal of the obvious inclusion.) We define $\overline{b}(\tilde{\tau},\mu)$ to be the element of $T(\mathbb{A}_L^f)/T(L)$ corresponding to $b_0(\tilde{\tau},\mu)$. Lemma 3.5 shows that it lies in $(T(\mathbb{A}_L^f)/T(L))^{\mathrm{Gal}(L/\mathbb{Q})}$ and hence gives rise to an element $c(\tilde{\tau},\mu) \in H^1(L/\mathbb{Q},\ T(L))$ through the boundary map in the exact sequence

$$1 \to T(\mathbb{Q}) \longrightarrow T(\mathbb{A}^f) \longrightarrow (T(\mathbb{A}_L^f)/T(L))^{\mathrm{Gal}(L/\mathbb{Q})} \longrightarrow H^1(L/\mathbb{Q},T(L)).$$

<u>Lemma 3.7.</u> The cohomology class $c(\tilde{\tau},\mu)$ depends only on the image of $\tilde{\tau}$ in $\mathrm{Gal}(L/\mathbb{Q})$.

Proof. Suppose $\tilde{\tau}'$ and $\tilde{\tau}$ have the same image in $\mathrm{Gal}(L/\mathbb{Q})$; then $\tilde{\tau}' = u\tilde{\tau}$ with $u \in C_L$, and $c_{\sigma,\tilde{\tau}'} = \sigma(u)c_{\sigma,\tilde{\tau}}$. Thus $b_0(\tilde{\tau},\mu)$ is multiplied by $\Pi\sigma(u)^{\sigma\mu} = NR(u)$, where NR is the map of algebraic groups $L^\times \xrightarrow{\mathrm{Res}(\mu)} \mathrm{Res}_{L/\mathbb{Q}} T_L \xrightarrow{N_{L/\mathbb{Q}}} T$. Choose an element $\tilde{u} \in \mathbb{A}_L^f$ such that $\tilde{u}$ and u represent the same element in $C_L/L_\infty^\times$. Then $NR(\tilde{u}) \in T(\mathbb{A}^f)$ has the same image as $NR(u)$ in $T(C_L)/T(L_\infty^\times)$, and we see that $\bar{b}(\tilde{\tau}',\mu) = \overline{NR}(\tilde{u})\ \bar{b}(\tilde{\tau},\mu)$ where $\overline{NR}(\tilde{u})$ denotes the image of $NR(\tilde{u})$ in $T(\mathbb{A}^f)/T(\mathbb{Q}) \subset T(\mathbb{A}_L^f)/T(L)$. Hence $c(\tilde{\tau},\mu) = c(\tilde{\tau}',\mu)$.

Thus we can write $c(\tau,\mu)$ for $c(\tilde{\tau},\mu)$ where $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ (or even $\mathrm{Gal}(L/\mathbb{Q})$).

Lemma 3.8. Up to multiplication by an element of the closure $T(\mathbb{Q})^\wedge$ of $T(\mathbb{Q})$ in $T(\mathbb{A}^f)$, $\bar{b}(\tilde{\tau},\mu)$ depends only on τ (and not $\tilde{\tau}$).

Proof. From (3.2) we see that $\tilde{\tau}$ can be multiplied only by an element u of the identity component of C_L. An argument as in the proof of (3.7) shows that multiplying $\tilde{\tau}$ by u corresponds to multiplying $\bar{b}(\tilde{\tau},\mu)$ by $\overline{NR}(\tilde{u})$, where $\tilde{u}$ is a lifting of u to $\mathbb{A}_L^f$. But $\tilde{u}$ is in the closure of $L^\times \subset (\mathbb{A}_L^f)^\times$, and so $\overline{NR}(\tilde{u})$ is in the closure of $T(\mathbb{Q})$.

Thus, for any $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ satisfying (3.3), there is a well-defined element $\bar{b}(\tau,\mu) \in T(\mathbb{A}_L^f)/T(L)\ T(\mathbb{Q})^\wedge$.

Example 3.9. For any T and μ, $\bar{b}(\iota,\mu)$ is defined; we show that it is 1. We can take $\tilde{\iota} = w_\iota$. If $\sigma \in H$ (see 3.4), then $w_\sigma\tilde{\iota} = w_\sigma w_\iota = w_{\sigma\iota}$, and $c_{\sigma,\tilde{\iota}} = 1$; moreover $w_{\sigma\iota}\tilde{\iota} = w_\sigma w_\iota w_\iota = w_\sigma a_{\iota,\iota} = \sigma(a_{\iota,\iota})w_\sigma$ and $c_{\sigma\iota,\tilde{\iota}} = \sigma(a_{\iota,\iota}) \in L_\infty^\times$. Clearly $b_0(\tilde{\iota},\mu) = 1$.

Proposition 3.10. Let $h : \mathbb{S} \to T_{\mathbb{R}}$ be a homomorphism and $\mu = \mu_h$ be the corresponding cocharacter. Assume that μ is defined over $E \subset L$. Then $\bar{b}(\tau,\mu)$ is defined for all $\tau \in \mathrm{Gal}(L^{ab}/E)$ and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(L^{ab}/E) & \xrightarrow{\bar{b}(-,\mu)} & T(\mathbb{A}_L^f)/T(L)\ T(\mathbb{Q})^\wedge \\ \downarrow \text{rest} & & \uparrow \\ \mathrm{Gal}(E^{ab}/E) & \xrightarrow{r_E(T,h)^{-1}} & T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge \end{array}$$

in which $r_E(T,h)$ is the reciprocity morphism (Deligne [2,2.2.3]). In particular, $c(\tau,\mu)$ is trivial.

Proof. Let $\tau \in \mathrm{Gal}(L^{ab}/E)$. Then τ fixes μ, and so (3.3) is satisfied and $\bar{b}(\tau,\mu)$ is defined. We may choose the section w to $W_{L/\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q})$ in such a way that $w_\tau = \tilde{\tau}$ maps to τ in $\mathrm{Gal}(L^{ab}/\mathbb{Q})$. Then $c_{\sigma,\tilde{\tau}} = a_{\sigma,\tau}$. Let R be a set of representatives for $\mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/E)$. We have

$$\begin{aligned} b_0(\tilde{\tau},\mu) &= \prod_{\rho\in R}\ \prod_{\sigma\in \mathrm{Gal}(L/E)} a_{\rho\sigma,\tau}^{\rho\mu} \quad \text{(since } \sigma\mu = \mu) \\ &= \prod_{\rho\in R} \Big(\prod_\sigma (\rho a_{\sigma,\tau}\cdot a_{\rho,\sigma\tau}\cdot a_{\rho,\sigma}^{-1})\Big)^{\rho\mu} \\ &= \prod_{\rho\in R} (\rho a)^{\rho\mu}, \quad \text{where } a = \prod_\sigma a_{\sigma,\tau}. \end{aligned}$$

To evaluate a, we use the commutative diagram (Tate[2, W_3])

$$\begin{array}{ccc} C_E & \xrightarrow{r_E} & W_E^{ab} \\ \downarrow & & \downarrow t \\ C_L & \xrightarrow{r_L} & W_L^{ab} \end{array}$$

where t is the transfer map arising from the inclusion $W_L \hookrightarrow W_E$. Clearly $r_L(a) = \prod r_L(a_{\sigma,\tau}) = t(\tilde{\tau} W_E^c)$. Thus a is an element of C_E that maps to $\tau | E^{ab}$ in $Gal(E^{ab}/E)$. Let $\tilde{a} \in \mathbb{A}_E^f$ represent the same element in $C_E/E_\infty^\times$ as a. Then $\bar{b}(\tau,\mu)$ is the image of $\tilde{a}$ under $\mathbb{A}_E^{f\times} \xrightarrow{NR} T(\mathbb{A}^f)/T(\mathbb{Q})^\wedge$, and this equals $r_E(T,h)(\tau | E^{ab})^{-1}$.

We now apply the above theory to construct the Taniyama group of a finite Galois extension L of $\mathbb{Q}$, $L \subset \bar{\mathbb{Q}}$. To do so, we take the torus T to be S^L and μ to be the canonical co-character of S^L (see §1). Since $S^L(\mathbb{Q})$ is closed in $S^L(\mathbb{A}^f)$ the above constructions give a map $Gal(L^{ab}/\mathbb{Q}) \longrightarrow (S^L(\mathbb{A}_L^f)/S^L(L))^{Gal(L/\mathbb{Q})}$ which we denote by $\bar{b}$ (or $\bar{b}^L$).

Proposition 3.11. The map $\bar{b}$ satisfies the conditions of (2.7) and so defines an extension

$$1 \longrightarrow S^L \longrightarrow \underset{\sim}{T}^L \longrightarrow Gal(L^{ab}/\mathbb{Q}) \longrightarrow 1$$

together with a continuous splitting over $\mathbb{A}^f$.

<u>Proof</u>. We have already observed (Lemma 3.5) that $\bar{b}(\tau)$ is fixed by $\mathrm{Gal}(L/\mathbb{Q})$ for all τ. To show $\bar{b}$ satisfies (2.7b), let $\tau = \tau_1\tau_2$ and lift τ_1 and τ_2 to elements $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of $W_{L/\mathbb{Q}}$. We take $\tilde{\tau}_1\tilde{\tau}_2$ to be the lift of $\tau = \tau_1\tau_2$. Then we have

$$c_{\sigma,\tilde{\tau}} = c_{\sigma,\tilde{\tau}_1}\, c_{\sigma\tau_1,\tilde{\tau}_2} \quad \text{for all } \sigma \in \mathrm{Gal}(L/\mathbb{Q}) .$$

Hence

$$b_o(\tilde{\tau},\mu) = \prod_\sigma (c_{\sigma,\tilde{\tau}_1})^{\sigma\mu} \cdot \prod_\sigma (c_{\sigma\tau_1,\tilde{\tau}_2})^{\sigma\mu} .$$

The first factor is $b_o(\tilde{\tau}_1,\mu)$, and the second one is $\prod_\sigma (c_{\sigma,\tilde{\tau}_2})^{\sigma\tau_1^{-1}\mu}$, which is $\tau_1(b_o(\tilde{\tau}_2,\mu))$ (recall that the action of τ_1 on S^L is the 'algebraic' one, see §1.8). Thus $\bar{b}(\tau_1\tau_2) = \bar{b}(\tau_1)\,\tau_1(\bar{b}(\tau_2))$. To prove (2.7c), consider the diagram

$$\begin{array}{ccc}
\mathrm{Gal}(L^{ab}/\mathbb{Q}) & \xrightarrow{\ \bar{b}\ } & S^L(\mathbb{A}^f_L)/S^L(L) \\
\uparrow & & \uparrow \\
\mathrm{Gal}(L^{ab}/L) & \xrightarrow{\ b\ } & S^L(\mathbb{A}^f)
\end{array}$$

where b is the map defined in (2.10). The diagram commutes because of (3.10). It is easy to extend b to a continuous map $\mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}^f)$ lifting $\bar{b}$ (see the proof of 2.10). Then b satisfies (2.7c) because its restriction to $\mathrm{Gal}(L^{ab}/F)$ is a homomorphism, where F is the finite extension of L defined in the proof of (2.10).

The extension, together with the splitting, is the Taniyama group of L. The next lemma implies that the Taniyama groups for varying L form a projective system: we have an extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by S in the sense of §2.

Lemma 3.12. If $L' \supset L$ then

$$\begin{array}{ccccc}
\mathrm{Gal}(L'^{ab}/\mathbb{Q}) & & \xrightarrow{\quad \bar{b}^{L'} \quad} & & S^{L'}(\mathbb{A}^f_{L'})/S^{L'}(L') \\
\downarrow \text{rest.} & & & & \downarrow N_{L'/L} \\
\mathrm{Gal}(L^{ab}/\mathbb{Q}) & \xrightarrow{\bar{b}^{L}} & S^{L}(\mathbb{A}^f_{L})/S^{L}(L) & \hookrightarrow & S^{L}(\mathbb{A}^f_{L'})/S^{L}(L')
\end{array}$$

commutes.

Proof. We discuss the case $\mathrm{Gal}(L'/L) \cap \mathrm{Gal}(L'_v/\mathbb{Q}_v) = \{1\}$ first. Let R be a set of representatives for the coset space

$$\mathrm{Gal}(L'/L)\backslash \mathrm{Gal}(L'/\mathbb{Q})/\mathrm{Gal}(L'_v/\mathbb{Q}_v) .$$

We choose R such that $1 \in R$. For elements ξ in $\mathrm{Gal}(L'/L) \cup R \cup \mathrm{Gal}(L'_v/\mathbb{Q}_v)$, choose $w'_\xi \in W_{L'/\mathbb{Q}}$ lifting ξ; we choose $w'_1 = 1$ and for $\rho \in \mathrm{Gal}(L'_v/\mathbb{Q}_v)$, choose w'_ρ to be in $W_{L'_v/\mathbb{Q}_v}$. Write an element σ of $\mathrm{Gal}(L'/\mathbb{Q})$ uniquely as $\sigma = \zeta\eta\rho$ with $\zeta \in \mathrm{Gal}(L'/L)$, $\eta \in R$ and $\rho \in \mathrm{Gal}(L'_v/\mathbb{Q}_v)$, and put

$$w'_\sigma = w'_\zeta \, w'_\eta \, w'_\rho \quad .$$

Then $\sigma \mapsto w'_\sigma$ is a section of $W_{L'/\mathbb{Q}} \to \mathrm{Gal}(L'/\mathbb{Q})$ satisfying (3.4). We choose a section $\sigma \mapsto w_\sigma$ of $W_{L/\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q})$ as follows: for $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$, σ extends to a unique $\eta\rho$ in $\mathrm{Gal}(L'/\mathbb{Q})$ with $\eta \in R$ and $\rho \in \mathrm{Gal}(L'_v/\mathbb{Q}_v)$; we take w_σ to be the image of $w'_{\eta\rho} = w'_\eta \, w'_\rho$ in $W_{L/\mathbb{Q}}$.

Let τ be an element of $\mathrm{Gal}(L'^{ab}/\mathbb{Q})$. We lift $\tau|L'$ to $\tilde{\tau}'$ in $W_{L'/\mathbb{Q}}$, and let $\tilde{\tau}$ be the image of $\tilde{\tau}'$ in $W_{L/\mathbb{Q}}$. Suppose $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ lifts to $\eta\rho \in \mathrm{Gal}(L'/\mathbb{Q})$ and $\sigma\tau \in \mathrm{Gal}(L/\mathbb{Q})$ lifts to $\eta'\rho' \in \mathrm{Gal}(L'/\mathbb{Q})$. Then

$$w'_\eta \, w'_\rho \, \tilde{\tau}' = d' \, w'_{\eta'} \, w'_{\rho'} \quad ,$$

with $d' \in W_{L'/L} \subset W_{L'/\mathbb{Q}}$. This shows that under the homomorphism $W_{L'/\mathbb{Q}} \to W_{L/\mathbb{Q}}$, the image d of d' belongs to $W_{L/L} = W_L^{ab}$ and is the image of $c_\sigma(\tilde{\tau})$ under the isomorphism $r_L : C_L \to W_L^{ab}$. On the other hand, for $\zeta \in \mathrm{Gal}(L'/L)$, there is a unique $\zeta' \in \mathrm{Gal}(L'/L)$ such that $\zeta\eta\rho\tau = \zeta'\eta'\rho'$. By definition

$$w'_\zeta \, w'_\eta \, w'_\rho \, \tilde{\tau}' = c_{\zeta\eta\rho}(\tilde{\tau}') \, w'_{\zeta'} \, w'_{\eta'} \, w'_{\rho'} \quad .$$

It follows that

$$w'_\zeta \, d' = c_{\zeta\eta\rho}(\tilde{\tau}') \, w'_{\zeta'} \qquad (3.13)$$

This is an equation in $W_{L'/L}$. Let $t : W_L^{ab} \cong (W_{L'/L})^{ab} \to W_{L'}^{ab} = (W_{L'/L'})^{ab}$ be the transfer homomorphism arising from $W_{L'/L'} \hookrightarrow W_{L'/L}$. We have an exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_{L'} & \longrightarrow & W_{L'/L} & \longrightarrow & \mathrm{Gal}(L'/L) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\approx} & & & & & & \\ & & W_{L'/L'} & & & & & & \end{array}$$

and $\zeta \mapsto w'_\zeta$ is a section of $W_{L'/L} \to \mathrm{Gal}(L'/L)$; thus (3.13) shows that $t(d) = r_{L'}\Big(\prod_{\zeta \in \mathrm{Gal}(L'/L)} c_{\zeta\eta\rho}(\tilde{\tau}')\Big)$. Since

$$\begin{array}{ccc} C_{L'} & \xrightarrow{\ r_{L'}\ } & W_{L'}^{ab} \\ \uparrow\cup & & \uparrow t \\ C_L & \xrightarrow{\ r_L\ } & W_L^{ab} \end{array}$$

commutes (Tate [2, §1, W_3]) and $r_L(c_\sigma(\tilde{\tau})) = d$, $c_\sigma(\tilde{\tau})$ regarded as an element of $C_{L'}$ is $\prod_{\zeta \in \mathrm{Gal}(L'/L)} c_{\zeta\eta\rho}(\tilde{\tau}')$. Now under $N_{L'/L} : X_*(S^{L'}) \to X_*(S^L)$, $\zeta\eta\rho\mu$ maps to $\sigma\mu$ for all $\zeta \in \mathrm{Gal}(L'/L)$. Therefore

$$b_o(\tilde{\tau}',\mu) = \prod_{\zeta,\eta,\rho} c_{\zeta\eta\rho}(\tilde{\tau}')^{\zeta\eta\rho\mu} \in C_{L'} \otimes X_*(S^{L'})$$

maps to

$$\prod_{\substack{\eta,\rho \\ \eta\rho\to\sigma}} \Big(\prod_{\zeta} c_{\zeta\eta\rho}(\tilde\tau')\Big)^{\sigma\mu} \in C_{L'} \otimes X_*(S^L) .$$

But

$$\prod_{\substack{\eta,\rho \\ \eta\rho\to\sigma}} \Big(\prod_{\zeta} c_{\zeta\eta\rho}(\tilde\tau')\Big)^{\sigma\mu} = \prod_{\sigma} (c_\sigma(\tilde\tau))^{\sigma\mu} = b_o(\tilde\tau,\mu) \in C_L \otimes X_* (S^L) .$$

Hence the diagram in the Lemma commutes.

Now suppose $Gal(L'/L) \cap Gal(L'_v/\mathbb{Q}_v) \neq \{1\}$. This happens only if $L_v = \mathbb{R}$. Thus in this case $X_*(S^L) \simeq \mathbb{Z}$ and the Galois group acts on it trivially. Let $\sigma \mapsto w_\sigma$ be an arbitrary section of $W_{L/\mathbb{Q}} \to Gal(L/\mathbb{Q})$, not necessarily satisfying (3.4). Define $c_\sigma \in C_L$ by $w_\sigma\tilde\tau = c_\sigma w_{\sigma\tau}$. Then $\prod_\sigma c_\sigma^{\sigma\mu} = \prod_\sigma c_\sigma^{\mu} \in C_L$ is independent of the choice of the section $\sigma \mapsto w_\sigma$, for in replacing w_σ by $e_\sigma w_\sigma$, $e_\sigma \in C_L$, $\prod_\sigma c_\sigma^\mu$ is multiplied by the factor $\prod_\sigma (e_\sigma e_{\sigma\tau}^{-1})^\mu$, which is 1. In particular, $b_o(\tilde\tau,\mu) = \prod_\sigma c_\sigma^{\sigma\mu}$. Similarly, let $\rho \mapsto w'_\rho$ be an arbitrary section of $W_{L'/\mathbb{Q}} \to Gal(L'/\mathbb{Q})$, and define $c'_\rho \in C_L$, by $w'_\rho \tilde\tau' = c'_\rho w'_{\rho\tau}$. Then the image of $\prod_\rho (c'_\rho)^{\rho\mu}$ in $S^L(C_{L'})$ is independent of the choice of $\rho \mapsto w'_\rho$; in particular, it is the image of $b_o(\tilde\tau,\mu)$. For our purpose, we choose $\sigma \mapsto w_\sigma$ and $\rho \mapsto w_\rho$ as follows. Let R be a set of representatives for the coset space $Gal(L'/L)\backslash Gal(L'/\mathbb{Q})$. Fix $w'_\xi \in W_{L'/\mathbb{Q}}$ projecting to ξ for each ξ in $Gal(L'/L) \cup R$. For $\rho \in Gal(L'/\mathbb{Q})$, write $\rho = \zeta\eta$ with $\zeta \in Gal(L'/L)$ and $\eta \in R$, then put $w'_\rho = w'_\zeta w'_\eta$. For $\sigma \in Gal(L/\mathbb{Q})$, let η be the unique element of R extending σ, and let w_σ be the image of w'_η in $W_{L/\mathbb{Q}}$. As before, we have $c_\sigma = \prod_{\zeta \in \mathrm{Gal}(L'/L)} c'_{\zeta\eta}$ if $\eta \in R$ maps to σ in $\mathrm{Gal}(L/\mathbb{Q})$. It follows that the image of $b_o(\tilde{\tau}',\mu)$ is $b_o(\tilde{\tau},\mu)$.

Proposition 3.14. Let T be a torus over $\mathbb{Q}$, let $\mu \in X_*(T)$, and let τ be an automorphism of $\mathbb{C}$. Assume (3.3) holds, so that $c(\tau,\mu) \in H^1(L/\mathbb{Q}, T(L))$ is defined for L a sufficiently larger number field. The image of $c(\tau,\mu)$ in $H^1(L_v/\mathbb{Q}_v, T(L_v))$ is represented by $\mu(-1)/\tau^{-1}\mu(-1) \in \mathrm{Ker}(1 + \iota : T(\mathbb{C}) \to T(\mathbb{C}))$.

Proof. The image of $c(\tau,\mu)$ in $H^1(\mathbb{C}/\mathbb{R}, T(\mathbb{C}))$ is the cup-product of the local fundamental class in $H^2(\mathbb{C}/\mathbb{R}, \mathbb{C}^\times)$ with the element of $H^{-1}(\mathbb{C}/\mathbb{R}, X_*(T))$ represented by $(1 - \tau^{-1})\mu$. (See Langlands [1, p. 225]). Thus the proposition is a consequence of the following easy lemma.

Lemma 3.15. For any torus T over $\mathbb{R}$, the map $H^{-1}(\mathbb{C}/\mathbb{R}, X_*(T)) \to H^1(\mathbb{C}/\mathbb{R}, T)$ induced by cupping with the fundamental class in $H^2(\mathbb{C}/\mathbb{R}, \mathbb{C}^\times)$ sends the class represented by $\chi \in X_*(T)$ to the class represented by $\chi(-1)$.

Remark 3.16. Thus $c(\tau,\mu)$ has the following property: For any finite prime p of $\mathbb{Q}$ and extension of p to L, $c(\tau,\mu)$ has image 1 in $H^1(L_{\mathfrak{P}}/\mathbb{Q}_p, T(L_{\mathfrak{P}}))$, and the image of $c(\tau,\mu)$

in $H^1(\mathbb{C}/\mathbb{R}\,,\ T(\mathbb{C}))$ is represented by $(1-\tau^{-1})\,\mu(-1)$. When $T = S^L$, (2.5) shows that this property determines $c(\tau,\mu)$ uniquely. On the other hand, it is not difficult to construct directly a cohomology class having the property. Consider the exact commutative diagram

$$\begin{array}{ccccc}
H^1(L/\mathbb{Q},T(L)) & \longrightarrow & \oplus_p\, H^1(L_{\mathfrak{P}}/\mathbb{Q}_p,T(L_{\mathfrak{P}})) & \longrightarrow & H^1(L/\mathbb{Q},T(C_L)) \\
 & & \uparrow\wr & & \uparrow\wr \\
 & & \oplus_p\, H^{-1}(L_{\mathfrak{P}}/\mathbb{Q}_p,X_*(T)) & \longrightarrow & H^{-1}(L/\mathbb{Q},X_*(T))
\end{array}$$

in which the vertical maps are the Tate-Nakayama isomorphisms (Tate [1]). For a finite group G and G-module M, $H^{-1}(G,M) = (\operatorname{Ker} N : M \to M)/\Sigma(\sigma-1)M$. Thus (3.3) shows that $(1-\tau^{-1})\mu$ defines an element $\alpha_\infty \in H^{-1}(\mathbb{C}/\mathbb{R}\,,\ X_*(T))$, and we let $\alpha = (\alpha_p) \in \oplus_p\, H^{-1}(L_{\mathfrak{P}}/\mathbb{Q}_p,X_*(T))$ with $\alpha_p = 0$ for $p \neq \infty$ and α_∞ the element just defined. Note that the image of α_∞ in $H^1(\mathbb{C}/\mathbb{R},T(\mathbb{C}))$ is represented by $(1-\tau^{-1})\mu(-1)$. The image of α in $H^{-1}(L/\mathbb{Q},X_*(T))$ is represented by $(1-\tau^{-1})\mu$, and is therefore zero. It follows that the image of α in $\oplus_p\, H^1(L_{\mathfrak{P}}/\mathbb{Q}_p\,,\ T(L_{\mathfrak{P}}))$ arises from an element of $H^1(L/\mathbb{Q},T(L))$, and this is the class sought.

The next property of the Taniyama group will be needed in showing that the zeta function of an abelian variety of potential CM-type is the L-series of a representation of the Weil group.

<u>Proposition 3.17</u>. For any finite Galois extension L of $\mathbb{Q}$ that is not totally real, there is a homomorphism $\phi : W_{L/\mathbb{Q}} \to \underset{\sim}{T}^L(\mathbb{C})$ making

$$\begin{array}{ccccccccc} & & & & & & W_{L/\mathbb{Q}} & & \\ & & & & & \phi\swarrow & \downarrow & & \\ 1 & \longrightarrow & S^L(\mathbb{C}) & \dashrightarrow & \underset{\sim}{T}^L(\mathbb{C}) & \longrightarrow & \mathrm{Gal}(L^{ab}/\mathbb{Q}) & \longrightarrow & 1 \end{array}$$

commute. If ϕ' is a second such homomorphism then $\phi' = \phi \cdot \alpha$ with α a 1-cocycle for $W_{L/\mathbb{Q}}$ with values in $S^L(\mathbb{C})$.

<u>Proof</u>. We have to show that the 2-cocycle (d_{τ_1,τ_2}) defining the extension (see 2.1) becomes trivial when inflated to $H^2(W_{L/\mathbb{Q}}, S^L(\mathbb{C}))$. Choose a section $\sigma \mapsto w_\sigma$ to $W_{L/\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q})$ as in (3.4) and a map $b : \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to S^L(\mathbb{A}_L^f)$ lifting the map $\bar{b}$ defined above and satisfying (2.7c). For $w \in W_{L/\mathbb{Q}}$ mapping to $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ define $c_{\sigma,w} \in C_L$ by the condition $w_\sigma w = c_{\sigma,w}\, w_{\sigma\tau}$ and set

$$b_o(w) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} c_{\sigma,w}^{\sigma\mu} \in S^L(C_L) .$$

A calculation as in the proof of (3.11) shows that $b_o(w_1 w_2) = b_o(w_1) \cdot \tau_1(b_o(w_2))$, where τ_1 as the image of w_1 in $\mathrm{Gal}(L^{ab}/\mathbb{Q})$. Choose a mapping $b : W_{L/\mathbb{Q}} \to S^L(\mathbb{A}_L)$ making

$$\begin{array}{ccc} W_{L/\mathbb{Q}} & \xrightarrow{\ b\ } & S^L(\mathbb{A}_L) \\ & \underset{b_o}{\searrow} & \downarrow b \\ & & S^L(C_L) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W_{L/\mathbb{Q}} & \twoheadrightarrow & \mathrm{Gal}(L^{ab}/\mathbb{Q}) \\ \downarrow b & & \downarrow \\ S^L(\mathbb{A}_L) & \twoheadrightarrow & S^L(\mathbb{A}_L^f) \end{array}$$

commute. Then $b(w_1) \cdot \tau_1 b(w_2) \cdot b(w_1 w_2)^{-1}$ lies in $S^L(L) \subset S^L(\mathbb{A}_L)$, and projects onto d_{τ_1,τ_2} in $S^L(\mathbb{A}_L^f)$. It is therefore equal to d_{τ_1,τ_2}. Let v be an infinite prime of L such that $L_v = \mathbb{C}$, and let $b_v(w) \in S^L(L_v) = S^L(\mathbb{C})$ be the component of $b(w)$ at v. Then $w \mapsto b_v(w)$ is a 1-cochain whose coboundary is (d_{τ_1,τ_2}).

<u>Remark 3.18</u>. In V we shall need to use the following notations. For any $\mathbb{Q}$-rational torus T, split by L, and cocharacter μ satisfying (3.3) relative to $\tau \in \mathrm{Gal}(L^{ab}/\mathbb{Q})$ we have defined an element $\overline{b}(\tau,\mu) \in T(\mathbb{A}_L^f)/T(L)\ T(\mathbb{Q})^\wedge$. It is natural also to define $\overline{\beta}(\tau,\mu) = \overline{b}(\tau^{-1},\mu)^{-1}$ and $\gamma(\tau,\mu) = c(\tau^{-1},\mu)^{-1}$ (c.f. 2.9). If μ satisfies the stronger condition (1.1) then there is a unique homomorphism $\rho_\mu : S^L \to T$ such that $\rho_\mu \circ \mu^L = \mu$, and we have $\overline{\beta}(\tau,\mu) = \rho_\mu(\overline{\beta}(\tau))$ and $\gamma(\tau,\mu) = \rho_\mu(\gamma(\tau))$.

References

Deligne, P.

1. La conjecture de Weil pour les surfaces K3, Invent. Math. 15 (1972) 206-226.
2. Variétés de Shimura : interpretation modulaire, et techniques de construction de modèles canoniques. Proc. Symp. Pure Math., A.M.S., 33 (1979) part 2, 247-290.

Demazure, M. et Gabriel, P.

1. Groupes Algébriques, Tome I: Géométrie algébriques, generalités, groupes commutatifs. Masson, Paris, 1979.

Langlands, R.

1. Automorphic representations, Shimura varieties, and motives. Ein Märchen. Proc. Symp. Pure Math., A.M.S., 33 (1979), part 2, 205-246.

Serre, J.-P.

1. Abelian ℓ-adic representations and elliptic curves. Benjamin, New York, 1968.

Tate, J.

1. The cohomology groups of tori in finite Galois extensions of number fields, Nagoya Math. J. 27 (1966) 709-719.
2. Number theoretic background, Proc. Symp. Pure Math. A.M.S., 33 (1979) part 2, 3-26.