Progress in Math 35
Birkhäuser 1983

1983a

The Action of an Automorphism of C On a Shimura Variety and its Special Points

J. S. Milne

To I.R. Shafarevich

In [8, pp. ~~222–223~~ 232–233] Langlands made a very precise conjecture describing how an automorphism of **C** acts on a Shimura variety and its special points. The results of Milne-Shih [15], when combined with the result of Deligne [5], give a proof of the conjecture (including its supplement) for all Shimura varieties of abelian type (this class excludes only those varieties associated with groups having factors of exceptional type and most types D). Here the proof is extended to cover all Shimura varieties. As a consequence, one obtains a complete proof of Shimura's conjecture on the existence of canonical models. The main new ingredients in the proof are the results of Kazhdan [7] and the methods of Borovoi [2].

In the preprint [7], Kazhdan shows that the conjugate (by an automorphism of **C**) of the quotient of a Hermitian symmetric domain by an arithmetic group is a variety of the same form. (For a precise statement of what we use from [7], see (3.2).) In sections 2 and 3 we apply this result to prove the following weak form of Langlands's conjecture:

(0.1) let $M^\circ(G, X^+)$ be the connected Shimura variety defined by a simply-connected semi-simple algebraic group G and Hermitian symmetric domain X^+; then, for any automorphism τ of **C**, there is a connected Shimura variety $M^\circ(G', X'^+)$ for which there exist compatible isomorphisms

$$\varphi\colon \tau M^\circ(G, X^+) \longrightarrow M^\circ(G', X'^+)$$
$$\psi\colon G_{\mathbf{A}_f} \longrightarrow G'_{\mathbf{A}_f}.$$

In [2] Borovoi shows that the analogue of (0.1) for non-connected Shimura varieties implies the existence of canonical models for all Shimura varieties. We adapt his methods to show, in section 4, 5, and 6, that (0.1) implies Langlands's conjecture for all connected Shimura varieties.

In the final section we review the main consequences of this result: Langlands's conjecture in its original form; the existence of canonical models in the sense of Deligne; Langlands's conjecture describing the action of complex conjugation on a Shimura variety with a real canonical model; the existence of canonical models in the sense of Shimura. An expository account of this, and related material, can be found in [11].

I am indebted to P. Deligne for several valuable conversations and, especially, for suggestions that led to the elimination of a hypothesis on the congruence nature of arithmetic subgroups in the statement of the main theorem.

Notations. The notations are the same as [4]. In particular, a reductive group G is connected with centre $Z(G)$. A superscript $+$ denotes a topological connected component, and $G(\mathbf{Q})_+$ denotes the inverse image of $G^{ad}(\mathbf{R})^+$ under $G(\mathbf{Q}) \to G^{ad}(\mathbf{R})$. The symbol $\mathbf{S}$ denotes $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m$, and, for any homomorphism $h\colon \mathbf{S} \longrightarrow G$, μ_h denotes the restriction of $h_{\mathbf{C}}$ to the first factor in $\mathbf{S}_{\mathbf{C}} = \mathbf{G}_m \times \mathbf{G}_m$.

If (G, X) and (G', X') are pairs defining Shimura varieties, then a map $(G, X) \to (G', X')$ is a homomorphism $G \to G'$ carrying X into X'. An inclusion $(T, h) \quad \to (G, X)$ will always mean that T is a maximal torus of G.

If G is a group over $\mathbf{Q}$, then $G_l = G_{\mathbf{Q}_l}$ .By a homomorphism $G_{\mathbf{A}^f} \to G'_{\mathbf{A}^f}$ we mean a family of homomorphisms of algebraic groups $G_l \to G'_l$ whose product maps $G(\mathbf{A}^f)$ into $G'(\mathbf{A}^f)$. (As usual, $\mathbf{A}^f = (\varprojlim \mathbf{Z}/m\mathbf{Z}) \otimes \mathbf{Q}$.) The closure of $G(\mathbf{Q})$ in $G(\mathbf{A}^f)$ is denoted by $G(\mathbf{Q})^-$.

We say that a group G satisfies the Hasse principle for H^i if the map of Galois cohomology groups $H^i(\mathbf{Q}, G) \to \prod_l H^i(\mathbf{Q}_l, G)$ is injective, where l runs through all primes of $\mathbf{Q}$ including $l = \infty$.

If V is an algebraic variety over a field k, and $\tau\colon k \quad \to K$ is an inclusion of fields, then τV denotes $V \otimes_{k,\tau} K = V \times_{\mathrm{spec}\, k} \mathrm{spec}\, K$.

The main definitions concerning connected Shimura varieties are reviewed in an Appendix.

§1. Statement of the First Theorem

To a pair (G, X^+) satisfying (C) (see the Appendix), a special point $h \in X^+$, and an automorphism τ of $\mathbf{C}$, Langlands [8] associates another pair $({}^{\tau}G, {}^{\tau}X^{+})$ also satisfying (C), a special point ${}^{\tau}h \in {}^{\tau}X^{+}$, and an isomorphism $\psi_{\tau} = (g \mapsto {}^{\tau}g)\colon G(\mathbf{A}^{f}) \longrightarrow {}^{\tau}G(\mathbf{A}^{f})$. (See also [14]; in general, we shall use the definitions of [14] and [15] rather than [8].)

Theorem 1.1. *Assume that G is simply-connected; then, with the above notations, there exists an isomorphism*

$$\varphi_{\tau}\colon \tau M^{o}(G, X^{+}) \longrightarrow M^{o}({}^{\tau}G, {}^{\tau}X^{+})$$

such that

$$(a)\ \varphi_{\tau}(\tau[h]) = [{}^{\tau}h] \quad (\textit{for the particular special } h)$$
$$(b)\ \varphi_{\tau}(\tau(gx)) = {}^{\tau}g\varphi_{\tau}(x), \quad \textit{all } x \in M^{o}(G, X^{+}),\ g \in G(\mathbf{A}^{f}).$$

Remark 1.2. This is a weak form of part (a) of Conjecture C^{o} [15, p. 340]. In §6 we shall see that it leads to a proof of the full conjecture.

Remark 1.3. The real approximation theorem [3, 0.4] shows that $G(\mathbf{Q})_{+}$ is dense in $G(\mathbf{R})_{+}$. Therefore, for any $x \in X^{+}$, $G(\mathbf{Q})_{+}x$ is (real) dense in X^{+}, and its image in $\Gamma \backslash X^{+}$ is Zariski dense. It follows that there is at most one map φ_{τ} satisfying the conditions of the theorem.

Remark 1.4. Let G be a semi-simple group over $\mathbf{Q}$, and let $i\colon T \to G$ be the inclusion of a maximal torus. Then $\mathrm{Aut}(G, i) = \overline{T} \overset{\mathrm{df}}{=} T/Z(G)$. Fix a finite Galois extension $L/\mathbf{Q}$ and consider triples (G', i', ψ) where $(G', T \xrightarrow{i'} G')$ is isomorphic to (G, i) over L, and ψ is an isomorphism $(G, i)_{\mathbf{A}^{f}} \overset{\approx}{\to} (G', i')_{\mathbf{A}^{f}}$ (i.e., an isomorphism $G_{\mathbf{A}^{f}} \to G'_{\mathbf{A}^{f}}$ carrying $i_{\mathbf{A}^{f}}$ into $i'_{\mathbf{A}^{f}}$). Given such a triple, choose an $a\colon (G, i)_{L} \overset{\approx}{\to} (G', i')_{L}$ and define $\beta = \beta(G', i', \psi) \in \overline{T}(\mathbf{A}^{f}_{L})$ by the equation $\psi \circ \beta = a$. Let $\overline{\beta}$ be the image of β in $\overline{T}(\mathbf{A}^{f}_{L})/\overline{T}(L)$. Then $(G', i', \psi) \mapsto \overline{\beta}(G', i', \psi)$ defines a one-to-one correspondence

$$\{\text{isomorphism classes of triples } (G', i', \psi)\} \leftrightarrow \left(\overline{T}(\mathbf{A}^{f}_{L}/\overline{T}(L)\right)^{\mathrm{Gal}(L/\mathbf{Q})}.$$

Consider now $(T, h) \overset{i}{\hookrightarrow} (G, X^{+})$. Using the element $\overline{\beta}(\tau, \mu_{h})$ explicitly defined in [14, 3.18], one obtains from this correspondence a pair $({}^{\tau}G, {}^{\tau}i\colon T \hookrightarrow {}^{\tau}G)$ together with an isomorphism $\psi_{\tau}\colon G_{\mathbf{A}^{f}} \longrightarrow {}^{\tau}G_{\mathbf{A}^{f}}$ carrying

i into ${}^\tau i$. These are the objects in (1.1). The map ${}^\tau h\colon \mathbf{S} \to T \xrightarrow{{}^\tau i} {}^\tau G$ is that whose associated cocharacter is $\tau\mu_h$, and ${}^\tau X^+$ is the ${}^\tau G^{ad}(\mathbf{R})^+$-conjugacy class containing ${}^\tau h$.

One other fact we shall need concerns the class γ of $({}^\tau G, {}^\tau i)$ in $H^1(\mathbf{Q}, \overline{T})$ (equal to the image of $\overline{\beta}(\tau, \mu_h)$ under

$$(\overline{T}(\mathbf{A}_L^f)/\overline{T}(L))^{\mathrm{Gal}(L/\mathbf{Q})} \xrightarrow{d} H^1(\mathrm{Gal}(L/\mathbf{Q}), \overline{T}(L)) \to H^1(\mathbf{Q}, \overline{T}).$$

The existence of ψ shows that the image of γ in $H^1(\mathbf{Q}_l, \overline{T})$ is zero, for all finite l; its image in $H^1(\mathbf{R}, \overline{T})$ is represented by $\tau\mu(-1)/\mu(-1)$ [14, 3.14].

Remark 1.5. Theorem 1.1 (in fact, Conjecture C^o) is proved in ([5], [15]) for Shimura varieties of abelian type. Since we shall need to make use of this result for groups of type A, we outline the main steps in its proof. For pairs (G, X^+) with G the symplectic group and X^+ the Siegel upper half-space, (1.1) is shown in [15, 7.17] to be a consequence of a statement about abelian varieties of CM-type. This statement is proved in [5]. Let G be of type A, and suppose that G is almost simple over $\mathbf{Q}$. Then G can be embedded into a symplectic group ([4, 2.3.10]), and the following easy lemma can be applied.

Lemma 1.6. *Let (G, X^+) satisfy (C), and let $\overline{H}$ be a reductive subgroup of G^{ad}. Suppose that some $h \in X^+$ factors through $\overline{H}_{\mathbf{R}}$, and let X_H^+ be the $\overline{H}^{ad}(\mathbf{R})^+$-conjugacy class containing the composite h' of h with $\overline{H} \to \overline{H}^{ad}$. Assume that $\overline{H}^{ad}$ satisfies (C_3) and let H be the simply connected covering group of $\overline{H}^{ad}$. Then (H, X_H^+) satisfies (C), and there is an embedding $M^o(H, X_H^+) \hookrightarrow M^o(G, X^+)$ compatible with $H(\mathbf{A}^f) \hookrightarrow G(\mathbf{A}^f)$ under which $[h'] \mapsto [h]$. If h is special, so also is h', and if (1.1) holds for (G, X^+) and h, then it does also for (H, X_H^+) and h'.*

§2. Morphisms of Shimura Varieties

A morphism $\varphi\colon M^o(G, X^+) \longrightarrow M^o(G', X'^+)$ will be said to be finite and étale if, for any $\Gamma' \in \sum(G')$, there exists a $\Gamma \in \sum(G)$ such that

$$\varphi_{\Gamma',\Gamma}\colon \Gamma \backslash X^+ \longrightarrow \Gamma' \backslash X'^+$$

is finite and étale.

Proposition 2.1. *Let (G,X^+) and (G',X'^+) satisfy (C), and let $\psi\colon G_{\mathbf{A}^f}\longrightarrow G'_{\mathbf{A}^f}$ be an isomorphism such that $\psi(G(\mathbf{Q})_+^-)=G'(\mathbf{Q})_+^-$. For any finite étale morphism $\varphi\colon M^o(G,X^+)\longrightarrow M^o(G',X'^+)$ compatible with ψ, there exists an element $g\in G(\mathbf{Q})_+^-$ and an isomorphism $\psi_o\colon G\overset{\approx}{\rightarrow}G'$ such that $\varphi=M^o(\psi_o)\circ g$. In particular, φ is an isomorphism.*

Proof. Choose an $h\in X^+$ and write $\varphi([h])=g'[h']$, some $h'\in X'^+$, $g'\in G'(\mathbf{Q})_+^-$ (see the Appendix). Let $g^{-1}=\psi^{-1}(g')$; then

$$\varphi\circ g\colon M^o(G,X^+)\longrightarrow M^o(G',X'^+)$$

and

$$\psi\circ\underline{ad}\,g\colon G_{\mathbf{A}^f}\longrightarrow G'_{\mathbf{A}^f}$$

satisfy the same conditions as φ and ψ,and $\varphi\circ g[h]=[h']$. It therefore suffices to prove the following proposition.

Proposition 2.2. *In addition to the hypotheses of (2.1), suppose there exist $h\in X^+$ and $h'\in X'^+$ such that $\varphi([h])=[h']$. Then ψ is defined over $\mathbf{Q}$ and $\varphi=M^o(\psi)$.*

Proof. There exists a unique isomorphism $\tilde\varphi\colon X^+\overset{\approx}{\rightarrow}X'^+$ lifting all $\varphi_{\Gamma',\Gamma}\colon \Gamma\setminus X^+\longrightarrow\Gamma'\setminus X'^+$ and sending h to h'. Let $\tilde\varphi_*$ be the map $\alpha\mapsto\tilde\varphi\circ\alpha\circ\tilde\varphi^{-1}\colon\mathrm{Aut}(X^+)\to\mathrm{Aut}(X'^+)$. For any

$$\alpha\in G^{ad}(\mathbf{Q})^+\subset\mathrm{Aut}(X^+),$$

$\tilde\varphi_*(\alpha)$ induces an automorphism of $M^o(G',X'^+)$; in particular, it lies in the commensurability group of any $\Gamma'\in\sum(G')$ and therefore belongs to $G'^{ad}(\mathbf{Q})$ (see [1, Thm. 2]). Consider a $q\in G(\mathbf{Q})_+$, and write q_∞ and q_f for its images in $G^{ad}(\mathbf{Q})_+$ and $G(\mathbf{Q})_+^-$. Then q_∞ and q_f define the same automorphism of $M^o(G,X^+)$, and so $\tilde\varphi_*(q_\infty)$ and $\psi(q_f)$ define the same automorphism of $M^o(G',X'^+)$. They therefore have the same image in $G'^{ad}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\,G')=G'(\mathbf{Q})_+^-{}_{\ast\, G'(\mathbf{Q})_+}G'^{ad}(\mathbf{Q})^+$. Therefore $\psi(q_f)\in G'(\mathbf{Q}_+)$ (and $\tilde\varphi_*(q_\infty)=\psi(q_f)$ in $G'^{ad}(\mathbf{Q})^+$). As $G(\mathbf{Q})_+$ is Zariski dense in G, we conclude that ψ is defined over $\mathbf{Q}$. Write ψ_o for ψ regarded as a $\mathbf{Q}$-rational map, and consider

$$M^o(\psi_o)^{-1}\circ\varphi\colon M^o(G,X^+)\longrightarrow M^o(G,X^+).$$

It remains to show that this map is the identity. We know that it is finite and étale, maps $[h]$ to $[\psi_o^{-1} \circ h']$, and commutes with the action of $G(\mathbf{Q})_+^-$. We have therefore to prove the proposition in the case that $G = G'$ and ψ is the identity map. The equality noted parenthetically in the last paragraph shows that in this case $\tilde{\varphi}_*(q) = q$ for $q \in G(\mathbf{Q})_+/Z(\mathbf{Q})$. The real approximation theorem [3, 0.4] states that $G(\mathbf{Q})_+$ is dense in $G(\mathbf{R})_+$, and so $\tilde{\varphi}_*$ is the identity map on $G^{ad}(\mathbf{R})^+$. As $\tilde{\varphi}_* = \underline{ad}\,\tilde{\varphi}$ this means that $\tilde{\varphi}$ centralizes $G^{ad}(\mathbf{R})^+$, which implies that $\tilde{\varphi} = id$ [18, II 2.6].

Corollary 2.3. *The map*

$$G(\mathbf{Q})_+^- \underset{G(\mathbf{Q})_+}{*} G^{ad}(\mathbf{Q})^+ \to \mathrm{Aut}(M^o(G, X^+))$$

identifies $\mathrm{Aut}_{G(\mathbf{Q})_+}(M^o(G, X^+))$ *with* $\{g * \alpha \mid \underline{ad}\, g = \alpha^{-1} \text{ in } G^{ad}(\mathbf{A}^f)\}$.

Proof. An automorphism of $M^o(G, X^+)$ commuting with the action of $G(\mathbf{Q})_+$ commutes (by continuity) with the action of $G(\mathbf{Q})_+^-$. It can therefore be written $g \circ M^o(\alpha)$ for some $g \in G(\mathbf{Q})_+^-$ and $\alpha \in \mathrm{Aut}(G)$. In order for this map to commute with the action of $G(\mathbf{Q})_+$, α^{-1} and $\underline{ad}\, g$ must be equal. In particular α must be an inner automorphism, $\alpha \in G^{ad}(\mathbf{Q})$, and so the map is that defined by $g * \alpha$.

Corollary 2.4. *If $Z = Z(G)$ satisfies the Hasse principle for H^1, then $Aut_{G(\mathbf{Q})_+}(M^o(G, X^+)) = Z(\mathbf{A}^f) \cap G(\mathbf{Q})_+^-/Z(\mathbf{Q})$; for example, if G is an adjoint group,* $\mathrm{Aut}_{G(\mathbf{Q})_+}(M^o(G, X^+)) = 1$.

Proof. The hypothesis implies that if an element of $G^{ad}(\mathbf{Q})^+$ lifts to an element of $G(\mathbf{A}^f)$, then it lifts to an element of $G(\mathbf{Q})$.

Example 2.5. The last corollary applies to the Shimura varieties defined by simply connected groups without factors of type A_n, $n \geq 8$ (see (3.8) below). For these groups $G(\mathbf{Q})_+^- = G(\mathbf{A}^f)$ and so

$$\mathrm{Aut}_{G(\mathbf{Q})_+}(M^o(G, X^+)) = Z(\mathbf{A}^f)/Z(\mathbf{Q}).$$

The propositions can also be used to compute the automorphism groups of non-connected Shimura varieties.

Corollary 2.6. *Let (G,X) satisfy [4, 2.1.1.1–2.1.1.3]; then the canonical map* $\left(G(\mathbf{A}^f)/Z(\mathbf{Q})^-\right) *_{G(\mathbf{Q})} G^{ad}(\mathbf{Q}) \to \operatorname{Aut}(M(G,X))$ *identifies*

$$\operatorname{Aut}_{G(\mathbf{A}^f)}(M(G,X))$$

with $\{g * \alpha \mid \underline{ad}(g) = \alpha^{-1} \text{ in } G^{ad}(\mathbf{A}^f)\}$.

Proof. Let $\varphi \in \operatorname{Aut}_{G(\mathbf{A}^f)}(M(G,X))$. Then there exists a $g \in G(\mathbf{A}^f)$ such that $g \circ \varphi$ maps $[h,1]$ to $[h',1]$ for some $h \in X^+$ and $h' \in X'^+$. Then $g \circ \varphi$ maps $M^o(G^{der},X^+)$ into $M^o(G^{der},X^+)$ and we can therefore apply 2.2.

Corollary 2.7. *Assume, in (2.6), that the centre Z of G satisfies the Hasse principle for H^1 for finite primes. Then*

$$\operatorname{Aut}_{G(\mathbf{A}^f)}(M(G,X)) = Z(\mathbf{A}^f)/Z(\mathbf{Q})^-.$$

Proof. This follows from (2.6) as (2.4) follows from (2.3).

§3. Proof of a Weak Form of (1.1)

This section is devoted to proving the following result.

Proposition 3.1. *Let (G,X^+) satisfy (C), and assume that G is simply connected; then, for any automorphism τ of $\mathbf{C}$, there is a pair (G',X'^+) satisfying (C) for which there exist compatible isomorphisms*

$$\varphi\colon \tau M^o(G,X^+) \longrightarrow M^o(G',X'^+)$$
$$\psi\colon G_{\mathbf{A}^f} \longrightarrow G'_{\mathbf{A}^f}.$$

We begin by recalling a theorem of Kazhdan. Let X^+ be a Hermitian symmetric domain, so that the identity component G of $\operatorname{Aut}(X^+)$ is a product of connected non-compact simple real Lie groups. For Γ an arithmetic subgroup of G, $\Gamma \backslash X^+$ carries a unique structure of an algebraic variety, and so $\tau(\Gamma \backslash X^+)$ is defined, $\tau \in \operatorname{Aut}(\mathbf{C})$.

Theorem 3.2 (Kazhdan). (a) *The universal covering space* X'^{+} *of* $\tau(\Gamma \setminus X^{+})$ *is a Hermitian symmetric domain.*

(b) *Let* G' *be the identity component of* $\mathrm{Aut}(X'^{+})$, *and identify the fundamental group* Γ' *of* $\tau(\Gamma \setminus X^{+})$ *with a subgroup of* G'; *then* Γ' *is a lattice in* G'.

Proof. The assumption that Γ is an arithmetic subgroup of G means that there exists a group G_1 over $\mathbf{Q}$ and a surjective homomorphism $f: G_1(\mathbf{R})^{+} \longrightarrow G$ with compact kernel carrying an arithmetic subgroup of G_1 into a group commensurable with Γ. If G_1 is the symplectic group, then the theorem follows from the theory of moduli varieties of abelian varieties. If G_1 has no $\mathbf{Q}$-simple factor G_0 such that $G_{0\mathbf{R}}$ is of type E_6 or E_7 or has factors of both types $D^{\mathbf{R}}$ and $D^{\mathbf{H}}$, then the $\mathbf{Q}$-simple factors of G_1 can be embedded into symplectic groups, and this case follows from the last case. When $\Gamma \setminus X^{+}$ is compact (so that G_1 has $\mathbf{Q}$-rank zero), the theorem is proved in [6] (it also follows from Yau's theorem [21] on the existence of Einstein metrics). The remaining cases are treated in [7].

Remark 3.3. If Γ is irreducible (for example, if G is $\mathbf{Q}$-simple) then Γ' is also irreducible because otherwise $\tau(\Gamma \backslash X^{+})$, and hence $\Gamma \backslash X^{+}$, would have a finite étale covering that was a product. Consequently, when $\mathrm{rank}_{\mathbf{R}}\, G' > 1$, Margulis's theorem [10, Thm. 1] shows that Γ' is arithmetic.

Let (G, X^{+}) be as in the statement of (3.1). In proving the proposition, we can assume that G is almost simple over $\mathbf{Q}$ and is not of type A (because when G is of type A we know much more — see (1.5)). This last assumption implies that G_l is not compact for any l.

Choose a compact open subgroup K of $G(\mathbf{A}^f)$ containing $Z(\mathbf{Q})$, and let $\Gamma = G(\mathbf{Q}) \cap K$ be the corresponding congruence subgroup. Then

$$M_K^{\circ}(G, X^{+}) = G(\mathbf{Q}) \setminus X^{+} \times G(\mathbf{A}^f)/K = \Gamma \setminus X^{+}.$$

On applying (3.2), one obtains a Hermitian symmetric domain X_1^{+}, a real Lie group G' such that $G' = \mathrm{Aut}(X_1^{+})^{+}$, and an irreducible lattice Γ' in G' such that $\tau M_K^{\circ}(G, X^{+}) = \Gamma' \setminus X_1^{+}$.

For any $g \in G(\mathbf{Q})$, let $\Gamma_g = \Gamma \cap g^{-1}\Gamma g$. There are two obvious maps $1, g: \Gamma_g \setminus X^{+} \rightrightarrows \Gamma \setminus X^{+}$, namely the projection map and the projection map preceded by left multiplication by g. On applying τ, we obtain maps

$$\tau(1), \tau(g): \tau(\Gamma_g \setminus X^{+}) \rightrightarrows \tau(\Gamma \setminus X^{+}) = \Gamma' \setminus X_1^{+}.$$

Choose $X_1^+ \to \tau(\Gamma_g \setminus X^+)$ so as to make the following diagram commute with the upper arrows, and choose $\tilde{g}$ to make it commute with the lower arrows:

$$\begin{array}{ccc} X_1^+ & \overset{id}{\underset{\tilde{g}}{\rightrightarrows}} & X_1^+ \\ \downarrow & & \downarrow \\ \tau(\Gamma_g \setminus X^+) & \overset{\tau(1)}{\underset{\tau(g)}{\rightrightarrows}} & \Gamma' \setminus X_1^+ \end{array} \tag{3.3.1}$$

The double coset $\Gamma'\tilde{g}\Gamma' \subset \mathrm{Aut}(X_1^+)$ is well-defined, and we let

$$\Gamma_0 = \bigcup \Gamma'\tilde{g}\Gamma', \qquad g \in G(\mathbf{Q}).$$

Then Γ_0 is a subgroup of $\mathrm{Aut}(X_1^+)$ and is independent of the choice of K. (In [7] it is denoted by G^σ.)

There is a map $\gamma \mapsto \gamma_f \colon \Gamma_0 \longrightarrow G(\mathbf{A}^f)/Z(\mathbf{Q})$ that can be characterized as follows: for all $\Gamma = K \cap G(\mathbf{Q})$ (as above), and all $g \in G(\mathbf{Q}) \cap \gamma_f K$, the diagram (3.3.1) commutes with $\tilde{g}$ replaced by γ. We have therefore a canonical embedding

$$\gamma \mapsto (\gamma_\infty, \gamma_f) \colon \Gamma_0 \longrightarrow \mathrm{Aut}(X_1^+) \times G(\mathbf{A}^f)/Z(\mathbf{Q}).$$

Both γ_∞ and γ_f act on $\tau M^\circ(G, X^+)$, the first through its action on X_1^+ and the second through its action on $M^\circ(G, X^+)$. These actions are equal.

Lemma 3.4. *Regard* Γ_0 *as a subgroup of* $G(\mathbf{A}^f)/Z(\mathbf{Q})$.

(a) Γ_0 *is dense in* $G(\mathbf{A}^f)/Z(\mathbf{Q})$.

(b) $\Gamma_0 \cap Z(\mathbf{A}^f)/Z(\mathbf{Q}) = 1$.

(c) *For any compact open subgroup* K *of* $G(\mathbf{A}^f)$ *containing* $Z(\mathbf{Q})$,

$$\tau M_K^\circ(G, X^+) = (\Gamma_0 \cap K/Z(\mathbf{Q})) \setminus X_1^+ = \Gamma_0 \setminus X_1^+ \times G(\mathbf{A}^f)/K.$$

Proof. (a) it is clear from the definition of $\gamma \mapsto \gamma_f$ that, for any K, $\Gamma_0 \to G(\mathbf{A}^f)/Z(\mathbf{Q})K$ is surjective.

(b) Suppose $\gamma \in \Gamma_0$ is such that $\gamma_f \in Z(\mathbf{Q}^f)/Z(\mathbf{Q})$. Then the remark preceding the statement of the lemma shows that γ_∞ centralizes Γ_0 in G'. As Γ_0 has finite covolume in G', this implies that γ_∞ is in the centre of G' [17, 5.4, 5.18], and so $\gamma_\infty = 1$.

(c) We can assume that K is the group used in the construction of Γ_0. It is then clear that $\Gamma' = \Gamma_0 \cap K/Z(\mathbf{Q})$. The second equality follows from the first and (a).

Later we shall show that Γ_0 is contained in the identity component G' of $\mathrm{Aut}(X_1^+)$, but for the present we define $\Gamma_0^+ = \Gamma_0 \cap G'$.

We now fix an integral structure for G and define, for any finite set S of finite primes, $G(\mathbf{A}_S^f) = \prod_{l \in S} G(\mathbf{Q}_l) \times \prod_{l \notin S} G(\mathbf{Z}_l)$. Let $\Gamma_{0,S} = \Gamma_0 \cap G(\mathbf{A}_S^f)$; then $\Gamma_{0,S}^+ \stackrel{\mathrm{df}}{=} \Gamma_{0,S} \cap \Gamma_0^+$ can be regarded as a subgroup of $G'_S \stackrel{\mathrm{df}}{=} G' \times \prod_{l \in S} G_l$.

Lemma 3.5. *The group $\Gamma_{0,S}^+$ is an irreducible lattice in G'_S.*

Proof. It follows from (3.4c) that Γ_0 is a discrete subgroup of $G' \times G(\mathbf{A}^f)$, and therefore $\Gamma_{0,S}^+$ is a discrete subgroup of $G' \times G(\mathbf{A}_S^f)$. As $\prod_{l \notin S} G(\mathbf{Z}_l)$ is compact, the projection $G'_S \times \prod_{l \notin S} G(\mathbf{Z}_l) \to G'_S$ takes discrete groups to discrete groups [20, p. 4], and so $\Gamma_{0,S}^+$ is discrete in G'_S.

Let U be a compact open subgroup of $\prod_{l \in S} G_l$, and let

$$\Gamma = (U \times \prod_{l \notin S} G(\mathbf{Z}_l)) \cap \Gamma_0^+.$$

Then $U\Gamma_{0,S} \setminus G'_S = \Gamma' \setminus G'$, which carries an invariant finite measure. It follows that $\Gamma_{0,S}$ is of finite covolume in G'_S.

Our assumption that G is almost simple over $\mathbf{Q}$ implies that $\Gamma_{0,S}$ is irreducible (cf. 3.3).

Now assume that S is nonempty and sufficiently large that $\mathrm{rank}(G'_S) \geq 2$. Then Margulis's theorem [10, Thm. 7] shows that $\Gamma_{0,S}^+$ is arithmetic. More precisely, there is the following result.

Lemma 3.6. *There exists an algebraic group G_1 over $\mathbf{Q}$ and a map $\psi_S\colon G_{1S} \longrightarrow G'_S$, where $G_{1S} = G_{1\mathbf{R}} \times \prod_{l \in S} G_{1,l}$, having the following properties:*

(a) *there exists an S-arithmetic subgroup Γ_S in $G_1(\mathbf{Q})$ such that $\psi_S(\Gamma_S)$ is commensurable with $\Gamma_{0,S}^+$;*

(b) *write $\psi_S = \psi_\infty \times \prod_{l \in S} \psi_l$; then ψ_∞ is surjective with a compact kernel, and each ψ_l is an isomorphism.*

Moreover, (G_1, ψ_S) is uniquely determined by the conditions (a) and (b).

Proof. Margulis's theorem gives us a pair (G_1, ψ_S) satisfying (a) and such that ψ_S is surjective with a compact kernel. We can suppose that Γ_S is irreducible. Then G_1 is almost simple over $\mathbf{Q}$ and so cannot be of type A. Therefore G_{1l} does not have any compact factors and ψ_l must be an isomorphism.

Let Λ be a subgroup of $\Gamma_{0,S}$ of finite index, and let $A_\Lambda = \{(f_l) \in \prod_{l \in S} \Gamma(G_l, \mathcal{O}_{G_l}) \mid f_l(\lambda) \in \mathbf{Q},\ f_l(\lambda) = f_{l'}(\lambda), \text{ all } l, l' \in S, \lambda \in \Lambda\}$. Then, for all sufficiently small Λ, A_Λ is independent of Λ and $\operatorname{Spec} A_\Lambda = G_1$. This shows the uniqueness.

When we enlarge S, to S' say, then G_1 does not change and $\psi_{S'}|G_{1S} = \psi_S$. We therefore get a map

$$\psi = \psi_\infty \times \psi^f : G_{1\mathbf{R}} \times G_{1\mathbf{A}^f} \longrightarrow G' \times G_{\mathbf{A}^f}$$

such that for all finite S, $\psi^f(G_{1,S})$ is commensurable with $\Gamma_{0,S}$, where $G_{1,S} = G_1(\mathbf{Q}) \cap G_1(\mathbf{A}_S^f)$. Let G_1^{comp} be the product of the anisotropic factors of $G_{1\mathbf{R}}$, and consider

$$\begin{array}{ccc} G^1(\mathbf{Q}) & \hookrightarrow & (G_1/G_1^{comp})(\mathbf{R}) \times G_1(\mathbf{A}^f) \\ & & \downarrow \overline{\psi} \\ \Gamma_0 & \hookrightarrow & \operatorname{Aut}(X_1^+) \times G(\mathbf{A}^f)/Z(\mathbf{Q}). \end{array}$$

For any finite set S, $\tilde{\Gamma}_{0,S} \overset{\mathrm{df}}{=} \overline{\psi}^{-1}(\Gamma_{0,S})$ is commensurable with $G_1(\mathbf{Q})_S$. Ultimately we shall show that $\Gamma_0 = G_1(\mathbf{Q})/Z(\mathbf{Q})$, but we begin with a weaker result.

Lemma 3.7. *The group* $\Gamma_0 \subset \psi(G_1(\mathbf{Q})) \cdot Z(\mathbf{A}_S^f)$.

Proof. Let $\gamma \in \Gamma_0$, and let $\tilde{\gamma} \in \operatorname{Aut}(X_1^+) \times G_1(\mathbf{A}^f)$ map to γ regarded as an element of $\operatorname{Aut}(X_1^+) \times G(\mathbf{A}^f)/Z(\mathbf{Q})$. For large enough S, $\gamma \in \Gamma_{0,S}$, and $\underline{ad}\,\tilde{\gamma}$ maps $\tilde{\Gamma}_{0,S}$ into $\tilde{\Gamma}_{0,S}$.

Choose an irreducible representation T: $G_1^{ad} \hookrightarrow GL_n$ of G_1^{ad}, and consider the representation $\underline{ad}\,\tilde{\gamma} \circ T$ of $\tilde{\Gamma}_{0,S} \cap G_1(\mathbf{Q})$. The Zariski closure of $(\underline{ad}\,\tilde{\gamma} \circ T)(\tilde{\Gamma}_{0,S} \cap G_1(\mathbf{Q}))$ in GL_n is $T(G_1^{ad})$, and so [10, Thm. 8] can be applied to show that there is a morphism $\overline{\alpha}$: $G_1 \longrightarrow G_1^{ad}$ whose restriction to $\tilde{\Gamma}_{0,S}$ is $\underline{ad}\,\tilde{\gamma}$. Lift $\overline{\alpha}$ to an isomorphism α: $G_1 \longrightarrow G_1$. Then α and $\underline{ad}\,\tilde{\gamma}$ agree on a subgroup of $\tilde{\Gamma}_{0,S}$ of finite index (therefore also on a subgroup of $G_1(\mathbf{Q})_S$ of finite index). As $\Gamma_{0,S}$ is dense in $\prod_{l \in S} G(\mathbf{Q}_l)/Z(\mathbf{Q})$, this shows that

$\alpha = \underline{ad}\,\gamma$ on $\prod G(\mathbf{Q}_l)/Z(\mathbf{Q})$. In particular, α is an inner automorphism, i.e., $\alpha \in G_1^{ad}(\mathbf{Q})$. Moreover, α has the property that it lifts to $G_1(\mathbf{Q}_l)$ for all $l \in S$, and hence for all l because we can extend S. The next lemma shows that this implies that α lifts to an element $\alpha_1 \in G_1(\mathbf{Q})$. This element α_1 has the same image as γ in $\mathrm{Aut}(X_1^+) \times G^{ad}(\mathbf{A}^f)$, which completes the proof.

Lemma 3.8. *Let G be a simply connected semi-simple group over a number field k, and let $Z = Z(G)$. Then $H^1(k, Z) \to \prod_{v\, finite} H^1(k_v, Z)$ is injective, provided G has no factors of type A_n, $n \geq 4$.*

Proof. We can assume G is absolutely almost simple. Then $Z(\bar{k}) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/n\mathbf{Z}$, $n \leq 4$. If $Z(\bar{k}) = Z(k)$, then the result is obvious from class field theory. In any case, $Z(\bar{k}) = Z(L)$ for L a Galois extension of k with Galois group S_3 or $\mathbf{Z}/n\mathbf{Z}$, $n \leq 3$. The exact sequence

$$0 \to H^1(L/k, Z) \to H^1(k, Z) \to H^1(L, Z)$$

shows that is suffices to prove that $H^1(L/k, Z) \to \prod H^1(L_v/k_v, Z)$ is injective. In fact it suffices to do this with k replaced by the fixed field of a Sylow subgroup of $\mathrm{Gal}(L/k)$. But then the Galois group is cyclic, and the result is obvious.

Lemma 3.7 implies that Γ_0 and $G_1(\mathbf{Q})$ have the same image in $G^{ad}(\mathbf{A}^f) = G_1^{ad}(\mathbf{A}^f)$. If we form the quotient of $M^o(G, X^+)$ by the action of $Z(\mathbf{A}^f)$, we get $Z(\mathbf{A}^f) \setminus M^o(G, X^+) = M^o(G^{ad}, X^+)$. Therefore,

$$\begin{aligned}\tau M^o(G^{ad}, X^+) &= \Gamma_0 Z(\mathbf{A}^f) \setminus X_1^+ \times G(\mathbf{A}^f) \\ &= G_1(\mathbf{Q}) Z_1(\mathbf{A}^f) \setminus X_1^+ \times G_1(\mathbf{A}^f) = M^o(G_1^{ad}, X_1^+).\end{aligned}$$

We have proved (3.1) with G replaced by G^{ad}. An argument of Borovoi (see 5.2a below; it is not necessary to assume there that G is simply connected) shows that the map $G^{ad}_{\mathbf{A}^f} \to G^{ad}_{1\mathbf{A}^f}$ defined by ψ identifies G_1^{ad} with an inner form of G^{ad}. Therefore $\psi^f\colon G_{\mathbf{A}^f} \longrightarrow G_{1\mathbf{A}^f}$ has the same property, and so $\psi^f | Z(G)$ is defined over $\mathbf{Q}$, i.e., ψ^f identifies $Z \overset{\mathrm{df}}{=} Z(G)$ with $Z_1 \overset{\mathrm{df}}{=} Z(G_1)$.

From (3.4) we know that $\Gamma_0 \cap Z(\mathbf{A}^f)/Z(\mathbf{Q}) = 1$. Therefore any element g of $G_1(\mathbf{Q})/Z(\mathbf{Q})$ can be written uniquely as $g = \gamma \cdot z_g$, $\gamma \in \Gamma_0$,

$z_g \in Z(\mathbf{A}^f)/Z(\mathbf{Q})$. The map $g \mapsto z_g$ is a homomorphism

$$G_1(\mathbf{Q})/Z(\mathbf{Q}) \to Z(\mathbf{A}^f).$$

If we knew, as is conjectured, that $G_1(\mathbf{Q})/Z(\mathbf{Q})$ is simple, then this homomorphism would have to be zero, and we would have achieved our immediate goal of showing that $\Gamma_0 \supset G_1(\mathbf{Q})/Z(\mathbf{Q})$. Instead, we argue as follows.

Let F be a totally real Galois extension of $\mathbf{Q}$ with Galois group Δ; let $G_* = \mathrm{Res}_{F/\mathbf{Q}} G_F$ and let X_*^+ be such that there is an embedding

$$(G, X^+) \hookrightarrow (G_*, X_*^+).$$

Then Δ acts on G_* and X_*^+, and G is the unique subgroup of G_* such that $\prod_{\delta\in\Delta} G_F \xrightarrow{(\delta,\dots)} (G_*)_F$ is an isomorphism. The group Δ as continues to act when we make the above constructions for (G_*, X_*^+). In particular, Δ acts on G_{1*} and there is an inclusion $G_1 \quad \to G_{1*}$ that identifies G_{1*} with $\mathrm{Res}_{F/\mathbf{Q}} G_1$. We can conclude:

Lemma 3.9. *For any F as above, the diagram*

$$\begin{array}{ccc} g \mapsto z_g\colon G_1(\mathbf{Q})/Z(\mathbf{Q}) & \to & Z(\mathbf{A}^f)/Z(\mathbf{Q}) \\ \big\downarrow\cap & & \big\downarrow\cap \\ g \mapsto z_g\colon G_{1*}(\mathbf{Q})/Z_*(\mathbf{Q}) & \to & Z_*(\mathbf{A}^f)/Z_*(\mathbf{Q}) \end{array}$$

commutes, where $Z_* = \mathrm{Res}_{F/\mathbf{Q}} Z$.

Let $\gamma \in G_1(\mathbf{Q})$; we shall show that $\gamma\,(\mathrm{mod}\ Z(\mathbf{Q})) \in \Gamma_0$.

Lemma 3.10. *There exist fundamental maximal tori $T_i \subset G_1$, $i = 1, \dots, k$, and elements $\gamma_i = T_i(\mathbf{Q})$ such that $\gamma = \gamma_1 \dots \gamma_k$.*

Proof. Let U be the set of $g \in G_1(\mathbf{R})$ such that the centralizer of g is a compact maximal torus. Then, the usual argument using the Lie algebra, shows that U is open in $G_1(\mathbf{R})$. Moreover, U generates $G_1(\mathbf{R})$. Let $\gamma = \gamma_1' \dots \gamma_k'$ with $\gamma_i' \in U$. According to the real approximation theorem, the set $G_1(\mathbf{Q}) \cap U$ is dense in U. We can therefore choose $\gamma_i \in G_1(\mathbf{Q}) \cap U$,

$i = 2, 3, \ldots, k$, so close to γ_i' that $\gamma_1 \stackrel{\text{df}}{=} \gamma(\gamma_2 \ldots \gamma_k)^{-1}$ also lies in U. As $\gamma_1 \in G_1(\mathbf{Q})$, the elements $\gamma_1, \ldots, \gamma_k$ fulfill the requirements of the lemma.

Thus we can assume $\gamma \in T(\mathbf{Q}) \subset G(\mathbf{Q})$, where T is a maximal torus such that $T(\mathbf{R})$ is compact. This last condition on T implies that T splits over a CM-field L, which can be chosen to be Galois over $\mathbf{Q}$. Let F be the maximal totally real subfield of L and let $G_* = \operatorname{Res}_{F/\mathbf{Q}} G$. The construction of Borovoi recalled below in §4, gives a reductive group H_α of type A_1 such that $T_* \subset H_\alpha \subset G_{1*}$. After possibly extending F we can assume no $(H_\alpha)_l$ is anisotropic. Consider

$$\begin{array}{ccc} \gamma \in G_1(\mathbf{Q})/Z(\mathbf{Q}) \rightarrow & Z(\mathbf{A}^f)/Z(\mathbf{Q}) \\ \cap & \cap \\ G_{1*}(\mathbf{Q})/Z_*(\mathbf{Q}) \rightarrow & Z_*(\mathbf{A}^f)/Z_*(\mathbf{Q}). \end{array}$$

Note that $\gamma \in H_\alpha(\mathbf{Q})/Z_*(\mathbf{Q}) \cap H_\alpha(\mathbf{Q})$. A theorem of Platonov and Rapinčuk [16] shows that $H_\alpha(\mathbf{Q})$ has no non-central normal subgroup. Therefore, the lower map is zero on $H_\alpha(\mathbf{Q})/Z_*(\mathbf{Q}) \cap H_\alpha(\mathbf{Q})$ and so γ maps to zero. It therefore lies in Γ_0.

We have shown $\Gamma_0 \supset G_1(\mathbf{Q})/Z(\mathbf{Q})$. Therefore, for any compact open $K \subset G(\mathbf{A}^f)$ containing $Z(\mathbf{Q})$, we have a finite étale map

$$\Gamma_0 \setminus X_1^+ \times G(\mathbf{A}^f)/K \rightarrow G_1(\mathbf{Q}) \setminus X_1^+ \times G_1(\mathbf{A}^f)/K.$$

We therefore have a finite étale map

$$\tau M^o(G, X^+) \rightarrow M^o(G_1, X_1^+)$$

which is $G(\mathbf{A}^f)$-equivariant when we identify $G_{\mathbf{A}^f}$ with $G_{1\mathbf{A}^f}$ by means of ψ^f. When we apply τ^{-1} to this map, and repeat the whole of the above construction for (G_1, X_1^+) and τ^{-1}, we obtain maps

$$M^o(G, X^+) \rightarrow \tau^{-1} M^o(G_1, X_1^+) \rightarrow M^o(G_2, X_2^+).$$

The composite map satisfies the hypotheses of (2.1) and therefore is an isomorphism. This implies that the first map

$$M^o(G, X^+) \rightarrow \tau^{-1} M^o(G_1, X_1^+)$$

is an isomorphism and, on applying τ^{-1}, we get that

$$\tau M^o(G, X^+) \rightarrow M^o(G_1, X_1^+)$$

is an isomorpism; this is what we had to prove.

Corollary 3.11. *Suppose in (3.2) that Γ is a congruence group, i.e., that there exists a simply-connected semi-simple group G_1 over $\mathbf{Q}$ and a surjective homomorphism $f\colon G_1(\mathbf{R}) \longrightarrow G$ with compact kernel carrying a congruence subgroup of $G(\mathbf{Q})$ into a subgroup of Γ with finite index. Then $\tau(\Gamma \backslash X^+)$ is the quotient of a Hermitian symmetric domain by a congruence group.*

§4. Embedding Forms of SL_2 into G

In this section, we recall some results of Borovoi [2]. Let (G, X^+) satisfy (C), and let $(T, h) \subset (G, X)$. Throughout the section, we assume the following conditions hold:

(4.1a) $G = \mathrm{Res}_{F/\mathbf{Q}}\, G'$, where F is totally real and G' is absolutely almost simple;

(4.1b) the maximal torus $T' \subset G'$ such that $\mathrm{Res}_{F/\mathbf{Q}}\, T' = T$ splits over a quadratic, totally imaginary extension L of F.

Remark 4.2. The condition (b) implies that T', and any subtorus, is a product of one-dimensional tori, and therefore satisfies the Hasse principle for H^1.

As T'_L is split, we can write

$$\mathrm{Lie}\, G'_L = \mathrm{Lie}\, T'_L \bigoplus_{\alpha \in R} (\mathrm{Lie}\, G'_L)_\alpha$$

where $R = R(G'_{\mathbf{C}}, T'_{\mathbf{C}})$. An $\alpha \in R$ will be said to be *totally compact* if it is a compact root of $(G' \otimes_{F,\sigma} \mathbf{R})_{\mathbf{C}}$ for all embeddings $\sigma\colon F \hookrightarrow \mathbf{R}$. Let R^{ntc} denote the set of roots that are not totally compact. Then, for $\alpha \in R^{ntc}$,

$$\mathfrak{H}'_\alpha \overset{\mathrm{df}}{=} \mathrm{Lie}\, T'_L \oplus (\mathrm{Lie}\, G'_L)_\alpha \oplus (\mathrm{Lie}\, G'_L)_{-\alpha}$$

is defined over F, and we let H'_α be the corresponding connected subgroup of G'. Write $H_\alpha = \mathrm{Res}_{F/\mathbf{Q}}\, H'_\alpha$ and $Z_\alpha = Z(H_\alpha)$.

Proposition 4.3 (Borovoi, Oniščic). $Z(G) = \bigcap_\alpha Z_\alpha$, $(\alpha \in R^{ntc})$.

Proof. From the formula $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, valid whenver $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of a non-compact real Lie algebra, it follows that $R^{ntc} + R^{ntc} \supset R$. Thus

$$\bigcap_{\alpha \in R^{ntc}} Z(H'_\alpha) = \bigcap_{\alpha \in R^{ntc}} \mathrm{Ker}(\alpha) = \bigcap_{\alpha \in R} \mathrm{Ker}(\alpha) = Z(G'),$$

and the proposition follows by applying $\mathrm{Res}_{F/\mathbf{Q}}$.

Corollary 4.4. *Let* $\overline{T} = T/Z$ *and* $\overline{Z}_\alpha = Z_\alpha/Z$ *where* $Z = Z(G)$. *Regard* $\overline{Z}_\alpha(\mathbf{A}^f)/\overline{Z}_\alpha(\mathbf{Q})$ *as a subgroup of* $\overline{T}(\mathbf{A}^f)/\overline{T}(\mathbf{Q})$. *Then*

$$\bigcap_\alpha \overline{Z}_\alpha(\mathbf{A}^f)/\overline{Z}_\alpha(\mathbf{Q}) = 1, (\alpha \in R^{ntc}).$$

Proof. This follows easily from the fact that $\bigcap \overline{Z}_\alpha = 1$.

§5. Completion of the Proof of (1.1)

In this section, we use the methods of Borovoi [2] to deduce (1.1) from (3.1).

Let (G, X^+) satisfy (C), and consider $(T, h) \subset (G, X^+)$. For any $q \in T(\mathbf{Q})$, $q[h] = [h]$, and so there is a representation ρ_h of T on the tangent space $\mathfrak{t}_h$ to $M^o(G, X^+)$ at $[h]$.

Lemma 5.1 (cf. [2, 3.2, 3.3]). (a) $\rho_h \sim \bigoplus \alpha$, $\alpha \in R(G_\mathbf{C}, T_\mathbf{C})$, $\langle \alpha, \mu_h \rangle = 1$.

(b) *Suppose that* (G, X'^+) *also satisfies* (C), *and that*

$$(T', h') \subset (G, X'^+);$$

if $\rho_h \sim \rho_{h'}$, *then* $h = h'$ *(and* $X'^+ = X^+$*).*

Proof. (a) According to (C_1) there is a decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$

$$\mathfrak{g}_\mathbf{C} = \mathfrak{g}^{o,o} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

such that $Ad\,\mu(z)$ acts trivially on $\mathfrak{g}^{o,o}$, as z on $\mathfrak{p}^+$, and as z^{-1} on $\mathfrak{p}^-$. Thus $\mathfrak{g}_\alpha \subset \mathfrak{p}^+$ if and only if $\alpha(\mu(z)) = z$, i.e., $\langle \alpha, \mu \rangle = 1$. The canonical maps $\mathfrak{p} \xrightarrow{\approx} \mathfrak{t}_h$ and $\mathfrak{p} \hookrightarrow \mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ induce a $\mathbf{C}$-linear, equivariant isomorphism $\mathfrak{p}^+ \xrightarrow{\approx} \mathfrak{t}_h$ (see [4, 1.1.14]).

(b) It follows from [4, 1.2.7] that $X'^+ = X^+$. Since

$$\{\alpha \mid \langle \alpha, \mu_h \rangle = 1\} = \{\alpha \mid \langle \alpha, \mu_{h'} \rangle = 1\},$$

h and h' define the same Hodge filtration on $\mathrm{Lie}(G)$, but this implies that they are equal [4, p. 254].

Proposition 5.2 (Borovoi [2]). *Let (G, X^+), (G', X'^+), and τ, φ, and ψ be as in (3.1). Consider $i\colon (T, h) \hookrightarrow (G, X^+)$.*

(a) *There exists an inclusion $i'\colon T \hookrightarrow G'$ and a $g \in G(\mathbf{A}^f)$ such that $\psi \circ \underline{ad}(g) \circ i_{\mathbf{A}^f} = i'_{\mathbf{A}^f}$.*

(b) *There exists an $h' \in X'$ factoring through i' such that $\mu_{h'} = \tau\mu_h$ (as maps into T).*

(c) *The pair (G', j') is a form of (G, j); its class in $H^1(\mathbf{Q}, \overline{T})$, $\overline{T} \stackrel{\mathrm{df}}{=} T/Z(G)$, has image zero in $H^1(\mathbf{Q}_l, \overline{T})$ for all $l \neq \infty$ and is represented by $\tau\mu_h(-1)/\mu_h(-1)$ for $l = \infty$.*

Proof. (a) Let $\varphi(\tau[h]) = g'[h']$, $h' \in X'^+$, $g' \in G(\mathbf{A}^f)$ (see the Appendix). For any $t \in i(T(\mathbf{Q}))$, $t[h] = [h]$, and so $\psi(t)g'[h'] = g'[h']$, i.e.,

$$(g'^{-1}\psi(t)g')[h'] = [h'].$$

This implies that $g'^{-1}\psi(t)g' \in G'(\mathbf{Q})$. Let $g = \psi^{-1}(g'^{-1})$; then $\psi \circ \underline{ad}\, g \circ i\colon T_{\mathbf{A}^f} \longrightarrow G'_{\mathbf{A}^f}$ maps $T(\mathbf{Q})$ into $G'(\mathbf{Q})$. As $T(\mathbf{Q})$ is Zariski dense in T, this means that $\psi \circ \underline{ad}\, g \circ i$ is defined over $\mathbf{Q}$; we denote it by i'.

(b) Let h' be as in (a); then $\rho_{h'} = \tau\rho_h$ because $(\varphi \circ g)(\tau[h]) = [h']$ and $\varphi \circ g$ is $T(\mathbf{Q})$-equivariant. Therefore (b) follows from (5.1b).

(c) As $\psi \circ \underline{ad}\, g$ is an isomorphism $(G, i)_{\mathbf{A}^f} \xrightarrow{\approx} (G', i^p)_{\mathbf{A}^f}$, it is clear that the class of (G, i) in $H^1(\mathbf{Q}_l, \overline{T})$ is trivial for all finite l. The element $\underline{ad}\,\mu(-1)$ is a Cartan involution on G; therefore the class of $\mu(-1)$ in $H^1(\mathbf{R}, \overline{T})$ corresponds to the (unique) compact form of $G_{\mathbf{C}}$. Similarly, the class of $\tau\mu(-1)$ corresponds to the compact form of $G'_{\mathbf{C}} = G_{\mathbf{C}}$. The conclusion is now clear.

Remark 5.3. If in (5.2) we replace φ by $\varphi \circ g$ and ψ by $\psi \circ \underline{ad}(g)$, then φ and ψ are still compatible, but now $\varphi(\tau[h]) = [h']$ and $\psi \circ i = i'$.

Corollary 5.4. *Consider $(T,h) \xrightarrow{i} (G,X^+)$, and assume that G is simply-connected and that $\overline{T} \stackrel{\text{df}}{=} T/Z(G)$ satisfies the Hasse principle for H^1. Let $(T,{}^\tau h) \xhookrightarrow{{}^\tau i} ({}^\tau G,{}^\tau X^+)$ be as in §1. Then there exist isomorphisms*

$$\varphi: \tau M^o(G,X^+) \longrightarrow M^o({}^\tau G,{}^\tau X^+)$$
$$\psi: G_{\mathbf{A}^f} \longrightarrow {}^\tau G_{\mathbf{A}^f}$$

such that

(a) $\varphi(\tau[h]) = [{}^\tau h]$;
(b) φ *and* ψ *are compatible;*
(c) $\psi \circ i_{\mathbf{A}^f} = {}^\tau i_{\mathbf{A}^f}$.

Proof. Let (G',X'^+) be as in (3.1), and let $i'\colon T' \hookrightarrow G'$ and $h' \in X'^+$ be as in (5.2a) and (5.2b). Choose φ and ψ as in (5.3). Then (5.2c) shows that there exists an isomorphism $(G',i') \overset{\approx}{\rightarrow} ({}^\tau G,{}^\tau i)$ (cf. (1.4)), and (5.2b) shows that the isomorphisms carries h' into ${}^\tau h$. The corollary is now clear.

We now prove (1.1). Let (G,X^+), h, and τ be as in the statement of (1.1). We can assume that G is almost simple over $\mathbf{Q}$. Then $G = \mathrm{Res}_{F_1/\mathbf{Q}}\, G_1$ for some totally real field F_1 and absolutely almost-simple group G_1. Let $(T,h) \subset (G,X^+)$ and let T_1 be the maximal torus in G_1 such that $\mathrm{Res}_{F_1/\mathbf{Q}}\, T_1 = T$. As $T_{\mathbf{R}}$ is anisotropic, T_1 splits over a CM-field $L \supset F$. Let F be the maximal totally real subfield of L and let $G_* = \mathrm{Res}_{F/\mathbf{Q}}\, G'$ where $G' = G_{1F}$. Let h_* be the composite of h with the canonical inclusion $G \hookrightarrow G_*$. Then the $G_*(\mathbf{R})^+$-conjugacy class X_* containing h_*, together with G_*, satisfy (C), and (1.6) shows that it suffices to prove (1.1) for (G_*,X_*) and h_*. This allows us to assume that the original objects, $(T,h) \hookrightarrow (G,X^+)$, satisfy (4.1).

Let φ and ψ be isomorphisms as in (5.4). As both ψ and ψ_τ (see §1) are isomorphisms $(G,i)_{\mathbf{A}^f} \overset{\approx}{\rightarrow} ({}^\tau G,{}^\tau i)_{\mathbf{A}^f}$, there exists a $t \in \overline{T}(\mathbf{A}^f)$ such that $\psi_\tau = \underline{ad}(t) \circ \psi$. To any non totally compact root α, there corresponds a subgroup H_α of G containing T (see §4). Let $H'_\alpha = H_\alpha^{der}$, and let X_α^+ be the $H_\alpha^{ad}(\mathbf{R})^+$-conjugacy class containing $h_\alpha \stackrel{\text{df}}{=} (\mathbf{S} \xrightarrow{h} H_\alpha \rightarrow H_\alpha^{ad})$. Then φ maps $\tau M^o(H'_\alpha,X_\alpha^+)$ into $M^o({}^\tau H_\alpha,{}^\tau X_\alpha^+)$ because it maps $\tau(g[h_\alpha])$ to $\underline{ad}\, t^{-1}(\psi_\tau(g))[{}^\tau h_\alpha]$ for all $g \in H'_\alpha(\mathbf{A}^f)$ and $H'_\alpha(\mathbf{A}^f).[h_\alpha]$ is dense in $M^o(H'_\alpha,X_\alpha^+)$. Since H'_α is of type A_1, we know (1.1) for it: there exists an isomorphism $\varphi_\tau\colon \tau M^o(H'_\alpha,X_\alpha^+) \longrightarrow M^o({}^\tau H'_\alpha,{}^\tau X_\alpha^+)$, compatible with ψ_τ, and taking $\tau[h_\alpha]$ to $[{}^\tau h_\alpha]$. The map

$$\tau^{-1}(\varphi_\tau^{-1}\circ\varphi)\colon M^o(H'_\alpha, X_\alpha^+)\longrightarrow M^o(H'_\alpha, X_\alpha^+)$$

fixes $[h_\alpha]$ and is compatible with $\underline{ad}\,t\colon H'_\alpha(\mathbf{A}^f)\longrightarrow H'_\alpha(\mathbf{A}^f)$. Therefore (2.2) shows that $\underline{ad}\,t$, regarded as element of $H_\alpha^{ad}(\mathbf{A}^f)$, lies in $H_\alpha^{ad}(\mathbf{Q})$, i.e., $t\in \overline{Z}_\alpha(\mathbf{A}^f)/\overline{Z}_\alpha(\mathbf{Q})$. Now (4.4) shows that $t\in\overline{T}(\mathbf{Q})$, and so $\varphi\circ t$ and $\underline{ad}(t)\circ\psi$ fulfill the requirements of the theorem.

§6. Compatibility of the Maps φ_τ for Different Special Points

Let (G, X^+) satisfy (C) and let $(T, h)\subset(G, X^+)$. Then Langlands's constructions lead to the definition of a map

$$\psi_\tau = (g\mapsto {}^\tau g)\colon G^{ad}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\,G)\longrightarrow {}^\tau G^{ad}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\,{}^\tau G)$$

that is compatible with the maps of the same name, $G(\mathbf{A}^f)\to {}^\tau G(\mathbf{A}^f)$, $G^{ad}(\mathbf{A}^f)\to {}^\tau G^{ad}(\mathbf{A}^f)$ (see [14, §8]).

Proposition 6.1. *Assume that G is simply connected, and let φ_τ be an isomorphism $\tau M^o(G, X^+)\to M^o({}^\tau G, {}^\tau X^+)$ with the properties (a) and (b) of (1.1). Then (1.1b) holds for all $g\in G^{ad}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\,G)$.*

Proof. Let $G_* = \mathrm{Res}_{F/\mathbf{Q}}(G_F)$ for some totally real field F, and let h_* be the composite of h with $G\hookrightarrow G_*$. Then there is a commutative diagram

$$\begin{array}{ccc}
\tau M^o(G, X^+) & \overset{\varphi_\tau}{\rightarrow} & M^o({}^\tau G, {}^\tau X^+)\\
\big\downarrow\cap & & \big\downarrow\cap\\
\tau M^o(G_*, X_*^+) & \overset{\varphi_\tau}{\rightarrow} & M^o({}^\tau G_*, {}^\tau X_*^+).
\end{array}$$

$(({}^\tau G, {}^\tau X^+)$ and $({}^\tau G_*, {}^\tau X_*^+)$ defined using h and h_*). This shows that φ_τ (for G) is compatible with the action of g for all $g\in G_*(\mathbf{A}^f)$ (any F) ~~and all $g\in H_\alpha(\mathbf{Q})^+$ (any $H_\alpha\subset G_*$; see §4)~~, but these elements ~~generate~~ form a dense subgroup of $G^{ad}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\,G)$.

cf Kott P... WP p51

Remark 6.2. Theorem 1.1 and (6.1) prove part (a) of Conjecture C^o (see [14, p. 340]) for simply connected groups.

We shall now need to consider two special points h and h' for a given (G, X^+). To distinguish the objects ${}^\tau G, {}^\tau X^+, \varphi_\tau, \ldots$ constructed relative to h from the similar objects constructed relative to h', we shall write the former ${}^{\tau,h}G, {}^{\tau,h}X^+, \varphi_{\tau,h}, \ldots$.

Let $\overline{\beta}(\tau, h)$ and $\overline{\beta}(\tau, h')$ be the elements of $\left(G^{ad}(\mathbf{A}_L^f)/G^{ad}(L)\right)^{\mathrm{Gal}(L/\mathbf{Q})}$ corresponding to h and h' respectively (see (1.4)). The image of $\overline{\beta}(\tau, h)$ and $\overline{\beta}(\tau, h')$ in $H^1(\mathbf{Q}, G^{ad})$ are trivial at the finite primes and are equal at the infinite primes (see [15, p. 315–316]). As G^{ad} satisfies the Hasse principal for H^1 ([9, VII. 6]) this shows that $\overline{\beta}(\tau, h)$ and $\overline{\beta}(\tau, h')$ have the same image in $H^1(\mathbf{Q}, G^{ad})$ and therefore $B \overset{\mathrm{df}}{=} \overline{\beta}(\tau, h')\overline{\beta}(\tau, h)^{-1}$ lies in $G^{ad}(\mathbf{A}^f)/G^{ad}(\mathbf{Q})$. Moreover, there is an isomorphism $f\colon {}^{\tau,h}G \longrightarrow {}^{\tau,h'}G$ such that

$$f_{\mathbf{A}^f} = \psi_{\tau,h'} \circ B \circ \psi_{\tau,h}^{-1}\colon {}^{\tau,h}G(\mathbf{A}^f) \longrightarrow {}^{\tau,h'}G(\mathbf{A}^f).$$

Define

$$\varphi^o(\tau; h', h) : M^o({}^{\tau,h}G, {}^{\tau,h}X^+) \overset{\approx}{\rightarrow} M^o({}^{\tau,h'}G, {}^{\tau,h'}X^+)$$

to be $({}^{\tau,h'}B)^{-1} \circ M^o(f)$. It carries the action of ${}^{\tau,h}g$, $g \in G(\mathbf{A}^f)$, into the action of ${}^{\tau,h'}g$. (See [15, p. 312–318].)

Theorem 6.3. *Assume that G is simply connected; then for any special $h, h' \in X^+$, the diagram*

$$\begin{array}{ccc} \tau M^o(G, X^+) & \xrightarrow{\varphi_{\tau,h}} & M^o({}^{\tau,h}G, {}^{\tau,h}X^+) \\ & \searrow^{\varphi_{\tau,h'}} & \downarrow{\scriptstyle \varphi(\tau; h', h)} \\ & & M^o({}^{\tau,h'}G, {}^{\tau,h'}X^+) \end{array}$$

commutes.

Proof. We first prove this under the assumption that $h' = \underline{ad}\, q \circ h, q \in G^{ad}(\mathbf{Q})^+$. In this case $B = q$ (see [15, 9.1]) so that $M^o(f) = ({}^{\tau,h'}q) \circ \phi^o(\tau; h', h) = \phi^o(\tau; h', h) \circ ({}^{\tau,h}q)$.

Consider the diagram

$$\begin{array}{ccccc}
\tau M^o(G,X^+) & \overset{\phi_{\tau,h}}{\rightarrow} & M^o({}^{\tau,h}G, {}^{\tau,h}X^+) & & \\
\downarrow \tau(g) & & \downarrow {}^{\tau,h}g & & \\
\tau M^o(G,X^+) & \overset{\phi_{\tau,h}}{\rightarrow} & M^o({}^{\tau,h}G, {}^{\tau,h}X^+) & \Bigg\downarrow & M^o(f) \\
& \searrow^{\varphi_{\tau,h'}} & \downarrow \overset{\circ}{\phi}(\tau;h',h) & & \\
& & M^o({}^{\tau,h'}G, {}^{\tau,h'}X^+) & &
\end{array}$$

The upper square commutes because of (6.1). The outside of the diagram commutes because both maps send $[h]$ to $[{}^{\tau}h'] = [f \circ {}^{\tau}h]$ and the action of $\tau(g)$ to that of ${}^{\tau,h'}g$. Thus the lower triangle commutes.

Before proving the general case, we need some lemmas.

Lemma 6.4. *Consider an inclusion $i\colon (G,X) \hookrightarrow (G',X')$. Theorem (6.3) holds for h and h' (as elements of X) if and only if it holds for $i \circ h$ and $i \circ h'$.*

Proof. Since i induces an embedding $M^o(G,X^+) \hookrightarrow M^o(G',X'^+)$ and the maps $\phi_{\tau,i\circ h}$, $\phi_{\tau,i\circ h'}$, and $\phi(\tau; i\circ h', i\circ h)$ restrict to $\phi_{\tau,h}$, $\phi_{\tau,h'}$, and $\phi(\tau;h',h)$ on $M^o(G,X^+)$, the sufficiency is clear. The necessity follows from the facts that $G'(\mathbf{Q})_+[i\circ h]$ is dense in $M^o(G',X'^+)$ and $\phi_{\tau,i\circ h'}$ and $\phi(\tau;i\circ h', i\circ h)\circ\phi_{\tau,i\circ h}$ have the same behaviour with respect to the Hecke operators.

Lemma 6.5. *If (6.3) is true for the pairs (h,h') and (h',h''), then it is also true for the pair (h,h'').*

Proof. This is immediate, since $\phi(\tau;h'',h')\circ\phi(\tau;h',h) = \phi(\tau;h'',h)$.

Lemma 6.6 (Borovoi). *Let G be a simple noncompact group over $\mathbf{R}$ and let X be a $G(\mathbf{R})$-conjugacy class of homomorphisms $\mathbf{S}\to G$ satisfying (C_1) and (C_2). Suppose $h,h'\in X$ factor through the same maximal torus $T\subset G$. Then $h'=h$ of $h'=h^{-1}$.*

Proof. Let K_∞ be the centralizer of $h(\mathbf{S})$. Then [4, 1.2.7], K_∞ is a maximal connected compact subgroup of G. The pair (G,T) determines the set of compact roots in $R(G_{\mathbf{C}}, T_{\mathbf{C}})$, which determines K_∞. The centre Z of K_∞ is 1-dimensional (loc. cit.), and h defines an isomorphism $\mathbf{S}/\mathbf{G}_m \overset{\approx}{\to} Z$; it is therefore determined up to sign.

We now complete the proof of (6.3). As usual, we can assume that G is almost simple over $\mathbf{Q}$ and therefore that $G = \operatorname{Res}_{F/\mathbf{Q}} G_1$, where F is totally real and G_1 is absolutely almost simple. Let $(T, h) \subset (G, X^+)$ and $(T', h') \subset (G, X^+)$, and let T_1, $T'_1 \subset G_1$ be such that $\operatorname{Res}_{F/\mathbf{Q}} T_1 = T$ and $\operatorname{Res}_{F/\mathbf{Q}} T'_1 = T'$. There exists a CM-field L splitting both T_1 and T'_1. After replacing G with $G_* = \operatorname{Res}_{F'/\mathbf{Q}} G_{1,F'}$, where F' is the maximal totally real subfield of L, and using (6.4), we can assume that L is a quadratic extension of F. As $T_{1,L}$ and $T'_{1,L}$ are split, there exists a $\beta \in G_1(L)$ such that $\beta T_1 \beta^{-1} = T'_1$. For each real prime v: $F \hookrightarrow \mathbf{R}$ of F, choose an extension (also denoted by v) of v to L and write H_v for $H \otimes_{F,v} \mathbf{R}$, any F-group H. As $T_{1,v}$ and $T'_{1,v}$ are compact, there exists a $\gamma_v \in G_v(\mathbf{R})$ such that $\gamma_v T_{1,v} \gamma_v^{-1} = T'_{1,v}$. Let $c_\sigma = \beta^{-1} \cdot \sigma\beta$, where σ generates $\operatorname{Gal}(L/F)$. Then $c_\sigma \in N(L)$, where N is the normalizer of T, and so it defines a class $c \in H^1(L/F, N)$. As $\gamma_v^{-1} \cdot v(\beta) \in N(\mathbf{C})$ and

$$v(c_\sigma) = \left(\gamma_v^{-1} \cdot v(\beta)\right)^{-1} \cdot \iota\left(\gamma_v^{-1} \cdot v(\beta)\right),$$

where ι denotes complex conjugation, we see that c maps to 1 in $H^1(L_v/F_v, N)$. Let ω_v be the image of $\gamma_v^{-1} \cdot v(\beta)$ in $W(L_v)$, where $W = N/T$. The image ω of c_σ in $W(L)$ is ι because $v(\omega) = \omega_v^{-1} \cdot \iota\omega_v$ and ι acts trivially on $W(L_v)$ (see [15, p. 307]). Thus $c_\sigma \in T(L)$ and

$$c \in \operatorname{Ker}\left(H^1(L/F, T) \to \bigoplus_v H^1(L_v/F_v, N)\right).$$

The following diagram is useful:

$$\begin{array}{ccccccc}
N(F) & \to & W(F) & \to & H^1(L/F,T) & \to & H^1(L/F,N) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_v N(F_v) & \to & \bigoplus_v W(F_v) & \to & \bigoplus_v H^1(L_v/F_v,T) & \to & \bigoplus_v H^1(L_v/F_v,N)
\end{array}$$

We now prove (6.3) by induction on $l = \sum_v l(\omega_v)$, where $l(\omega_v)$ is the length of ω_v as an element of $W(\mathbf{C})$. Suppose first that $l = 0$. Then $\gamma_v^{-1} \cdot v(\beta) \in T(L_v)$ and so c maps to 1 in $H^1(L_v/F_v, T)$ for all v. Note that

$T \approx U^r$ some r, where U is the unique one-dimensional non-split F-torus split by L, and therefore $H^1(L/F,T) \approx (F^\times/NL^\times)^r$. The penultimate assertion shows that c is represented by a family $(c_1,\ldots,c_r)$ of totally positive elements of $F^\times$. After adjoining $\sqrt{c_i}$ to F, $i=1,\ldots,r$, we can assume $c=1$. Then $c_\sigma = t^{-1}\cdot\sigma t$ some $t\in T(L)$ and so, after replacing β with βt^{-1}, we can assume it lies in $G_1(F)$. Regard β as an element of $G(\mathbf{Q})$. Lemma (6.5) and the first part of the proof show that we need only prove the theorem for $\underline{ad}\,\beta\circ h$ and h'. This means that we can assume that h and h' factor through the same torus. But then they must be equal because, in the context of (6.6), h^{-1} does not lie in the same connected component as h.

Finally, suppose $l(\omega_{v_o})\neq 0$, say $\omega_{v_o}=s_\alpha\omega'_{v_o}$ with $l(\omega'_{v_o})<l(\omega_{v_o})$ and s_α the reflection corresponding to the root α. If α is compact at v, then s_α lifts to $\gamma_\alpha\in N(F_v)$ (see [15, p. 308]) and we can replace γ_{v_o} with $\gamma_{v_o}\gamma_\alpha^{-1}$. Then ω_{v_o} is replaced with ω'_{v_o} and we can apply the induction hypothesis. Suppose therefore that α is not compact at v and define $H'_\alpha\subset G_1$, $H'_\alpha\supset T_1$, H'_α of type A_1, as in §4. Let H_α be the derived group of H'_α and let $T_\alpha = H_\alpha\cap T$. Then $T_\alpha\approx U$, which implies that $H^1(L/F,T_\alpha)\to\bigoplus_v H^1(L_v/F_v,T_\alpha)$ is surjective. Choose $c_\alpha\in H^1(L/F,T_\alpha)$ mapping to (c_v) where $c_v=1$ for $v\neq v_o$ and $c_{v_o}=\delta(s_\alpha)$ where δ is the boundary map

$$W_\alpha(F_{v_o})\to H^1(L_{v_o}/F_{v_o},T_\alpha),\quad W_\alpha = N_\alpha/T_\alpha,$$

$N_\alpha = \mathrm{Norm}(T_\alpha)$. Then c_α maps to 1 in $\bigoplus H^1(L_v/F_v,N_\alpha)$. The Hasse principle therefore shows that c_α splits in $H^1(L/F,H_\alpha): (c_\alpha)_\sigma = g^{-1}\cdot\sigma g$, $g\in H_\alpha(L)$. Lift s_α to $n_\alpha\in N_\alpha(L)$; then $v_o\big((c_\alpha)_\sigma\big) = n_\alpha^{-1}\cdot\iota n_\alpha$ and so $gn_\alpha^{-1}\in H_\alpha(\mathbf{R})$. Since we know the theorem for $\mathrm{Res}_{F/\mathbf{Q}}H_\alpha$, (6.4) allows us to replace (T,h) with $\big(\underline{ad}\,g\circ T, \underline{ad}(gn_\alpha^{-1})\circ h\big)$. This replaces β with βg^{-1}, γ_{v_o} with $\gamma_{v_o}n_\alpha g^{-1}$, γ_v with $\gamma_v g^{-1}$, $v\neq v_o$, ω_{v_o} with $gs_\alpha^{-1}\omega_{v_o}g^{-1} = g\omega'_{v_o}g^{-1}$, and ω_v with $g\omega_v g^{-1}$, $v\neq v_o$. Thus $\sum l(\omega_v)$ is diminished, and we can apply the induction hypothesis.

§7. Conclusions

We have shown that Conjecture C^o of [15, p. 340–341] is true for (G,X^+) whenever G is simply connected. As is remarked in [15, 9.6], this implies the general case.

Theorem 7.1. *The conjecture of Langlands [8, p. 232–233] (see also [15, p. 311]) is true for all Shimura varieties.*

Proof. In [15, 9.4] it is shown that this conjecture is equivalent to Conjecture C^o.

Theorem 7.2. *Canonical models (in the sense of [4, 2.2.5]) exist for all Shimura varieties.*

Proof. This is a consequence of (7.1) (see [15, §7]).

Theorem 7.3. *The conjecture of Langlands describing the action of complex conjugation on a Shimura variety having a real canonical model [8, p. 234] is true.*

Proof. This again follows from (7.1)

Theorem 7.4. *The main theorems of [13], viz. (4.6) and (4.9), are true for all Shimura varieties.*

Proof. They are proved in [13] under the assumption that G is classical and the canonical model exists, but the first assumption is only used to simplify the proof of [13, 1.3], and we can instead deduce this theorem from Proposition 2.1 above.

Remark 7.5. Theorem 7.4 gives a definitive answer to the question of Shimura [19, p. 347].

Remark 7.6. For Shimura varieties of Abelian type, (7.1), (7.2), and (7.3) were first proved in ([5], [15]), [4], and [12] respectively.

Appendix

We say that (G, X^+) satisfies (C) if G is a semi-simple group over $\mathbf{Q}$ and X^+ is a $G(\mathbf{R})^+$-conjugacy class of maps $\mathbf{S} \to G^{ad}_{\mathbf{R}}$ for which the following hold:

(C_1) for all $h \in X^+$, the Hodge structure on Lie($G_\mathbf{R}$) defined by h is of type $\{(-1,1),(0,0),(1,-1)\}$;

(C_2) $\underline{ad}\, h(i)$ is a Cartan involution on $G^{ad}_\mathbf{R}$;

(C_3) G^{ad} has no non-trivial factors defined over $\mathbf{Q}$ that are anisotropic over $\mathbf{R}$.

Such a (G, X^+) defines a connected Shimura variety $M^o(G,X^+)$. The topology $\tau(G)$ on $G^{ad}(\mathbf{Q})$ is that for which the images of the congruence subgroups of $G(\mathbf{Q})$ form a fundamental system of neighbourhoods of the identity, and

$$M^o(G,X^+) = \varprojlim \Gamma \setminus X^+$$

where the limit is over the set $\sum(G)$ of torsion-free arithmetic subgroups of $G^{ad}(\mathbf{Q})$ that are open relative to the topology $\tau(G)$. For $h \in X^+$, $[h]$ and $[h]_\Gamma$ denote the images of h in $M^o(G,X^+)$ and ${}_\Gamma M^o(G,X^+) \overset{\mathrm{df}}{=} \Gamma \setminus X^+$.

Any $\alpha \in G^{ad}(\mathbf{Q})^+$ acts on $M^o(G,X^+)$ by transport of structure: $\alpha[h]_\Gamma = [\alpha \circ h]_{\alpha(\Gamma)}$. Any $g \in G(\mathbf{Q})^{-}_{+}$ acts as follows: Let $\Gamma \in \sum(G)$ and let K be a compact open subgroup of $G(\mathbf{A}^f)$ such that Γ contains the image of $K \cap G(\mathbf{Q})_+$; then $g \in qK$ some $q \in G(\mathbf{Q})_+$, and $g[h]_\Gamma \overset{\mathrm{df}}{=} [\underline{ad}\, q \circ h]_{q\Gamma q^{-1}}$. These actions combine to give an action of $G(\mathbf{Q})^{-}_{+} \times G^{ad}(\mathbf{Q})^+$ (semi-direct product for the obvious action of $G^{ad}(\mathbf{Q})^+$ on $G(\mathbf{Q})^{-}_{+}$). The map $q \mapsto (q, \underline{ad}\, q^{-1})$ identifies $G(\mathbf{Q})_+$ with a normal subgroup of the product, and the quotient

$$G(\mathbf{Q})^{-}_{+} *_{G(\mathbf{Q})_+} G^{ad}(\mathbf{Q})^+ \overset{\mathrm{df}}{=} G(\mathbf{Q})^{-}_{+} \times G^{ad}(\mathbf{Q})^+ / G(\mathbf{Q})_+$$

continues to act on $M^o(G,X^+)$. The completion of $G^{ad}(\mathbf{Q})^+$ for the topology $\tau(G)$, $G^{ad}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G)$, is equal to $G(\mathbf{Q})^{-}_{+} *_{G(\mathbf{Q})_+} G^{ad}(\mathbf{Q})^+$, and this identification is compatible with the actions of the groups on $M^o(G,X^+)$ (see [4, 2.1.6.2]).

Any $x \in M^o(G,X^+)$ can be written $x = g[h]$ for some $g \in G(\mathbf{Q})^{-}_{+}$ and $h \in X^+$. For suppose $x_\Gamma = [h]_\Gamma$; then, for any $\Gamma_1 \subset \Gamma$, $x_{\Gamma_1} = \gamma_{\Gamma_1}[h]$ some $\gamma_{\Gamma_1} \in \Gamma$; let $\gamma = \lim_{\Gamma_1 \to 1} \gamma_{\Gamma_1}$, and let $\gamma = g * \alpha$; then $x = \gamma[h] = g[\alpha(h)]$.

If G is simply connected, then $G(\mathbf{Q})^{-}_{+} = G(\mathbf{A}^f)$; moreover

$${}_\Gamma M^o(G,X^+) = \Gamma \setminus X^+ = G(\mathbf{Q}) \setminus X^+ \times G(\mathbf{A}^f)/K$$

if K is a compact open subgroup of $G(\mathbf{A}^f)$ containing $Z(\mathbf{Q})$, and Γ is the image of $K \cap G(\mathbf{Q})$ in $G^{ad}(\mathbf{Q})^+$.

References

[1] A. Borel, Density and maximality of arithmetic subgroups. J. Reine Angew. Math. *224* (1966), 78–89.

[2] M. Borovoi, Canonical models of Shimura varieties. Handwritten notes dated 26/5/81.

[3] P. Deligne, Travaux de Shimura. Sém. Bourbaki Février 71, Exposé 389, Lecture Notes in Math., 244, Springer, Berlin, 1971.

[4] P. Deligne, Variétés de Shimura: interpretation modulaire, et techniques de construction de modéles canoniques. Proc. Symp. Pure Math., A.M.S., *33* (1979), part 2, 247–290.

[5] P. Deligne, Motifs et groupes de Taniyama. Lecture Notes in Math., 900, Springer, Berlin, 1982, pp. 261–279.

[6] D. Kazhdan, On arithmetic varieties. *Lie Groups and Their Representations*. Budapest, 1971, pp. 151–217.

[7] D. Kazhdan, On arithmetic varieties, II. Preprint.

[8] R. Langlands, Automorphic representations, Shimura varieties, and motives. Ein Märchen. Proc. Symp. Pure Math., A.M.S. *33* (1979), part 2, 205–246.

[9] R. Langlands, Les débuts d'une formule des traces stables. Publ. Math. Univ. Paris VII. (To appear).

[10] G. Margulis, Discrete groups of motions of manifolds of nonpositive curvature. Amer. Math. Soc. Transl. *109* (1977), 33–45.

[11] J. Milne, The arithmetic of automorphic functions. In preparation.

[12] J. Milne and K-y. Shih, The action of complex conjugation on a Shimura variety. Annals of Math., *113* (1981), 569–599.

[13] J. Milne and K-y. Shih, Automorphism groups of Shimura varieties and reciprocity laws. Amer. J. Math., *103* (1981), 911–935.

[14] J. Milne and K-y. Shih, Langlands's construction of the Taniyama group. Lecture Notes in Math., 900, Springer, Berlin, 1982, 229–260.

[15] J. Milne and K-y. Shih, Conjugates of Shimura varieties. Lecture Notes in Math., 900, Springer, Berlin, 1982, pp. 280–356.

[16] V. Platonov and A. Rapinčuk, On the group of rational points of three-dimensional groups. Soviet Math. Dokl. *20* (1979), 693–697.

[17] M. Ragunathan, *Discrete Subgroups of Lie Groups*. Erg. Math. *68*, Springer, Berlin, 1972.

[18] I. Satake, *Algebraic Structures of Symmetric Domains*. Publ. Math. Soc. Japan 14, Princeton University Press, Princeton, 1980.

[19] G. Shimura, On arithmetic automorphic functions. Actes, Congrès Intern. Math. (1970) Tom 2, 343–348.

[20] G. Shimura, *Arithmetic Theory of Automorphic Functions.* Publ. Math. Soc. Japan 11, Princeton University Press, Princeton, 1971.

[21] S-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. Comm. Pure Appl. Math., *31* (1978), 339–411.

Received April 29, 1982

This work was completed while the author was at The Institute for Advanced Study and was supported in part by NSF grant MCS 8103365.

Professor J. S. Milne
Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109