See also
Mok/Yeung 1983
Nori/Rag [illegible]

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Kazhdan's Theorem on Arithmetic Varieties

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Let X be a nonsingular algebraic variety over $\mathbb{C}$ whose universal covering manifold $\tilde{X}$ is a symmetric Hermitian domain (i.e., a symmetric Hermitian space without Euclidean or compact factors). Then the group $\mathrm{Aut}(\tilde{X})$ of automorphisms of $\tilde{X}$ (as a complex manifold) has only finitely many connected components, and its identity component $\mathrm{Aut}(\tilde{X})^+$ is a real semisimple Lie group with no compact factors. The variety X will be said to be <u>arithmetic</u> if the group Γ of covering transformations of $\tilde{X}$ over X is torsion-free and is an arithmetic subgroup of $\mathrm{Aut}(\tilde{X})$ in the sense that there is a linear algebraic group G over $\mathbb{Q}$ and a surjective homomorphism $f: G(\mathbb{R})^+ \longrightarrow \mathrm{Aut}(X)^+$ with compact kernel carrying $G(\mathbb{Z}) \cap G(\mathbb{R})^+$ into a group commensurable with Γ.

The theorem in question is the following:

<u>Theorem</u> 0.1. If X is an arithmetic variety, then, for all automorphisms σ of $\mathbb{C}$, σX is also an arithmetic variety.

The result was first stated, with some indications of a proof, by Kazhdan in his talk at the Nice Congress in 1970 [, Tome 2, p321-325], but, in fact, it is only recently that the proof has been completed.

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In the case that X is compact, a detailed proof of (0.1) was given by Kazhdan in his talk at the Budapest conference in 1971 (published 1975, [ ]).

There is now an alternative approach to this case of the theorem: roughly speaking, the first part of Kazhdan's paper, where the nondegeneracy of the Bergmann metric on the universal covering manifold of σX is established, can be replaced by an appeal to Yau's theorem on the existence of Kähler-Einstein metrics; the second group-theoretic part, of the paper, can be replaced by an appeal to the general results of Margulis (see (2.7) below).

Let $X' \to X$ be a finite étale morphism. If X is arithmetic then clearly so also is X', and the converse assertion is also true provided the fundamental group of X is torsion-free. From this it follows that it suffices to prove (0.1) under the hypothesis:

(0.2) There exists an almost-$\mathbb{Q}$-simple, simply connected algebraic group G over $\mathbb{Q}$, a surjective homomorphism $f: G(\mathbb{R}) \to \mathrm{Aut}(\tilde{X})^+$ with compact kernel, and a congruence subgroup $\Gamma \subset G(\mathbb{Q})$ such that X is equal to $f(\Gamma)\backslash\tilde{X}$ with its unique algebraic structure.

The group G will then be of the form $G = \mathrm{Res}_{F/\mathbb{Q}} G'$ with G' an absolutely almost-simple group over a totally real number field F. The type of $G_{\mathbb{R}}$ (A, B, C, $D^{\mathbb{R}}$, $D^{\mathbb{H}}$, mixed type ($D^{\mathbb{R}}$, $D^{\mathbb{H}}$), E_6, or E_7; cf. Deligne [ ]) will be called the *type* of X. Corvallis article

If X is as in (0.2) and of type A, B, C, $D^{\mathbb{R}}$ or $D^{\mathbb{H}}$, then it is a moduli variety for a class of abelian varieties with a family of Hodge cycles and a level structure — see for example [, §2]. (In the case $D^{\mathbb{H}}$ one should also assume that the image of Γ in $G^{ad}(\mathbb{Q})$ is a congruence subgroup there.) For any automorphism σ of $\mathbb{C}$, the conjugate variety σX will be a moduli variety for the conjugate class of abelian varieties with extra structure; and so is again an arithmetic variety. Thus (0.1) is true for arithmetic varieties of these types. SLN 900 IV

(Note that this argument makes use of Deligne's theorem [ ] that a Hodge cycle on an abelian variety is absolutely SLN 900, I

Hodge.)

Israel J M 44 1983 139-159 Referred to as K II. In [ ], Kazhdan proves the following result: let X be as in (0.2), noncompact, and of type E_6, E_7, or mixed type $(D^{\mathbb{R}}, D^{\mathbb{H}})$; then, if one assumes a certain result in the theory of group representations (Conjecture 3.17 below), (0.1) is true for X. A weak form of (3.17), sufficient for the purpose of proving (0.1), has recently been established by L. Clozel. Thus this last result of Kazhdan, together with the proofs for compact varieties and moduli varieties, suffices to prove (0.1).

For his proof in these last cases, Kazhdan shows, case-by-case, that X contains an arithmetic subvariety X' associated with a subgroup G' of G such that

$$\dim X > \dim X' > \dim (X^* - X)$$

where X^* is the canonical (Baily-Borel) compactification of X. Because (0.1) is known for curves (they are all of type A_1), in proving (0.1) for X he can assume by induction on the dimension that it holds for X'.

In these notes, I modify Kazhdan's arguments to give a uniform proof that (0.1) holds for all X (satisfying (0.2)) such that

(0.3a) $\operatorname{codim}(X^* - X) \geq 3$;

(0.3b) G' contains a maximal torus T' that splits over an imaginary quadratic extension of F and is such that $T'(\mathbb{R})$ is compact.

The proof does not use the classification and avoids treating the compact varieties separately, but it does assume (0.1) for the case of arithmetic varieties of type A_1 (for these varieties it is easily proved using moduli varieties). It also, of course, makes use of Clozel's result.

Let X satisfy (0.2). A construction of Piatetsky-Shapiro and Borovoi allows one to embed X into an arithmetic variety X' satisfying also (0.3). I do not know whether it is possible to prove directly that (0.1) for X' implies (0.1) for X, but in any case one has the implications:

(0.1) for all arithmetic varieties satisfying (0.2) and (0.3)

Corvallis $\Rightarrow$ Langlands's conjecture [, p232-33] for all Shimura varieties

$\Rightarrow$ (0.1) for all arithmetic varieties.

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(See [ ] or [ ].)

The first section of these notes contains generalities on arithmetic varieties. Also a group $\mathcal{G}$ is defined that will play the role of $G(\mathbb{Q})/Z(\mathbb{Q})$ for the conjugate variety σX. Criteria are given in §2 for a variety to be arithmetic; in conjunction with Yau's theorem, they suffice to show that σX is arithmetic in the case that X is compact. In §3 Clozel's result is used to show that the Bergmann volume form on a certain covering manifold of σX is not identically zero. In the next section, the subbundles of the tangent bundle on $\tilde{X}$ are studied and, finally, in §5 the main theorem is proved.

1. <u>Generalities on arithmetic varieties; definition of $\tilde{X}^\sigma$ and Q</u>

Let G be a simply connected semisimple algebraic group over $\mathbb{Q}$, and let $\tilde{X}$ be a symmetric Hermitian domain on which $G(\mathbb{R})$ acts in such a way that $\tilde{X} \approx G(\mathbb{R})/(\text{maximal compact subgroup})$. We always assume that G has no $\mathbb{Q}$-factor that is anisotropic over $\mathbb{R}$. For any compact open subgroup K of $G(\mathbb{A}^f)$, $K \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$, and we let $\Gamma = \Gamma(K)$ be its image in $G^{ad}(\mathbb{Q})$. Clearly $\Gamma(K) = \Gamma(K.Z(\mathbb{Q}))$, where $Z = Z(G)$, and so we can always assume that $K \supset Z(\mathbb{Q})$. Let X_K, or X_Γ, denote $\Gamma(K)\backslash\tilde{X}$ regarded as an algebraic variety. The strong approximation theorem shows that $G(\mathbb{Q})$ is dense in $G(\mathbb{A}^f)$, and it follows that

$$X_K(\mathbb{C}) \stackrel{df}{=} \Gamma\backslash\tilde{X} = G(\mathbb{Q})\backslash\tilde{X} \times G(\mathbb{A}^f)/K.$$

The actions in the right hand term are

$$q(x,a)k = (qx, qak)\ ,\quad q \in G(\mathbb{Q}),\quad x \in \tilde{X}\ ,\quad a \in G(\mathbb{A}^f),\ k \in K.$$

Let $\mathcal{S}$ denote the set of compact open subgroups K of $G(\mathbb{A}^f)$ containing $Z(\mathbb{Q})$ and such that $\Gamma(K)$ is torsion-free. The groups $\Gamma(K)$, $K \in \mathcal{S}$, have the following properties: each $\Gamma(K)$ is an arithmetic group; $\bigcap_{K\in\mathcal{S}} \Gamma(K) = \{1\}$; if $K_1, \dots, K_m$ are in $\mathcal{S}$, then $K = \bigcap K_i$ is in $\mathcal{S}$ and $\Gamma(K) = \bigcap \Gamma(K_i)$; if $q \in G^{ad}(\mathbb{Q})$ and $K \in \mathcal{S}$, $q\,\Gamma(K)\,q^{-1} = \Gamma(qKq^{-1})$ with $qKq^{-1} \in \mathcal{S}$. We shall often identify the elements of $\mathcal{S}$ with their images in $G(\mathbb{A}^f)/Z(\mathbb{Q})$.

Consider the projective system $(X_K)_{K\in\mathcal{S}}$ of algebraic varieties. There is a left action of $G(\mathbb{A}^f)$ on this system:

$$g : X_K \longrightarrow X_{gKg^{-1}}\ ,\quad [x,a] \longmapsto [x, ag^{-1}]\ ,\quad x \in \tilde{X},\ a, g \in G(\mathbb{A}^f).$$

Let $\hat{X} = \varprojlim X_K$ — this is an irreducible scheme over $\mathbb{C}$ (not of finite-type!)

<u>Lemma</u> 1.1. There are equalities (of sets)

$$\hat{X}(\mathbb{C}) = \varprojlim X_K(\mathbb{C}) = G(\mathbb{Q})\backslash \tilde{X} \times G(\mathbb{A}^f) = (G(\mathbb{Q})/Z(\mathbb{Q}))\backslash \tilde{X} \times (G(\mathbb{A}^f)/Z(\mathbb{Q})).$$

<u>Proof</u>: Only the middle equality requires proof. Recall the following result [Bourbaki, Topologie Générale, III 7.2]: consider a projective system (G_α) of topological groups acting continuously and compatibly on a topological space S; then the canonical map $S/\varprojlim G_\alpha \longrightarrow \varprojlim S/G_\alpha$ is bijective if

(a) the isotropy group in G_α of each $s \in S$ is compact, and

(b) the orbit $G_\alpha s$ of each $s \in S$ is compact.

Apply this with $S = G(\mathbb{Q})\backslash \tilde{X} \times G(\mathbb{A}^f)$ and $(G_\alpha) = (K)$. Then (a) holds because each isotropy group is $Z(\mathbb{Q})$, and (b) holds because K is compact. As $\varprojlim K = \bigcap K = 1$, we have that

$$G(\mathbb{Q})\backslash \tilde{X} \times G(\mathbb{A}^f) \longrightarrow \varprojlim G(\mathbb{Q})\backslash \tilde{X} \times G(\mathbb{A}^f)/K$$

is bijective, as required.

The action of $G(\mathbb{A}^f)$ on the system (X_K) defines an action of $G(\mathbb{A}^f)$ on $\hat{X}$: for $g \in G(\mathbb{A}^f)$, $g : \hat{X} \longrightarrow \hat{X}$ is $[x,a] \mapsto [x, ag^{-1}]$. The knowledge of $\hat{X}$ with this action is equivalent to the knowledge of the projective system (X_K) together with the action of $G(\mathbb{A}^f)$ on it; in particular $X_K = K\backslash\hat{X}$. We shall sometimes use $\hat{X}$ to denote the projective system (X_K) rather than its limit. Note that the action of $G(\mathbb{A}^f)/Z(\mathbb{Q})$ on $\hat{X}$ is effective.

There is an action of $G^{ad}(\mathbb{Q})^+$ on $\hat{X}$:

$$\alpha[x,a] = [\alpha x, \alpha(a)], \quad \alpha \in G^{ad}(X)^+, \; x \in \tilde{X}, \; a \in G(\mathbb{A}^f).$$

The actions of $G(\mathbb{Q})$ on $\hat{X}$ defined by the maps $G(\mathbb{Q}) \longrightarrow G^{ad}(\mathbb{Q})$ and $G(\mathbb{Q}) \longrightarrow G(\mathbb{A}^f)$ are equal:

$$[qx, qaq^{-1}] = [x, aq^{-1}], \quad q \in G(\mathbb{Q}), \; x \in \tilde{X}, \; a \in G(\mathbb{A}^f).$$

Let (X_α) be a projective system of smooth complex algebraic varieties such that the transition maps $X_\beta \to X_\alpha$, $\beta \geq \alpha$, are étale. Then $\hat{X} = \varprojlim X_\alpha$ has a canonical structure as a complex manifold: a basis for the atlas on $\hat{X}$ is formed by pairs (U, φ) for which there exists an α such that the projection $p_\alpha : \hat{X} \longrightarrow X_\alpha$ is injective on U and $(p_\alpha(U), \varphi \circ p_\alpha^{-1})$ is an open chart for X_α. We write $\hat{X}^{an}$ for $\hat{X}$ with this structure. Note that the topology on $\hat{X}^{an}$ is, in general, strictly finer than the projective limit of the topologies on the X_α^{an}. (However, the Zariski topology on $\hat{X}$ is the projective limit of the Zariski topologies on the X_α.)

In the situation considered above, if $G(\mathbb{A}^f)$ is given the discrete topology then

$$\hat{X}^{an} = G(\mathbb{Q}) \backslash \tilde{X} \times G(\mathbb{A}^f).$$

Note that the map

$$\tilde{X} \longrightarrow \hat{X}^{an}, \quad x \longmapsto [x, 1]$$

is injective (because $\bigcap \Gamma(K) = \{1\}$), and is an isomorphism of $\tilde{X}$ (as a complex manifold) onto a connected component of $\hat{X}^{an}$. We shall use the following notations for the maps:

$$\begin{tikzcd} \tilde{X} \arrow[r, hook, "p"] \arrow[dr, "p_K"'] & \hat{X}^{an} \arrow[d, "q_K^{an} \ (\text{or } q_K)"] & \hat{X} \arrow[d, "q_K"] \\ & X_K^{an} & X_K \end{tikzcd}$$

If $x \in \tilde{X}$, we often write x_K and $\hat{x}$ for $p_K(x)$ and $p(x)$.

<u>Lemma</u> 1.2. Let $g \in G(\mathbb{A}^f)/Z(\mathbb{Q})$; if g stabilizes $\tilde{X} \subset \hat{X}$, then it belongs to $G(\mathbb{Q})/Z(\mathbb{Q})$.

<u>Proof</u> Let $x \in \tilde{X}$. As g stabilizes $\tilde{X}$, $[x, g] = [x', 1]$ for some $x' \in \tilde{X}$. This means that there exists a $q \in G(\mathbb{Q})/Z(\mathbb{Q})$ such that $(qx', q) = (x, g)$ as elements of $\tilde{X} \times G(\mathbb{A}^f)/Z(\mathbb{Q})$. In particular, $q = g \in G(\mathbb{Q})/Z(\mathbb{Q})$.

Now fix an automorphism σ of $\mathbb{C}$. The discussion above shows that there is a canonical structure of a complex manifold on $\sigma\hat{X}$. Choose a connected component $\tilde{X}^\sigma$ of $(\sigma\hat{X})^{an}$, and let p^σ and p^σ_K be the inclusion $\tilde{X}^\sigma \hookrightarrow (\sigma\hat{X})^{an}$ and the composite $(\sigma q_K)^{an} \circ p^\sigma$ respectively: thus

$$\begin{tikzcd} \tilde{X}^\sigma \arrow[r, hook, "p^\sigma"] \arrow[dr, "p^\sigma_K"'] & (\sigma\hat{X})^{an} \arrow[d, "(\sigma q_K)^{an}"] \\ & \sigma X_K \end{tikzcd} \qquad \begin{tikzcd} \sigma\hat{X} \arrow[d, "\sigma q_K"] \\ \sigma X_K . \end{tikzcd}$$

The group $G(\mathbb{A}^f)$ continues to act on $\sigma\hat{X}$, and this action is compatible with the complex structure on $(\sigma\hat{X})^{an}$. Define $Q \subset G(\mathbb{A}^f)/Z(\mathbb{Q})$ to be the stabilizer of $\tilde{X}^\sigma$. Note that $\tilde{X}^\sigma \longrightarrow \sigma X^{an}$ is a local isomorphism. Let $x \in \tilde{X}^\sigma$, and let M be the universal covering manifold of σX^{an}. For any $m \in M$, there is a unique map $M \longrightarrow \sigma\hat{X}$ such that m maps to $\hat{x}$ and the composites $M \to \sigma\hat{X} \to \sigma X_K^{an}$ are all analytic. Clearly $M \to \sigma\hat{X}$ is analytic, and so its image is contained in $\tilde{X}^\sigma$. This shows that $\tilde{X}^\sigma \to X^{an}$ is surjective and that $\tilde{X}^\sigma$ is a covering manifold of X^{an}. In particular, $\tilde{X}^\sigma$ is Zariski dense in $\sigma\hat{X}$, and so the action of Q on $\tilde{X}^\sigma$ is effective.

Exercise: Show $\tilde{X}^\sigma$ is universal for finite covering manifolds of σX?

<u>Lemma</u> 1.3. For any K in $\mathcal{S}$, the map

$$(k, x) \longmapsto kx : K \times \tilde{X}^\sigma \longrightarrow (\sigma\hat{X})^{an}$$

is surjective

Proof: Let $x \in \sigma\hat{X}^{an}$. By what we have just proved, there exists an $\tilde{x} \in \tilde{X}^{\sigma}$ such that $\tilde{x}$ and x have the same image in σX_K^{an}. Let $\hat{x} = p^{\sigma}(\tilde{x})$. Then $(\sigma q_K)(\hat{x}) = (\sigma q_K)(x)$, and so $q_K(\sigma^{-1}\hat{x}) = q_K(\sigma^{-1}x)$ in $X_K^{an} = K\backslash\hat{X}^{an}$. Therefore there exists a $k \in K$ such that $k(\sigma^{-1}\hat{x}) = \sigma^{-1}x$, and so $k\hat{x} = x$.

Proposition 1.4. For any K in $\mathcal{S}$, $KQ = G(\mathbb{A}^f)/Z(\mathbb{Q})$. In particular, Q is dense in $G(\mathbb{A}^f)/Z(\mathbb{Q})$

Proof. For any $g \in G(\mathbb{A}^f)$, $g\tilde{X}^{\sigma}$ is a connected component of $\sigma\hat{X}^{an}$, and so the lemma shows there exists a k in K such that $kg\tilde{X}^{\sigma} = \tilde{X}^{\sigma}$. By definition, $kg \in Q$, and therefore $g \in k^{-1}Q \subset KQ$.

Proposition 1.5. (a) The map

$$[x,g] \longmapsto gx \;:\; Q\backslash\tilde{X}^{\sigma} \times G(\mathbb{A}^f) \longrightarrow \sigma\hat{X}^{an}$$

is an isomorphism of complex manifolds ($G(\mathbb{A}^f)$ with the discrete topology).

(b) For every K, the map

$$[x] \longmapsto p_K^{\sigma}(x) \;:\; Q\cap K\backslash\tilde{X}^{\sigma} \longrightarrow \sigma X_K^{an}$$

is an isomorphism of complex manifolds.

Proof: (a) The lemma shows that $\tilde{X}^{\sigma} \times G(\mathbb{A}^f) \longrightarrow \sigma\hat{X}^{an}$ is surjective, and it is obvious that the fibres of the map are the orbits of Q.

(b) As $\sigma X_K = K\backslash\sigma\hat{X}$, it follows from (a) that there is an isomorphism $Q\backslash\tilde{X}^{\sigma} \times G(\mathbb{A}^f)/K \longrightarrow \sigma X_K^{an}$. The preceding proposition can be used to show that $Q\cap K\backslash\tilde{X}^{\sigma} \to Q\backslash\tilde{X}^{\sigma} \times G(\mathbb{A}^f)/K$ is an isomorphism.

We write $\Gamma^{\sigma}(K)$ for $Q\cap K$. Thus the proposition shows that $\Gamma^{\sigma}(K)\backslash\tilde{X}^{\sigma} \xrightarrow{\approx} \sigma(\Gamma(K)\backslash\tilde{X})$.

<u>Proposition</u> 1.6. For any $K \in \mathcal{S}$, $\mathcal{Q}$ is contained in the commensurability group of $\Gamma^\sigma(K)$

<u>Proof</u>: Let $g \in \mathcal{Q}$; we have to show that $g\,\Gamma^\sigma(K)\,g^{-1}$ is commensurable with $\Gamma^\sigma(K)$. Note that $g\,\Gamma^\sigma(K)g^{-1} = \Gamma^\sigma(gKg^{-1})$. Let $K' = K \cap gKg^{-1}$. The diagram of finite étale maps

$$\begin{array}{ccc} & X_{K'} & \\ \swarrow & & \searrow \\ X_K & & X_{gKg^{-1}} \end{array}$$

gives rise to a similar diagram

$$\begin{array}{ccc} & \sigma X_{K'} = \Gamma^\sigma(K')\backslash \tilde{X}^\sigma & \\ \swarrow & & \searrow \\ \sigma X_K = \Gamma^\sigma(K)\backslash\tilde{X}^\sigma & & \sigma X_{gKg^{-1}} = \Gamma^\sigma(gKg^{-1})\backslash\tilde{X}^\sigma. \end{array}$$

This shows that $\Gamma^\sigma(K')$ has finite index in both $\Gamma^\sigma(K)$ and $\Gamma^\sigma(gKg^{-1}) = g\,\Gamma^\sigma(K)g^{-1}$

<u>Remark</u> 1.7. It follows from (1.3) that $G(\mathbb{A}^f)$ acts transitively on the space of connected components of $(\sigma\hat{X})^{an}$. Thus every connected component is of the form $g\tilde{X}^\sigma$, $g \in G(\mathbb{A}^f)$, and has a stabilizer that is conjugate to $\mathcal{Q}$.

<u>Proposition</u> 1.8. Let Z be a nonempty $\mathcal{Q}$-invariant subset of $\tilde{X}^\sigma$. Assume that Z is a complex analytic subset of an open submanifold of $\tilde{X}^\sigma$ and that $p^\sigma_K(Z)$ is an algebraic subvariety of σX_K for some K in $\mathcal{S}$; then $Z = \tilde{X}^\sigma$

<u>Proof</u>. The hypotheses imply that $\hat{Z} \overset{df}{=} (p^\sigma_K(Z))_{K\in\mathcal{S}}$ is a pro-algebraic subvariety of $\sigma\hat{X}$ and is $\mathcal{Q}$-invariant. As $\mathcal{Q}$ is dense in $G(\mathbb{A}^f)/Z(\mathbb{Q})$, $\hat{Z}$ is invariant under $G(\mathbb{A}^f)/Z(\mathbb{Q})$,

and so $\sigma^{-1}(\hat{Z})$ is a $G(\mathbb{A}^f)/Z(\mathbb{Q})$-invariant subscheme of $\hat{X}$. Let $\tilde{Z} = \tilde{X} \cap \sigma^{-1}(\hat{Z})$. Then $\tilde{Z}$ is $G(\mathbb{Q})$-invariant and, as $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ and $G(\mathbb{R})$ acts transitively, this shows that $\tilde{Z} = \tilde{X}$.

For any K in $\mathcal{S}$, let X_K^* be the canonical (Baily-Borel) compactification of X_K. The dimension of $X_K^* - X_K$ is independent of K, and we shall write it $\dim(\partial X)$.

Corollary 1.9. Let Z be a nonempty Q-invariant analytic subset of $\tilde{X}^\sigma$ such that $\dim Z > \dim(\partial X)$; then $Z = \tilde{X}^\sigma$.
Proof: Let $d = \dim Z$, and let $Z^{(d)}$ be the set of $z \in Z$ such that Z has a component of dimension d at z. Then $Z^{(d)}$ is an
SLN25 analytic subset of Z [Narasimhan, p67] and is Q-invariant. Therefore we can assume $Z = Z^{(d)}$. The image Z' of Z in X_K (any K) is analytic and such that $Z' = Z'^{(d)}$. As $\dim(X_K^* - X_K) < d$, the theorem of Remmert-Stein [loc. cit., p123] shows that the closure $\bar{Z}'$ of Z in X_K^* is analytic, and Chow's theorem [loc. cit., p125] shows that $\bar{Z}'$ is algebraic. Therefore $\bar{Z}' \cap X_K = Z'$ is algebraic, and the proposition applies.

A point $\tilde{x} \in \tilde{X}$ is said to be special if there exists a torus $T \subset G$ such that $T_{\mathbb{C}}$ is maximal and $T(\mathbb{R})\tilde{x} = \tilde{x}$.

Proposition 1.10. Let $x \in \tilde{X}$ be special, and let $x^\sigma \in \tilde{X}^\sigma$ be any point such that $\hat{x}^\sigma = g(\sigma\hat{x})$ for some $g \in G(\mathbb{A}^f)$. Consider the σ-linear map of tangent spaces $\alpha : T_x(\tilde{X}) \longrightarrow T_{x^\sigma}(\tilde{X}^\sigma)$ defined by

$$\begin{tikzcd}
x \arrow[d, mapsto] & \tilde{X} \arrow[d] & & \sigma\hat{X} \arrow[d] \arrow[r, "g"] & \sigma\hat{X} & g^{-1}(x^\sigma) \arrow[d, mapsto] \arrow[r, mapsto] & x^\sigma . \\
x_K & X_K \arrow[rr, "\sigma"] & & \sigma X_K & & \sigma x_K &
\end{tikzcd}$$

Then there exists a homomorphism $j : T(\mathbb{Q}) \longrightarrow Q$ such that $j(T(\mathbb{Q}))$ fixes x and α commutes with the actions of $T(\mathbb{Q})$ on the two tangent spaces. The closure of $j(T(\mathbb{Q}))$ in $\mathrm{Aut}(T_{x^\sigma}(\tilde{X}^\sigma))$ for the real topology contains an element inducing multiplication by $\sqrt{-1}$.

<u>Proof</u>: Let $t \in T(\mathbb{Q}) \subset G(\mathbb{A}^f)$; then $t\hat{x} = \hat{x}$ and so $t\sigma\hat{x} = \sigma\hat{x}$. As $\sigma\hat{x} = g^{-1}\hat{x}^\sigma$, this means that $(gtg^{-1})\hat{x}^\sigma = \hat{x}^\sigma$. In particular, gtg^{-1} maps one point of $\tilde{X}^\sigma$ into $\tilde{X}^\sigma$ and so stabilizes $\tilde{X}^\sigma$. Thus $gtg^{-1} \in Q$, and we can define $j(t) = gtg^{-1}$. By construction $j(t)$ fixes x^σ, and it is routine to check that α carries the action of t on $T_x(\tilde{X})$ into the action of $j(t)$ on $T_{x^\sigma}(\tilde{X}^\sigma)$ ($\alpha = dg \circ (dp_K^\sigma)^{-1} \circ d\sigma \circ dp_K$; the actions of t on $\tilde{X}$, X_K, and σX_K commute with p_K, σ, and p_K^σ, and the action of t on $\tilde{X}^\sigma$ is transformed by $g : \tilde{X}^\sigma \to \tilde{X}^\sigma$ into the action of gtg^{-1}. To prove the last assertion of the proposition, we need a lemma.

<u>Lemma</u> 1.11. Let T be a torus over $\mathbb{Q}$. For each automorphism σ of $\mathbb{C}$, there is a unique automorphism $t \mapsto t^\sigma$ of $T(\mathbb{C})$ such that $\chi(t^\sigma) = \sigma\chi(t)$, all $t \in T(\mathbb{C})$, $\chi \in X^*(T)$. The map $t \mapsto t^\sigma$ is continuous and, if $T(\mathbb{R})$ is compact, takes

$T(\mathbb{R})$ into itself. For $t \in T(\mathbb{Q})$, $\sigma(\chi(t)) = \chi(t^\sigma)$

<u>Proof</u>: There is an isomorphism

$$t \mapsto (\chi \mapsto \chi(t)) : T(\mathbb{C}) \longrightarrow \mathrm{Hom}(X^*(T), \mathbb{C}^\times),$$

and we define $t \mapsto t^\sigma$ on $T(\mathbb{C})$ to correspond to the map on $\mathrm{Hom}(X^*(T), \mathbb{C}^\times)$ induced by the $\mathbb{Z}$-linear map $\sigma : X^*(T) \longrightarrow X^*(T)$. Clearly $\chi(t^\sigma) = (\sigma\chi)(t)$, and $t \mapsto t^\sigma$ is the unique map with this property. The continuity is obvious. If $T(\mathbb{R})$ is compact, it is the unique maximal compact subgroup of $T(\mathbb{C})$, and so it is preserved by $t \mapsto t^\sigma$. For $t \in T(\mathbb{Q})$, $\sigma\chi(t) = (\sigma\chi)(t) = \chi(t^\sigma)$.

$\sigma V = V \otimes_{\mathbb{C},\sigma} \mathbb{C}$

<u>Proof of 1.10 continued</u>: The σ-linear map $\alpha : T_x(\tilde{X}) \longrightarrow T_{x^\sigma}(\tilde{X}^\sigma)$ induces a $\mathbb{C}$-linear isomorphism $\beta : \sigma T_x(\tilde{X}) \longrightarrow T_{x^\sigma}(\tilde{X}^\sigma)$. Moreover,

$$\beta \circ \sigma(\rho_x(t)) = \rho_{x^\sigma}(j(t)) \circ \beta \quad , \quad t \in T(\mathbb{Q})$$

where ρ_x and ρ_{x^σ} denote the representations of G and $\mathbb{Q}$ on the tangent spaces at x and x^σ. Clearly $(\sigma\rho_x)(t) = \rho_x(t^\sigma)$, and so

$$\beta \circ \rho_x(t^\sigma) \circ \beta^{-1} = \rho_{x^\sigma}(j(t)) \quad , \quad t \in T(\mathbb{Q}).$$

As $T(\mathbb{Q})$ is dense in $T(\mathbb{R})$, it follows that for any $\gamma \in T(\mathbb{R})$ there exists a γ' in the closure of $\rho_{x^\sigma}(j(T(\mathbb{Q})))$ in $\mathrm{Aut}(T_{x^\sigma}(\tilde{X}^\sigma))$ such that γ' acts as $\beta \circ \rho_x(t^\sigma) \circ \beta^{-1}$ on $T_{x^\sigma}(\tilde{X}^\sigma)$. It is known (see, for example, [Helgason, VIII, 4.5]) that there is a γ in $T(\mathbb{R})$ acting as $\sqrt{-1}$ on $T_x(\tilde{X})$; therefore

$$\begin{aligned} \gamma' &= \beta \circ \rho_x(\gamma) \circ \beta^{-1} \\ &= \beta \circ (\text{multiplication by } \sqrt{-1}) \circ \beta^{-1} \\ &= \text{multiplication by } \sqrt{-1}. \end{aligned}$$

2 A criterion to be an arithmetic variety

Let (M, g) be an oriented Riemannian manifold. There is a unique volume element μ on M having value $+1$ on any orthonormal frame. In local coordinates

$$g = \sum g_{ij}\, dx^i \otimes dx^j$$
$$\mu = \sqrt{\det(g_{ij})}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m .$$

Let ∇ be the connection defined by g. The <u>curvature tensor</u> is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} .$$

In terms local coordinates

check the constants

$$R\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right) \tfrac{\partial}{\partial x^l} = \sum_k R^k_{lij} \tfrac{\partial}{\partial x^k} .$$

The <u>Ricci tensor</u> $r(X, Y)$ is determined in local coordinates by

$$r(X,Y) = \sum R_{ij}\, dx^i \otimes dx^j , \quad R_{ij} = \sum_k R^k_{ijk} .$$

Assume further that M is a complex manifold, with multiplication by $\sqrt{-1}$ being described by the tensor J. Then g is said to be <u>Hermitean</u> if $g(JX, JY) = g(X, Y)$, all X, Y. In this case

$$h(X,Y) \overset{df}{=} g(X,Y) + i\, g(X, JY)$$

is a positive-definite Hermitian form. The form $\Phi(X,Y) = g(X, JY)$ is skew-symmetric. It is called the fundamental 2-form of the Hermitean metric h, and if it is closed, h is said to be <u>Kählerian</u>.

<u>Lemma</u> 2.1. Let M be a connected complex manifold of dimension n with Kählerian Riemannian structure g. If the associated volume element is described in terms of local coordinates by

$$\mu = k\left(\tfrac{i}{2}\right)^n dz_1 \wedge d\bar z_1 \wedge \cdots \wedge d\bar z_n$$

then the Ricci tensor is described by

$$r = \sum_{1\le i,j\le n} \frac{\partial^2 \log k}{\partial z_i \, \partial \bar z_j}\, dz_i \otimes d\bar z_j \,. \qquad (2.1.1)$$

Proof: [Helgason, VIII 2.5]

In general, if $\mu = k\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar z_1 \wedge \cdots \wedge d\bar z_n$ is a volume element on a complex manifold, then the tensor defined by (2.1.1) will be called the Ricci tensor, $\mathrm{Ric}(\mu)$, of μ.

Lemma 2.2 Let M be a connected complex manifold in which a group G acts transitively, and let μ be a G-invariant volume element on M such that $g = \mathrm{Ric}(\mu)$ is positive definite (and therefore is a Riemannian metric). Then g is equal to its Ricci tensor (and therefore is an Einstein metric).

Proof: Let μ' be the volume form associated with g. Then $\mu' = f\mu$ for some positive function f on M. As μ is G-invariant, so also are g, μ', and f. Therefore f is constant, and so the Ricci tensor of g is $\mathrm{Ric}(f\mu) = \mathrm{Ric}(\mu) = g$.

Theorem 2.3. Let M be a connected complex manifold on which a unimodular Lie group G acts effectively and transitively. Assume there is a G-invariant volume element μ on M such that $\mathrm{Ric}(\mu)$ is positive definite. Then M is a Hermitian symmetric domain, and $G \supset \mathrm{Aut}(M)^+$.

Proof: This is proved in [Koszul, São Paulo notes 1959, p61]. Alternatively one can combine the following results:

A complex homogeneous manifold with an invariant volume form whose Ricci tensor is positive definite is isomorphic to a homogeneous bounded domain [Piatetski-Shapiro, Automorphic Functions ... (1969), p48]

A connected unimodular Lie group acting effectively and transitively on a bounded domain is semisimple [Hano, Am. J. M. 69 (1957) 885-900]

A bounded domain admitting a transitive semisimple group of automorphisms is symmetric [Borel, Proc N. A. S. 40 (1954) 1147-1151] [Koszul, Can. J. Math 7 (1955) 562-576].

Check Ref.

Theorem 2.4. Let M be a connected complex manifold, and endow $\mathrm{Aut}(M)$ with the compact-open topology. Let $\Gamma \subset Q$ be subgroups of $\mathrm{Aut}(M)$ with Γ discrete and torsion free, and assume that an orbit of Q in M is dense. Assume also that there is a Q-invariant volume form μ on M such that

(a) $\mathrm{Ric}(\mu)$ is positive definite

(b) $\int_{\Gamma\backslash M} \mu < \infty$.

Then M is a Hermitian symmetric domain, and the closure $\bar{Q}$ of Q in $\mathrm{Aut}(M)$ is a semisimple Lie group whose identity component is $\mathrm{Aut}(M)^+$. If moreover Q is contained in the commensurability group of Γ in $\bar{Q}$, then Γ is arithmetic and so $\Gamma\backslash M$ is an arithmetic variety.

Proof: Let $g = \mathrm{Ric}(\mu)$. By assumption, (M, g) is a Riemannian manifold, and so the group $\mathrm{Is}(M,g)$ of its isometries is a

Lie group [Kobayashi, Ergeb. 70, II Thm 1.1]. As μ is Q-invariant, Q is contained in $Is(M,g)$, and its closure G is a Lie subgroup of $Is(M,g)$. Clearly $G = \bar{Q}$.

Lemma 2.5. Let (M,g) be a Riemannian manifold. Then for all $m \in M$, the map $Is(M,g) \longrightarrow M$, $\alpha \mapsto \alpha m$, is proper.
Proof: Let $O(M)$ be the bundle of orthonormal frames over M. Every automorphism α of M defines a compatible automorphism $\bar{\alpha}$ of $O(M)$. Let $u \in O(M)$ and let m be its image in M. The mapping $\alpha \mapsto \bar{\alpha}(u)$ embeds $Is(M,g)$ as a closed submanifold of $O(M)$ [ibid, p41]. The projection $O(M) \to M$ is clearly proper, and its restriction to $Is(M,g)$ is the map $\alpha \mapsto \alpha m$.

Let m be a point of M such that Qm is dense in M. Then the lemma shows that Gm is closed and so equals M. The orbit under G^+

~~Lie group [Montgomery-Zippin, p208]. As μ is Q-invariant, Q, and its closure $\bar{Q}$, are contained in $Is(M,g)$. Write $G = \bar{Q}$ — it is a Lie group, and we are assuming that it~~ acts ~~transitively on M~~. ~~The orbit under G^+ of m~~ of m contains an open set in M, and so equals M [Kobayoshi-Nomizu, p178, Cor 4.8]; thus G^+ also acts transitively on M. The isotropy group K at the point m is compact. Thus the fact that $M = G^+/K$ carries a G^+-invariant measure such that $\Gamma \backslash M$ has finite volume implies that G^+ carries left invariant measure relative to which $\Gamma \cap G^+ \backslash G^+$ has finite volume. Hence $\Gamma \cap G^+$ is a lattice in G^+, and so G is unimodular (the image of the modulus function Δ_G in $\mathbb{R}_{>0}$ is a subgroup with finite measure, and so is $\{1\}$ — see [Raghunathan, 1.9]). The preceding theorem now shows that M is a Hermitean symmetric domain and $G^+ = \mathrm{Aut}(M)^+$.

The final statement of the theorem is a consequence of the following theorem of Margulis: Let G be a semisimple connected real Lie group with no compact factors, and let Γ be a lattice in G ; if the commensurability group of Γ in G is dense in G, then Γ is an arithmetic subgroup of G [Margulis, AMS Transl 109 (1977) 33–45, Thm 9].

Remark 2.5. In the statement of the theorem just applied, Margulis assumes that Γ is irreducible, but this is unnecessary. If Γ is reducible, then there exist connected normal subgroups G_i of G such that

$G = \prod G_i$ (almost direct product), $\Gamma_i \overset{df}{=} \Gamma \cap G_i$ is an irreducible lattice in G_i, and $\prod \Gamma_i$ is a subgroup of finite index in Γ. Clearly the commensurability group of Γ in G is the product of the commensurability groups of the Γ_i : $\mathrm{Comm}(\Gamma) = \prod \mathrm{Comm}(\Gamma_i)$. Thus if $\mathrm{Comm}(\Gamma)$ is dense in G, each group $\mathrm{Comm}(\Gamma_i)$ is dense in G_i, and Margulis's statement can be applied to show that each Γ_i is arithmetic.

<u>Corollary</u> 2.6 Let X be an arithmetic variety as in §1, and suppose that there exists a Q-invariant volume element μ on $\tilde{X}^\sigma$ satisfying the conditions (2.4a) and (2.4b). Assume that there exists a finite family of arithmetic subvarieties $i_\alpha : X_\alpha \hookrightarrow X$ such that for some special point x of X, $x_\alpha \overset{df}{=} i_\alpha^{-1}(x)$ is special on X_α, all α. If each σX_α is arithmetic, and the subspace of $T_x(X)$ generated by the $T_{x_\alpha}(X_\alpha)$ has dimension $> \dim(\partial X)$, then σX is arithmetic.

<u>Proof</u>: Let $g = \mathrm{Ric}(\mu)$, and let M be the closure of the orbit $Q\tilde{x}$ for some $\tilde{x} \in \tilde{X}^\sigma$ lifting x Then (M, g) is a Riemannian manifold, and the closure $\bar{Q}$ of Q in $Is(M, g)$ acts transitively on M (by 2.5). Since $q \mapsto q\tilde{x}$, $\bar{Q} \to \tilde{X}^\sigma$, is a proper map, it is an embedding and M is a regular closed submanifold of $\tilde{X}^\sigma$ By (1.10), M has a $\bar{Q}$-invariant complex structure — it is therefore a complex analytic subset of $\tilde{X}^\sigma$. Since it contains $\sigma\tilde{X}_\alpha$ for each α, it has dimension $> \dim(\partial X)$, and so (1.9) shows that $M = \tilde{X}^\sigma$.

Now (2.4) can be applied.

<u>Remark</u> 2.7. It is now possible to complete the proof of Theorem 0.1 in the case that X is compact. In this case, $\dim(\partial X) = -1$ and so the map $x \hookrightarrow X$ will serve for the family $(X_\alpha \hookrightarrow X)$. Recall the following theorem of Yau [Comm. Pure Appl. M 31 (1978)]: let V be a smooth projective algebraic variety over $\mathbb{C}$; then there exists a unique Kähler metric on V^{an} such that $\mathrm{Ric}(g) = g$. Apply this to σX and let μ be the inverse image on $\tilde{X}^\sigma$ of the volume element associated with g. The uniqueness of g shows

Expand that μ is invariant under all automorphisms of $\tilde{X}^\sigma$. Since $\mathrm{Ric}(\mu) = \tilde{g}$ it satisfies (a), and it satisfies (b) because X is compact. Thus 2.5 applies.

In the remainder of this section we shall show that the condition (2.4b) is automatically satisfied in the context of (2.6).

A complex manifold M will be said to be <u>compactifiable</u> if it can be embedded as an open dense subset of a compact analytic space $\bar{M}$ in such a way that $\bar{M} - M$ is an

Check. analytic subset. Hironaka's theorem [ ] then shows that $\bar{M}$ can be chosen to be a manifold such that $\bar{M} - M$ is a divisor with normal crossings. Clearly, if $M = V^{an}$ with V a quasi-projective algebraic variety, then M is compactifiable (in fact, not even quasi-projectivity is necessary).

<u>Proposition</u> 2.8. Let M be a compactifiable complex manifold of dimension n, and let μ be a volume form on M such that

$$\left.\begin{array}{l} \mathrm{Ric}(\mu) \text{ is positive-definite} \\ \mathrm{Ric}(\mu)^n \geq \mu \end{array}\right\} \quad (2.8.1)$$

Then $\int_M \mu < \infty$.

Let $D(a) = \{z \in \mathbb{C} \mid |z| < a\}$

$D^*(a) = \{z \in \mathbb{C} \mid 0 < |z| < a\}$.

Lemma 2.9. For $\delta < 1$,

$$\int_{D(\delta)} \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} = \frac{2\delta^2}{1-\delta^2}$$

$$\int_{D^*(\delta)} \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2} = -1/\log\delta .$$

Proof: Put $z = te^{i\theta}$, $\bar{z} = te^{-i\theta}$. Then $dz \wedge d\bar{z} = 2it\, d\theta \wedge dt$.
Thus,

$$\int_{D(\delta)} \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} = \int_0^\delta \frac{i}{\cancel{\pi}} \cdot 2\cancel{\pi} \cdot \frac{2it}{(1-t^2)^2}\, dt = \frac{2}{1-t^2}\Big|_0^\delta = \frac{2\delta^2}{1-\delta^2}$$

$$\int_{D^*(\delta)} \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2} = \int_0^\delta \frac{i}{\cancel{\pi}} \cancel{2}\cancel{\pi} \frac{2it}{t^2 \cancel{4}(\log t)^2}\, dt = -\int_0^\delta \frac{dt}{t(\log t)^2} = \frac{-1}{\log t}\Big|_0^\delta$$

$$= -1/\log\delta$$

Remark 2.10. Let $\mu_D = i/\pi \dfrac{dz \wedge d\bar{z}}{(1-|z|^2)^2}$ and $\mu_{D^*} = \dfrac{i}{\pi} \dfrac{dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2}$.

Then μ_D is the Poincaré metric on the unit disc, and $Ric(\mu_D) = \mu_D$. The inverse image of μ_{D^*} relative to the covering map $D \longrightarrow D^*$ is μ_D, and so $Ric(\mu_{D^*}) = \mu_{D^*}$ also. (Cf. [Griffiths, p47]).

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Lemma 2.11. Let μ be a volume element on $D(1)^r \times D^*(1)^{n-r}$ satisfying the estimates (2.8.1). Then

$$\mu \le \mu_D^r \times \mu_{D^*}^{n-r}.$$

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Proof: If $r = n$, this is precisely Ahlfors's lemma [Griffiths, 2.21]. Consider the covering map

$$D(1)^n \xrightarrow{\varphi} D(1)^r \times D^*(1)^{n-r}.$$

Then $\varphi^*(\mu)$ satisfies (2.8.1) and so $\varphi^*(\mu) \le \mu_D^n = \varphi^*(\mu_D^r \times \mu_{D^*}^{n-r})$. This implies that $\mu \le \mu_D^r \times \mu_{D^*}^{n-r}$.

Lemma 2.12. Let $U = D(1)^n$ and let $U' = D(1)^r \times D^*(1)^{n-r}$. Then there exists an open neighbourhood V of $U - U'$ in U such that $|\int_V \mu_D^r \times \mu_{D^*}^{n-r}| < \infty$

Proof: A stronger statement is proved in (3.11) below.

Proof of 2.8. Embed M in a compact manifold $\bar{M}$ in such a way that $N \overset{df}{=} \bar{M} - M$ is a divisor with normal crossings. Then there is a finite family of open subsets U_i of $\bar{M}$ such that $N \subset \bigcup U_i$ and, for each i, the pair $(U_i, U_i \cap M)$ is isomorphic to $(D(1)^n, D(1)^r \times D^*(1)^{n-r})$. For each i, choose $V_i \subset U_i$ to correspond to an open subset of $D(1)^n$ satisfying the conditions of (2.12). Then the complement C of $\bigcup V_i$ in $\bar{M}$ is compact, and it is contained in M. Thus

$$\int_M \mu \le \int_C \mu + \sum \int_{V_i \cap M} \mu \le \int_C \mu + \sum \int_{V_i \cap M} \mu_D^r \times \mu_{D^*}^{n-r} < \infty.$$

Corollary 2.13. Let X be an arithmetic variety (as in §1), and let U be an open dense Γ-invariant submanifold of $\tilde{X}^\sigma$ such that $\tilde{X}^\sigma - U$ is an analytic subset of $\tilde{X}^\sigma$. Assume that there is a Γ-invariant volume element on U such that $\mathrm{Ric}(\mu) > 0$. Suppose further that there exists a finite family of arithmetic subvarieties $i_\alpha : X_\alpha \hookrightarrow X$ such that for some special point x of X such that σx is in the image of $\mathcal{U} \to \sigma X$, $i_\alpha^{-1}(x)$ is special on X_α all α. If each σX_α is arithmetic, and the subspaces $T_{x_\alpha}(X_\alpha)$ of $T_x(X)$ generate it, then σX is arithmetic.

Proof. Let $\tilde{x} \in \mathcal{U}$ map to $\sigma x \in \sigma X$, and let $g = \mathrm{Ric}(\mu)$ — it is a Riemannian metric on $\mathcal{U}$. Define M to be the closure in $\mathcal{U}$ of the orbit $Q\tilde{x}$. As in the proof of (2.6), $M = \bar{Q}\tilde{x}$, where $\bar{Q}$ is the closure of Q in $\mathrm{Is}(M, g)$, and it is a complex analytic subset of $\mathcal{U}$. As it contains the X_α for each α, it is also open in $\mathcal{U}$, and so $M = \mathcal{U}$.

We know μ on M satisfies condition (2.4a) and we now check that it satisfies (2.4b). Suppose $\mathrm{Ric}(\mu)^n = c\mu$ at some point of $\check{X}^\sigma$. Then $c > 0$, and because $\bar{Q}$ acts transitively on M and μ is $\bar{Q}$-invariant, we must have $\mathrm{Ric}(\mu)^n = c\mu$ holding on all of M. Since $\mathrm{Ric}(c\mu) = \mathrm{Ric}(\mu)$, this shows that $c\mu$ satisfies the estimates (2.8.1). Therefore so also does the volume element induced by $c\mu$ on $\Gamma^\sigma \backslash M$, and so Proposition 2.8 implies that $\int_{\Gamma^\sigma \backslash M} c\mu < \infty$.

We can now apply (2.4) to M, Q, Γ, and we find that $\Gamma \backslash M$ is an arithmetic variety. In particular, it is an open algebraic subvariety of σX, and so (1.9) implies that $M = \tilde{X}^\sigma$. We conclude that $\sigma X = \Gamma \backslash M$, and so it is an arithmetic variety.

i^{n^2}

3. The Bergmann metric on $\tilde{X}^{\sigma}$

Let M be a complex manifold of dimension n, and let

$$\mathcal{H}(M) = \{\omega \in \Gamma(M, \Omega_M^{n,0}) \mid \; |\int_M \omega \wedge \bar{\omega}| < \infty\}$$

Then $\mathcal{H}(M)$ is a separable Hilbert space with inner product

$$(\omega_1|\omega_2) = i^{n^2}\int_M \omega_1 \wedge \bar{\omega}_2 .$$

Let $\omega_0, \omega_1, \ldots$ be an orthonormal basis for $\mathcal{H}(M)$, and let

$$\mu_M = \Sigma\, \omega_i \wedge \bar{\omega}_i .$$

This is a nonnegative C^∞ $2n$-form on M, called the Bergmann volume form. When it has no zeros it is a volume element. Note that

$$\int_M \mu_M = \dim \mathcal{H}(M).$$

Proposition 3.1 (a) Let $m \in M$; then

$$\mu_M(m) = \sup_{\substack{\omega\in\mathcal{H}(M)\\(\omega|\omega)=1}} (\omega\wedge\bar{\omega})(m)$$

(b) All automorphisms of M leave μ_M invariant.

(c) If M' is a connected open submanifold of M, then $\mu_M|M' = c\,\mu_{M'}$ where c is a function on M', $0 \le c \le 1$; moreover, if $M-M'$ is a complex analytic subvariety of dimension $\le n-1$, then $\mathcal{H}(M) \longrightarrow \mathcal{H}(M')$ is bijective and so $\mu_M|M' = \mu_{M'}$.

(d) If M_1 and M_2 are complex manifolds of dimensions n_1 and n_2, then

$$\mu_{M_1\times M_2} = (-1)^{n_1 n_2}\mu_{M_1}\wedge\mu_{M_2} .$$

Trans AMS 92 (1959) Proof: [Kobayashi, 2.1, 2.2, 2.3, 2.4, 2.5]

Consider the condition:

(3.2) for every $m \in M$, there exists an $\omega \in \mathcal{H}(M)$ such that $\omega(m) \neq 0$.

When this condition holds, μ_M is a volume element, and we let $h_M = \mathrm{Ric}(\mu_M)$ be the associated Hermitian tensor.

Consider the condition:

(3.3) for all $m \in M$ and $Z \in T_m(M)$, there exists an $\omega = f\, dz_1 \wedge \cdots \wedge dz_n$ in $\mathcal{H}(M)$ such that $f(z)=0$, $Z(f) \neq 0$.

<u>Proposition</u> 3.4. The form h_M is invariant under all automorphisms of M and is positive semi-definite. It is positive definite if and only if (3.3) holds.

<u>Proof</u>. [Ibid., 3.1]

<u>Remark</u> 3.5. Let $\mathbb{P}(\mathcal{H}(M)^\vee)$ be the (possibly infinite dimensional) projective space of lines in the dual Hilbert space to $\mathcal{H}(M)$, and assume that M satisfies (3.2). Then there is a canonical map $j: M \longrightarrow \mathbb{P}(\mathcal{H}(M)^\vee)$ such that, if ω_0 is nowhere zero on $U \subset M$, then $j(m)$, $m \in U$, is the class of the map $\omega \longmapsto (\omega/\omega_0)(m)$. It is possible to regard $\mathbb{P}(\mathcal{H}(M)^\vee)$ as an infinite-dimensional complex manifold, and the usual construction in the finite dimensional case generalizes to give a complete Kähler metric on $\mathbb{P}(\mathcal{H}(M)^\vee)$. The Bergmann Hermitian form h_M on M is the inverse image under j of this canonical metric on $\mathbb{P}(\mathcal{H}(M)^\vee)$. The map j is an immersion, and h_M is a metric, if and only if (3.3) holds. (See [Kobayashi, §7, 8] for this.)

<u>Remark</u> 3.6. Let X be an arithmetic variety and assume there exists a family $(X_\alpha \hookrightarrow X)$ as in (2.6). Then it follows from (2.6) and (2.12) that σX is an arithmetic variety if

the Bergmann Hermitian form on $\tilde{X}^\sigma$ is a metric; conversely, if σX is an arithmetic variety, then $\tilde{X}^\sigma$ is a bounded domain and so Bergmann's original theorem says that it has a nondegenerate Bergmann metric.

The main result of this section is a first step towards proving that $\tilde{X}^\sigma$ has a nondegenerate Bergmann metric.

Theorem 3.7 With the notations of §1, $\mathcal{H}(\tilde{X}^\sigma) \neq \{0\}$

Proposition 3.8 Let M be a compactifiable complex manifold. For all $\varepsilon > 0$, there exists an open subset U_ε in M with compact complement such that, for any étale covering $\varphi : N \longrightarrow M$, $\left| \int_{\varphi^{-1}(U_\varepsilon)} \mu_N \right| < \varepsilon \deg \varphi$ where μ_N is the Bergmann volume form on N.

The proof is based on the following elementary result.

Lemma 3.9. Let $\varphi_m : D^*(1) \longrightarrow D^*(1)$ be the map $z \longmapsto z^m$. For all $\varepsilon > 0$, there exists a δ such that

$$\left| \int_{\varphi_m^{-1}(D^*(\delta))} \mu_{D^*(1)} \right| < \varepsilon m, \text{ all } m.$$

Proof: The Bergmann metric on $D(1)$ is the Poincaré metric, and so from (3.1c) and (2.9) we find that

$$\int_{\varphi_m^{-1}(D^*(\delta))} \mu_{D^*(1)} = \int_{D^*(\delta^{1/m})} \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} = \frac{2}{(1/\delta)^{2/m} - 1}$$

As $x - 1 > \log x$ for all $x > 1$, we see that

$$\frac{2}{(1/\delta)^{2/m}-1} < \frac{2}{\log(1/\delta)^{2/m}} = \frac{m}{\log(1/\delta)}$$

from which the sublemma is obvious.

<u>Remark</u> 3.10. Note that the notation in this section conflicts with that in (2.10) — $\frac{i}{\pi}\frac{dz\wedge d\bar z}{|z|^2(\log|z|^2)^2}$ is not the Bergmann volume element on $D^*(1)$. Denote this element by $\nu_{D^*(1)}$. Then Ahlfors's lemma applied on the covering space $D(1)$ of $D^*(1)$ shows that $\nu_{D^*(1)} \geq \mu_{D^*(1)}$ (cf. 2.11). Thus we find again that

$$\int_{D^*(\delta^{1/m})} \mu_{D^*(1)} \leq \int_{D^*(\delta^{1/m})} \nu_{D^*(1)} \overset{(2.9)}{=} 1/\log(1/\delta)^m = m/\log(1/\delta).$$

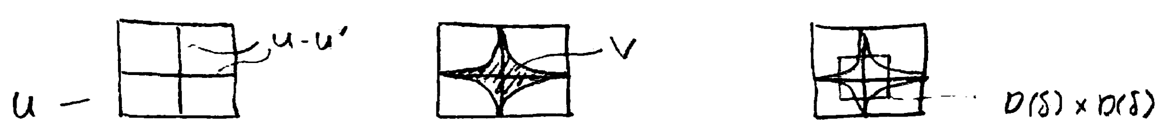
i.e. finite étale covering

<u>Lemma</u> 3.11. Let $U = D(1)^n$ and $U' = D(1)^r \times D^*(1)^{n-r}$. For all $\varepsilon>0$, there exists an open neighbourhood V of $U - U'$ in U such that for any étale covering $\varphi : N \to U'$, $\left|\int_{\varphi^{-1}(V)} \mu_N\right| < \varepsilon \deg\varphi$.

<u>Proof</u>: The fundamental group of U' is $\mathbb{Z}^{n-r}$, and so every étale covering of it is of the form

$$N = D(1)^r \times D^*(1)^{n-r} \longrightarrow D(1)^r \times D^*(1)^{n-r},\ (z_1,\dots)\mapsto(z_1,\dots,z_r,z_{r+1}^{m_1},\dots).$$

The proof is too messy to write out in detail. Consider for example the case $n=2$, $U' = D^*(1)^2$. Then



(a) Look at $V \cap D(\delta)\times D(\delta)$ — apply (3.9)

(b) Each other connected component V_0 of V is simply connected. Thus

$\varphi^{-1}(V_0)$ = disjoint union of degree(φ) copies of V_0

Hence $\int_{\varphi^{-1}(V_0)} \mu_N \leq \int_{\varphi^{-1}(V_0)} \mu_{\varphi^{-1}(V_0)} = \deg(\varphi)\int_{V_0}\mu_{V_0}$, and so one only has to arrange things so that $\int_{V_0}\mu_{V_0} < \varepsilon/100$. $\left(|z_1| \cdot |z_2| ?\right) < \varepsilon/100)$

<u>Proof</u> of 3.8. Embed M in a compact manifold $\bar{M}$ in such a way that $\bar{M}-M$ is a divisor with normal crossings. Then there is a finite family of open subsets $(U_i)_{1\le i\le s}$ of $\bar{M}$ such that $\bar{M}-M \subset \bigcup U_i$ and, for each i, the pair $(U_i, U_i \cap M)$ is isomorphic to $(D(1)^n, D(1)^r \times D^*(1)^{n-r})$. For each i, choose a neighbourhood V_i of $U_i - U_i \cap M$, as in the sublemma for ε/s. Then the complement of $U \overset{\text{df}}{=} \bigcup V_i$ in $\bar{M}$ is compact, and it is contained in M. Moreover, for any étale covering $\varphi : N \longrightarrow M$,

$$\left| \int_{\varphi^{-1}(U)} \mu_N \right| \le \sum_{i=1}^{s} \left| \int_{\varphi^{-1}(V_i)} \mu_N \right| < s\left(\tfrac{\varepsilon}{s} \deg \varphi\right) = \varepsilon \deg \varphi.$$

The next result gives a criterion for showing $\mathcal{H}(\tilde{M})$ is nonzero.

<u>Proposition</u> 3.12. Let M be a compactifiable manifold of dimension n, and let $p : \tilde{M} \longrightarrow M$ be an infinite Galois covering with Galois group Γ. Assume there is a sequence of normal subgroups of finite index in Γ,

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_i \supset \cdots \supset \{1\}$$

such that $\bigcap \Gamma_i = \{1\}$.

Let $M_i = \Gamma_i \backslash \tilde{M}$ and let $h_i = \dim \mathcal{H}(M_i)$. Then

(a) $h_i < \infty$;

(b) $\{h_i / (\Gamma : \Gamma_i)\}$ is bounded;

(c) if the sequence $h_i/(\Gamma:\Gamma_i)$ does not tend to zero, then $\mathcal{H}(\tilde{M}) \neq 0$.

<u>Remark</u> 3.13. In general, if $\varphi : Y \to X$ is an étale covering

one can show that

$$(\deg\varphi)\,\varphi^*(\mu_X) \geq \mu_Y \geq \left(\frac{1}{\deg\varphi}\right)\varphi^*(\mu_X).$$

Hence

$$(\deg\varphi)^2\int_X \mu_X \geq \int_Y \mu_Y \geq \int_X \mu_X$$

For example, if Y is a disjoint union of copies of X, then $\mu_Y = \varphi^*(\mu_X)$ and $\int_Y \mu_Y = \deg\varphi \int_X \mu_X$.

In the situation of the proposition, one has the trivial estimates

$$(\Gamma:\Gamma_i)\,h_0 \geq \frac{h_i}{(\Gamma:\Gamma_i)} \geq \frac{h_0}{(\Gamma:\Gamma_i)}$$

with $h_i = (\Gamma:\Gamma_i)$ in the case that $\tilde{M}$ is a trivial covering of M.

<u>Proof</u> of 3.12. As the Bergmann volume form μ_{M_i} on M_i is Γ-invariant, it induces a form μ_i on M such that $q_i^*\mu_i = \mu_{M_i}$ where q_i is the covering map $M_i \longrightarrow M$. Clearly,

$$\int_M \mu_i = \frac{1}{(\Gamma:\Gamma_i)}\int_{M_i}\mu_{M_i} = h_i/(\Gamma:\Gamma_i). \qquad (3.12.1)$$

According to (3.8), there exists an open subset $U = U_1$ with compact complement, $C = M - U$, such that

$$\left|\int_{\varphi^{-1}(U)}\mu_N\right| < \deg\varphi$$

for any étale covering $\varphi: N \longrightarrow M$. Because C is compact, there exists a finite set $(U_r)_{1\leq r\leq s}$ of open subsets of M such that (a) there exist isomorphisms $\varphi_r : U_r \xrightarrow{\approx} D(1)^n$ (of complex manifolds);

(b) $\bigcup_{r=1}^{s} \varphi_r^{-1}(D(\tfrac{1}{2})^n) \supset C$.

Because $q_i^{-1}(U_r)$ is a disjoint union of copies of U_r, we have $q_i^*(\mu_{U_r}) = \mu_{q_i^{-1}(U_r)} \geq \mu_{M_i}$ and so $\mu_{U_r} \geq \mu_i | U_r$. Note that

$\mu_{U_r} = \varphi_r^*(\mu^n_{D(1)})$ (see 3.1d). We conclude that
$$\int_C \mu_i \leq s \int_{D(\frac{1}{2})^n} \mu_{D(1)^n} = s\left(\int_{D(\frac{1}{2})} \mu_{D(1)}\right)^n = B < \infty,$$

and that
$$h_i/(\Gamma:\Gamma_i) \overset{(3.12.1)}{=} \int_M \mu_i = \int_C \mu_i + \int_U \mu_i \leq B+1$$

This proves both (a) and (b).

Now assume that the sequence $h_i/(\Gamma:\Gamma_i)$ does not tend to zero. By passing to a subsequence, we can in fact assume that for some $a>0$, $h_i/(\Gamma:\Gamma_i) \geq a$ all i. Choose $U \subset M$ as in (3.8) with $\varepsilon = a/2$, and let $C = M - U$; then
$$\int_C \mu_i = \int_M \mu_i - \int_U \mu_i = h_i/(\Gamma:\Gamma_i) - \int_{\varphi^{-1}(U)} \mu_{M_i} \geq a - \frac{a}{2} = \frac{a}{2}.$$

Let $\nu = \sum_{r=1}^{s} \varphi_r^*(\mu_{D(1)^n})$. We showed above that, on U_r $\mu_i \leq \mu_{U_r} = \varphi_r^*(\mu_{D(1)^n})$, and so on C, $\mu_i \leq \nu$.

<u>Lemma</u> 3.14. There exists an $x_0 \in C$ such that a subsequence of $(\mu_i/\nu)(x_0)$ converges to some $b>0$.

<u>Proof</u>: Suppose $\sup_{x\in C}((\mu_i/\nu)(x)) \to 0$; then $\int_C \mu_i \to 0$, which contradicts the assertion above that $\int_C \mu_i = a/2$. Hence the sequence $\sup((\mu_i/\nu)(x))$ does not tend to zero, and so we can choose a subsequence of i's for which $\sup_x((\mu_i/\nu)(x)) \to b > 0$. Choose x_i such that $|(\mu_i/\nu)(x_i) - \sup((\mu_i/\nu)(x))| < 1/i$. Then $(\mu_i/\nu)(x_i) \to b$. Now take a subsequence of the x_i converging to a limit x_0. Then clearly, $(\mu_i/\nu)(x_0) \to b$.

<u>Lemma</u> 3.15 Let $\bar{x}_0 \in \bar{M}$. Then there exists a sequence of open neighbourhoods

$\cdots \subset N_i \subset N_{i+1} \subset \cdots$ of $\tilde{x}_0$ such that $\tilde{M} = \cup N_i$ and the restriction of $p_i : \tilde{M} \longrightarrow M_i$ to N_i is an isomorphism of N_i with a dense open subset of M_i

Proof: Choose a Riemannian metric ρ on M, and let $\tilde{\rho}$ be the metric it induces on $\tilde{X}^\sigma$. Define

$$N_i = \{\tilde{x} \in \tilde{M} \mid \tilde{\rho}(\tilde{x}_0, \tilde{x}) < \tilde{\rho}(\tilde{\gamma}_0, \tilde{x}), \text{ all } \gamma \in \Gamma_i, \gamma \neq 1\}$$

Check that these sets have the right properties (cf. Koz I, p167)

We now prove (c) of the proposition. Let x_0 be as in (3.4), and let $\tilde{x}_0 \in \tilde{M}$ map to it. Then, from (3.1a) we know that there exists an $\omega_i \in \mathcal{L}(M_i)$ such that $(\omega_i | \omega_i) = 1$ and

$$(\omega_i \wedge \bar{\omega}_i)(p_i(\tilde{x}_0)) \geq \tfrac{1}{2} \mu_{M_i}(p_i(\tilde{x}_0)) \qquad (3.15.1)$$

Lemma 3.16. There exists a compact neighbourhood $\tilde{U}$ of $\tilde{x}_0$ such that for some $c > 0$, $\int_{p_i(\tilde{U})} \omega_i \wedge \bar{\omega}_i > c$, all i.

Proof: Obvious from (3.14) and (3.15.1). (Cf. Koz II p153)

Now let $\hat{\mathcal{L}}$ be the Hilbert space of measurable square-integrable sections η of $\Omega^1_{\tilde{M}}$. For each i, define $\eta_i \in \hat{\mathcal{L}}$ by

$$\begin{cases} \eta_i \mid N_i = p_i^*(\omega_i) \\ \eta_i \mid \tilde{M} - N_i = 0 \end{cases}$$

As $(\eta_i | \eta_i) = 1$, there exists a weakly convergent subsequence of the η_i tending to $\eta \in \hat{\mathcal{L}}$. Now η is holomorphic on $\cup N_i = \tilde{M}$, and $\int_{\tilde{U}} \eta \wedge \bar{\eta} > c/2$, and so $\eta \neq 0$. Thus $\mathcal{L}(\tilde{M}) \neq 0$.

Conjecture 3.17. Let $\bar{G}$ be a semisimple real Lie group, Γ an arithmetic subgroup of $\bar{G}$, and

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \quad \supset \Gamma_i \supset \cdots \supset \{1\}$$

a sequence of normal subgroups of Γ of finite index such that $\bigcap \Gamma_i = \{1\}$. Let W be an irreducible cuspidal representation of $\bar{G}$ and define $h_i(W) = \dim(\mathrm{Hom}_{\bar{G}}(W, L^2(\Gamma_i\backslash\bar{G}))$. Then $\dfrac{h_i(W)}{(\Gamma:\Gamma_i)}$ does not tend to zero as $i \to \infty$

Kaz II In [, p156], Kazhdan calls this "Theorem A, which we will prove in another paper". At present the statement still seems to be unproven, but Clozel has recently shown the following weaker result which is sufficient for our purposes: let G be a simply connected semisimple algebraic group over $\mathbb{Q}$, and let p_0 be a prime such that $G(\mathbb{Q}_{p_0})$ has a supercuspidal representation π_{p_0} (e.g., take p_0 to be any prime such that G is split over $\mathbb{Q}_{p_0}$); let K_{p_0} be a compact open subgroup

Here $\mathcal{H}$ = Hecke alg. of $G(\mathbb{Q}_{p_0})$ such that $\mathcal{H}(\pi_{p_0})^{K_{p_0}} \neq 0$; let $K_0 \supset K_1 \supset$ be a sequence of compact open subgroups of $G(\mathbb{A}^f)$ such that

a) $(K_i)_{p_0} = K_{p_0}$

b) there exists a finite set S of primes such that $(K_i)_p$ is maximal for $p \notin S$;

c) $\bigcap K_i = \{1\}$.

Then for any irreducible cuspidal representation W of $G(\mathbb{R})$, there exists a constant $a > 0$ such that

$$\frac{h_i(W)}{(\Gamma_0:\Gamma_i)} \geq a \quad , \quad \Gamma_i = G(\mathbb{Q}) \cap K_i .$$

Now return to the situation in the statement of (3.7). Then $W \overset{df}{=} \mathcal{H}(\tilde{X})$ is in a natural way, a cuspidal representation of $G(\mathbb{R})$.

Lemma 3.19. There is an isomorphism

$$\mathrm{Hom}_{G(\mathbb{R})}(W, L^2(\Gamma\backslash\tilde{X})) \xrightarrow{\simeq} \mathcal{H}(X).$$

Proof. Well known — see Kaz II, p156-157.

Lemma 3.20. The dimensions of $\mathcal{H}(X)$ and $\mathcal{H}(\sigma X)$ are equal.

Proof: Let $\bar{X}$ be a smooth variety containing X as a dense open subvariety. Then (3.12) shows that $\mathcal{H}(X) \xrightarrow{\simeq} \mathcal{H}(\bar{X})$, and $\mathcal{H}(\bar{X}) = \Gamma(\bar{X}, \Omega^1_{X,alg})$. The Lemma is now obvious.

We are now ready to prove (3.7). Choose a family K_i satisfying the conditions of Clozel's theorem. Then the theorem and (3.19) show that for some $a>0$, $\dim \mathcal{H}(X_i)/(\Gamma_0 : \Gamma_i) \geq a$ all i, where $X_i = \Gamma(K_i)\backslash\tilde{X}$. Now (3.20) shows that $\dim \mathcal{H}(\sigma X_i)/(\Gamma_0^\sigma : \Gamma_i^\sigma) \geq a$, where $\Gamma_i = \Gamma^\sigma(K_i)$, and (3.12) implies that $\mathcal{H}(\tilde{X}^\sigma) \neq 0$.

4. Subbundles of $T(\tilde{X})$

Let G be a simply connected semisimple algebraic group over $\mathbb{Q}$, and let $\tilde{X}$ be a symmetric Hermitian domain on which $G(\mathbb{R})$ acts in such a way that $\tilde{X} \approx G(\mathbb{R})/(\text{maximal compact subgroup})$.
Let

$$\tilde{X} = X_1 \times \cdots \times X_s \quad , \; X_i \text{ irreducible symmetric Hermitian domain}$$

Choose a point $x = (x_1, \dots, x_s) \in \tilde{X}$. Then

$$G_{\mathbb{R}} = G_1 \times \cdots \times G_s \times K_0, \quad G_i \text{ noncompact}, \; G_i(\mathbb{R})x = G_i(\mathbb{R})x_i = X_i,$$

K_0 compact.
Let

$$X_i' = \{x_1\} \times \cdots \times X_i \times \cdots \times \{x_s\} \subset \tilde{X}.$$

Then $T(X_i') \subset T(\tilde{X})$, and $T(\tilde{X}) = \bigoplus_{i=1}^{s} T(X_i')$.
For $I \subset \{1, \dots, s\}$, define

$$T^I(\tilde{X}) = \bigoplus_{i \in I} T(X_i') = \{(x,v) \in T(\tilde{X}) \mid v \in \oplus T_{x_i}(X_i)\}$$

Then $T^I(\tilde{X})$ is a subbundle of $T(\tilde{X})$, stable under $G(\mathbb{R})$.

<u>Lemma</u> 4.1. Every $G(\mathbb{Q})$-stable complex subbundle of $T(\tilde{X})$ is of the form $T^I(\tilde{X})$ for some $I \subset \{1, \dots, s\}$.

<u>Proof</u>: Let W be such a subbundle. As $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ (real approximation theorem) and $G(\mathbb{R})$ acts transitively on $\tilde{X}$, it will suffice to show that $W_x = T^I(\tilde{X})_x$, some I, for a fixed $x \in \tilde{X}$. Let K_x be the isotropy group at x. Then W_x is a K_x-stable subspace of $T(\tilde{X})_x = \oplus T(X_i)_{x_i}$. With the usual notations, let

$$\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad , \; \mathfrak{k} = \mathrm{Lie}(K_x) \; , \; \mathfrak{p} = T_x(\tilde{X})$$

$$\mathrm{Lie}(G_i) = \mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i \quad , \; \mathfrak{p}_i = T_{x_i}(\tilde{X}_i)$$

Then $\mathfrak{k} = \mathrm{Lie}(K_0) \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_s$ and $\mathfrak{p} = \oplus \mathfrak{p}_i$. Note that

W_x is a $\mathfrak{K}$-stable subspace of $\mathfrak{p}$. Almost by definition of what it means for X_i to be irreducible [Helgason, VIII.5, p377], the action of $\mathfrak{K}_i$ on $\mathfrak{p}_i$ is irreducible. Thus $W_x = \bigoplus_{i \in I} \mathfrak{p}_i$, some I.

<u>Lemma</u> 4.2. Let G be a simply connected, almost simple, algebraic group over a number field F. Let S_∞ be the set of infinite primes of F, and let $v \in S_\infty$ be such that $G(F_v)$ is not compact. Then for any congruence group $\Gamma \subset G(F)$, $G(F_v)\Gamma$ is dense in $\prod_{v \in S_\infty} G(F_v)$.

<u>Proof</u>: For some compact open subgroup K of $G(\mathbb{A}_F^f)$, $\Gamma \supset K \cap G(F)$. The strong approximation theorem shows that $G(F_v)G(F)$ is dense in $G(\mathbb{A}_F) = \prod_{v \in S_\infty} G(F_v) \times G(\mathbb{A}_F^f)$ and so, for any open subset U of $\prod_{v \in S_\infty} G(F_v)$, there exist elements $\alpha \in G(F_v)$ and $\beta \in G(F)$ such that $\alpha\beta \in U \times K$. Clearly $\beta \in \Gamma$, and $\alpha\beta \in U$; thus $G(F_v)\Gamma$ is dense in $\prod_{v \in S_\infty} G(F_v)$.

<u>Remark</u> 4.3. Let $G' = \mathrm{Res}_{F/\mathbb{Q}} G$ with G and F as in the Lemma. Then the conclusion of the lemma can be restated as follows: let G'_v be a noncompact factor of the Lie group $G'(\mathbb{R})$, and let Γ be a congruence subgroup of $G'(\mathbb{Q})$; then $G'_v.\Gamma$ is dense in $G'(\mathbb{R})$.

Since $T^I(\tilde{X})$ is stable under Γ, it defines a subbundle $T^I(X)$ of $T(X)$. By construction, $T^I(\tilde{X})$ is involutive, and so $T^I(X)$ is an involutive subbundle of $T(X)$.

<u>Proposition</u> 4.4. Assume G is almost simple over $\mathbb{Q}$. If the foliation defined by $T^I(X)$ on X has a closed leaf, then $I = \Phi$ or $I = \{1, \dots, s\}$

<u>Proof</u>: Let Z be the closed leaf, and let $\tilde{Z} = p^{-1}(Z)$. Then $\tilde{Z}$ is a closed submanifold of $\tilde{X}$ stable under Γ and all $G_i(\mathbb{R})$ for $i \in I$. If $I \neq \Phi$, then (4.3) shows that $\Gamma . \prod G_i(\mathbb{R})$ is dense in $G(\mathbb{R})$, which acts transitively on $\tilde{X}$. Therefore $\tilde{Z} = \tilde{X}$ and $Z = X$, whence $T^I(X) = T(X)$.

5 Completion of the proof

Let X, $\tilde{X}$, G, ... be as in §1 and assume condition (0.3a), i.e., that $\operatorname{codim}(\partial X) \geq 3$. Recall the notations,

$$\begin{array}{ccccc} \tilde{X} & \xhookrightarrow{p} & \hat{X}^{an} & & \hat{X} \\ & \searrow^{p_K} & \downarrow & & \downarrow q_K \\ & & X_K^{an} & & X_K = X \end{array} \qquad \begin{array}{ccccc} \tilde{X}^\sigma & \xhookrightarrow{p^\sigma} & (\sigma\hat{X})^{an} & & \sigma\hat{X} \\ & \searrow^{p_K^\sigma} & \downarrow & & \downarrow \sigma q_K \\ & & (\sigma X)^{an} & & \sigma X \end{array}.$$

Let μ be the Bergmann volume form on $\tilde{X}^\sigma$, and let Z_0 be the set on which μ is zero. As we saw in §3, Z_0 is a proper subset of $\tilde{X}^\sigma$. Clearly it is a complex analytic subset, and is Q-invariant. The complement $\tilde{U}_0$ of $\tilde{Z}_0$ in $\tilde{X}^\sigma$ is also Q-invariant, and (see 3.5) there is a map $\gamma : \tilde{U}_0 \longrightarrow \mathbb{P}(\mathcal{H}(\tilde{X}^\sigma)^\vee)$ such that the Bergmann Hermitian form on $\tilde{U}_0$ is the inverse image of the canonical metric on $\mathbb{P}(\mathcal{H}(\tilde{X}^\sigma)^\vee)$. Note that Q acts on $\mathbb{P}(\mathcal{H}(\tilde{X}^\sigma)^\vee)$ through its action on $\tilde{X}^\sigma$, and that γ is Q-equivariant map. Define

$$\tilde{Z}_1 = \{z \in \tilde{U}_0 \mid \operatorname{rank}(d\gamma)_z < \max_x \operatorname{rank}(d\gamma)_x\}$$

and $\tilde{Z} = \tilde{Z}_0 \cup \tilde{Z}_1$ — this is again a Q-invariant complex analytic subset of $\tilde{X}^\sigma$ not equal to $\tilde{X}^\sigma$, and so (1.9) shows that $\operatorname{codim}(\tilde{Z}) \geq \operatorname{codim}(\partial X) \geq 3$. The complement $\tilde{U} = \tilde{X}^\sigma - \tilde{Z}$ of $\tilde{Z}$ is also Q-invariant. We have a diagram

Check
Should observe also that $d\gamma$ is not identically zero — if it was then γ would be "constant" — could not have $\int \omega \wedge \bar{\omega} < \infty$

$$\begin{array}{ccc} T(\tilde{U}) & \longrightarrow & T(\mathbb{P}) \\ \downarrow & & \downarrow \\ \tilde{U} & \xrightarrow{\gamma} & \mathbb{P} \end{array}$$

Define

$$\tilde{W} = \operatorname{Ker}(T(\tilde{U}) \longrightarrow \gamma^* T(\mathbb{P})).$$

Because it has constant rank, $\tilde{W}$ is a subbundle of $T(\tilde{U})$. It is Q-invariant.

As in §4, let $\tilde{X} = X_1 \times \cdots \times X_s$, and, for each $I \subset \{1, \cdots, s\}$ define $T^I(\tilde{X}) \subset T(\tilde{X})$ and $T^I(X) \subset T(X)$. Recall the following result: let $\bar{V}$ be a complete algebraic variety over $\mathbb{C}$, and let V be a nonsingular subvariety such that $\bar{V} - V$ has codimension ≥ 3; then $F \mapsto F^{an}$ induces an equivalence between the category of coherent locally free algebraic sheaves on V and that of coherent locally free analytic sheaves on V^{an} (see, for example, [Hartshorne,
SLN 156 p222-223]). Thus $T^I(X)$ is an algebraic subbundle of $T(X)$, and $\sigma T^I(X) \subset \sigma T(X) = T(\sigma X)$ is defined. Set $T^I(\tilde{X}^\sigma)$ equal to the inverse image of $\sigma T^I(X)$ on $\tilde{X}^\sigma$.

<u>Lemma</u> 5.1. For some $I \subset \{1, 2, \cdots, s\}$, $\tilde{W} = T^I(\tilde{X}^\sigma) \mid \tilde{U}$.

<u>Proof</u>: Since both $\tilde{U}$ and $\tilde{W}$ are $\mathcal{G}$-invariant and $\Gamma^\sigma \subset \mathcal{G}$, we can pass to the quotient and obtain $U = \Gamma^\sigma \backslash \tilde{U} \subset \sigma X$ and $W = \Gamma^\sigma \backslash \tilde{W}$ a subbundle of $T(U)$. As $\operatorname{codim}(\sigma X^* - U) \geq 3$, the result recalled above shows that W is an algebraic subbundle of $T(U)$. Let j be the inclusion $U \hookrightarrow \sigma X$. Regard W as a sheaf on U, and form $j_* W$ — this is a coherent algebraic sheaf on σX. Let Y be the subset of σX where $j_* W$ is not locally free. Then Y is an algebraic subset of σX and its inverse image on $\tilde{X}^\sigma$ is $\mathcal{G}$-invariant ($(p^\sigma)^{-1} Y$ is the set where $\tilde{j}_* \tilde{W}$ is not locally free). Therefore (1.8) shows that Y is empty, and so $j_* W$ is locally free. A similar argument applied to the support of the kernel of $j_* W \to T(\sigma X)$ shows that $j_* W$ is a locally free subsheaf of $T(\sigma X)$. The fact recalled above shows that it

is algebraic. Similar arguments apply to each variety in the projective system $\sigma\hat{X}$, and so we obtain an algebraic subbundle $\hat{W} \subset T(\sigma\hat{X})$ invariant under Q, and therefore under $G(\mathbb{A}^f)$. Hence $\sigma^{-1}\hat{W}$ is a subbundle of $T(\hat{X})$ invariant under $G(\mathbb{A}^f)$, and $(\sigma^{-1}\hat{W})^{an} \mid \tilde{X}$ is $G(\mathbb{R})$-invariant. Now (4.1) shows that $(\sigma^{-1}\hat{W})^{an} \mid \tilde{X} = T^{I}(\tilde{X})$ for some I and, because $G(\mathbb{A}^f)$ acts transitively on the set of connected components of $\hat{X}^{an}$, this implies that $\sigma^{-1}\hat{W} = T^{I}(\hat{X})$. This completes the proof.

The condition that a subbundle of the tangent bundle to an algebraic variety be involutive is algebraic. Thus $T^{I}(\tilde{X}^{\sigma})$ is involutive

<u>Lemma</u> 5.2 The foliation of $U = \Gamma^{\sigma}\backslash\tilde{U}$ defined by $W = \Gamma^{\sigma}\backslash\tilde{W}$ has a closed leaf

<u>Proof</u>: The reader is invited to check for himself the proof of Kazhdan (Kaz II, p 153 – 156; — essentially the leaves of the foliation are the equivalence classes for the relation defined on p 153). Alternatively, it may be possible to give a proof along the following lines: as we observed above, there is a Q-equivariant map $\tilde{\delta}: \tilde{U} \longrightarrow \mathbb{P}(\mathcal{L}(\tilde{X}^{\sigma})^{\vee})$; if we can pass to the quotient by Γ^{σ}, we get a map $U = \Gamma^{\sigma}\backslash\tilde{U} \longrightarrow \Gamma^{\sigma}\backslash\mathbb{P}(\mathcal{L}(\tilde{X}^{\sigma})^{\vee})$ whose fibres are the leaves of the foliation.

We now assume also the condition (0.3b). Then $G = \mathrm{Res}_{F/\mathbb{Q}} G'$ with G' absolutely simple and F totally real, and for some special point $\tilde{x} \in \tilde{X}$, the maximal torus $T \subset G$ fixing $\tilde{x}$ is of the form $\mathrm{Res}_{F/\mathbb{Q}} T'$ where T' splits over a quadratic imaginary extension L of F.

As T'_L is split, we can write

$$\mathfrak{g}'_L = \mathfrak{t}'_L \oplus \bigoplus_{\alpha \in R} (\mathfrak{g}'_L)_\alpha \;, \quad \mathfrak{g}'_L = \mathrm{Lie}(G'_L)\;, \quad \mathfrak{t}'_L = \mathrm{Lie}(T'_L)\,,$$

where R is the set of roots $R(G'_{\mathbb{C}}, T'_{\mathbb{C}})$. For each $\alpha \in R$, let

$$\mathfrak{h}'_\alpha = \mathfrak{t}'_L \oplus (\mathfrak{g}'_L)_\alpha \oplus (\mathfrak{g}'_L)_{-\alpha}.$$

It is defined over F, and we let H'_α be the corresponding connected subgroup of G'. Let $H_\alpha = \mathrm{Res}_{F/\mathbb{Q}} H'_\alpha$ — it is a reductive group of type A_1 and H_α^{ad} is $\mathbb{Q}$-simple.

Let $\tilde{X}_\alpha$ be the orbit of $\tilde{x}$ under the action of $H_\alpha(\mathbb{R})$. Then $\tilde{X}_\alpha$ is a Hermitian symmetric domain (possibly consisting of one element) with $\mathrm{Aut}(\tilde{X}_\alpha)^+ = (H_\alpha(\mathbb{R})/(\text{max'l compact normal subgroup}))^+$.

Bourbaki talk, 389 It follows from [Deligne, , 1.15] that the projective system of arithmetic varieties $\hat{X}_\alpha$ embeds into $\hat{X}$. We assume Theorem 0.1 for the families $\hat{X}_\alpha$; in particular $\hat{X}_\alpha$ is the family associated with a $\mathbb{Q}$-group H_α^σ.

<u>Lemma</u> 5.3. The tangent space $T_{\tilde{x}}(\tilde{X})$ is generated by the subspaces $T_{\tilde{x}}(\tilde{X}_\alpha)$. (Note, $\tilde{x}$ is as defined above.)

<u>Proof</u>: Easy.

<u>Lemma</u> 5.4. Let $\tilde{x}^\sigma$ be any point of $\tilde{X}^\sigma$ such that $\tilde{x}^\sigma = g(\sigma\hat{x})$ for some $g \in G(\mathbb{A}^f)$, where $\hat{x}$ is the image of $\tilde{x}$ (see above) in $\hat{X}$. Then $\tilde{x}^\sigma \in \tilde{\mathcal{U}}$.

<u>Proof</u>: Otherwise $\tilde{Z} \supset \mathbb{Q}\,\tilde{x}^\sigma \supset H_\alpha^\sigma(\mathbb{Q})\,\tilde{x}^\sigma$, and so $\tilde{Z} \supset \tilde{X}_\alpha^\sigma$ for all α. This implies that $\dim \tilde{Z} = \dim \tilde{X}^\sigma$, which contradicts

an earlier statement.

Let $J = \{\tau : F \longrightarrow \mathbb{R} \mid G'_\tau$ is not compact$\}$, where $G'_\tau = G \otimes_{F,\tau} \mathbb{R}$. Then J indexes the irreducible components of $\tilde{X}$, and we shall use it for this purpose rather than $\{1, \dots, s\}$. Thus I is now a subset of J. Let $J_\alpha = \{\tau : F \longrightarrow \mathbb{R} \mid H'_{\alpha,\tau}$ is not compact$\}$.

<u>Lemma</u> 5.5. Either $J_\alpha \cap I = \Phi$ or $J_\alpha \subset I$.

<u>Proof</u>. The set $\tilde{Z} \cap \tilde{X}^\sigma_\alpha$ is stable under $\mathcal{G} \cap H_\alpha(\mathbb{A}^f) = H_\alpha(\mathbb{Q})/Z_\alpha(\mathbb{Q})$ and so is either empty or all of $\tilde{X}^\sigma_\alpha$. The last lemma shows that it is not equal to $\tilde{X}^\sigma_\alpha$, and so we have that $\tilde{X}^\sigma_\alpha \subset \check{U}$.

From (4.1), (5.2), and (4.4) we know that $W \mid \sigma X_\alpha$ is 0 or $T(\sigma X_\alpha)$ (recall that we are assuming σX_α is arithmetic). Thus $\sigma^{-1} W$ on X_α is 0 or $T(X_\alpha)$. But (by 5.1), $\sigma^{-1} W = T^I(X)$ for some $I \subset J$. We conclude that $I \supset J_\alpha$ (and $\sigma^{-1} W \mid X_\alpha = T(X_\alpha)$) or $I \cap J_\alpha = \Phi$ (and $\sigma^{-1} W \mid X_\alpha = 0$).

Note that it is not possible for I to contain all J_α, for then $I = J$ and the Bergmann Hermitian form is identically zero (cf. p 5.1). On the other hand, if $I \cap J_\alpha = \Phi$ for all α, then $I = \Phi$, $\tilde{W} = 0$, and we can apply (2.13) to complete the proof of (0.1).

Thus it remains to show that the hypothesis that $I \cap J_\alpha = \Phi$ for some, but not all, α leads to a contradiction

<u>Lemma</u> 5.6. Let $\mathfrak{h}'_I = \sum_{J_\alpha \subset I} \mathfrak{h}'_\alpha$; then $\mathfrak{h}'_I$ is a sub-Lie-algebra of $\mathfrak{g}'$.

<u>Proof</u>: Let $\mathfrak{g}'_\tau = \mathfrak{k}'_\tau \oplus \mathfrak{p}'_\tau$ as usual. Then

$$J_\alpha \subset I \iff \forall \tau \notin I, \quad \mathfrak{h}'_{\alpha,\tau} \subset \mathfrak{k}'_\tau .$$

As $\mathfrak{k}'_\tau$ is a subalgebra of $\mathfrak{g}'_\tau$, the result is now obvious.

Let H'_I be the subgroup of G' corresponding to $\mathfrak{h}'_I$. Then: for $\tau \notin I$, $H'_{I,\tau}$ is compact ($\Rightarrow H'_I$ reductive)

for $\tau \in I$, $H'_{I,\tau}(\mathbb{R})\,\tilde{x}_\tau \supset (\tilde{X}_\alpha)_\tau$ for all α, and so $H'_{I,\tau}(\mathbb{R})\,\tilde{x}_\tau = \tilde{X}_\tau$ (apply [Kobayashi-Nom, I, p178, 4.8] if it is not obvious). (We are writing $\tilde{x} = (\cdots, \tilde{x}_\tau, \cdots) \in \cdots \times \tilde{X}_\tau \times \cdots$)

This shows that $H'_{I,\tau} = G'_\tau$ (see 2.3), and so $H'_I = G'$.

This is the contradiction that proves the theorem.