LEFSCHETZ CLASSES ON ABELIAN VARIETIES

J.S. MILNE

Contents

Introduction. Let $\sim$ be an adequate equivalence relation on algebraic cycles, for example, rational equivalence (rat), homological equivalence with respect to some Weil cohomology theory (hom), or numerical equivalence (num). For a smooth projective variety X, $\mathcal{Z}^s(X)$ will denote the group of algebraic cycles on X of codimension s, and

$$\mathcal{C}^s_\sim(X) = (\mathcal{Z}^s(X)/\sim) \otimes \mathbb{Q}.$$

Then $\mathcal{C}_\sim(X) \overset{\text{df}}{=} \oplus_s \mathcal{C}^s_\sim(X)$ becomes a graded $\mathbb{Q}$-algebra under the intersection product, and we define $\mathcal{D}_\sim(X)$ to be the $\mathbb{Q}$-subalgebra of $\mathcal{C}_\sim(X)$ generated by the divisor classes:

$$\mathcal{D}_\sim(X) = \mathbb{Q}[\mathcal{C}^1_\sim(X)].$$

The elements of $\mathcal{D}_\sim(X)$ will be called the *Lefschetz classes* on X (for the relation $\sim$). They are the algebraic classes on X expressible as linear combinations of intersections of divisor classes (including the empty intersection, X).

Our main theorem states that, for any Weil cohomology theory $X \mapsto H^*(X)$ and any abelian variety A over an algebraically closed field, there is a reductive algebraic group $L(A)$ (not necessarily connected) such that the cycle class map induces an isomorphism

$$\mathcal{D}^s_{\text{hom}}(A^r) \otimes_{\mathbb{Q}} k \to H^{2s}(A^r)(s)^{L(A)}$$

for all integers $r, s \geq 0$; moreover, $\mathcal{D}^s_{\text{num}}(A^r) = \mathcal{D}^s_{\text{hom}}(A^r)$. Here $A^r = A \times \cdots \times A$ (r copies), k is the coefficient field for the cohomology theory, and "(s)" denotes a Tate

Date: 30/12/96; 1/9/97; 21/2/99.

1991 *Mathematics Subject Classification.* 14C25; 11G10; 14K15.
A part of the work on this article has been supported by the National Science Foundation.
Duke Math. J. 96 (1999), pp. 639-675.

twist. In comparison with the results of Tanke'ev (1982), Ribet (1983), Murty (1984), Hazama (1984), Ichikawa (1991), and Zarhin (1994), the main novelty of our theorem is that it is completely general, applying to all abelian varieties over all algebraically closed ground fields and to all Weil cohomology theories, and that it necessarily allows the group $L(A)$ to be nonconnected.

This theorem is an existence result for Lefschetz classes. It implies that, if the cohomology classes $t_1, \ldots, t_m$ are Lefschetz, i.e., lie in the k-subspace spanned by the cohomology classes of Lefschetz classes, then any class fixed by the group fixing them is also Lefschetz. As a consequence, we obtain that for abelian varieties over algebraically closed fields:

- the various classes predicted to be algebraic by Grothendieck's standard conjectures (Grothendieck 1969) are not only algebraic, but even Lefschetz;
- if γ is Lefschetz, and there exists a cohomology class γ' such that $\gamma\cdot\gamma' \neq 0$, then there exists a Lefschetz cohomology class with this property;
- for any regular map $\phi\colon A \to B$ of abelian varieties,

$$\phi_*\colon H^{2r}(A)(r) \to H^{2r+2c}(B)(r+c), \quad c = \dim B - \dim A,$$

  sends Lefschetz classes to Lefschetz classes;
- the question of whether the $\mathbb{Q}$-algebra of Hodge classes (or the $\mathbb{Q}_\ell$-algebra of Tate classes) on a given abelian variety is generated by divisor classes becomes a question of whether two reductive groups are equal.

These results are special to abelian varieties: in general ϕ_* will *not* preserve Lefschetz classes when ϕ is a regular map of arbitrary smooth projective varieties.

The group $L(A)$ given by the theorem is called the *Lefschetz group* of A (for the given Weil cohomology).

In a later work, we shall show that the theorem implies that there are sufficiently many Lefschetz classes for there to exist a good theory of "Lefschetz motives" based on abelian varieties. From this perspective, the Lefschetz groups for the various Weil cohomology theories are the different realizations of the fundamental group of a single Tannakian category.

By definition, $L(A)$ is an algebraic group over the coefficient field of the Weil cohomology. In two important cases it is naturally defined over $\mathbb{Q}$: when the ground field is $\mathbb{C}$ and the cohomology is the Betti cohomology (because, then the coefficient field *is* $\mathbb{Q}$); and when the ground field is the algebraic closure of a finite field (because then the fundamental group in question is commutative). I do not expect there to exist a naturally defined algebraic group $L(A)_0$ over $\mathbb{Q}$ giving rise by base change to the Lefschetz groups $L(A)_\ell$ attached to the ℓ-adic étale cohomologies when A is an abelian variety in characteristic $p \neq 0$ whose moduli are transcendental.

In later work, we shall give applications of the theory developed in this article to the Tate conjecture for abelian varieties over finite fields and to the points on Shimura varieties over finite fields (Milne 1996, 1995).

Notations and conventions. For a vector space V,

$$\begin{aligned} rV &= \text{direct sum of } r \text{ copies of } V, \\ V^\vee &= \text{dual of } V, \\ \bigotimes V &= \text{tensor algebra on } V, \\ \bigwedge V &= \text{exterior algebra on } V. \end{aligned}$$

For a ring R, $R^\times$ denotes the group of invertible elements of R, and $M_m(R)$ denotes the ring of $m \times m$ matrices with coefficients in R.

An *involution* on a k-algebra R (not necessarily commutative) is a bijective k-linear map $\alpha \mapsto \alpha^\dagger\colon R \to R$ such that $(\alpha\beta)^\dagger = \beta^\dagger\alpha^\dagger$ and $\alpha^{\dagger\dagger} = \alpha$ for $\alpha, \beta \in R$. A *(skew-) Hermitian form* on a left R-module V is a k-bilinear map $\phi\colon V \times V \to R$ such that

$$\phi(\alpha x, \beta y) = \alpha \cdot \phi(x,y) \cdot \beta^\dagger, \text{ all } \alpha, \beta \in R, \quad x, y \in V$$

and

$$\phi(y,x)^\dagger = (-)\phi(x,y).$$

When R is commutative, the identity map is an involution, and (skew-) Hermitian forms for the identity involution are called (skew-) symmetric.

A *CM-field* is a finite extension K of $\mathbb{Q}$ admitting a nontrivial involution ι such that $\rho(\iota x) = \overline{\rho(x)}$ for all homomorphisms $\rho\colon K \to \mathbb{C}$. The involution ι is unique. We sometimes denote ιx by $\bar{x}$.

"Algebraic group" means "affine algebraic group". For such a group G, $G(K)$ is the set of points on G with coordinates in K, and G_K or $G_{/K}$ is $G \times_{\operatorname{Spec} k} \operatorname{Spec} K$. For an algebraic group G over a field K and a subfield k of K such that K has finite degree over k, $\operatorname{Res}_{K/k} G$ denotes the algebraic group over k obtained from G by restriction of scalars.

For a Hermitian (or skew-Hermitian) form ϕ, $\mathrm{U}(\phi) \overset{\text{df}}{=} \operatorname{Aut}(\phi)$ is the unitary group; for a skew-symmetric form ϕ, $\mathrm{Sp}(\phi) \overset{\text{df}}{=} \operatorname{Aut}(\phi)$ and $\mathrm{GSp}(\phi)$ are the groups of symplectic automorphisms and symplectic similitudes respectively; and for a symmetric form ϕ, $\mathrm{O}(\phi) \overset{\text{df}}{=} \operatorname{Aut}(\phi)$ is the orthogonal group.

For abelian varieties A and B, $\operatorname{Hom}^0(A,B) = \operatorname{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$. By an "isogeny $A \to B$" we mean an element of $\operatorname{Hom}^0(A,B)$ that admits an inverse in $\operatorname{Hom}^0(B,A)$.

To signify that objects X and Y are isomorphic, we write $X \approx Y$; when a particular isomorphism is given (or there is a canonical or preferred isomorphism), we write $X \cong Y$. Also, $X \overset{\text{df}}{=} Y$ means that X is defined to be Y, or that $X = Y$ by definition.

1. Definition of $C(A)$ and $S(A)$

In this section, we attach a k-algebra $C(A)$ with involution † to an abelian variety A and a Weil cohomology theory with coefficient field k. The elements γ of $C(A)$ such that $\gamma^\dagger\gamma = 1$ are the points of a reductive group $S(A)$ over k whose fixed tensors in the cohomology of A will be shown (in Section 3) to be precisely the Lefschetz classes on A.

Preliminaries. We fix an algebraically closed field Ω and a Weil cohomology theory $X \mapsto H^*(X)$ taking a variety X over Ω to a graded algebra $H^*(X)$ over a field k. In the appendix, we list the axioms to which such cohomology theory must submit, but the reader may prefer simply to note the following examples:

	Ω	k	$H^s(X)$
Betti cohomology	$\mathbb{C}$	$\mathbb{Q}$	$H^s(X(\mathbb{C}),\mathbb{Q})$
étale cohomology	arbitrary	$\mathbb{Q}_\ell$, $\ell \neq \mathrm{char}(\Omega)$	$H^s(X_{\mathrm{et}},\mathbb{Q}_\ell)$
de Rham cohomology	char $= 0$	Ω	$\mathbb{H}^s(X_{\mathrm{Zar}},\Omega^{\cdot}_{X/\Omega})$
crystalline cohomology	char $\neq 0$	ff$(W))$	$H^s_{\mathrm{crys}}(X/W)\otimes_W k$

In the bottom row, W is the ring of Witt vectors with coefficients in Ω and ff(W) is its field of fractions. When it is necessary to specify the cohomology theory, we add subscripts B, ℓ, dR, or crys.

For an abelian variety A over Ω, we define $V(A)$ to be the dual of $H^1(A)$. For example,

$$\begin{aligned} V_B(A) &= H_1(A(\mathbb{C}),\mathbb{Q}), \\ V_\ell(A) &= (\text{Tate module of } A)\otimes_{\mathbb{Z}}\mathbb{Q}, \\ V_{\mathrm{crys}}(A) &= \text{covariant Dieudonné module of } A. \end{aligned}$$

Let $k(1) = H^2(\mathbb{P}^1)^\vee$. For example, for the Betti cohomology theory $\mathbb{Q}(1) = 2\pi i\mathbb{Q}$, and for the étale cohomology theory $\mathbb{Q}_\ell(1) = (\varprojlim \mu_{\ell^n}(\Omega))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$. In every case, $k(1)$ is a one-dimensional vector space over k.

Cup-product defines an isomorphism of graded k-algebras

$$\bigwedge H^1(A) \to H^*(A).$$

A divisor D on A defines a class $cl(D)$ in $H^2(A)(1)$ and, via the isomorphisms,

$$H^2(A)(1) \cong (\bigwedge^2 H^1(A))(1) \cong \mathrm{Hom}(\bigwedge^2 V(A), k(1)),$$

a skew-symmetric pairing

$$e^D\colon V(A)\times V(A)\to k(1).$$

When D is ample, e^D is nondegenerate, and we let $\beta^\dagger$ denote the adjoint with respect to e^D of a k-linear endomorphism β of $V(A)$:

$$e^D(\beta x, y) = e^D(x,\beta^\dagger y), \quad \text{all } x,y\in V(A).$$

Then $\beta \mapsto \beta^\dagger$ is an involution of the k-algebra $\mathrm{End}_k(V(A))$ whose restriction to $\mathrm{End}^0(A)$ is the Rosati involution defined by D. If D' is a second divisor on A, then $e^{D'} = e^D\circ(\alpha\times 1)$ for some $\alpha\in\mathrm{End}^0(A)$ with $\alpha^\dagger = \alpha$ (Mumford 1970, p208), and the involution defined by D' (if also ample) is $\beta\mapsto\alpha^{-1}\beta^\dagger\alpha$.

The k-algebra $C(A)$. For an abelian variety A over Ω, we define $C(A)$ to be the centralizer of $\mathrm{End}^0(A)$ in $\mathrm{End}_k(V(A))$:

$$C(A) = \mathrm{End}_{\mathrm{End}^0(A)\otimes_{\mathbb{Q}}k}(V(A)).$$

Then $C(A)$ is a k-algebra stable under the involution † defined by an ample divisor D, and the restriction of † to $C(A)$ is independent of the choice of D.

An isogeny $\alpha\colon A \to B$ defines an isomorphism

$$\gamma \mapsto V(\alpha) \circ \gamma \circ V(\alpha)^{-1}\colon C(A) \to C(B)$$

of k-algebras with involution, which is independent of the choice of α. Therefore $C(A)$, as a k-algebra with involution, depends only on the isogeny class of A (up to a canonical isomorphism).

For any positive integer r, $V(A^r) = rV(A)$, and the diagonal action of $C(A)$ on $rV(A)$ identifies $C(A)$ with $C(A^r)$ (as k-algebras with involution).

Let $A = A_1 \times \cdots \times A_s$. Then

$$C(A) \subset C(A_1) \times \cdots \times C(A_s),$$

with equality holding if and only if $\mathrm{Hom}(A_i, A_j) = 0$ for all i, j, $i \neq j$. Moreover, if D_i is an ample divisor on A_i, $i = 1, \dots, s$, then

$$D = \sum_i A_1 \times \cdots \times A_{i-1} \times D_i \times A_{i+1} \times \cdots \times A_s$$

is an ample divisor on A, and the involution it defines on $C(A)$ is the restriction of the product of the involutions on the $C(A_i)$ defined by the D_i.

On combining the remarks in the last three paragraphs, we obtain the following statement:

Proposition 1.1. *Let $A_1, \dots, A_s$ be a set of representatives for the simple isogeny factors of A, so that there exists an isogeny $A_1^{r_1} \times \cdots \times A_s^{r_s} \to A$ for some $r_i > 0$. Any such isogeny induces an isomorphism*

$$C(A_1) \times \cdots \times C(A_s) \to C(A)$$

of k-algebras with involution, which is independent of the choice of the isogeny.

Remark 1.2. Because $\mathrm{End}^0(A)$ is a semisimple $\mathbb{Q}$-algebra whose centre is separable over $\mathbb{Q}$, $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} k$ is a semisimple k-algebra. Therefore, the centralizer of $C(A)$ in $\mathrm{End}_k(V(A))$ is $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} k$.

Proposition 1.3. *The skew-symmetric k-bilinear forms $\psi\colon V(A) \times V(A) \to k(1)$ such that*

$$\psi \circ (\gamma \times 1) = \psi \circ (1 \times \gamma^\dagger), \text{ all } \gamma \in C(A),$$

are exactly the k-linear combinations of forms e^D with D a divisor on A.

Proof. Let D_0 be an ample divisor on A, and let † be the involution it defines on $\mathrm{End}_k(V(A))$. As we noted above, if D is a second divisor on A, then $e^D = e^{D_0} \circ (\alpha \times 1)$ for some $\alpha \in \mathrm{End}^0(A)$. This implies that $e^D \circ (\gamma \times 1) = e^D \circ (1 \times \gamma^\dagger)$ for all $\gamma \in C(A)$.

Conversely, because e^{D_0} is non-degenerate, any k-bilinear form $\psi\colon V(A) \times V(A) \to k(1)$ can be written $\psi = e^{D_0} \circ (\beta \times 1)$ for some $\beta \in \mathrm{End}_k(V(A))$. If ψ is skew-symmetric, then $\beta = \beta^\dagger$, and

$$\psi \circ (\gamma \times 1) = \psi \circ (1 \times \gamma^\dagger), \forall \gamma \in C(A) \implies \beta\gamma = \gamma\beta, \forall \gamma \in C(A) \overset{(1.2)}{\implies} \beta \in \mathrm{End}^0(A)_{\mathbb{Q}}k.$$

Therefore, any ψ as in the statement of the proposition is of the form $e^{D_0} \circ (\beta \times 1)$ for some $\beta \in \mathrm{End}^0(A) \otimes_{\mathbb{Q}} k$ with $\beta = \beta^\dagger$. Hence $\beta = \sum c_i\beta_i$ with $c_i \in k$, $\beta_i \in \mathrm{End}^0(A)$,

$\beta_i^\dagger = \beta_i$. According to (Mumford 1970, p208), $e^{D_0} \circ (\beta_i \times 1)$ is of the form e^{D_i} for a divisor D_i on A, which completes the proof. □

REMARK 1.4. Let A be an abelian variety over $\mathbb{C}$, and let $V = V_B(A)$. To give $C(A)$ with its action on V is the same as to give $\mathrm{End}^0(A)$ with its action on V, and to give the involution † on $C(A)$ is the same as to give the set

$$\{e^D \mid D \text{ a divisor on } A\}.$$

Moreover, if D_0 is one ample divisor, then the cone $\{e^D \mid D \text{ ample}\}$ is exactly the set of forms $e^{D_0} \circ (\alpha \times 1)$ with α an element of $\mathrm{End}^0(A)^\times$ such that $\alpha = \alpha^\dagger$ and α is a square in $\mathbb{Q}[\alpha] \otimes_{\mathbb{Q}} \mathbb{R}$ (the first condition implies $\mathbb{Q}[\alpha] \otimes_{\mathbb{Q}} \mathbb{R}$ is a product of copies of $\mathbb{R}$, and the second is equivalent to the condition that the roots of the characteristic polynomial of α on A are positive).

The group $S(A)$. For an abelian variety A over Ω, we define $S(A)$ to be the algebraic subgroup of $\mathrm{GL}(V(A))$ such that, for all commutative k-algebras R,

$$S(A)(R) = \{\gamma \in C(A) \otimes_k R \mid \gamma^\dagger \gamma = 1\}.$$

Thus, for any ample divisor D on A, $S(A)$ is the largest algebraic subgroup of $\mathrm{Sp}(e^D)$ whose elements commute with the endomorphisms of A. Clearly $S(A)$ depends only on the isogeny class of A (up to a unique isomorphism). It is a reductive group (not necessarily connected) over k whose nonabelian simple quotients are classical groups (cf. Weil 1960).

PROPOSITION 1.5. *Let $A_1, \ldots, A_s$ be a set of representatives for the simple isogeny factors of A, so that there exists an isogeny $A_1^{r_1} \times \cdots \times A_s^{r_s} \to A$ for some $r_i > 0$. Any such isogeny induces an isomorphism*

$$S(A_1) \times \cdots \times S(A_s) \to S(A),$$

which is independent of the choice of the isogeny.

PROOF. This is an immediate consequence of Proposition 1.1. □

REMARK 1.6. If $X \mapsto H^*(X)$ is a Weil cohomology theory with coefficient field k, and k' is a field containing k, then $X \mapsto H^*(X) \otimes_k k'$ is a Weil cohomology theory with coefficient field k'. If $C'(A)$ and $S'(A)$ denote the objects defined relative to the second theory, then there are canonical isomorphisms

$$C'(A) \cong C(A) \otimes_k k', \quad S'(A) \cong S(A)_{/k'}.$$

Comparison isomorphisms. We write $S_B(A)$, $S_\ell(A)$, $S_{\mathrm{dR}}(A)$, or $S_{\mathrm{crys}}(A)$ for $S(A)$ when we wish to indicate its dependence on the Weil cohomology theory used.

The comparison theorems between the various cohomology theories, together with (1.6), show that, when A is an abelian variety over $\mathbb{C}$, there are canonical isomorphisms

$$S_B(A)_{/\mathbb{Q}_\ell} \cong S_\ell(A), \quad S_B(A)_{/\mathbb{C}} \cong S_{\mathrm{dR}}(A).$$

Abelian varieties over $\mathbb{F}$. Let A be an abelian variety over an algebraic closure $\mathbb{F}$ of a finite field, and let $C_0(A)$ be the centre of the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$—it is a product of fields, each of which is either a CM-field or $\mathbb{Q}$. Every Rosati involution $\dagger$ preserves each factor of $C_0(A)$ and acts on it as complex conjugation. Define $S_0(A)$ to be the algebraic group over $\mathbb{Q}$ such that, for all commutative $\mathbb{Q}$-algebras R,

$$S_0(A)(R) = \{\gamma \in C_0(A) \otimes_{\mathbb{Q}} R \mid \gamma^\dagger \gamma = 1\}.$$

PROPOSITION 1.7. *The action of* $\mathrm{End}^0(A)$ *on* $V_\ell(A)$ *induces an isomorphism*

$$C_0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to C_\ell(A)$$

of $\mathbb{Q}_\ell$*-algebras with involution, and hence an isomorphism of algebraic groups*

$$S_0(A)_{/\mathbb{Q}_\ell} \to S_\ell(A).$$

PROOF. Both $C_0(A)$ and $C_\ell(A)$ satisfy the statement of Proposition 1.1, and so we may suppose that A is simple. Let A_0 be a model of A over a subfield $\mathbb{F}_q$ of $\mathbb{F}$. After possibly replacing $\mathbb{F}_q$ with a larger field, we may suppose that $\mathrm{End}(A_0) = \mathrm{End}(A)$. Let $\pi \in \mathrm{End}(A_0)$ be the Frobenius endomorphism of A_0 relative to $\mathbb{F}_q$. Tate's theorem (Tate 1966) shows that $C_0(A) = \mathbb{Q}[\pi]$ and that the centralizer $C_\ell(A)$ of $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ in $\mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(A))$ is $\mathbb{Q}[\pi] \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Therefore the action of $\mathrm{End}^0(A_0)$ on $V_\ell(A)$ induces an isomorphism $C_0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to C_\ell(A)$ of $\mathbb{Q}_\ell$-algebras, which clearly preserves the involutions. □

Similarly, the action of $\mathrm{End}^0(A)$ on $V_{\mathrm{crys}}(A)$ induces an isomorphism

$$S_0(A)_{/\mathrm{ff}(W(\mathbb{F}))} \to S_{\mathrm{crys}}(A).$$

2. CALCULATION OF $C(A)$ AND $S(A)$.

We wish to calculate $C(A)$ and $S(A)$, together with their actions on $V(A)$, for an arbitrary abelian variety A. After Propositions 1.1 and 1.5 it suffices to do this in the case that A is simple, in which case it is an exercise[1] in linear algebra.

Recall (e.g. Mumford 1970, p201) that the simple abelian varieties fall into four classes according to the type of their endomorphism algebra $E \overset{\mathrm{df}}{=} \mathrm{End}^0(A)$:

Type I: E is a totally real field, and the Rosati involutions are trivial.

Type II: E is a totally indefinite quaternion division algebra over a totally real field F. If $\alpha \mapsto \alpha' \overset{\mathrm{df}}{=} \mathrm{Trd}_{E/F}\,\alpha - \alpha$ denotes the standard involution on E, then the Rosati involutions are the maps $\alpha \mapsto a\alpha' a^{-1}$ with a an element of E such that a^2 lies in F and is totally negative.

Type III: E is a totally definite quaternion algebra over a totally real field F, and the Rosati involutions are the maps $\alpha \mapsto a\alpha' a^{-1}$ with a a totally positive element of F; in particular, the standard involution itself is a Rosati involution.

Type IV: E is a division algebra with centre a CM-field K. For a finite prime v of K,

$$\begin{cases} \mathrm{Inv}_v(E) = 0 \text{ if } \iota v = v, \\ \mathrm{Inv}_v(E) + \mathrm{Inv}_{\iota v}(E) = 0 \text{ if } \iota v \neq v. \end{cases}$$

[1]There are similar calculations in Hazama 1984 and Murty 1984 for the case $\Omega = \mathbb{C}$ and the Betti cohomology theory.

There exists an isomorphism

$$E \otimes_{\mathbb{Q}} \mathbb{R} \to M_d(\mathbb{C}) \times \cdots \times M_d(\mathbb{C}),$$

carrying a Rosati involution into $(\ldots, (a_{ij}), \ldots) \mapsto (\ldots, (\bar{a}_{ji}), \ldots)$.

Preliminaries. As usual, we fix an algebraically closed field Ω and a Weil cohomology theory with coefficient field k. We shall use the following notations:

$$\begin{aligned} A &= \text{a simple abelian variety of dimension } g \text{ over } \Omega, \\ E &= \mathrm{End}^0(A) \text{ (a division algebra)}, \\ K &= \text{the centre of } E \text{ (a field)}, \\ F &= \text{the subfield of } K \text{ on which the Rosati involutions act trivially}, \\ f &= [F\colon \mathbb{Q}],\ d = \sqrt{[E\colon K]}. \end{aligned}$$

The field F is totally real, and K equals F except when A is of type IV, in which case it is a CM-field of degree 2 over F. Always $fd|g$, and when A is of type II, $2fd|g$.

We fix an isomorphism $k(1) \approx k$, so that an ample divisor D on A now defines a skew-symmetric k-bilinear form

$$e^D\colon V(A) \times V(A) \to k.$$

If † denotes the Rosati involution with respect to D, then

$$e^D(\alpha x, y) = e^D(x, \alpha^\dagger y), \quad \text{all } \alpha \in E, \quad x, y \in V(A).$$

Let L be a semisimple k-subalgebra of $E \otimes_{\mathbb{Q}} k$ stable under †. There exists[2] a unique skew-Hermitian pairing of L-modules

$$\phi\colon V(A) \times V(A) \to L$$

such that $\mathrm{Tr}_{L/k} \circ \phi = e^D$. From the uniqueness of ϕ, one deduces that an L-linear automorphism of $V(A)$ fixes ϕ if and only if it fixes e^D.

Let

$$F \otimes_{\mathbb{Q}} k = F_1 \times \cdots \times F_t,$$

be the decomposition of $F \otimes_{\mathbb{Q}} k$ into a product of fields, and let

$$1 = e_1 + \cdots + e_t,$$

be the corresponding decomposition of 1 into a sum of orthogonal idempotents. Then

$$V(A) = V_1 \oplus \cdots \oplus V_t, \quad V_i \overset{\text{df}}{=} e_i V = V \otimes_{F \otimes_{\mathbb{Q}} k} F_i.$$

Proposition 2.1 below shows that V_i has dimension $2g/f$ over F_i. Any k-linear map $\alpha\colon V \to V$ commuting with the action of F decomposes into

$$\alpha = \alpha_1 \oplus \cdots \oplus \alpha_t, \quad \alpha_i\colon V_i \to V_i, \quad F_i\text{-linear}.$$

[2]The trace pairing $(a, b) \mapsto \mathrm{Tr}_{L/k}(ab)\colon L \times L \to k$ is nondegenerate. Let $x, y \in V(A)$. If ϕ exists, then

$$\mathrm{Tr}_{L/k}(a\phi(x, y)) = \psi(ax, y) \quad \text{for all } a \in L.$$

Define $\phi(x, y)$ to be the unique element of L satisfying this equation for all $a \in L$, and check that the map $(x, y) \mapsto \phi(x, y)$ has the required properties.

Let D be an ample divisor on A, and let $\phi\colon V\times V\to F\otimes_{\mathbb{Q}}k$ be the skew-symmetric $F\otimes_{\mathbb{Q}}k$-bilinear form such that $\mathrm{Tr}_{F\otimes_{\mathbb{Q}}k/k}\,\phi=e^{D}$. Because $\phi\circ(\alpha\times 1)=\phi\circ(1\times\alpha)$ for $\alpha\in F$, ϕ decomposes into

$$\phi=\phi_1\oplus\cdots\oplus\phi_t,\quad \text{where } \phi_i\colon V_i\times V_i\to F_i,\quad \text{is skew-symmetric and } F_i\text{-bilinear.}$$

Proposition 2.1. *Let A be an abelian variety, and let L be a subfield of* $\mathrm{End}^0(A)$ *containing the identity map. Then $V(A)$ is a free $L\otimes_{\mathbb{Q}}k$-module of rank* $2\dim A/[L\colon\mathbb{Q}]$.

Proof. If $L\otimes_{\mathbb{Q}}k$ is again a field, then there is nothing to prove. In general it will decompose into a product of fields $L\otimes_{\mathbb{Q}}k=\prod L_i$ and correspondingly $V(A)\approx\oplus L_i^{m_i}$, some $m_i\geq 0$, as an $L\otimes_{\mathbb{Q}}k$-module. Our task is to show that the m_i are all equal (in fact, to $2\dim A/[L\colon\mathbb{Q}]$).

Let $\alpha\in L$. The characteristic polynomial $P_{A,\alpha}(X)$ of α as an endomorphism of A is monic of degree $2\dim A$ with coefficients in $\mathbb{Q}$, and it is equal to the characteristic polynomial of $V(\alpha)$ acting on the k-vector space $V(A)$ (see the appendix).

From the decomposition $L\otimes_{\mathbb{Q}}k=\prod L_i$ we find that

$$P_{L/\mathbb{Q},\alpha}(X)=\prod_i P_{L_i/k,\alpha}(X)$$

where $P_{L/\mathbb{Q},\alpha}(X)$ (resp. $P_{L_i/k,\alpha}(X)$) denotes the characteristic polynomial of α in the field extension $L/\mathbb{Q}$ (resp. L_i/k).

From the isomorphism of $L\otimes_{\mathbb{Q}}k$-modules $V(A)\approx\oplus L_i^{m_i}$ we find that

$$P_{A,\alpha}(X)=\prod P_{L_i/k,\alpha}(X)^{m_i}.$$

If we assume that α generates L as a field extension of $\mathbb{Q}$, so that $P_{L/\mathbb{Q},\alpha}(X)$ is irreducible, then the two equations show that any monic irreducible factor of $P_{A,\alpha}(X)$ in $\mathbb{Q}[X]$ shares a root with $P_{L/\mathbb{Q},\alpha}(X)$, and therefore equals it. It follows that $P_{A,\alpha}(X)=P_{L/\mathbb{Q},\alpha}(X)^m$ for some integer m and that each $m_i=m$. On equating the degrees, we find that $2\dim A=m[L\colon\mathbb{Q}]$. □

Remark 2.2. Let k be a field, and let k' be an étale k-algebra of degree 2 (so that either $k'=k\times k$ or k' is a field of degree 2 over k). Let ϕ be a nondegenerate skew-Hermitian form on a k'-vector space V relative to the nontrivial involution of k' fixing k. Let Ω be a field containing k and large enough so that there exist distinct k-homomorphisms $\sigma_1,\sigma_2\colon k'\to\Omega$. Write

$$V\otimes_k\Omega=V_1\oplus V_2,\quad V_i\stackrel{\mathrm{df}}{=}V\otimes_{k',\sigma_i}\Omega,\quad i=1,2.$$

The form ϕ extends to an $\Omega\times\Omega$-bilinear form on $V\otimes_k\Omega$, which we again denote ϕ. Then

$$\phi|V_1\times V_1=0=\phi|V_2\times V_2,$$

and there is a nondegenerate Ω-bilinear form $\phi_1\colon V_1\times V_2\to\Omega$ such that

$$\phi((x_1,x_2),(y_1,y_2))=(\phi_1(x_1,y_2),-\phi_1(x_2,y_1)),\quad x_1,y_1\in V_1,\ x_2,y_2\in V_2.$$

Therefore, the map

$$\alpha\mapsto\alpha|V_1\colon \mathrm{U}(\phi)_\Omega\to\mathrm{GL}(V_1)$$

is an isomorphism, and the representation of $\mathrm{U}(\phi)_\Omega$ on $V\otimes_k\Omega$ becomes the direct sum of the representation of $\mathrm{GL}(V_1)$ on V_1 (standard representation) and the representation of $\mathrm{GL}(V_1)$ on V_2 (contragredient of the standard representation).

Remark 2.3. Consider the abstract situation: k is a field, E is a simple k-algebra with centre a field K of finite degree over k, and V is a left E-module. Then the centralizer $C(E)$ of E in $\mathrm{End}_k(V)$ is equal to its centralizer in $\mathrm{End}_K(V)$, and the canonical homomorphism

$$E\otimes_K C(E)\to\mathrm{End}_K(V)$$

is an isomorphism. Let S be a simple E-module—any two such modules are isomorphic. Then $\Delta\overset{\mathrm{df}}{=}\mathrm{End}_E(S)$ is a division algebra, and $E=\mathrm{End}_\Delta(S)$. The choice of a Δ-linear isomorphism $S\to\Delta^r$ determines an isomorphism $E\to M_r(\Delta^{\mathrm{opp}})$, and the choice of an E-linear isomorphism $V\to S^t$ determines an isomorphism $C(E)=\mathrm{End}_E(V)\to M_t(\Delta)$. In short:

$$E\approx M_r(\Delta^{\mathrm{opp}})\implies C(E)\approx M_t(\Delta),\quad t=\frac{\dim_K(V)}{r[\Delta\colon K]}.$$

Note that if Δ admits a K-involution $*$, then $\alpha^{\mathrm{opp}}\leftrightarrow\alpha^*$ is an isomorphism $\Delta^{\mathrm{opp}}\leftrightarrow\Delta$.

Remark 2.4. Let R be a k-algebra with an involution †, and endow $M_m(R)$ with the involution $(a_{ij})\mapsto(a_{ji}^\dagger)$. Let e be the matrix with 1 in the $(1,1)$ position and zeros elsewhere. The map $(V,\phi)\mapsto(eV,\phi|eV)$ defines an equivalence from

> the category of pairs (V,ϕ) consisting of a finitely-generated projective $M_m(R)$-module V and a (skew) Hermitian form $\phi\colon V\times V\to M_m(R)$,

to

> the category of pairs (V,ϕ) consisting of a finitely-generated projective R-module V and a (skew) Hermitian form $\phi\colon V\times V\to R$.

A quasi-inverse is provided by the functor sending (V,ϕ) to the pair

$$(R^m\otimes_R V,(e_i\otimes x,e_j\otimes y)\mapsto\phi(x,y)E_{ij}),$$

where $\{e_1,\dots,e_m\}$ is the standard basis for R^m and E_{ij} is the matrix with 1 in the (i,j) position and zero elsewhere (Knus 1991, I.9.5).

Simple abelian variety of type I. In this case $E=F$. Let D be an ample divisor on A, and let $\phi\colon V(A)\times V(A)\to F\otimes_{\mathbb{Q}}k$ be the skew-symmetric form such that $\mathrm{Tr}_{F\otimes_{\mathbb{Q}}k/k}\circ\phi=e^D$. Then ϕ is independent of the choice of D up to multiplication by a nonzero element of F.

Corresponding to the decomposition $F\otimes_{\mathbb{Q}}k=\prod_{i=1}^t F_i$ of $F\otimes_{\mathbb{Q}}k$ into a product of fields, there is a decomposition

$$(V(A),\phi)=(V_1,\phi_1)\oplus\cdots\oplus(V_t,\phi_t),\quad (V_i,\phi_i)\overset{\mathrm{df}}{=}(V(A),\phi)\otimes_{F\otimes_{\mathbb{Q}}k}F_i.$$

Here ϕ_i is a nondegenerate skew-symmetric form on the F_i-vector space V_i. Therefore,

$$C(A)=C_1\times\cdots\times C_t,\quad C_i\overset{\mathrm{df}}{=}\mathrm{End}_{F_i}(V_i)\approx M_{\frac{2g}{f}}(F_i)$$

and the involution sends an element of C_i to its adjoint with respect to ϕ_i. Moreover,

$$S(A)=S_1\times\cdots\times S_t,\quad S_i\overset{\mathrm{df}}{=}\mathrm{Res}_{F_i/k}\,\mathrm{Sp}(\phi_i).$$

Similarly, if

$$F \otimes_{\mathbb{Q}} k^{\mathrm{al}} = \prod_{\sigma : F \hookrightarrow k^{\mathrm{al}}} k_\sigma, \quad k_\sigma = k^{\mathrm{al}},$$

is the decomposition of $F \otimes_{\mathbb{Q}} k^{\mathrm{al}}$ into a product of fields, then

$$(V(A), \phi) \otimes_k k^{\mathrm{al}} = \oplus_{\sigma : F \hookrightarrow k^{\mathrm{al}}} (V_\sigma, \phi_\sigma), \quad (V_\sigma, \phi_\sigma) \overset{\mathrm{df}}{=} (V(A), \phi) \otimes_{F,\sigma} k^{\mathrm{al}}$$

and

$$S(A)_{k^{\mathrm{al}}} \cong \prod_{\sigma : F \to k^{\mathrm{al}}} \mathrm{Sp}(\phi_\sigma).$$

Simple abelian variety of type II. In this case E is a totally definite quaternion algebra over a totally real field F. Therefore, there exists a basis $1, \alpha, \beta, \alpha\beta$ for E with

$$\begin{aligned} \alpha^2 &= a \in F, \text{ totally negative,} \\ \beta^2 &= b \in F, \text{ totally negative,} \\ \alpha\beta &= -\beta\alpha. \end{aligned}$$

Let D be an ample divisor on A whose Rosati involution is $\gamma \mapsto \gamma^\dagger = \alpha\gamma'\alpha^{-1}$, and let $\phi \colon V(A) \times V(A) \to E \otimes_{\mathbb{Q}} k$ be the skew-Hermitian form such that $\mathrm{Tr}_{E \otimes k/k} \circ \phi = e^D$.

Corresponding to the decomposition $F \otimes_{\mathbb{Q}} k = \prod_{i=1}^t F_i$, there are decompositions

$$\begin{aligned} E \otimes_{\mathbb{Q}} k &= E_1 \times \cdots \times E_t, \quad E_i \overset{\mathrm{df}}{=} E \otimes_F F_i, \\ (V(A), \phi) &= (V_1, \phi_1) \oplus \cdots \oplus (V_t, \phi_t), \quad (V_i, \phi_i) \overset{\mathrm{df}}{=} (V(A), \phi) \otimes_{E \otimes_{\mathbb{Q}} k} E_i. \end{aligned}$$

Here E_i is a quaternion algebra (possibly split) over F_i, and ϕ_i is a skew-Hermitian form $V_i \times V_i \to E_i$. Therefore,

$$C(A) = C_1 \times \cdots \times C_t, \quad C_i \overset{\mathrm{df}}{=} \mathrm{End}_{E_i}(V_i)$$

and the involution on C_i sends an E_i-endomorphism of V_i to its adjoint with respect to ϕ_i.

Let $L = F[\alpha]$. Then $E = L \cdot 1 \oplus L \cdot \beta$, and so we can write

$$\phi(x, y) = \phi_1(x, y) + \phi_2(x, y)\beta, \quad \phi_1(x, y), \phi_2(x, y) \in L \otimes_{\mathbb{Q}} k.$$

Then ϕ_1 is a skew-Hermitian form $V \times V \to L \otimes_{\mathbb{Q}} k$, and ϕ_2 is a skew-symmetric form. Any $L \otimes_{\mathbb{Q}} k$-linear automorphism of $V(A)$ fixing ϕ_1 and ϕ_2 fixes ϕ and is $E \otimes_{\mathbb{Q}} k$-linear[3], and so

$$\mathrm{Aut}_{E \otimes_{\mathbb{Q}} k}(V, \phi) = \mathrm{Aut}_{L \otimes_{\mathbb{Q}} k}(V, \phi_1) \cap \mathrm{Aut}_{L \otimes_{\mathbb{Q}} k}(V, \phi_2) \quad (\text{inside } \mathrm{Aut}_{L \otimes_{\mathbb{Q}} k}(V)).$$

Corresponding to the decomposition of $F \otimes_{\mathbb{Q}} k$, there are decompositions

$$\begin{aligned} L \otimes_{\mathbb{Q}} k &= \prod_{i=1}^t L_i, \quad L_i \overset{\mathrm{df}}{=} L \otimes_F F_i, \text{ and} \\ (V(A), \phi_1, \phi_2) &= \oplus_i (V_i, \phi_{1,i}, \phi_{2,i}), \quad (V_i, \phi_{1,i}, \phi_{2,i}) \overset{\mathrm{df}}{=} (V(A), \phi_1, \phi_2) \otimes_{F \otimes k} F_i. \end{aligned}$$

[3]Let $\gamma \colon V(A) \to V(A)$ be an $L \otimes k$-linear automorphism fixing ϕ. Then

$$\phi(\beta\gamma x, \gamma y) = \beta\phi(\gamma x, \gamma y) = \phi(\beta x, y) = \phi(\gamma\beta x, \gamma y), \quad \text{all } x, y \in V(A),$$

which implies that $\beta\gamma = \gamma\beta$, and hence that γ commutes with the action of $L[\beta] = E$.

Here $\phi_{1,i}\colon V_i \times V_i \to L_i$ is skew-Hermitian and $\phi_{2,i}$ is skew-symmetric. Therefore,

$$S(A) = \prod_{i=1}^{t} \mathrm{Res}_{F_i/k} \mathrm{U}(\phi_{1,i}) \cap \mathrm{Res}_{L_i/k} \mathrm{Sp}(\phi_{2,i}).$$

Similarly,

$$S(A)_{/k^{\mathrm{al}}} = \prod_{\sigma: F \hookrightarrow k^{\mathrm{al}}} \mathrm{U}(\phi_{1,\sigma}) \cap \mathrm{Sp}(\phi_{2,\sigma}), \quad (V_\sigma, \phi_{1,\sigma}, \phi_{2,\sigma}) = (V(A), \phi_1, \phi_2) \otimes_{F,\sigma} k^{\mathrm{al}}.$$

Let $\sigma_1, \sigma_2\colon L \hookrightarrow k^{\mathrm{al}}$ be the extensions of σ to L. Then

$$V_\sigma = V_{\sigma_1} \oplus V_{\sigma_2}, \quad V_{\sigma_i} \stackrel{\mathrm{df}}{=} V \otimes_{L,\sigma_i} k^{\mathrm{al}},$$

and (see 2.2) $\gamma \mapsto \gamma | V_{\sigma_1}$ identifies $\mathrm{U}(\phi_{1,\sigma}) \cap \mathrm{Sp}(\phi_{2,\sigma})$ with $\mathrm{Sp}(\phi_{2,\sigma_1})$. The representation of $\mathrm{Sp}(\phi_{2,\sigma_1})$ on V_{σ_1} is its standard representation, and its representation on V_{σ_2} is the contragredient of the standard representation (which is isomorphic to the standard representation).

Simple abelian variety of type III. This is similar to the preceding case, except that $E = \left(\frac{a,b}{F}\right)$ with $a \in F$ totally negative (as before) but b totally positive, and the ample divisor D is chosen so that its Rosati involution is the standard involution. Again we let $L = F[\alpha]$, where $\alpha^2 = a$, and $L_i = L \otimes_{\mathbb{Q}} F_i$. In this case $\beta^\dagger = -\beta$, and

$$S(A) = \prod_{i=1}^{t} \mathrm{Res}_{F_i/k} \mathrm{U}(\phi_{1,i}) \cap \mathrm{Res}_{L_i/k} \mathrm{O}(\phi_{2,i})$$

with $\phi_{1,i}$ a skew-Hermitian form $V_i \times V_i \to L_i$ on V_i, and $\phi_{2,i}$ an L_i-bilinear symmetric form. Moreover

$$S(A)_{k^{\mathrm{al}}} \cong \prod \mathrm{O}(\phi_{2,\sigma_1})$$

where the product is indexed by the embeddings $\sigma\colon F \hookrightarrow k^{\mathrm{al}}$ of F into k^{al} and σ_1 is an extension of σ to L. The representation of $\mathrm{O}(\phi_{2,\sigma_1})$ on $V_{\sigma_1} \stackrel{\mathrm{df}}{=} V(A) \otimes_{L,\sigma_1} k^{\mathrm{al}}$ is its standard representation, and its representation on V_{σ_2} is the contragredient of the standard representation (which is isomorphic to the standard representation).

Simple abelian variety of type IV. In this case E is a division algebra whose centre is a CM-field K. We shall compute $C(A)$ and $S(A)$ over k^{al} only.

Recall that $d = [E\colon K]^{\frac{1}{2}}$ and that $fd|g$. Let S be a simple $M_d(k^{\mathrm{al}})$-module, and let

$$V_1 = S \oplus \cdots \oplus S, \quad (g/df \text{ copies}).$$

Then V_1 is a k^{al}-vector space of dimension g/f. Let $V_2 = V_1^\vee$. It has a natural structure of a right $M_d(k^{\mathrm{al}})$-module, which we turn into a left module structure by using the involution $\alpha \mapsto \alpha^{\mathrm{tr}}$. The bilinear form

$$(x_1, x_2) \mapsto \phi_0(x_1, x_2) \stackrel{\mathrm{df}}{=} x_2(x_1)\colon V_1 \times V_2 \to k$$

has the property that $\phi_0(\alpha x_1, x_2) = \phi_0(x_1, \alpha^{\mathrm{tr}} x_2)$. Set:

$$\begin{aligned}
\bar{V} &= V_1 \oplus V_2 \\
\bar{E} &= M_d(k^{\mathrm{al}}) \times M_d(k^{\mathrm{al}}), \quad (\alpha,\beta)^\dagger = (\beta^{\mathrm{tr}}, \alpha^{\mathrm{tr}}), \\
\bar{K} &= k^{\mathrm{al}} \times k^{\mathrm{al}}, \\
\bar{F} &= \{(a,a) \mid a \in \bar{K}\} \cong k^{\mathrm{al}}, \\
\bar{\phi} &: \bar{V} \times \bar{V} \to \bar{K}, \quad \bar{\phi}((x_1,x_2),(y_1,y_2)) = (\phi_0(x_1,y_2), -\phi_0(y_1,x_2)).
\end{aligned}$$

Let $(\alpha,\beta) \in \bar{E}$ act on $\bar{V}$ according to the rule:

$$(\alpha,\beta)(x_1,x_2) = (\alpha x_1, \beta x_2).$$

Then † is an involution on $\bar{E}$, $\bar{K}$ is the centre of $\bar{E}$, and $\bar{F}$ is the set of elements in $\bar{K}$ fixed by †. Moreover, $\bar{\phi}$ is a Hermitian form on $\bar{V}$ regarded as a free $\bar{K}$-module, and

$$\phi(\alpha\bar{x}, \bar{y}) = \phi(\bar{x}, \alpha^\dagger \bar{y}), \quad \text{all } \alpha \in \bar{E},\ \bar{x},\bar{y} \in \bar{V}.$$

Let D be an ample divisor on A, and let $\phi\colon V(A) \times V(A) \to K \otimes_{\mathbb{Q}} k$ be the skew-Hermitian form such that $\mathrm{Tr}_{K/\mathbb{Q}} \circ \phi = e^D$. Corresponding to the decomposition

$$F \otimes_{\mathbb{Q}} k^{\mathrm{al}} = \prod_{\sigma: F \hookrightarrow k^{\mathrm{al}}} k_\sigma, \quad k_\sigma = k^{\mathrm{al}},$$

of $F \otimes_{\mathbb{Q}} k^{\mathrm{al}}$ into a product of fields, there are the decompositions

$$\begin{aligned}
E \otimes_{\mathbb{Q}} k^{\mathrm{al}} &= \prod_\sigma E_\sigma, \quad E_\sigma = E \otimes_{F,\sigma} k^{\mathrm{al}} \\
K \otimes_{\mathbb{Q}} k^{\mathrm{al}} &= \prod_\sigma K_\sigma, \quad K_\sigma = E \otimes_{F,\sigma} k^{\mathrm{al}} \\
(V(A),\phi) \otimes_k F \otimes_{\mathbb{Q}} k^{\mathrm{al}} &= (V_\sigma, \phi_\sigma), \quad (V_\sigma,\phi_\sigma) = (V(A),\phi) \otimes_{F,\sigma} k^{\mathrm{al}}.
\end{aligned}$$

Using Remarks 2.2, 2.3, and 2.4, we find that, for each σ, there exist compatible isomorphisms

$$E_\sigma \to \bar{E}, \quad (V_\sigma, \phi_\sigma) \to (\bar{V}, \bar{\phi}),$$

the first of which carries the Rosati involution on E_σ into the involution † on $\bar{E}$, and therefore maps K_σ isomorphically onto $\bar{K}$ and F_σ isomorphically onto $\bar{F}$.

Consequently,

$$C(A) \otimes_k k^{\mathrm{al}} = \prod C_\sigma,$$

where

$$C_\sigma \approx \mathrm{End}_{\bar{E}}(\bar{V}) \approx M_{\frac{g}{fd}}(k^{\mathrm{al}}) \times M_{\frac{g}{fd}}(k^{\mathrm{al}}).$$

Moreover,

$$S(A)_{/k^{\mathrm{al}}} = \prod S_\sigma$$

where

$$S_\sigma \approx \mathrm{Aut}_{M_d(k^{\mathrm{al}})}(V_1) \approx \mathrm{GL}_{\frac{g}{fd}}(k^{\mathrm{al}}).$$

The representation of S_σ on V_σ is isomorphic to the direct sum of d copies of the standard representation of $\mathrm{GL}_{\frac{g}{fd}}(k^{\mathrm{al}})$ and d copies of its contragredient.

Summary. The following table summarizes the properties of the reductive groups $S(A)$.

Type	Group	Semisimple	Connected	Dimension	Rank
I	$\mathrm{Sp}_{\frac{2g}{f}}$	Yes	Yes	$\frac{2g^2}{f}+g$	g
II	$\mathrm{Sp}_{\frac{g}{f}}$	Yes	Yes	$\frac{g^2}{2f}+\frac{g}{2}$	$\frac{g}{2}$
III	$\mathrm{O}_{\frac{g}{f}}$	Yes	No	$\frac{g^2}{2f}-\frac{g}{2}$	$\frac{g}{2}-\epsilon$
IV	$\mathrm{GL}_{\frac{g}{df}}$	No	Yes	$\frac{g^2}{d^2f}$	$\frac{g}{d}$

The group $S(A)_{/k^{\mathrm{al}}}$ is isomorphic to f copies of the group listed in the second column. The third and fourth columns indicate whether the group is semisimple or connected, and the remaining columns give its dimension and rank (over k^{al}). The "ϵ" is 0 or $-\frac{f}{2}$ according as $\frac{g}{f}$ is even or odd. Note that, except when ϵ is nonzero, the rank of $S(A)$ is $\frac{g}{d}$.

3. The Cohomology Classes Fixed by $S(A)$.

In Section 1, we attached a reductive group $S(A)$ to an abelian variety A and a Weil cohomology theory. In this section, we prove that the space of vectors in $H^s(A^r)$ fixed by $S(A)$ is generated by cup-products of divisor classes[4]. After the calculations in Section 2, we are able to derive the statement from standard results on invariant theory (Weyl 1946). Because $S(A)$ need not be connected, the definition of its space of fixed vectors requires care.

The space of fixed vectors of an algebraic group. Throughout this subsection, k will be a field of characteristic zero. Let G be an algebraic group over k, and let $G \to \mathrm{GL}(V)$ be a representation of G on a finite-dimensional k-vector space V. A vector in V is said to be *fixed by* G if its image in $V \otimes k^{\mathrm{al}}$ is fixed by all $g \in G(k^{\mathrm{al}})$. The set of vectors in V fixed by G is a subspace of V, which we denote V^G. When $G(k)$ is Zariski dense in $G_{/k^{\mathrm{al}}}$, for example, when G is connected (Borel 1991, 18.3), V^G is the space of vectors in V fixed by the elements of $G(k)$.

The next lemma shows that the formation of the space of fixed vectors commutes with extension of scalars and with products.

Lemma 3.1. *Let G and H be algebraic groups over k acting on finite-dimensional k-vector spaces V and W respectively. Then*

$$(V^G) \otimes_k k^{\mathrm{al}} = (V \otimes_k k^{\mathrm{al}})^{G_{k^{\mathrm{al}}}}$$

and

$$(V \otimes_k W)^{G \times H} = V^G \otimes_k W^H.$$

Proof. Let $I = (V \otimes_k k^{\mathrm{al}})^{G_{k^{\mathrm{al}}}}$. Then I is a subspace of $V \otimes_k k^{\mathrm{al}}$ stable under the action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$, and a standard lemma (cf. Serre 1959, V.20, Lemme 26) shows that $I^{\mathrm{Gal}(k^{\mathrm{al}}/k)} \otimes_k k^{\mathrm{al}} = I$. As $I^{\mathrm{Gal}(k^{\mathrm{al}}/k)} = I \cap V = V^G$, this proves the first equality.

[4]For simple complex abelian varieties not of type III and the Betti cohomology theory, this is proved in Murty 1984, 3.6.

Clearly, $(V \otimes_k W)^{G\times H} \supset V^G \otimes_k W^H$, and so it suffices to prove the equality with k replaced with k^{al}, i.e., we may assume k to be algebraically closed. We then have to prove the same result with G and H replaced by $G(k)$ and $H(k)$. Let $\{f_1, f_2, \ldots\}$ be a basis for W. Any element x of $V \otimes_k W$ can be written uniquely $x = \sum_i a_i \otimes f_i$ with $a_i \in V$. If x is fixed by all $(g,1) \in G(k) \times H(k)$, then $ga_i = a_i$ for all i, and so $x \in V^G \otimes W$. This shows that $(V \otimes W)^{G\times 1} = V^G \otimes W$, and similarly $(V^G \otimes W)^{1\times H} = V^G \otimes W^H$. Therefore

$$(V \otimes W)^{G\times H} \subset V^G \otimes W^H.$$

□

Statement of the theorem. As in Section 1, we fix an algebraically closed field Ω and a Weil cohomology theory with coefficient field k. From the canonical isomorphisms

$$H^1(A) \overset{\mathrm{df}}{=} V(A)^\vee, \quad H^1(A^r) \cong rH^1(A), \quad H^*(A^r) \cong \bigwedge H^1(A^r),$$

we obtain an action of $S(A)$ on $H^*(A^r)$ for all r.

We fix an isomorphism $k \to k(1)$, and we use it to identify $H^2(A)(1)$ with $H^2(A)$. Thus the cohomology class of a divisor D on A now resides in $H^2(A)$.

THEOREM 3.2. *For any abelian variety A over Ω and integer $r \geq 0$, the k-algebra $H^*(A^r)^{S(A)}$ is generated by divisor classes.*

This will be a consequence of the following two propositions.

PROPOSITION 3.3. *For any abelian variety A over Ω and integer $r \geq 0$, the k-vector space $H^2(A^r)^{S(A)}$ is generated by divisor classes.*

For a k-algebra R and a subset W of R, we let $k[W]$ denote the smallest k-subalgebra of R containing W.

PROPOSITION 3.4. *For any abelian variety A over Ω and any integer r, $H^*(A^r)^{S(A)} = k[H^2(A^r)^{S(A)}]$.*

Proof of Proposition 3.3. Clearly, $H^*(A^r)^{S(A)} \supset k[H^2(A^r)^{S(A)}]$, and so Lemma 3.1 allows us to assume that k is algebraically closed. We have to show that the space of skew-symmetric forms $\psi\colon V(A) \times V(A) \to k(1)$ invariant under the action of $S(A)$ on $\mathrm{Hom}(\Lambda^2 V(A), k(1))$ is generated by the forms e^D with D a divisor on A. But, because $\gamma^\dagger\gamma = 1$ for $\gamma \in S(A)(k)$, ψ is invariant under $S(A)$ if and only if

$$\psi \circ (\gamma \times 1) = \psi \circ (1 \times \gamma^\dagger), \quad \text{all } \gamma \in S(A)(k).$$

The next lemma shows that the k-algebra $C(A)$ is generated by the $\gamma \in S(A)(k)$, and so Proposition 3.3 follows from Proposition 1.3.

LEMMA 3.5. *Any semisimple algebra with involution $(R, ^\dagger)$ of finite dimension over an algebraically closed field k is generated (as a k-algebra) by the subset U of elements u satisfying $u^\dagger u = 1$.*

PROOF. Each pair $(R, ^\dagger)$ is a product of pairs of the following types:

(a) $R = M_n(k) \times M_n(k)$ and $(A,B)^\dagger = (B^{\mathrm{tr}}, A^{\mathrm{tr}})$;
(b) $R = M_n(k)$ and $A^\dagger = A^{\mathrm{tr}}$;

(c) $R = M_{2n}(k)$ and $A^\dagger = J^{-1}A^{\mathrm{tr}}J$ with J an invertible skew-symmetric matrix.

We may assume $(R, ^\dagger)$ is one of the above pairs. Let $S = k[U]$. In each case, there is a natural representation of R on a finite-dimensional k-vector space, which is semisimple when regarded as a representation of U (hence also as a representation of S). The ring of k-linear endomorphisms commuting with the action of U (hence also of S) is $k \times k$ in the first case, and k in the remaining cases. Now the following standard result implies that $S = R$:

> Let S be a finite-dimensional algebra over a field k, and let $S \to \mathrm{End}_k(V)$ be a faithful semisimple representation of S; let $C(S)$ and $C(C(S))$ be the centralizers of S and $C(S)$ respectively in $\mathrm{End}_k(V)$; then $S = C(C(S))$.

□

Start of the proof of Proposition 3.4. It follows from Proposition 1.5 that $S(A) = S(A^r)$, and so it suffices to prove the statement with $r = 1$. Moreover, an isogeny

$$A \to A_1^{r_1} \times \cdots \times A_s^{r_s}$$

with the A_i simple and pairwise nonisogenous defines isomorphisms

$$\begin{aligned} H^*(A) &\to H^*(A_1^{r_1}) \otimes \cdots \otimes H^*(A_s^{r_s}) \\ S(A) &\to S(A_1) \times \cdots \times S(A_s) \qquad \text{(see 1.5)} \end{aligned}$$

and hence (by 3.1 and induction) an isomorphism

$$H^*(A)^{S(A)} \to H^*(A_1^{r_1})^{S(A_1)} \otimes \cdots \otimes H^*(A_s^{r_s})^{S(A_s)}.$$

If the proposition is true for each variety $A_i^{r_i}$, so that each k-algebra $H^*(A_i^{r_i})^{S(A_i)}$ is generated by the vectors of degree 2 fixed by $S(A_i)$, then it is clear that it is also true for A. Thus, it suffices to prove the proposition for a simple abelian variety A. This we shall do after reviewing some invariant theory.

Invariant theory. The results reviewed in this subsection can be found in Weyl 1946 or, more conveniently, in Appendix F of Fulton and Harris, 1991.

In this subsection, V will be a finite-dimensional vector space over an algebraically closed field k of characteristic zero. We consider a representation $G \to \mathrm{GL}(V)$ of an algebraic group G on V, and we wish to determine the space of fixed vectors $(H^{\otimes m})^G$ where H is the dual of V equipped with the contragredient representation. Note that $H^{\otimes m}$ can be identified with the space of m-linear forms on V. The tensor product $f \otimes g$ of a p-linear form f with a q-linear form g is the $(p+q)$-linear form

$$(x_1, \ldots, x_{p+q}) \mapsto f(x_1, \ldots, x_p)g(x_{p+1}, \ldots, x_{p+q}).$$

The symplectic group. Let ϕ be a nondegenerate skew-symmetric bilinear form on V, and let $G = \mathrm{Sp}(\phi)$. Note that ϕ can be identified with an element of $H \otimes H$. For odd m, $(H^{\otimes m})^G = 0$, and for even m, $(H^{\otimes m})^G$ is generated as a $k[S_m]$-algebra by $\phi \otimes \phi \otimes \cdots \otimes \phi$ ($\frac{m}{2}$ copies) (Fulton and Harris 1991, F.13). Here S_m is the symmetric group on $\{1, 2, \ldots, m\}$ acting on $H^{\otimes m}$ according to the rule: $\sigma(x_1 \otimes \cdots \otimes x_m) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}$.

The orthogonal group. Let ϕ be a nondegenerate symmetric bilinear form on V, and let $G = \mathrm{O}(\phi)$. For odd m, $(H^{\otimes m})^G = 0$, and for even m, $(H^{\otimes m})^G$ is generated as a $k[S_m]$-module by $\phi \otimes \phi \otimes \cdots \otimes \phi$ ($\frac{m}{2}$ copies) (ibid. F.16).

The general linear group. Let $G = \mathrm{GL}(V)$. Then $(V^{\otimes m} \otimes H^{\otimes n})^G = 0$ for $m \neq n$, and $(V^{\otimes m} \otimes H^{\otimes m})^G$ is generated as a k-vector space by the tensors $t(\sigma)$, $\sigma \in S_m$, where $t(\sigma)$ is the element

$$f_1 \otimes \cdots \otimes f_m \otimes x_1 \otimes \cdots \otimes x_m \mapsto f_1(x_{\sigma(1)}) \cdots f_m(x_{\sigma(m)})$$

of $(H^{\otimes m} \otimes V^{\otimes m})^\vee = V^{\otimes m} \otimes H^{\otimes m}$ (ibid. F.20).

PROPOSITION 3.6. *With the above notations,*

$$(\bigotimes rH)^G = k[(\otimes^2 rH)^G] \text{ all } r \geq 1,$$

in each of the following cases:

(a) $G = \mathrm{Sp}(\phi)$ *with* ϕ *a nondegenerate skew-symmetric form on* V*;*
(b) $G = \mathrm{O}(\phi)$ *with* ϕ *a nondegenerate symmetric form on* V*;*
(c) $G = \mathrm{GL}(W)$ *and* $V = W \oplus W^\vee$ *(*G *acts on* $W^\vee$ *via the contragredient representation).*

PROOF. We first prove (a) and (b). Let $V_1, V_2, \ldots, V_r$ be the copies of V, and let $H_1, \ldots, H_r$ be their duals. Then

$$(H_1 \oplus \cdots \oplus H_r)^{\otimes m}$$

is a direct sum of spaces of the form

$$H_1^{\otimes i_1} \otimes \cdots \otimes H_r^{\otimes i_r}, \quad i_1 + \cdots + i_r = m,$$

with G acting through its action on each H_i. But $H_1^{\otimes i_1} \otimes \cdots \otimes H_r^{i_r} = H^{\otimes m}$, and so it contains no G-invariant tensors unless m is even, in which case the G-invariant tensors are linear combinations of the forms

$$\phi \otimes \cdots \otimes \phi \in (X_1 \otimes X_2) \otimes \cdots \otimes (X_{m-1} \otimes X_m)$$

where each X_j is an H_ℓ and each H_ℓ occurs exactly i_ℓ times. Clearly such a form is a product of G-invariant forms in

$$(H_1 \oplus \cdots \oplus H_r)^{\otimes 2} = \bigoplus_{1 \leq i,j \leq r} H_i \otimes H_j.$$

Next we prove (c). The same argument as in the previous case shows that we need only consider a G-invariant tensor in a space

$$X_1 \otimes X_2 \otimes \cdots \otimes X_m$$

in which each X_i is either W or $W^\vee$. But this space contains no nonzero G-invariant tensor unless m is even and there are $\frac{m}{2}$ copies each of W and $W^\vee$. After applying a permutation, we may assume our space is $W^{\vee \otimes \frac{m}{2}} \otimes W^{\otimes \frac{m}{2}}$. The G-invariant tensors in this space are linear combinations of forms

$$(f_1, \ldots, f_{\frac{m}{2}}, x_1, \ldots, x_{\frac{m}{2}}) \mapsto \prod f_i(x_{\sigma(i)}), \quad \sigma \in S_{\frac{m}{2}},$$

each of which is visibly a product of 2-forms. □

Remark 3.7. If the k-algebra $(\bigotimes H)^G$ is generated by the G-invariant tensors of degree 2, then the same is true of $(\bigwedge H)^G$.

There are two approaches to proving this statement. For the first, we assume that G is reductive. Since (by assumption) k has characteristic zero, this implies that the representations of G are all semisimple. By definition, $\bigwedge^m H$ is the quotient of $\bigotimes^m H$ by the subspace generated by elements of the form $v_1 \otimes \cdots \otimes v_m$ with two of the v_i equal, and so the semisimplicity shows that $(\bigwedge H)^G$ is a quotient of $(\bigotimes H)^G$.

For the second approach, which is both more elementary and more general, we refer the reader to Ribet 1983, p530.

Completion of the proof of Proposition 3.4. Recall that it remains to prove Proposition 3.4 in the case of a simple abelian variety A and with the assumption that k is algebraically closed. Let $H = V(A)^\vee$. Then $H^*(A^r) = (\bigwedge rH)$. After Remark 3.7, it suffices to show that the k-algebra of $S(A)$-invariant tensors in $\bigotimes rH$ is generated by those of degree 2. Let F be the largest totally real subfield of the centre of $\mathrm{End}^0(A)$. Then (see Section 2) there are decompositions

$$H = \oplus_{\sigma: F \hookrightarrow k} H_\sigma, \quad S(A) = \prod_{\sigma: F \hookrightarrow k} S_\sigma,$$

and so (see 3.1),

$$(\bigotimes(rH))^{S(A)} = \bigotimes_\sigma (\bigotimes rH_\sigma)^{S_\sigma}.$$

Now the explicit description of S_σ and its representation on H_σ, together with Proposition 3.6, show that each of the k-algebras $(\bigotimes rH_\sigma)^{S_\sigma}$ is generated by tensors of degree 2, which completes the proof.

Abelian varieties over the algebraic closure of a finite field. Because it is so much more elementary than the general case, we explain the proof of Theorem 3.2 in the case of an abelian variety over the algebraic closure $\mathbb{F}$ of a finite field. We shall first need a lemma concerning a torus T acting on a vector space V over an algebraically closed field k. For each character χ of T, V_χ denotes the subspace of V on which T acts through χ. Then $V = \oplus V_\chi$, and the χ for which $V_\chi \neq 0$ are called the *weights* of T in V.

Lemma 3.8. *Let Ξ be the set of weights of T in V, and assume that the elements of Ξ can be numbered $\xi_1, \ldots, \xi_{2m}$ in such a way that the $\mathbb{Z}$-module of relations among the ξ_i is generated by the relations*

$$\xi_i + \xi_{m+i} = 0, \quad i = 1, \ldots, m.$$

Then $(\bigwedge V)^T$ is generated as a Ω-algebra by $(\bigwedge^2)^T$.

Proof. After Remark 3.7, it suffices to prove the analogous result for the tensor algebra $\bigotimes V$ of V. Fix an integer $n \geq 1$. For a $2m$-tuple $\Sigma = (n_1, \ldots, n_{2m})$ of nonnegative integers with sum n, let $[\Sigma]$ be the character $\sum n_i \xi_i$ of T, and let $V(\Sigma) = \bigotimes_{i=1}^{2m} V_{\xi_i}^{\otimes n_i}$. Then $V^{\otimes n} = \oplus_\Sigma V(\Sigma)$, and T acts on $V(\Sigma)$ through the character $[\Sigma]$. Therefore, $(V^{\otimes n})^T = \oplus_{[\Sigma]=0} V(\Sigma)$. By assumption, the character $[\Sigma]$ is zero if and only if $n_i = n_{m+i}$ for $i = 1, \dots, m$, and so

$$[\Sigma] = 0 \implies V(\Sigma) = \bigotimes_{i=1}^{m} (V_{\xi_i} \otimes V_{\xi_{m+i}})^{\otimes n_i}.$$

But $V_{\xi_i} \otimes V_{\xi_{m+i}} \subset (V^{\otimes 2})^T$ because $\xi_i + \xi_{m+i} = 0$ in $X^*(T)$, and so $(\bigotimes^n V)^T$ is generated as a k^{al}-algebra by $(\bigotimes^2 V)^T$. □

Let A be a simple abelian variety over $\mathbb{F}$, and let K be the centre of $\mathrm{End}^0(A)$.

If A is a supersingular elliptic curve, then $K = \mathbb{Q}$, $S(A) = \mu_2$, and $-1 \in \mu_2(\mathbb{Q})$ acts as multiplication by $(-1)^i$ on $H^i(A^r)$. Hence $(H^*(A^r))^{S(A)} = \oplus_i H^{2i}(A^r)$. Since Tate's theorem (Tate 1966) implies that $H^2(A^r)$ is spanned by divisor classes, this proves the theorem in this case.

If A is not a supersingular elliptic curve, then K is a CM-field, and $S(A)$ is a torus. Every embedding $\sigma\colon K \hookrightarrow k^{\mathrm{al}}$ defines a character ξ_σ of $S(A)$, and the character group of $S(A)$ is the quotient of $\oplus \mathbb{Z}\xi_\sigma$ by the subgroup generated by the elements $\xi_\sigma + \xi_{\iota\sigma}$.

Lemma 2.1 shows that $H^1(A^r) \otimes k^{\mathrm{al}}$ is a free $K \otimes_{\mathbb{Q}} k^{\mathrm{al}}$-module. Therefore the weights of $S(A)$ in $H^1(A) \otimes k^{\mathrm{al}}$ are precisely the characters ξ_σ, $\sigma \in \mathrm{Hom}(K, k^{\mathrm{al}})$, and each has the same multiplicity. Choose a CM-type $\{\varphi_1, \cdots, \varphi_m\}$ for K. Then the character group of $S(A)$ has generators $\{\xi_{\varphi_1}, \dots, \xi_{\varphi_m}, \xi_{\iota\varphi_1}, \dots, \xi_{\iota\varphi_m}\}$ and defining relations

$$\xi_{\varphi_i} + \xi_{\iota\varphi_i} = 0, \quad i = 1, \dots, m.$$

Here ι denotes complex conjugation. We can now apply Lemma 3.8 to deduce that $H^*(A^r)^{S(A)}$ is generated as a k-algebra by $H^2(A^r)^{S(A)}$. But Tate's theorem (Tate 1966) implies that this space is spanned by divisor classes.

4. Lefschetz Classes and the Lefschetz Group.

In this section, we define the Lefschetz group $L(A)$ of an abelian variety A and prove that it plays the same role for Lefschetz classes that the Hodge (alias, Mumford-Tate) group plays for Hodge classes. By relating $L(A)$ to $S(A)$, we are able to use the results of Section 2 to compute $L(A)$ for any abelian variety.

As in the previous sections, we fix an algebraically closed field Ω and a Weil cohomology theory with coefficient field k. For any smooth projective variety X over Ω, we define

$$H^{2*}(X)(*) = \oplus_r H^{2r}(X)(r).$$

Cup-product makes $H^{2*}(X)(*)$ into a k-algebra. Since we are now keeping track of the Tate twists, we shall have to distinguish between the canonical pairing

$$e^D\colon V(A) \times V(A) \to k(1)$$

defined by a divisor D on an abelian variety A and its composite E^D with a fixed isomorphism $k(1) \to k$.

Lefschetz classes for homological equivalence. Two algebraic cycles on a smooth complete variety X over Ω are said to be *homologically equivalent* (for the given Weil cohomology theory) if their classes in $H^{2*}(X)(*)$ are equal. Homological equivalence "hom" is an adequate equivalence relation, and so we obtain a $\mathbb{Q}$-algebra $\mathcal{D}_{\mathrm{hom}}(X)$ as in the Introduction. We define $\mathcal{D}_{\mathrm{hom}}(X)_k$ to be the k-subspace of $H^{2*}(X)(*)$ spanned by the image of the cycle class map $cl\colon \mathcal{D}_{\text{rat}}(X) \to H^{2*}(X)(*)$. The cycle class map induces an injection $\mathcal{D}_{\text{hom}}(X) \to \mathcal{D}_{\text{hom}}(X)_k$, and we shall prove later (Corollary 5.3) that, when X is an abelian variety, the induced map $\mathcal{D}_{\text{hom}}(A) \otimes_{\mathbb{Q}} k \to \mathcal{D}_{\text{hom}}(A)_k$ is an isomorphism.

PROPOSITION 4.1. *Let X and Y be smooth complete varieties over Ω. If the $\mathbb{Q}$-space* $\text{DC}(X,Y)$ *of divisorial correspondences between X and Y is zero, then the map*

$$x \otimes y \mapsto p^*x \cdot q^*y\colon \mathcal{D}_{\text{hom}}(X)_k \otimes \mathcal{D}_{\text{hom}}(Y)_k \to \mathcal{D}_{\text{hom}}(X \times Y)_k$$

is an isomorphism. Here p and q are the projection maps from $X \times Y$ to X and Y respectively.

PROOF. There is a canonical decomposition

$$\text{NS}(X \times Y) \cong \text{NS}(X) \oplus \text{NS}(Y) \oplus DC(X,Y)$$

which is compatible with the Künneth decomposition

$$H^2(X \times Y)(1) \cong H^2(X)(1) \oplus H^2(Y)(1) \oplus H^1(X) \otimes H^1(Y)(1).$$

By assumption $DC(X,Y) = 0$, and so we obtain a diagram

$$\begin{array}{ccc} \mathcal{D}^1_{\text{hom}}(X \times Y)_k & = & \mathcal{D}^1_{\text{hom}}(X)_k \oplus \mathcal{D}^1_{\text{hom}}(Y)_k \\ \cap & & \cap \\ H^2(X \times Y)(1) & \supset & H^2(X)(1) \oplus H^2(Y)(1) \end{array}$$

The subalgebras of $H^{2*}(X \times Y)(*) = H^{2*}(X)(*) \otimes H^{2*}(Y)(*)$ generated respectively by $\mathcal{D}^1_{\text{hom}}(X \times Y)_k$ and $\mathcal{D}^1_{\text{hom}}(X)_k \oplus \mathcal{D}^1_{\text{hom}}(Y)_k$ are $\mathcal{D}_{\text{hom}}(X \times Y)_k$ and $\mathcal{D}_{\text{hom}}(X)_k \otimes \mathcal{D}_{\text{hom}}(Y)_k$, which are therefore equal. □

COROLLARY 4.2. *An isogeny*

$$A \to A_1^{r_1} \times \cdots \times A_s^{r_s}$$

with the A_i simple and pairwise nonisogenous abelian varieties defines an isomorphism of $\mathcal{D}_{\text{hom}}(A)_k$ onto the subring $\mathcal{D}_{\text{hom}}(A_1^{r_1})_k \otimes \cdots \otimes \mathcal{D}_{\text{hom}}(A_s^{r_s})_k$ of $H^*(A) = H^*(A_1^{r_1}) \otimes \cdots \otimes H^*(A_s^{r_s})$.

PROOF. In general, $\text{DC}(X,Y) \cong \text{Hom}(A,B)$, where A is the Albanese variety of X and B is the Picard variety of Y. Therefore, if A and B are abelian varieties with $\text{Hom}(A,B) = 0$, then $\text{DC}(A,B) = 0$ and the projection maps define an isomorphism

$$\mathcal{D}_{\text{hom}}(A)_k \otimes \mathcal{D}_{\text{hom}}(B)_k \to \mathcal{D}_{\text{hom}}(A \times B)_k.$$

The general statement follows from this by induction. □

The Lefschetz group. Let A be an abelian variety over Ω. From the canonical isomorphisms

$$H^1(A) \cong \text{Hom}(V(A), k), \quad H^1(A^r) = rH^1(A), \quad H^*(A^r) \cong \bigwedge H^1(A^r).$$

we see that there is a natural left action of $\text{GL}(V(A))$ on $H^s(A^r)$ for all r, s. Using the identification $\mathbb{G}_m = \text{GL}(\mathbb{Q}(1))$, we extend this to an action of $\text{GL}(V(A)) \times \mathbb{G}_m$ on $H^{2s}(A^r)(m)$ for all r, s, m.

Definition 4.3. The *Lefschetz group* $L(A)$ of an abelian variety A over Ω is the largest algebraic subgroup of $\mathrm{GL}(V(A))\times\mathbb{G}_{m/k}$ fixing[5] the elements of $\mathcal{D}^s_{\mathrm{hom}}(A^r)_k\subset H^{2s}(A^r)(s)$ for all r,s.

Let $G(A)$ be the algebraic subgroup of $\mathrm{GL}(V(A))$ such that

$$G(A)(R)=\{\gamma\in C(A)\otimes R\mid \gamma^{\dagger}\gamma\in R^{\times}\}$$

for any k-algebra R. Thus, for any ample divisor D on A, $G(A)$ is the largest algebraic subgroup of $\mathrm{GSp}(E^D)$ commuting with the endomorphisms of A.

Theorem 4.4. *The map* $\gamma\mapsto(\gamma,\gamma^{\dagger}\gamma)\colon G(A)\to\mathrm{GL}(V(A))\times\mathbb{G}_m$ *sends* $G(A)$ *isomorphically onto* $L(A)$*.*

Proof. For any $\gamma\in G(A)(k)$ and divisor D on A,

$$e^D(\gamma x,\gamma y)=e^D(x,\gamma^{\dagger}\gamma x)=\gamma^{\dagger}\gamma\cdot e^D(x,y),\text{ all }x,y\in V(A),$$

and so $(\gamma,\gamma^{\dagger}\gamma)$ fixes e^D. It therefore fixes the class of D in $H^2(A)(1)$. More generally, any $\gamma\in G(A)(k^{\mathrm{al}})$ will fix all divisor classes on A^r, all r. This shows that $G(A)\subset L(A)$.

For the converse, note that Theorem 3.2 implies that $H^{2*}(A^r)(*)^{G(A)}=\mathcal{D}_{\mathrm{hom}}(A^r)_k$ for all r, and so a variant of Chevalley's theorem (Deligne 1982, 3.1) implies that $L(A)=G(A)$. □

Corollary 4.5. *For any abelian variety A and any $r\geq 0$, $H^{2*}(A^r)(*)^{L(A)}=\mathcal{D}_{\mathrm{hom}}(A^r)_k$.*

Proof. In the last proof, it was noted that the statement becomes true when $L(A)$ is replaced with $G(A)$. □

The projection map $\mathrm{GL}(V(A))\times\mathbb{G}_m\to\mathbb{G}_m$ defines a cocharacter of $L(A)$, which we denote $l(A)$ (or just l). The theorem shows that the kernel of $l(A)$, regarded as a subgroup of $\mathrm{GL}(V(A))$, equals $S(A)$.

It is clear from the theorem that the homomorphism

$$a\mapsto(a^{-1},a^{-2})\colon\mathbb{G}_m\to\mathrm{GL}(V(A))\times\mathbb{G}_m$$

takes values in $L(A)$. Therefore $L(A)$ has a canonical cocharacter w. Note that $l\circ w=-2$.

Definition 4.6. Consider a family $(G_i,t_i)_{i\in I}$ of pairs consisting of an algebraic group G_i over k and a homomorphism $t_i\colon G_i\to\mathbb{G}_m$. We define the product $\prod(G_i,t_i)$ of the family to be the pair (G,t) consisting of the largest subgroup of $\prod G_i$ on which the characters $(g_i)_{i\in I}\mapsto t_{i_0}(g_{i_0})$ agree and of the common restriction of these characters to G. It is universal with respect to the maps $(G,t)\to(G_i,t_i)$.

Corollary 4.7. *An isogeny*

$$A\to A_1^{r_1}\times\cdots\times A_s^{r_s}$$

with the A_i simple and pairwise nonisogenous defines an isomorphism

$$(L(A),l(A))\to\prod_{i=1}^{s}(L(A_i),l(A_i)),$$

[5]In the sense defined in the preceding section.

which is independent of the choice of the isogeny.

PROOF. Let $(L, l) = \prod(L(A_i), l(A_i))$, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & S(A) & \to & L(A) & \xrightarrow{l(A)} & \mathbb{G}_m & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \prod S(A_i) & \to & L & \xrightarrow{l} & \mathbb{G}_m & \to & 0. \end{array}$$

Because two of the vertical maps are isomorphisms, so also is the third. □

Application: Lefschetz classes and Hodge classes. A *Hodge class* on an abelian variety A over $\mathbb{C}$ is an element of type $(0,0)$ in $H_B^{2s}(A)(s)$ for some s. The Hodge classes on A form a $\mathbb{Q}$-subalgebra $\mathcal{H}(\mathcal{A})$ of $H_B^{2*}(A)(*)$. The *Hodge group* $Hg(A)$ of A is defined to be the largest algebraic subgroup of $\mathrm{GL}(V_B(A)) \times \mathbb{G}_m$ fixing all the Hodge classes on A and its powers. It has the property that $H^{2*}(A^r)(*)^{Hg(A)} = \mathcal{H}(A^r)$ for all r (Deligne 1982, proof of Proposition 3.4). Projection onto $\mathbb{G}_m$ defines a canonical character of $Hg(A)$, and we let $Hg'(A)$ denote its kernel.

The Betti cohomology theory has coefficient field $\mathbb{Q}$, and $\mathcal{D}_{\mathrm{hom}}(A) \cong \mathcal{D}_{\mathrm{hom}}(A)_{\mathbb{Q}}$. We drop the subscript in this case, and identify $\mathcal{D}_{\mathrm{hom}}(A)$ with a subspace of $H^{2*}(A)(*)$. Clearly $\mathcal{D}_{\mathrm{hom}}(A) \subset \mathcal{H}(A)$, and so $L(A) \supset Hg(A)$. A Hodge class not in $\mathcal{D}_{\mathrm{hom}}(A)$ will be said[6] to be *exotic*.

PROPOSITION 4.8. *The following conditions on an abelian variety A are equivalent:*

(a) *no power of A supports an exotic Hodge class;*
(b) $Hg(A) = L(A)$*;*
(c) $Hg'(A) = S(A)$*;*

PROOF. The groups $Hg(A)$ and $L(A)$ are the largest algebraic subgroups of $GL(H_1(A)) \times \mathbb{G}_m$ fixing respectively the Hodge classes and the Lefschetz classes on the powers of A, and conversely, these are precisely the classes fixed by the two groups. Hence

$$\mathcal{H}(A^r) = \mathcal{D}(A^r) \text{ for all } r \iff Hg(A) = L(A).$$

The proves the equivalence of the first two conditions, and the equivalence of the second two follows from applying the Five Lemma to the diagram:

$$\begin{array}{ccccccccc} 0 & \to & Hg'(A) & \to & Hg(A) & \to & \mathbb{G}_m & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & S(A) & \to & L(A) & \xrightarrow{l(A)} & \mathbb{G}_m & \to & 0. \end{array}$$

□

REMARK 4.9. When A has an isogeny factor of type III, the conditions in Proposition 4.8 always fail because $Hg'(A)$ is connected (Deligne 1982, p45) and $S(A)$ is not—see the table at the end of Section 2. In fact, a simple abelian variety A of type III supports an exotic Hodge class c such that $p^*(c) \cup q^*(c) \in H^{2*}(A \times A)(*)$ is

[6] Similar classes have been called *extraordinary* (Weil 1977, p424), *exceptional* (Murty 1984, p197), and *sporadic* (White 1993). In Ribet 1983, an abelian variety with no exotic Hodge class is said to *satisfy the* $(1,1)$*-criterion.* Other authors say that an abelian variety is *nondegenerate* if it has no exotic Hodge classes, and *stably nondegenerate* if no power of it has an exotic Hodge class.

Lefschetz (Murty 1984, 3.2). The class c is fixed by the identity component of $S(A)$ but not by $S(A)$ itself.

When A does not have an factor of type III, then the conditions in Proposition 4.8 are equivalent to the following condition:

$$\operatorname{rank} Hg'(A) = \operatorname{rank} S(A).$$

This can be proved by directly verifying it for simple abelian varieties (Hazama 1984), and then applying the following statement to $\operatorname{Lie} Hg'(A) \subset \oplus_i \operatorname{Lie} S(A_i)$ where the A_i are the simple isogeny factors of A:

> Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ be reductive Lie algebras over $\mathbb{C}$, and let $\mathfrak{g}$ be a reductive subalgebra of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. If $\mathfrak{g}$ projects onto each $\mathfrak{g}_i$ and $\operatorname{rank}(\mathfrak{g}) = \sum \operatorname{rank}(\mathfrak{g}_i)$, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ (ibid., 3.1).

Example 4.10. Let A be an abelian variety over $\mathbb{C}$, and let $i\colon K \to \operatorname{End}^0(A)$ be a homomorphism (sending 1 to 1) from a CM-field K into $\operatorname{End}^0(A)$. The pair (A, i) is said to be of *Weil type* if the tangent space to A at zero is a free $K \otimes_{\mathbb{Q}} \mathbb{C}$-module. A *polarization* of (A, i) is a polarization λ of A whose Rosati involution stabilizes K and induces complex conjugation on it.

Let (A, i) be of Weil type, and let $m = \frac{\dim A}{[K:\mathbb{Q}]}$. Then the subspace $\bigwedge_K^m H_B^1(A)(m)$ of $H_B^m(A)(\frac{m}{2})$ consists of Hodge classes (Deligne 1982, 4.4)—they are called the *Weil classes* on A.

Let D be an ample divisor on A inducing a polarization of (A, i), and let $\phi\colon V(A) \times V(A) \to K$ be the skew-Hermitian form such that $\operatorname{Tr}_{K/\mathbb{Q}} \circ \phi = E^D$. Then $S(A) = U(\phi)$. When A is general, $Hg'(A) = SU(\phi)$, and the Weil classes are exotic. Very few of them have been shown to be algebraic (Schoen 1988).

When $Hg'(A) = SU(\phi)$, invariant theory shows that $\mathcal{H}(A)$ is generated as a $\mathbb{Q}$-algebra by the divisor classes and Weil classes on A.

The above statements are due to Weil (1977) in the case that K has degree 2 over $\mathbb{Q}$. The extension to arbitrary CM-fields is straightforward.

Application: Lefschetz classes and Tate classes. Let A be an abelian variety over an algebraically closed field Ω, and let ℓ be a prime different from the characteristic of Ω. Define

$$\mathcal{T}_\ell(A) = \cup_{A_0/K} H_\ell^{2*}(A)(*)^{\operatorname{Gal}(K^{\mathrm{al}}/K)}$$

where the union is over the models A_0 of A over subfields K of Ω that are finitely generated over the prime field (K^{al} is the algebraic closure of K in Ω). Define the *Tate group* $Tt(A)$ to be the largest algebraic subgroup of $\mathrm{GL}(V_\ell(A)) \times \mathbb{G}_{m/\mathbb{Q}_\ell}$ fixing the Tate classes on A and its powers. It has the property that $H_\ell^{2*}(A^r)(*)^{Tt(A)} = \mathcal{T}_\ell(A^r)$ for all r.

Clearly $\mathcal{D}_{\mathrm{hom}}(A)_{\mathbb{Q}_\ell} \subset \mathcal{T}_\ell(A)$, and so $L(A) \supset Tt(A)$. Moreover

$$\mathcal{D}_{\mathrm{hom}}(A^r)_{\mathbb{Q}_\ell} = \mathcal{T}_\ell(A^r) \text{ for all } r \iff L(A) = Tt(A).$$

Analogues of the results in (4.8), (4.9), and (4.10) hold.

Example 4.11. Let A be a simple abelian variety over $\mathbb{F}$. Let $A_0/\mathbb{F}_q$ be a model of A such that $\operatorname{End}^0(A_0) = \operatorname{End}^0(A)$, and let π be the Frobenius endomorphism of

$A_0/\mathbb{F}_q$. Let $L(A)_0$ be the group of multiplicative type over $\mathbb{Q}$ such that

$$L(A)_0 = \{\gamma \in \mathbb{Q}[\pi] \mid \gamma\bar{\gamma} \in \mathbb{Q}^\times\}.$$

Let $Tt(A)_0$ be the Zariski closure of π^m in $L(A)_0$ for m sufficiently divisible. For any $\ell \neq \mathrm{char}(\mathbb{F})$, $L_\ell(A) = L(A)_{0/\mathbb{Q}_\ell}$ and $Tt_\ell(A) = Tt(A)_{0/\mathbb{Q}_\ell}$. Therefore, no power of A has an exotic Tate class if and only if $Tt(A)_0 = L(A)_0$. This is an explicit criterion, which one can use to prove the Tate conjecture for some classes of abelian varieties (Zarhin 1991, Lenstra and Zarhin 1993), but Wei (1993) has shown that, in a certain precise sense, the criterion fails for most isogeny classes of abelian varieties over $\mathbb{F}$.

5. Existence of Lefschetz classes

In this section, we apply the results of the preceding sections to prove that many naturally occurring cohomology classes on abelian varieties are Lefschetz, and that the family of Lefschetz classes is stable under certain natural operations.

As usual, we fix an algebraically closed field Ω and a Weil cohomology theory with coefficient field k. A regular map $\phi\colon X \to Y$ of smooth projective varieties induces a homomorphism $\phi^*\colon H^*(Y) \to H^*(X)$ of graded k-algebras and homomorphisms

$$\phi_*\colon H^s(X)(r) \to H^{s+2c}(X)(r+c), \quad c = \dim Y - \dim X,$$

of k-vector spaces. Because ϕ^* is a homomorphism of graded k-algebras commuting with the cycle maps, it maps Lefschetz classes to Lefschetz classes. On the other hand, ϕ_* will not in general map Lefschetz classes to Lefschetz classes. For example let Z be a smooth closed connected subvariety of X whose cohomology class is not Lefschetz, and let ϕ be the inclusion map $Z \hookrightarrow X$. The space $H^0(Z)$ consists of Lefschetz classes, but $\phi_*(1)$ is not Lefschetz because it is the cohomology class of Z in $H^{2c}(X)(c)$, $c = \dim X - \dim Z$.

Proposition 5.1. *For any abelian variety A, $\mathcal{D}_{\mathrm{hom}}(A)_k$ is a graded subalgebra of $H^{2*}(A)(*)$, i.e., if $\alpha \in \mathcal{D}_{\mathrm{hom}}(A)_k$ and $\alpha = \sum \alpha_r$ with $\alpha_r \in H^{2r}(A)(r)$, then $\alpha_r \in \mathcal{D}_{\mathrm{hom}}(A)_k$.*

Proof. Choose an isomorphism $k(1) \to k$, and use it to identify $\mathcal{D}_{\mathrm{hom}}(A)_k$ with a subalgebra of $H^*(A)$—it suffices to prove that this is a graded subalgebra. Define $w\colon \mathbb{G}_m \to \mathrm{GL}(V(A))$ to be the homomorphism such that $w(c)x = c^{-1}x$ for all $c \in k^\times$ and $x \in V(A)$. Suppose $\alpha \in H^*(A)$ decomposes as $\alpha = \sum \alpha_r$, $\alpha_r \in H^r(A)$. Then $w(c)\alpha = \sum c^r\alpha_r$ all $c \in k^\times$, and this equation determines the α_r uniquely. Since the action of $w(\mathbb{G}_m)$ commutes with that of $S(A)$, this shows that if α is fixed by $S(A)$ then so also are the α_r. Now we can apply Theorem 3.2 to complete the proof. □

Proposition 5.2. *Let A be an abelian variety over Ω. For any nonzero $a \in \mathcal{D}_{\mathrm{hom}}(A)_k$, there exists $b \in \mathcal{D}_{\mathrm{hom}}(B)_k$ such that $a \cdot b \neq 0$.*

Proof. Because $L(A)$ acts semisimply, the nondegenerate pairing

$$H^{2s}(A)(s) \times H^{2g-2s}(A)(g-s) \to H^{2g}(A)(g) \cong k, \quad g = \dim A,$$

induces a nondegenerate pairing

$$\begin{array}{ccccc} H^{2s}(A)(s)^{L(A)} & \times & H^{2g-2s}(A)(g-s)^{L(A)} & \to & k \\ \| & & \| & & \\ \mathcal{D}^s_{\mathrm{hom}}(A)_k & & \mathcal{D}^{g-s}_{\mathrm{hom}}(A)_k & & \end{array}.$$

□

COROLLARY 5.3. (a) *The canonical*[7] *map* $\mathcal{D}^s_{\mathrm{hom}}(A) \to \mathcal{D}^s_{\mathrm{num}}(A)$ *is bijective.*
(b) *The canonical map* $\mathcal{D}_{\mathrm{hom}}(A) \otimes_{\mathbb{Q}} k \to \mathcal{D}_{\mathrm{hom}}(A)_k$ *is an isomorphism.*

PROOF. (a) This is an immediate consequence of the proposition.

(b) This can be proved by the same argument as in Tate 1994, 2.5. □

In particular, on Lefschetz cycles, homological equivalence is independent of the Weil cohomology and $\mathcal{D}_{\mathrm{num}}(A) \otimes_{\mathbb{Q}} k \cong \mathcal{D}_{\mathrm{hom}}(A)_k$.

Direct images of Lefschetz classes are Lefschetz. Let T be a finite set of distinct isogeny classes of simple abelian varieties, and let $L = \prod_{B\in T}(L(B), l(B))$ (product in the sense of (4.6)). For an abelian variety A whose simple isogeny factors lie in T, $L(A)$ is canonically a quotient $L \to L(A)$ of L, and so there is a natural action of L on $V(A)$. If A' is a second such abelian variety, then the homomorphism $V(\phi)\colon V(A) \to V(A')$ induced by a regular map $\phi\colon A \to A'$ is L-equivariant. Because $\phi^*\colon H^*(A') \to H^*(A)$ equals $\bigwedge V(\phi)^\vee$, it also is L-equivariant.

PROPOSITION 5.4. *For any regular map* $\phi\colon A \to B$ *of abelian varieties, the map* $\phi_*\colon H^{2s}(A)(s) \to H^{2s+2c}(B)(s+c)$ *commutes with the actions of* $L(A \times B)$.

PROOF. Let $g = \dim A$. Because $H^{2g}(A)(g)$ consists of Lefschetz classes—it is generated by the class of D^g for any ample divisor D on A—the action of $L(A)$ on it is trivial. A similar remark applies to B. Let $\alpha \in L(A \times B)$. On replacing x and y with αx and αy in the projection formula

$$\eta_B(\phi_*(x) \cup y) = \eta_A(x \cup \phi^*(y)), \quad x \in H^{2g-2s}(A)(g-s), \quad y \in H^{2s}(B)(s),$$

and using that the action of α commutes with cup-products (by definition) and the action of ϕ^*, we find that

$$\eta_B(\phi_*(\alpha x) \cup \alpha y) = \eta_A(\alpha x \cup \phi^*(\alpha y)) = \eta_A(x \cup \phi^*(y)) = \eta_B(\phi_*(x) \cup y).$$

Therefore

$$\eta_B(\alpha^{-1}\phi_*(\alpha x) \cup y) = \eta_B(\phi_*(x) \cup y).$$

Since this holds for all $y \in H^{2s}(B)(s)$, we find that $\alpha^{-1}\phi_*(\alpha x) = \phi_*(x)$ for all $x \in H^{2g-2s}(A)(g-s)$. □

COROLLARY 5.5. *For any regular map* $\phi\colon A \to B$ *of abelian varieties,* ϕ_* *maps Lefschetz classes on* A *to Lefschetz classes on* B.

PROOF. Because ϕ_* commutes with the actions of $L(A \times B)$, it maps classes fixed by $L(A)$ to classes fixed by $L(B)$, and we can apply Corollary 4.5. □

COROLLARY 5.6. *The graph of any regular map* $\alpha\colon A \to B$ *of abelian varieties is Lefschetz.*

PROOF. In fact, $\Gamma_\alpha = (\mathrm{id}_A, \alpha)_*(1_A)$, and $1_A \in H^0(A)$ is Lefschetz. □

---

[7] For abelian varieties over $\mathbb{C}$, Lieberman 1968 shows that (Betti) homological equivalence agrees with numerical equivalence on all algebraic cycles on abelian varieties.

Correspondences. Let X and Y be smooth projective varieties over Ω. The elements of $H^{2*}(X \times Y)(*)$ are called *cohomological correspondences* between X and Y. The map sending $u \in H^{2s}(X \times Y)(s)$ to the composite

$$H^*(X) \xrightarrow{p^*} H^*(X \times Y) \xrightarrow{v \mapsto v \cup u} H^{*+2s}(X \times Y)(s) \xrightarrow{q_*} H^{*+2s-2d}(Y)(s-d), \quad d = \dim X,$$

is an isomorphism

$$u \mapsto \bar{u}\colon H^*(X \times Y) \to \mathrm{Hom}(H^*(X), H^{*+2s-2d}(Y)(s-d)).$$

A cohomological correspondence $u \in H^*(X \times Y)$ is said to be *Lefschetz* if it lies in the subalgebra $\mathcal{D}_{\mathrm{hom}}(X \times Y)_k$ of $H^*(X \times Y)$.

PROPOSITION 5.7. *Let A and B be abelian varieties over Ω. A cohomological correspondence u between A and B is Lefschetz if and only if $\bar{u}\colon H^*(A) \to H^*(B)$ commutes with the actions of $L(A \times B)$. If u is Lefschetz, then $\bar{u}$ maps $\mathcal{D}(A)_k$ into $\mathcal{D}(B)_k$.*

PROOF. The map $u \mapsto \bar{u}$ is bijective and commutes with the action of $L(A \times B)$, whence the first statement. The maps ϕ^*, ϕ_*, and cupping with a Lefschetz class, all preserve Lefschetz classes, whence the second statement. □

COROLLARY 5.8. *For any abelian variety A over Ω, the Künneth components of the diagonal are Lefschetz.*

PROOF. The projection operator $H^*(A) \to H^s(A)$ commutes with the action of $L(A)$. □

Recall that a smooth projective variety X is said to satisfy the strong Lefschetz theorem (with respect to a given Weil cohomology theory) if, for any hyperplane section Z, the map

$$L\colon H^*(X) \to H^*(X)(1), \quad x \mapsto x \cup cl(Z)$$

has the property that, for $s \leq d = \dim X$,

$$L^{d-s}\colon H^s(X) \to H^{2d-s}(X)(d-s)$$

is an isomorphism. For such a variety, the *primitive elements* of $H^s(X)$ are those killed by L^{d-s+1}. Any $x \in H^s(X)$ can be written unquely in the form $x = \sum_{i \geq 0, s-d} L^i x_i$ with x_i a primitive element of $H^{s-2i}(X)$. For $x = \sum L^i x_i \in H^s(X)$, define

$$\begin{aligned} \Lambda x &= \sum_{i \geq s-d, 1} L^{i-1} x_i \\ {}^c\Lambda x &= \sum_{i \geq s-d, 1} i(d-s+i+1) L^{i-1} x_i \\ {}^* x &= \sum_{i \geq s-d, 0} (-1)^{(s-2i)(s-2i+1)/2} L^{d-s+i} x_i. \end{aligned}$$

THEOREM 5.9. *Let A be an abelian variety over Ω. The correspondences Λ, ${}^c\Lambda$, and $*$ between A and itself are all Lefschetz.*

PROOF. It is known (e.g., Kleiman 1968, p367) that Λ, regarded as a map of cohomology groups, is inverse to L. Since the latter is Lefschetz, it commutes with the action of $L(A)$, which implies that the same is true of Λ, which is therefore Lefschetz. Consequently, all elements of the $\mathbb{Q}$-algebra $\mathbb{Q}[L,\Lambda]$ are Lefschetz. Since this algebra contains ${}^c\Lambda$ and $*$ (Kleiman 1968, 1.4.4), this completes the proof. □

Most of (Kleiman 1968) can now be rewritten with "variety" replaced by "abelian variety" and "algebraic cycle" with "Lefschetz cycle".

Other adequate equivalence relations. In the above, we have proved that certain naturally occurring classes in $\mathcal{C}_{\text{hom}}(\cdot)$ actually lie in $\mathcal{D}_{\text{hom}}(\cdot)$. One can ask whether the same is true for other adequate equivalence relations. I do not know the answer in every case, but it is possible to extract the following theorem from Scholl 1994.

THEOREM 5.10. *For every adequate equivalence relation, the graph of a regular map of abelian varieties $\phi\colon A \to B$ is a Lefschetz class on $A \times B$.*

PROOF. It suffices to prove this for the finest adequate relation, namely, rational equivalence. Let A be an abelian variety of dimension g. Choose a symmetric ample divisor D on A, and let $M = m^*D - p^*D - q^*D$. Let λ_D be the polarization defined by D. For $0 \le i \le 2g$, define

$$p_i = \frac{(-1)^i}{\sqrt{\deg(\lambda_D)}} \sum_{\max(0,i-g)\le j\le \frac{i}{2}} \frac{1}{j!(g-1+j)!(i-2j)!} p^*([D^{g-i+j}])\cdot q^*([D^j])\cdot [M]^{i-2j}$$

Here $[*]$ denotes the class of $*$ in $\mathcal{C}^1_{\text{rat}}(\cdot) = \operatorname{Pic}(\cdot)\otimes\mathbb{Q}$, so that $p_i \in \mathcal{C}^g_{\text{rat}}(A\times A)$. Then

$$p_0 + p_1 + \cdots + p_{2g} = \Delta_A \quad (\text{identity in } \mathcal{C}^g_{\text{rat}}(A\times A))$$

(Scholl 1994, Section 5). Clearly each p_i is Lefschetz, and so Δ_A is Lefschetz. Similarly Δ_B is Lefschetz, and so the formula

$$(\phi\times id)^*(\Delta_B) = \Delta_B \circ \Gamma_\phi = \Gamma_\phi$$

(Fulton 1984, 16.1.1) shows that Γ_ϕ is also Lefschetz. □

Let p_i be the class in $\mathcal{C}^g_{\text{rat}}(A\times A)$ defined in the above proof. Then, for every Weil cohomology theory, $cl(p_i)$ is the i^{th} Künneth component of the diagonal.

REMARK 5.11. The referee points out that it is possible to show similarly that the correspondences Λ, ${}^c\Lambda$, $*$ etc. are Lefschetz for rational equivalence. Following (Scholl 1994, 5.9), define for $0\le i\le 2g$,

$$f_i = \sum_{\max(0,i-g)\le j\le \frac{i}{2}} \frac{1}{j!(g-i+j)!(i-2j)!} p^*([D^j])\cdot q^*([D^j])\cdot [M]^{i-2j}.$$

Then f_i is Lefschetz, and $\frac{(-1)^i}{\sqrt{\deg(\lambda_D)}} f_i$ is the inverse of the strong Lefschetz isomorphism "cup with $[D]^{g-i}$" (cf. ib. 5.9.1). Thus

$$\Lambda = \deg(\lambda_D)^{-1/2}\left(\sum_{2\le i\le g} (-1)^i f_{i-2}\cdot p^*[D^{g+1-i}] + \sum_{g<i\le 2g} (-1)^i f_{2g-i}\cdot q^*[D^{i-g-1}]\right).$$

Also, the Fourier transform correspondence (Künneman 1994, p193),

$$F = \exp[c_1(P)] \in \mathcal{C}_{\mathrm{rat}}(A \times A^{\vee}), \quad P = \text{Poincaré line bundle},$$

is Lefschetz.

Other varieties. The above results hold, not only for abelian varieties, but also for certain other (albeit still very special) varieties, for example, for algebraic varieties whose connected components are products of projective spaces with varieties admitting the structure of an abelian variety.

Appendix: Weil Cohomology

We fix an algebraically closed field Ω and a field k of characteristic zero. A contravariant functor $X \mapsto H^*(X)$ from the category of smooth projective varieties over Ω to the category of finite-dimensional, graded, anti-commutative k-algebras is said to be a *Weil cohomology theory* if it carries disjoint unions to direct sums and admits a Poincaré duality, a Künneth formula, and a cycle map.

Poincaré duality: Let X be a connected smooth projective variety over Ω of dimension d.

(a) The groups $H^s(X)$ are zero except for $0 \leq s \leq 2d$, and $H^{2d}(X)$ has dimension 1.

(b) Let $k(-1) = H^2(\mathbb{P}^1)$. For any k-vector space V and integer m, let $V(m) = V \otimes_k k(-1)^{\otimes -m}$ or $V \otimes_k k(-1)^{\vee \otimes m}$ according as m is positive or negative. Then, for each X, there is given a natural isomorphism $\eta_X \colon H^{2d}(X)(d) \to k$.

(c) The pairings

$$H^r(X) \times H^{2d-r}(X)(d) \to H^{2d}(X)(d) \cong k$$

induced by the product structure on $H^*(X)$ are non-degenerate.

Let $\phi \colon X \to Y$ be a regular map of smooth projective varieties over Ω, and let $\phi^* = H^*(\phi) \colon H^*(Y) \to H^*(X)$. Because the pairing in (c) is nondegenerate, there is a unique linear map

$$\phi_* \colon H^*(X) \to H^{*+2c}(Y)(c), \quad c = \dim Y - \dim X$$

such that the projection formula

$$\eta_Y(\phi_*(x) \cup y) = \eta_X(x \cup \phi^* y)$$

holds for all $x \in H^{2\dim X - 2s}(X)(\dim X - s)$, $y \in H^{2s}(Y)(s)$.

Künneth formula: Let $p, q \colon X \times Y \to X, Y$ be the projection maps. Then the map

$$x \otimes y \mapsto p^* x \cup q^* y \colon H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

is an isomorphism of graded k-algebras.

Cycle map: There are given homomorphisms

$$cl_X \colon C^r_{\mathrm{rat}}(X) \to H^{2r}(X)(r)$$

satisfying the following conditions:

(a) (functoriality) For any regular map $\phi \colon X \to Y$,

$$\phi^* \circ cl_Y = cl_X \circ \phi^*, \quad \phi_* \circ cl_X = cl_Y \circ \phi_*.$$

(b) (multiplicativity) For any X, Y

$$cl_{X\times Y}(Z\times W)=cl_X(Z)\otimes cl_Y(W).$$

(c) (non-triviality) If P is a point, so that $C^*_{\text{rat}}(P)=\mathbb{Q}$ and $H^*(P)=k$, then cl_P is the natural inclusion map.

In the functoriality statement, the ϕ^* and ϕ_* on the right of the equality signs refer to the standard operations on the $\mathbb{Q}$-algebras of algebraic cycles modulo rational equivalence (Fulton 1984, Chapter I).

Proposition (A.1). *Let $i\colon Z\hookrightarrow X$ be a smooth closed subvariety of X. Then $cl_X(Z)=i_*(1_Z)$.*

Proof. Let $P=\operatorname{Spec} k$ and let $\phi\colon Z\to P$ be the structure map. Then

$$1_Z=\phi^*(1_P)=\phi^*(cl_P(P))=cl_Z(\phi^*P)=cl_Z(Z).$$

Therefore

$$i_*(1_Z)=i_*(cl_Z(Z))=cl_X(i_*(Z))=cl_X(Z).$$

□

Proposition (A.2). *Let A be an abelian variety of dimension g over Ω.*

(a) *The dimension of $H^1(A)$ is $2g$, and the inclusion $H^1(A)\to H^*(A)$ extends to an isomorphism of k-algebras $\bigwedge H^1(A)\to H^*(A)$.*
(b) *For any endomorphism α of A, the characteristic polynomial $P_{A,\alpha}(T)$ of α on A is equal to the characteristic polynomial of α acting on $V(A)$.*

Proof. Statement (a) is proved in Kleiman 1968, 2A8.

For (b), it follows from the axioms that an isogeny $\gamma\colon A\to A$ acts on $H^{2g}(A)$ as multiplication by $\deg\gamma$. Let

$$P(T)\overset{\text{df}}{=}\det(H^1(\alpha)-T|H^1(A))$$

be the characteristic polynomial of α acting on $H^1(A)$. Then $P(n)=\det(\alpha-n)$ for all integers n. But $\alpha-n$ acts on $\Lambda^{2g}H^1(A)=H^{2g}(A)$ as multiplication by $\det(\alpha-n)$. Therefore, $P(n)=\deg(\alpha-n)$ for all integers. But this is the condition characterizing $P_{A,\alpha}(T)$, and so $P(T)=P_{A,\alpha}(T)$. Since α has the same characteristic polynomial on $V(A)$ as on $H^1(A)$ ($\operatorname{End}(A)$ acts on $V(A)$ on the left and on $H^1(A)$ on the right), this completes the proof. □

The field k is called the *coefficient field* for the Weil cohomology theory. Note that if $X\mapsto H^*(X)$ is a Weil cohomology theory with coefficient field k, and $k'\supset k$, then $X\mapsto H^*(X)\otimes_k k'$ is a Weil cohomology theory with coefficient field k'.

References

Borel, A., 1991. Linear Algebraic Groups, Springer, Heidelberg, (2nd Edition).

Deligne, P., 1982. Hodge cycles on abelian varieties (Notes by J.S. Milne). In: Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math. 900, Springer, Heidelberg, 9–100.

Fulton, W., 1984. Intersection Theory, Ergeb. Math. Grenzgeb. (3), Springer, Heidelberg.

Fulton, W., and Harris, J., 1991. Representation Theory, Springer, Heidelberg.

Grothendieck, A., 1969. Standard conjectures on algebraic cycles. In: Algebraic Geometry, Oxford University Press, London, 193–199.

Hazama, F., 1984. Algebraic cycles on certain abelian varieties and powers of special surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., 31, 487–520.

Ichikawa, Takashi, 1991. Algebraic groups associated with abelian varieties, Math. Ann., 289, 133-142.

Kleiman, S.L., 1968. Algebraic cycles and Weil conjectures. In: Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 359–386.

Knus, M-A., 1991. Quadratic and Hermitian Forms over Rings, Grundlehren Math. Wiss. 294, Springer, Heidelberg.

Künneman, K., 1994. On the Chow motive of an abelian scheme, In: Motives (Proceedings of the Summer Research Conference on Motives, University of Washington, Seattle, July 20–August 2, 1991), (Eds. Jannsen, Kleiman, Serre), AMS, Proc. Symp Pure Math. 55, 1994, Part 1, 189–205.

Lenstra, H.W., and Zarhin, Yu, G., 1993. The Tate conjecture for almost ordinary Abelian varieties over finite fields. In: Advances in Number Theory, Proceedings of the Third Conference of the Canadian Number Theory Association, Oxford University Press, 179–194.

Lieberman, D. I., 1968. Numerical and homological equivalence of algebraic cycles on Hodge manifolds, Amer. J. Math. 90, 366–374.

Milne, J. S., 1986. Abelian varieties. In: Arithmetic Geometry, Springer, Heidelberg, 103–150.

Milne, J. S., 1995. On the conjecture of Langlands and Rapoport. Preliminary version Sept. 25, 1995.

Milne, J. S., 1996. Lefschetz motives and the Tate conjecture. Preprint Dec. 31, 1996. To appear in Compositio Math.. (Available at `http://www.math.lsa.umich.edu/~jmilne/`.)

Mumford, D., 1970. Abelian Varieties, Oxford University Press.

Murty, K., 1984. Exceptional Hodge cycles on certain abelian varieties, Math. Ann., 266, 197–206.

Ribet, K., 1983. Hodge classes on certain types of abelian varieties, Amer. J. Math., 105, 523–538.

Schoen, C., 1988. Hodge classes on self-products of a variety with an automorphism, Compositio Math., 65, 3–32.

Scholl, A., 1994. Classical motives. In: Motives (Proceedings of the Summer Research Conference on Motives, University of Washington, Seattle, July 20–August 2, 1991), (Eds. Jannsen, Kleiman, Serre), AMS, Proc. Symp Pure Math. 55, 1994, Part 1, 163–187.

Serre, J-P., 1959. Groupes Algébriques et Corps de Classes, Hermann, Paris.

Tanke'ev, S.G., 1982. Cycles on simple abelian varieties of prime dimension, Izv. 46, 155–170.

Tate, J., 1966. Endomorphisms of abelian varieties over finite fields, Invent. Math. 2, 134–144.

Tate, J., 1994. Conjecture on algebraic cycles in ℓ-adic cohomology. In: Motives (Proceedings of the Summer Research Conference on Motives, University of Washington, Seattle, July 20–August 2, 1991), (Eds. Jannsen, Kleiman, Serre), AMS, Proc. Symp Pure Math. 55, 1994, Part 1, 71–83.

Wei, W., 1993. Weil numbers and generating large field extensions, Thesis, University of Michigan.

Weil, A., 1960. Algebras with involution and classical groups, J. Indian Math. Soc. 24, 589–623.

Weil, A., 1977. Abelian varieties and Hodge ring. In: Collected Papers, Vol. III, Springer, Heidelberg, 421–429.

Weyl, H., 1946. The Classical Groups, Princeton University Press, (2nd Edition).

White, S.P., 1993. Sporadic cycles on CM abelian varieties, Compositio Math. 88, 123–142.

Zarhin, Yu. G., 1991. Abelian varieties of $K3$ type and ℓ-adic representations, In: Algebraic Geometry and Analytic Geometry, Tokyo 1990, Springer, Tokyo, 231–255.

Zarhin, Yu. G., 1994. The Tate conjecture for non-simple abelian varieties over finite fields. In: Algebra and Number Theory, de Gruyter, 1994, 267–296.

Mathematics Department, University of Michigan, Ann Arbor, MI 48109.

E-mail address: `jmilne@umich.edu`