# TOWARDS A PROOF OF THE CONJECTURE OF LANGLANDS AND RAPOPORT

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## CONTENTS

## 1. INTRODUCTION

A reductive group $G$ over $\mathbb{Q}$ plus a $G(\mathbb{R})$-conjugacy class $X$ of homomorphisms $\mathbb{C}^\times \to G(\mathbb{R})$ satisfying certain axioms (Deligne 1979) defines a Shimura variety $\mathrm{Sh}(G, X)$, which is the projective system of the double coset spaces

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K,$$

with $K$ running over the compact open subgroups of $G(\mathbb{A}_f)$. The axioms imply that the $X$ has the structure of a disjoint union of bounded symmetric domains, and $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$ is a disjoint union of spaces of the form $\Gamma \backslash X^+$ with $X^+$ a connected component of $X$ and $\Gamma$ a congruence subgroup of $G^{\mathrm{der}}(\mathbb{R})$, and so $\mathrm{Sh}(G, X)$ is a projective system of analytic spaces. The theorem of Baily and Borel shows that $\mathrm{Sh}(G, X)$ is a projective system of algebraic varieties (not connected!) over $\mathbb{C}$. A theorem of Shimura, Deligne, et al. shows that $\mathrm{Sh}(G, X)$ has a canonical model over a certain number field $E(G, X)$, called the reflex field. Thus, we may reduce $\mathrm{Sh}(G, X)$ modulo a prime ideal $\mathfrak{p}_v$ of $E(G, X)$, for example, by embedding each $\mathrm{Sh}_K(G, X)$ in projective space and then scaling and reducing the equations modulo $\mathfrak{p}_v$, to obtain a projective family of varieties over the residue field $\kappa(v)$.

However, without any further conditions on $G$ and $X$, the reduced varieties may be very singular. To avoid this, we assume that a hyperspecial group $K_p$ has been given, and consider the system

$$\mathrm{Sh}_p(G, X) = \{\mathrm{Sh}_{K^p \cdot K_p}(G, X) \mid K^p \text{ compact open in } G(\mathbb{A}_f^p)\}.$$

Here $\mathbb{A}_f^p$ is the ring of finite adèles with the $p$-component omitted. The existence of $K_p$ implies that $G$ is unramified over $\mathbb{Q}_p$, and by considering only groups of the form $K^p \cdot K_p$ we are, in effect, imposing a level structure only away from $p$.

When we reduce modulo a prime $\mathfrak{p}_v$ dividing $p$, we obtain pro-variety $\mathrm{Sh}_p(G, X)_v$ over $\kappa(v)$. We write $\mathrm{Sh}_p(\mathbb{F})$ for the set of its points in the algebraic closure $\mathbb{F}$ of $\kappa(v)$. This is a set with an action of the Frobenius generator of $\mathrm{Gal}(\mathbb{F}/\kappa(v))$ and $G(\mathbb{A}_f^p)$. The conjecture of Langlands and Rapoport gives a description $\mathrm{Sh}_p(\mathbb{F})$, together with the two actions, directly in terms of the initial data $G, X, K_p$.

## 2. The Different Types of Shimura Varieties

We shall look at the conjecture in three cases. Generally, I shall regard abelian varieties as lying in the category of abelian varieties up to isogeny, i.e., the category whose objects are the abelian varieties but in which the Hom-sets have been tensored with $\mathbb{Q}$.

*Shimura varieties of PEL-type.* For an appropriate choice of a representation $G \hookrightarrow \mathrm{GL}(V)$, Shimura varieties of PEL-type become moduli varieties[1] in characteristic zero, namely, their points classify isomorphism classes of triples $(A, \eta^p, \Lambda_p)$ where $A$ is an abelian variety endowed with a polarization and an action of a fixed $\mathbb{Q}$-algebra $B$, $\eta^p$ is a prime-to-$p$ level structure on $A$, and $\Lambda_p$ is a lattice in $H_1(A_{\mathrm{et}}, \mathbb{Q}_p)$. The triples are required to satisfy certain conditions, for example, that the representation of the $\mathbb{Q}$-algebra $B$ on the tangent space to $A$ at zero lies in a fixed isomorphism class. An isomorphism of triples is an isogeny of abelian varieties (element of $\mathrm{Hom}(A, A') \otimes \mathbb{Q}$ with an inverse in $\mathrm{Hom}(A', A) \otimes \mathbb{Q}$) preserving all the structure.

*Shimura varieties of Hodge type.* This class has description similar to the preceding class, except that now (in characteristic zero) $\mathrm{Sh}_p(G, X)$ is the moduli variety for triples $(A, \eta^p, \Lambda_p)$ where $A$ is an abelian variety endowed with some Hodge classes (in the sense of Deligne 1982).

*Shimura varieties of abelian type.* This is the almost-general case, since it excludes only the Shimura varieties defined by groups with factors of type $E_6$, $E_7$, and certain mixed types $D$. Associated with the datum defining the Shimura variety there is a "weight" homomorphism $w_X \colon \mathbb{G}_m \to G$. When $w_X$ is defined over $\mathbb{Q}$, then (in characteristic zero) the choice of a representation $G \hookrightarrow \mathrm{GL}(V)$ realizes $\mathrm{Sh}_p(G, X)$ as the moduli variety for triples $(M, \eta^p, \Lambda_p)$ where $M$ is now a motive rather than an abelian variety (Milne 1994). When $w_X$ is not defined over $\mathbb{Q}$, then $Sh_p(G, X)$ is not a moduli variety.

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[1]Strictly, I should say pro-variety...

*Comments.* There are many more Shimura varieties of abelian type than of Hodge type, and many more of Hodge type than of PEL-type. The case of Hodge type is always a useful stepping stone to the almost-general case. The PEL case is included only because it is so much easier than the other cases, and studying it gives a guide to how to proceed in the other cases.

Surprisingly, the study of Shimura varieties of Hodge type turns out to be much more difficult than that of those PEL-type for two reasons. First, multilinear algebra is more difficult than linear algebra. For example, as I mentioned above, one of the conditions for a triple $(A, \eta^p, \Lambda_p)$ to lie in the family parametrized by $\mathrm{Sh}_p(G, X)$ is that the representation of the algebra $B$ on the tangent space be of a fixed isomorphism type. Stating such a condition for tensors in some spaces is more difficult. The second reason is that Deligne's Hodge classes are defined only in characteristic zero.

## 3. The statement of the conjecture of Langlands and Rapoport

Langlands and Rapoport first "define" a category of motives $\mathbf{Mot}(\mathbb{F})$ over $\mathbb{F}$ — the reason for the quotes will be explained below.

Assume that the weight $w_X$ for $\mathrm{Sh}_p(G, X)$ is defined over $\mathbb{Q}$. Then, the above discussion suggests that, if we fix a representation $G \hookrightarrow \mathrm{GL}(V)$, then there should be a one-to-one correspondence the points in $\mathrm{Sh}_p(\mathbb{F})$ and a set of isomorphism classes of triples $(M, \eta^p, \Lambda_p)$ with $M$ a motive over $\mathbb{F}$, $\eta^p$ a prime-to-$p$ level structure on $M$, and $\Lambda_p$ a lattice in $H^*_{\mathrm{crys}}(M)$.

When, we vary the representation of $G$, then this should become a one-to-one correspondence between $\mathrm{Sh}_p(\mathbb{F})$ and a set of isomorphism classes of triples $(\underline{M}, \eta^p, \Lambda_p)$ where $\underline{M}$ is now a functor $\mathbf{Rep}(G) \to \mathbf{Mot}(\mathbb{F})$.

When we choose a fibre functor $\omega$ for $\mathbf{Mot}(\mathbb{F})$ and let $\mathfrak{P}$ be the corresponding groupoid, $\mathfrak{P} =_{\mathrm{df}} \mathrm{Aut}^\otimes_\mathbb{Q}(\omega)$, then the theory of Tannakian categories shows that to give an $\underline{M}$ is the same as to give a morphism of groupoids $\phi \colon \mathfrak{P} \to \mathfrak{G}_G$.

Finally, Langlands and Rapoport define another groupoid $\mathfrak{Q}$ having $\mathfrak{P}$ as a quotient, to allow for the weight to be irrational. They define a set of triples $(\phi, \eta^p, \Lambda_p)$ (depending only on $G, X, K_p$) where $\phi$ is now a homomorphism $\mathfrak{Q} \to \mathfrak{G}_G$ and their conjecture states that the elements of $\mathrm{Sh}_p(\mathbb{F})$ should be in one-to-one correspondence with the set $LR(\mathbb{F})$ of isomorphism classes of these triples. There is a natural action of the Frobenius automorphism and of $G(\mathbb{A}^p_f)$ on the triples, and the correspondence should respect these actions.

REMARK 3.1. Of course, it is easy to guess that somehow a Shimura variety modulo a prime $\mathfrak{p}_v$ should parametrize isomorphism classes of motives with additional structure. The point of the paper of Langlands and Rapoport is to define a category of motives $\mathbf{Mot}(\mathbb{F})$ and then to state precise conditions on the triples $(\phi, \eta^p, \Lambda_p)$ that are to correspond to a given Shimura variety.

REMARK 3.2. I will discuss the definition of $\mathbf{Mot}(\mathbb{F})$ below. The statement of the precise conditions on the triples $(\phi, \eta^p, \Lambda_p)$ is quite complicated, especially that on $\Lambda_p$, and I refer the reader to the original paper or Milne 1992 for these. Here I will only make a few comments.

Langlands and Rapoport defined the notion of an *admissible* homomorphism $\phi \colon \mathfrak{Q} \to \mathfrak{G}_G$. To be admissible, a homomorphism must satisfy one condition for each

prime $l$ (including $l = \infty$) and a condition on the composite of $\phi$ with $\mathfrak{G}_G \to \mathfrak{G}_{G/G^{\text{der}}}$.

A special point in $X$ defines a homomorphism $\phi \colon \mathfrak{Q} \to \mathfrak{G}_G$, called *special*. Langlands and Rapoport show that, when $G^{\text{der}}$ is simply connected, a homomorphism is admissible if and only if it is isomorphic to a special homomorphism. They also show that if their conjecture is correct for groups with $G^{\text{der}}$ simply connected, then it can't be correct without this condition (so they stated their conjecture only for $G^{\text{der}}$ simply connected).

I showed, on the other hand, that if one replaces "admissible" with the condition "is isomorphic to a special homomorphism" then the conjecture is true for all Shimura varieties when it is true for those with $G^{\text{der}}$ simply connected. This allows us to state the conjecture for all Shimura varieties and (to some extent) reduces the problem of its proof to the case where $G^{\text{der}}$ is simply connected.

REMARK 3.3. The description given by the conjecture of Langlands and Rapoport may look too abstract and complicated to be of use, but, in fact, it fairly straightforward to derive formulas for the numbers of points in terms of orbital and twisted orbital integrals (as conjectured by Langlands and Kottwitz) from it. This is explained in Milne 1992.

*From now on, I will assume that $G^{\text{der}}$ simply connected.*

## 4. Improvements to the Statement of the Conjecture

*Canonical integral models.* One defect of the original conjecture is that it doesn't specify how to reduce the Shimura variety. Defining a reduction amounts to defining a model of $\text{Sh}_p(G, X)$ over the ring of integers in $E_v$. Evidently, if the conjecture is true for one integral model, it will be false for most others. I suggested (Milne 1992) that there should be a canonical integral model characterized by a certain Néron-type property. The existence of such a model has been proved by Vasiu for $p \geq 5$ (Vasiu 1999).

To the original conjecture, one should add that the reduction is that defined by the canonical integral model.

*The definition of* $\mathbf{Mot}(\mathbb{F})$. The Tate conjecture implies that the category of motives over $\mathbb{F}$ should be Tannakian with a certain specific protorus $P$ as its band. The Tannakian categories with $P$ as band are classified up to $P$-equivalence by the cohomology group $H^2(\mathbb{Q}, P)$, and Langlands and Rapoport showed that there is only one class in $H^2(\mathbb{Q}, P)$ giving a Tannakian category for which the correct fibre functors exist. They define $\mathbf{Mot}(\mathbb{F})$ to be any Tannakian category with band $P$ having this cohomology class. Thus, $\mathbf{Mot}(\mathbb{F})$ is only defined up to a nonunique $P$-equivalence. Because some $H^1$'s vanish, the category is a little better defined than one might expect but it still not possible to talk of objects in $\mathbf{Mot}(\mathbb{F})$.

For example, let $M_1$ be one model for $\mathbf{Mot}(\mathbb{F})$ and let $X$ be an object of $M_1$. If $M_2$ is a second model, then there is a $P$-equivalence $F \colon M_1 \to M_2$, and so $X$ corresponds to an object $FX$ in $M_2$. But, there is no special $F$, and if $F'$ is second $P$-equivalence $M_1 \to M_2$, then $X$ will correspond to a second object $F'X$ of $M_2$. The objects $FX$ and $F'X$ will be isomorphic, but there is no preferred isomorphism. Thus, all one

say is that isomorphism classes of objects in $M_1$ correspond to isomorphism classes in $M_2$. This is scarcely better than the information provided by the Honda-Tate theorem on the category of abelian varieties up to isogeny over $\mathbb{F}$: it classifies only the isomorphism classes and their endomorphism algebras.

Below I shall provide a more precise definition of $\mathbf{Mot}(\mathbb{F})$ for which, in the above discussion, $FX$ and $F'X$ will isomorphic with a *unique* isomorphism. In other words, the object in $M_2$ corresponding to $X$ in $M_1$ will be well-defined up to a unique isomorphism.

This more precisely defined category (which gives a $\mathfrak{P}$ and $\mathfrak{Q}$) should be the one used in the statement of the conjecture.

*Canonicalness.* Once one has the notion of a canonical integral model, the set with operators $\mathrm{Sh}_p(\mathbb{F})$ is *canonically* associated with $G, X, K_p$. Moreover, once one chooses a fibre functor on $\mathbf{Mot}(\mathbb{F})$, the set $LR(\mathbb{F})$ is also *canonically* associated with $G, X, K_p$. Clearly, one should require that the one-to-one correspondence in the statement of the conjecture of Langlands and Rapoport *be canonical.*

In fact, I expect one can prove a uniqueness statement of the following form: there is at most one family of bijections $LR(G, X, K_p)(\mathbb{F}) \to Sh_p(G, X)(\mathbb{F})$ having certain functoriality properties and giving the correct map for the Siegel modular variety.

Henceforth, by the conjecture of Langlands and Rapoport I mean the canonical conjecture.

## 5. The Conditional Proof of Langlands and Rapoport in the PEL-case

In their paper (1987), Langlands and Rapoport prove their conjecture for Shimura varieties of PEL-type under the assumption of:

(a) the Hodge conjecture for complex abelian varieties of CM-type;
(b) the Tate conjecture for abelian varieties over $\mathbb{F}$;
(c) Grothendieck's standard conjectures for abelian varieties over $\mathbb{F}$.

One of the difficulties is that their abstractly-defined category $\mathbf{Mot}(\mathbb{F})$ does not contain the category of abelian varieties (up to isogeny) in any natural way. When one assumes (b), one gets a well-defined category of motives containing the category abelian varieties (up to isogeny), namely the category of motives based on abelian varieties using the algebraic classes modulo numerical equivalence as correspondences.

Another difficulty is that Deligne's Hodge classes make sense only in characteristic zero. Let $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ be the category of motives based on abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ of CM-type using the Hodge classes as correspondences. When (a) is assumed, the Hodge classes will be algebraic, and therefore will reduce to algebraic classes. We then get a canonical functor $R\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}(\mathbb{F})$.

In summary, assuming (a) and (b) we get a canonical commutative diagram

$$
\begin{array}{ccc}
\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) & \overset{I}{\leftarrow} & \mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) \\
\downarrow R & & \downarrow R \\
\mathbf{Mot}(\mathbb{F}) & \overset{I}{\leftarrow} & \mathbf{LMot}(\mathbb{F})
\end{array}
$$

where $\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})$ is the category of motives based on abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$ using the Lefschetz classes[2] as correspondences, and $\mathbf{LMot}(\mathbb{F})$ is the similar category based on abelian varieties over $\mathbb{F}$.

Finally, recall that a Weil form on an object $X$ of a Tannakian category is a form (bilinear or sesquilinear according to context) that induces a positive involution on $\mathrm{End}(X)$, and that to give a polarization on a Tannakian category is to give a distinguished class of "positive" Weil forms for each object satisfying certain compatibility conditions. A polarization of an abelian variety $A$ in the sense of algebraic geometry defines a Weil form on $h_1 A$. Grothendieck's standard conjectures imply that there is a polarization on $\mathbf{Mot}(\mathbb{F})$ for which these geometric Weil forms are all positive.

With the assumption of (a), (b), (c), the proof of the conjecture of Langlands and Rapoport for Shimura varieties of PEL-type becomes fairly straightforward. The canonical integral model is, in this case, a moduli variety, and so a point in $\mathrm{Sh}_p(\mathbb{F})$ corresponds to an isomorphism class of triples $(A, \lambda^p, \Lambda_p)$ with $A$ an abelian variety endowed with a polarization and an action of a $\mathbb{Q}$-algebra $B$. The object $I(h_1 A)$ then defines (by the theory of Tannakian categories) a morphism $\phi_A \colon \mathfrak{P} \to \mathfrak{G}_G$, and Langlands and Rapoport verify that there is a canonical bijection between the set $S(A)$ of isomorphism classes of triples $(A, \lambda^p, \Lambda_p)$ (fixed $A$) and the set of isomorphism classes of triples $(\phi_A, \lambda^p, \Lambda_p)$ (fixed $\phi_A$). The abelian variety $A$ with its PE-structure is, almost by definition, the reduction (up to isogeny) of an abelian variety $\tilde{A}$ with PE-structure in the family parametrized by $\mathrm{Sh}_p$ in characteristic zero. A theorem of Zink's shows that $\tilde{A}$ can be chosen to be of CM-type. It therefore corresponds to a special point $x$ of $X$, and one verifies that $\phi_x \approx \phi_A$. Thus $\phi_A$ is admissible.

## 6. Towards an unconditional proof in the PEL-case

We wish to carry out the above argument without assuming (a), (b), or (c). In Milne 1999, I showed that (a) implies (b), and I can show that (a) implies at least the consequence of (c) needed for the above proof. Thus, instead of three conjectures we need to assume only one, namely, the Hodge conjecture for abelian varieties of CM-type. Unfortunately, the meagre progress made on the Hodge conjecture in the 50 years since the conjecture was made suggests that it will not be wise to wait for a proof of the Hodge conjecture, even for abelian varieties of CM-type.

First I explain my new construction of $\mathbf{Mot}(\mathbb{F})$. The conjecture (a) implies that we have a functor $R \colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}(\mathbb{F})$ bound by a map $P \hookrightarrow S$ of pro-tori. The group $P$ acts on the objects of $\mathbf{Mot}(\mathbb{F})$ and we let $\mathbf{Mot}(\mathbb{F})^P$ be the subcategory of objects on which $P$ acts trivially. Thus $\mathbf{Mot}(\mathbb{F})^P$ comprises the motives consisting entirely of algebraic classes. Let $\mathbb{1}$ be an identity object of $\mathbf{Mot}(\mathbb{F})$. Then $X \mapsto \mathrm{Hom}(\mathbb{1}, X)$ is a fibre functor $\mathbf{Mot}(\mathbb{F})^P \to \mathbf{Vec}_{\mathbb{Q}}$. Its composite with $R$ is a fibre functor $\omega_0$ on $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$, and I claim we can (essentially) reverse this procedure and reconstruct $\mathbf{Mot}(\mathbb{F})$ from $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ and $\omega_0$. I now drop all assumptions.

First, one shows that there is a fibre functor $\omega_0$, unique up to isomorphism, that when tensored with $\mathbb{Q}_l$ is in the "correct" isomorphism class for all $l \leq \infty$. Now define $\mathbf{Mot}(\mathbb{F})'$ as follows:

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[2] A Lefschetz class is an element of the $\mathbb{Q}$-algebra generated by divisor classes inside the $\mathbb{Q}$-algebra of algebraic classes modulo numerical equivalence (or inside a Weil cohomology — there is no difference for abelian varieties).

     – $\mathbf{Mot}(\mathbb{F})'$ has one object $\bar{X}$ for each object $X$ of $\mathbf{Mot}(\mathbb{F})$;

     – for objects $\bar{X}$, $\bar{Y}$ of $\mathbf{Mot}(\mathbb{F})$, define $\mathrm{Hom}(\bar{X}, \bar{Y}) = \omega_0(\underline{\mathrm{Hom}}(X, Y)^P)$.

Here $\underline{\mathrm{Hom}}(X, Y)$ is the internal Hom of $X$ and $Y$ in $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$, and $\underline{\mathrm{Hom}}(X, Y)^P$ is the largest subobject fixed by $P$. Now $\mathbf{Mot}(\mathbb{F})$ is obtained from $\mathbf{Mot}(\mathbb{F})'$ by adding the images of projectors, i.e., by taking the pseudo-abelian (Karoubian) hull. It is only an exercise, using the dictionary between Tannakian categories and gerbs, to show that this does gives a Tannakian category and that $X \mapsto \bar{X}$ defines a tensor functor $R\colon \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}(\mathbb{F})$ bound by $P \hookrightarrow S$.

Because $\omega_0$ is uniquely determined up to an isomorphism, which itself is determined up to a *unique* isomorphism, $\mathbf{Mot}(\mathbb{F})$ has the uniqueness property claimed.

To give a fibre functor on $\mathbf{Mot}(\mathbb{F})$ is to give a fibre functor $\omega$ on $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$ together with an isomorphism $\omega_0 \to \omega \circ R$. In this way, one obtains fibre functors $\omega_\ell\colon \mathbf{Mot}(\mathbb{F}) \to \mathbf{Vec}_{\mathbb{Q}_\ell}$ for each $\ell \neq p$, and a functor $\omega_p\colon \mathbf{Mot}(\mathbb{F}) \to Isoc(\mathbb{F})$, well-defined to isomorphism.

We thus have:

$$
\begin{array}{ccc}
\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) & \overset{I}{\leftarrow} & \mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) \\
\downarrow R & & \downarrow R \\
\mathbf{Mot}(\mathbb{F}) & & \mathbf{LMot}(\mathbb{F})
\end{array}
$$

The subcategory of $\mathbf{Mot}(\mathbb{F})$ of objects of weight 1 is certainly equivalent to the category of abelian varieties up to isogeny over $\mathbb{F}$, and it follows one does get a functor $I\colon \mathbf{LMot}(\mathbb{F}) \to \mathbf{Mot}(\mathbb{F})$. One can even choose it so that $\omega_l^M \circ I \approx \omega_l^{LM}$ for all $l$. However, without something extra, $I$ will not be canonical and the diagram may not quite commute (its failure to commute is measured by a class in $H^1(\mathbb{Q}, T)$, $T$ the fundamental group of $\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})$, that is trivial at all the finite primes).

On applying the method of Langlands and Rapoport described above, one obtains a description of $\mathrm{Sh}_p(\mathbb{F})$ as the set of isomorphism classes of triples $(\phi, \eta^p, \Lambda_p)$ exactly as conjectured, except that the $\phi$ need not be admissible (each $\phi$ may be a twist of an admissible $\phi$ by a cohomology class which may be chosen to come from the centre of $I_\phi$ and split at all the finite primes). Thus, one doesn't obtain the canonical LR conjecture. For that, one needs the following conjecture.

CONJECTURE 6.1 (A). *Let $A$ be an abelian variety of CM-type over $\mathbb{Q}^{al}$ (say), and let $\alpha$ be a Hodge class on $A$ (thus $\alpha$ lies in a certain $\mathbb{Q}$-vector space). Such an $\alpha$ defines cohomology classes $\alpha_l$ for all $l$ (in $H^{2*}(A_{et}, \mathbb{Q}_\ell(*))$ for $\ell \neq p$ and in $H^{2*}_{dR}(A)$ for $\ell = p$). Each $\alpha_l$ defines a cohomology class $\bar{\alpha}_l$ on the reduction $A_\mathbb{F}$ of $A$ ($\bar{\alpha}_p$ lies in the crystalline cohomology). Let $\alpha_f = (\alpha_l)$ and $\bar{\alpha}_f = (\bar{\alpha}_l)$.*

*If $\bar{\alpha}_f$ is in $\mathbb{A}_f$-span of the Lefschetz classes on $A_\mathbb{F}$, then it is a Lefschetz class (i.e., it is in the $\mathbb{Q}$-space of such classes).*

*Equivalently, if $\bar{\alpha}_l$ is in the $\mathbb{Q}_l$-span of the Lefschetz classes for all $l$, then $\bar{\alpha}_l$ is the cohomology class of a Lefschetz class for all $l$, which is independent of $l$.*

Roughly speaking, the conjecture says that Hodge classes on $A$ that look as though they should become Lefschetz on $A_\mathbb{F}$, do in fact become Lefschetz. It is a compatibility conjecture between a $\mathbb{Q}$-structure in characteristic zero and a $\mathbb{Q}$-structure in characteristic $p$.

When Conjecture A is assumed, $\mathbf{Mot}(\mathbb{F})$ becomes well-defined up to a unique equivalence, the diagram commutes, there are well-defined functors $\omega_l$ on $\mathbf{Mot}(\mathbb{F})$ composing correctly with $R$ and $I$, and there is a unique polarization on $\mathbf{Mot}(\mathbb{F})$ for which the Weil forms coming from algebraic geometry are positive. Thus, the situation is essentially as good as when one assumes (a), (b), (c), and the argument in Langlands and Rapoport does give a proof of the canonical form of their conjecture for Shimura varieties of PEL-type.

I hope that Conjecture A is susceptible to proof by the same methods that Deligne used to prove his result on Hodge classes. Specifically, I can show that it is true when $\alpha$ is algebraic (and so the conjecture is implied by the Hodge conjecture for abelian varieties of CM-type). Moreover, I can show that there is a subgroup $G$ of the Lefschetz group of $A$ such that a Hodge class $\alpha$ becomes ($\mathbb{Q}$-rationally) Lefschetz on $A_{\mathbb{F}}$ if and only if $\alpha$ is fixed by $G$. The next step will be to prove the statement for Hodge classes defined by Weil. As in SLN 900, an abelian variety with a space of Weil classes deforms smoothly in characteristic zero to a power of an elliptic curve, on which all Hodge classes are Lefschetz. The problem is that the family may not reduce smoothly to $\mathbb{F}$. The final step will be to show that this gives enough classes for which the conjecture is true that the group fixing them is the correct one.

## 7. The case of Shimura varieties of Hodge type

From now on I assume Conjecture A — it seems to me essential to have such a statement to obtain the canonical form of the conjecture Langlands and Rapoport, even for Shimura varieties of PEL-type.

Then I can prove that, for a Shimura variety of Hodge type, there is a canonical injection

$$LR(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$$

compatible with the actions of $G(\mathbb{A}_f^p)$ and the Frobenius automorphism.

The main difficulty in proving this statement involves the lattices in the $p$-cohomology. In characteristic zero, they lie in the $p$-adic étale cohomology, and in characteristic $p$, they lie in the crystalline cohomology. Fortunately, the relation between these cohomologies is now rather well understood, especially in the case of good reduction. The proof of the statement uses theorems of Blasius and Wintenberger, Wintenberger, and Fontaine and Messing.

I now need to assume another statement (Conjecture 0.1 of Milne 1995), which is proved in a manuscript of Vasiu (.... Part 2A). As of writing, the proof of Vasiu has not been checked. Appeal to Vasiu's paper can be avoided (I think) if one extends Deligne's theorem on Tannakian categories (that any two fibre functors are locally isomorphic) from Tannakian categories over fields to Tannakian categories over Dedekind domains. (I have no idea whether such an extension is possible, or even true, but it would be of considerable interest if it is).

Assuming this, the map $LR(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$ is surjective if and only if the following conjecture holds:

CONJECTURE 7.1 (B). *Zink's theorem holds for Shimura varieties of Hodge type.*

[[Restate in terms of Mumford-Tate groups. Equivalent statement that Hodge classes on abelian varieties (not necessarily of CM-type) reduce to rational Tate classes. Hence Conjecture B is implied by the Hodge conjecture for abelian varieties.]]

## 8. The case of Shimura varieties of abelian type.

Happily, the extension from Hodge type to abelian type has been taken care of by Pfau. Specifically, he shows that if a "refined" form of the conjecture of Langlands and Rapoport holds for Shimura varieties of Hodge type, then the same form of the conjecture holds for all Shimura varieties of abelian type.

To state the refined form of the conjecture, he defines a map $LR(\mathbb{F}) \to \pi_0(\mathrm{Sh}_p)$. Here $\pi_0(*)$ denotes the set of connected components of $*$. The refined form of the conjecture then states that there is a bijection $LR(\mathbb{F}) \to \mathrm{Sh}_p(\mathbb{F})$ compatible with the maps to $\pi_0(\mathrm{Sh}_p)$ (and the actions of the Frobenius automorphism and $G(\mathbb{A}_f^p)$).

Unfortunately, rather than a single well-defined map $LR(\mathbb{F}) \to \pi_0(\mathrm{Sh}_p)$ Pfau defines only a distinguished class of maps.

I claim that one gets a canonical such map, almost for free. For a Shimura variety $\mathrm{Sh}(G, X)$, let $T = G/G^{\mathrm{der}}$ and let $\bar{X} = T(\mathbb{R})/Im(Z(\mathbb{R}))$ where $Z$ is the centre of $G$. The image of $Z(\mathbb{R})$ in $T(\mathbb{R})$ contains the identity component, and so $\bar{X}$ is finite. Define $\mathrm{Sh}(T, \bar{X})$ to be the system $\{T(\mathbb{Q})\backslash \bar{X} \times G(\mathbb{A}_f)/K\}$ with $K$ running through the compact open subgroups of $T(\mathbb{A}_f)$. This is not a Shimura variety in the sense of Deligne's original definition, but Pink has pointed out that the study of the boundaries of Shimura varieties suggests that Deligne's definition be extended to allow $X$ to be finite covering of a conjugacy class of maps $\mathbb{C}^\times \to G(\mathbb{R})$. For this extended definition, $\mathrm{Sh}(T, \bar{X})$ is a Shimura variety. Now (under our continuing assumption that $G^{\mathrm{der}}$ is simply connected), $\pi_0(\mathrm{Sh}(G, X)) = \mathrm{Sh}(T, \bar{X})$.

The definition Langlands and Rapoport extends easily to give a set $LR(T, \bar{X})(\mathbb{F})$ and it is easy to prove the conjecture in this case: there is a canonical bijection $LR(T, \bar{X})(\mathbb{F}) \to Sh_p(T, \bar{X})(\mathbb{F})$. The canonical Langlands-Rapoport conjecture for $\mathrm{Sh}_p(G, X)$ will give a commutative diagram

$$
\begin{array}{ccc}
LR(G, X)(\mathbb{F}) & \to & \mathrm{Sh}_p(G, X)(\mathbb{F}) \\
\downarrow & & \downarrow \\
LR(T, \bar{X})(\mathbb{F}) & \to & \mathrm{Sh}_p(T, \bar{X})(\mathbb{F}).
\end{array}
$$

Using that $\mathrm{Sh}_p(T, \bar{X})(\mathbb{F}) = \pi_0(\mathrm{Sh}_p(G, X))$, we see that this implies Pfau's refined form of the conjecture. Now, Pfau's arguments show that the canonical Langlands-Rapoport conjecture for Shimura varieties of Hodge type implies the same conjecture for Shimura varieties of abelian type.

## References

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