Polarizations and Grothendieck's Standard Conjectures

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Abstract. We prove that Grothendieck's Hodge standard conjecture holds for abelian varieties in arbitrary characteristic if the Hodge conjecture holds for complex abelian varieties of CM-type. For abelian varieties with no exotic algebraic classes, we prove the Hodge standard conjecture unconditionally.

Contents

Introduction. In examining Weil's proofs (Weil 1948) of the Riemann hypothesis for curves and abelian varieties over finite fields, Grothendieck was led to state two "standard" conjectures (Grothendieck 1969), which imply the Riemann hypothesis for all smooth projective varieties over a finite field, essentially by Weil's original argument. Despite Deligne's proof of the Riemann hypothesis, the standard conjectures retain their interest for the theory of motives.

The first, the *Lefschetz standard conjecture* (Grothendieck 1969, §3), states that, for a smooth projective variety V over an algebraically closed field, the operators Λ rendering commutative the diagrams ($0 \leq r \leq 2n$, $n = \dim V$)

$$\begin{array}{ccc} H^{r}(V) & \xrightarrow[\approx]{L^{n-r}} & H^{2n-r}(V) \\ \downarrow{\scriptstyle\Lambda} & & \downarrow{\scriptstyle L} \\ H^{r-2}(V) & \xrightarrow[\approx]{L^{n-r+2}} & H^{2n-r+2}(V) \end{array}$$

are algebraic. Here H is a Weil cohomology theory and L is cup product with the class of a smooth hyperplane section (L^{n-r} is assumed to be an isomorphism for

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$n \geq r$, and $L^{n-r} = (L^{r-n})^{-1}$ for $n < r$). This conjecture is known for abelian varieties (Lieberman 1968, Kleiman 1968), surfaces and Weil cohomologies for which $\dim H^1(V) = 2 \dim \mathrm{Pic}^0(V)$ (Grothendieck), and a few other varieties (see Kleiman 1994, 4.3). For abelian varieties, it is even known that the operator Λ is defined by a Lefschetz class, i.e., a class in the $\mathbb{Q}$-algebra generated by divisor classes (Milne 1999a, 5.9).

The second, the *Hodge standard conjecture* (Grothendieck 1969, §4), states that, for $r \leq n/2$, the bilinear form

$$(x, y) \mapsto (-1)^r \langle L^{n-2r} x \cdot y \rangle \colon P^r(V) \times P^r(V) \to \mathbb{Q}$$

is positive-definite. Here $P^r(V)$ is the $\mathbb{Q}$-space of primitive algebraic classes of codimension r modulo homological equivalence. In characteristic zero, $\mathrm{Hdg}(V)$ is a consequence of Hodge theory (Weil 1958). In nonzero characteristic, $\mathrm{Hdg}(V)$ is known for surfaces (Segre 1937; Grothendieck 1958). An important consequence of the Hodge standard conjecture for abelian varieties, namely, the positivity of the Rosati involution, was proved in nonzero characteristic by Weil (1948, Théorème 38). Apart from these examples and the general coherence of Grothendieck's vision, there appears to have been little evidence for the conjecture in nonzero characteristic.

In this paper, we prove that the Hodge standard conjecture holds for abelian varieties in arbitrary characteristic if the Hodge conjecture holds for complex abelian varieties of CM-type.

Let $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$ be the category of motives based on abelian varieties over $\mathbb{F}$ using the numerical equivalence classes of algebraic cycles as correspondences. This is a Tannakian category (Jannsen 1992, Deligne 1990), and it is known that the Tate conjecture for abelian varieties over finite fields implies that it has all the major expected properties but one, namely, that the Weil forms coming from algebraic geometry are positive for the canonical polarization on $\mathbf{Mot}(\mathbb{F}; \mathcal{A})$ (see Milne 1994, especially 2.47).

In Milne 1999b it is shown that the Hodge conjecture for complex abelian varieties of CM-type is stronger than (that is, implies) the Tate conjecture for abelian varieties over finite fields. Here, we show that the stronger conjecture also implies the positivity of the Weil forms coming from algebraic geometry (Theorem 2.1). As a consequence, we obtain the Hodge standard conjecture for abelian varieties over finite fields, and a specialization argument then proves it over any field of nonzero characteristic (Theorem 3.3).

Most of the arguments in the paper hold with "algebraic cycle" replaced by "Lefschetz cycle". In fact, the analogue of the Hodge standard conjecture holds unconditionally for Lefschetz classes on abelian varieties. In particular, the Hodge standard conjecture is true for abelian varieties without exotic (i.e., non-Lefschetz) algebraic classes (3.7, 3.8).

In preparation for proving these results, we study in §1 the polarizations on a quotient Tannakian category.

Notations and Conventions. The algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted $\mathbb{Q}^{\mathrm{al}}$. We fix a p-adic prime on $\mathbb{Q}^{\mathrm{al}}$ and denote its residue field by $\mathbb{F}$.

By the Hodge conjecture for a variety V over $\mathbb{C}$, we mean the statement that, for all r, the $\mathbb{Q}$-space $H^{2r}(V,\mathbb{Q}) \cap H^{r,r}$ is spanned by the classes of algebraic cycles.

By the Tate conjecture for a variety V over a finite field $\mathbb{F}_q$ we mean the statement that, for all r, the order of the pole of the zeta function $Z(V,t)$ at $t = q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r on V (Tate 1994, 2.9). We say that a variety over $\mathbb{F}$ satisfies the Tate conjecture if all of its models over finite fields satisfy the Tate conjecture (equivalently, one model over a "sufficiently large" finite field).

For abelian varieties A and B, $\mathrm{Hom}(A,B)_{\mathbb{Q}} = \mathrm{Hom}(A,B) \otimes \mathbb{Q}$. An abelian variety A over $\mathbb{C}$ (or $\mathbb{Q}^{\mathrm{al}}$) is said to be of CM-type if, for each simple isogeny factor B of A, $\mathrm{End}(B)_{\mathbb{Q}}$ is a commutative field of degree $2\dim B$ over $\mathbb{Q}$. A polarization of A is the isogeny $A \to A^{\vee}$ from A to its dual defined by an ample divisor on A.

Let $\mathcal{S}$ be a set of smooth projective varieties over an algebraically closed field k satisfying the following condition:

> **(0.1):** the projective spaces are in $\mathcal{S}$ and $\mathcal{S}$ is closed under passage to a connected component and under the formation of products and disjoint unions.

For example, $\mathcal{S}$ could be the class $\mathcal{T}$ of all smooth projective varieties over k or the smallest class $\mathcal{A}$ satisfying (0.1) and containing the abelian varieties. Then $\mathbf{Mot}(k;\mathcal{S})$ is defined to be the category of motives based on the abelian varieties over k with the algebraic classes modulo numerical equivalence as correspondences.

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1. Polarizations on quotient categories

We refer the reader to Deligne 1989, §§5,6, for the theory of algebraic geometry in a Tannakian category $\mathbf{C}$. In particular, the fundamental group $\pi(\mathbf{C})$ of $\mathbf{C}$ is an affine group scheme in $\mathbf{C}$, such that, for any fibre functor ω on $\mathbf{C}$,

$$\underline{\mathrm{Aut}}^{\otimes}(\omega) \cong \omega(\pi(\mathbf{C})).$$

The fundamental group acts on the objects of $\mathbf{C}$. When H is a closed subgroup of $\pi(\mathbf{C})$, we let X^H denote the largest subobject of X on which the action of H is trivial, and we let $\mathbf{C}^H$ denote the full subcategory of $\mathbf{C}$ of objects on which the action of H is trivial. The functor $\mathrm{Hom}(\mathbb{1},-)$ is a tensor equivalence from $\mathbf{C}^{\pi(\mathbf{C})}$ to the category of finite-dimensional vector spaces over $F =_{\mathrm{df}} \mathrm{End}(\mathbb{1})$, which allows us to regard the objects of $\mathbf{C}^{\pi(\mathbf{C})}$ as vector spaces. When $\pi(\mathbf{C})$ is commutative, it lies in $\mathrm{Ind}(\mathbf{C}^{\pi(\mathbf{C})})$, and hence can be regarded as a group scheme in the usual sense.

We refer to Saavedra 1972, V 2.3.1, V 3.2.1, for the definitions of a Weil form and of a (graded) polarization on a Tate triple over $\mathbb{R}$. We define a polarization on a Tate triple $\mathbf{C}$ over $\mathbb{Q}$ to be a polarization on $\mathbf{C}_{(\mathbb{R})}$.

Remark 1.1. Let $(\mathbf{C}, w, \mathbb{T})$ be a Tate triple. In particular, $\mathbf{C}$ is a rigid tensor category, and so each object X has a dual $(X^{\vee}, \mathrm{ev}_X)$; moreover,

$$\underline{\mathrm{End}}(X)^{\vee} \cong (X^{\vee} \otimes X)^{\vee} \cong X \otimes X^{\vee} \cong \underline{\mathrm{End}}(X^{\vee}). \qquad (1.1.1)$$

Let X be an object of weight n in $\mathbf{C}$. A nondegenerate $(-1)^n$-symmetric bilinear form $\psi\colon X \otimes X \to \mathbb{1}(-n)$ on X defines an isomorphism $X \to X^{\vee}(-n)$, and hence an isomorphism

$$\underline{\mathrm{End}}(X) \to \underline{\mathrm{End}}(X^\vee(-n)) \cong \underline{\mathrm{End}}(X^\vee).$$

This, together with the pairing

$$\mathrm{ev}\colon \underline{\mathrm{End}}(X)^\vee \otimes \underline{\mathrm{End}}(X) \to \mathbb{1}$$

and the isomorphism (1.1.1), gives a symmetric bilinear form

$$T^\psi\colon \underline{\mathrm{End}}(X) \otimes \underline{\mathrm{End}}(X) \to \mathbb{1}.$$

On $\mathrm{End}(X) \subset \underline{\mathrm{End}}(X)$, T^ψ is the form $(u, v) \mapsto \mathrm{Tr}_X(u^\psi \cdot v)$, and so to say that ψ is a Weil form amounts to saying that the form induced by T^ψ on $\mathrm{End}(X)$ is positive-definite.

A morphism $F\colon (\mathbf{C}_1, w_1, \mathbb{T}_1) \to (\mathbf{C}_2, w_2, \mathbb{T}_2)$ of Tate triples is an exact tensor functor $F\colon \mathbf{C}_1 \to \mathbf{C}_2$ preserving the gradations together with an isomorphism $F(\mathbb{T}_1) \cong \mathbb{T}_2$. We say that such a morphism F maps a polarization Π_1 on $\mathbf{C}_1$ to a polarization Π_2 on $\mathbf{C}_2$ (denoted $F\colon \Pi_1 \mapsto \Pi_2$) if

$$\psi \in \Pi_1(X) \Rightarrow F\psi \in \Pi_2(FX),$$

in which case, for an X of weight n, $\Pi_1(X)$ consists of the bilinear forms $\psi\colon X \otimes X \to \mathbb{1}(-n)$ such that $F\psi \in \Pi_2(FX)$. In particular, given F and Π_2, there exists at most one polarization Π_1 on $\mathbf{T}_1$ such that $F\colon \Pi_1 \mapsto \Pi_2$.

Lemma 1.2. *Let $F\colon \mathbf{C} \to \mathbf{Q}$ be a morphism of Tate triples, and assume that every object of $\mathbf{Q}$ is a direct summand of an object in the image of F. Let Π_1 be a polarization on $\mathbf{C}$. There exists a polarization Π_2 on $\mathbf{Q}$ such that $F\colon \Pi_1 \mapsto \Pi_2$ if and only if, for all X in $\mathbf{C}$ and all $\psi \in \Pi_1(X)$, $F\psi$ is a Weil form on FX.*

Proof. $\Rightarrow$: This follows directly from the definitions.

$\Leftarrow$: For Y a direct summand of FX, define $\Pi_2(Y)$ to be the compatibility class of $(F\psi)|Y$ for some $\psi \in \Pi_1(X)$. It is straightforward to verify that the sets $\Pi_2(Y)$ are well-defined and form a polarization on $\mathbf{Q}$. □

Recall that an exact tensor functor $q\colon \mathbf{C} \to \mathbf{Q}$ of Tannakian categories defines a morphism $\pi(q)\colon \pi(\mathbf{Q}) \to q(\pi(\mathbf{C}))$ (Deligne 1990, 8.15.2).

Definition 1.3. Let $q\colon \mathbf{C} \to \mathbf{Q}$ be an exact tensor functor, and let H be a closed subgroup of $\pi(\mathbf{C})$. We say that $(\mathbf{Q}, q)$ is a *quotient of* $\mathbf{C}$ *by* H if $\pi(q)$ is an isomorphism of $\pi(\mathbf{Q})$ onto $q(H)$.

When $(\mathbf{Q}, q)$ is a quotient of $\mathbf{C}$ by $H \subset \pi(\mathbf{C})$, every object in $\mathbf{Q}$ is a subquotient of an object in the image of q. Moreover, q maps $\mathbf{C}^H$ into $\mathbf{Q}^{\pi(\mathbf{Q})}$, and so, for $X \in \mathbf{C}^H$, we can identify qX with the vector space $\mathrm{Hom}(\mathbb{1}, qX)$. With this identification, there is a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{Q}}(qX, qY) \cong q(\underline{\mathrm{Hom}}(X, Y)^H),\ X, Y \in \mathrm{ob}(\mathbf{C}).$$

Proposition 1.4. *Let $(\mathbf{C}, w, \mathbb{T})$ be a Tate triple over $\mathbb{R}$. Let $(\mathbf{Q}, q)$ be a quotient of $\mathbf{C}$ by $H \subset \pi(\mathbf{C})$, and let Π_1 be a polarization on $\mathbf{C}$. Suppose that $H \supset w(\mathbb{G}_m)$, so that $\mathbf{Q}$ inherits a Tate triple structure from that on $\mathbf{C}$, and that $\mathbf{Q}$ is semisimple. Assume:*

(*) *for all* X *in* $\mathbf{C}^H$ *and all* $\psi \in \Pi_1(X)$*,* $q\psi$ *is a positive-definite form on the vector space* qX*.*

Then there exists a polarization on Π_2 *on* $\mathbf{Q}$ *such that* $q\colon \Pi_1 \mapsto \Pi_2$*.*

PROOF. Because $\mathbf{Q}$ is semisimple, every object of $\mathbf{Q}$ is a direct summand of an object in the image of q. We shall check the condition in Lemma 1.2.

Let $\psi \in \Pi_1(X)$. Then T^ψ is positive for Π_1, and hence so also is its restriction $T^\psi|$ to $\underline{\mathrm{End}}(X)^H$. Therefore, (*) implies that $q(T^\psi|)$ is a positive-definite form on the vector space $q(\underline{\mathrm{End}}(X)^H)$. But $q(\underline{\mathrm{End}}(X)^H) \cong \mathrm{End}_{\mathbf{Q}}(qX)$ and $q(T^\psi|) \cong T^{q\psi}$, and so $q\psi$ is a Weil form, as required. □

REMARK 1.5. Instead of (*), it suffices to assume that there exists a single X in $\mathbf{C}^H$ such that $\pi(\mathbf{C})/H$ acts faithfully on X and a single $\psi \in \Pi(X)$ such that $q\psi$ is a positive-definite form on qX.

2. Polarizations on categories of motives over finite fields

Consider $\mathbf{Mot}(k;\mathcal{S})$ for $\mathcal{S}$ some class satisfying (0.1). For an abelian variety A in $\mathcal{S}$, a divisor D on A defines a pairing $\phi_D\colon h_1A \otimes h_1A \to \mathbb{T}$, which is a Weil form if D is ample (Weil 1948, Théorème 38). Such a Weil form will be said to be *geometric*.

Consider $\mathbf{Mot}(\mathbb{F};\mathcal{A})$. If the Tate conjecture holds for all abelian varieties over $\mathbb{F}$, then $\mathbf{Mot}(\mathbb{F};\mathcal{A})$ is a semsimple Tate triple over $\mathbb{Q}$ with the Weil number torus P as fundamental group (see, for example, Milne 1994, 2.26). Moreover, there exist two graded polarizations on $\mathbf{Mot}(\mathbb{F})$, and for exactly one of these (denoted Π^{Mot}) the geometric Weil forms on any supersingular elliptic curve are positive (ibid., 2.44).

Consider $\mathbf{Mot}(\mathbb{Q}^{\mathrm{al}};\mathcal{C})$ where $\mathcal{C}$ is the smallest class satisfying (0.1) and containing the abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$. It is a Tate triple over $\mathbb{Q}$ with the Serre group S as fundamental group, and it has a canonical polarization Π^{CM}. If the Hodge conjecture holds for complex abelian varieties of CM-type, then the Tate conjecture holds for abelian varieties over $\mathbb{F}$ (Milne 1999b, 7.1), and, corresponding to the p-adic prime we have fixed on $\mathbb{Q}^{\mathrm{al}}$, there is a reduction functor $R\colon \mathbf{Mot}(\mathbb{Q}^{\mathrm{al}};\mathcal{C}) \to \mathbf{Mot}(\mathbb{F};\mathcal{A})$ which realizes $\mathbf{Mot}(\mathbb{F};\mathcal{A})$ as the quotient of $\mathbf{Mot}(\mathbb{Q}^{\mathrm{al}},\mathcal{C})$ by the closed subgroup P of the Serre group S (a description of the inclusion $P \hookrightarrow S$ can be found, for example, in Milne 1994, 4.12). For a motive $X = h(A)(r)$ in $\mathbf{Mot}(\mathbb{Q}^{\mathrm{al}};\mathcal{C})$, $R(X^P)$ is the $\mathbb{Q}$-space of numerical equivalence classes of algebraic cycles of codimension r on the reduction $A_{\mathbb{F}}$ of A.

THEOREM 2.1. *If the Hodge conjecture holds for complex abelian varieties of CM-type, then* $R\colon \Pi^{CM} \mapsto \Pi^{Mot}$ *and all geometric Weil forms on all abelian varieties over* $\mathbb{F}$ *are positive for* Π^{Mot}*.*

PROOF. I claim that to prove the theorem it suffices to show:

(*) there exists a polarization Π on $\mathbf{Mot}(\mathbb{F})$ such that $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi$.

The geometric Weil forms are positive for Π^{CM} and every polarized abelian variety A over $\mathbb{F}$ lifts (up to isogeny) to a polarized abelian variety of CM-type over $\mathbb{Q}^{\mathrm{al}}$ (Zink 1983, 2.7), and so if $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi$, then every geometric Weil form is positive for Π. In particular, the geometric Weil forms on a supersingular elliptic curve are positive, and so $\Pi = \Pi^{\mathrm{Mot}}$. This proves the claim.

We now prove (*). Fix a CM-field $K \subset \mathbb{Q}^{\text{al}}$ such that

(a) K is finite and Galois over $\mathbb{Q}$, and
(b) K properly contains an imaginary quadratic field in which p splits.

Let $\mathcal{C}^K$ (resp. $\mathcal{A}^K$) be the smallest subset of $\mathcal{C}$ (resp. $\mathcal{A}$) satisfying (0.1) and containing the CM abelian varieties over $\mathbb{Q}^{\text{al}}$ with reflex field contained in K (resp. the abelian varieties over $\mathbb{F}$ with endomorphism algebra split by K), and let S^K and P^K be the corresponding quotients of S and P. It suffices to prove (*) for

$$R^K\colon \mathbf{Mot}(\mathbb{Q}^{\text{al}};\mathcal{C}^K) \to \mathbf{Mot}(\mathbb{F};\mathcal{A}^K).$$

Let A be the product of a set of representatives for the simple isogeny classes of abelian varieties in $\mathcal{C}^K$, and let $X = \underline{\text{End}}(h_1A)^P$. It follows from Milne 1999b that S^K/P^K acts faithfully on X: with the notations of that paper, T^K acts faithfully on h_1A as an object of the category of Lefschetz motives; therefore, T^K/L^K acts faithfully on $\underline{\text{End}}(h_1A)^{L^K}$, and (ibid. §6) the canonical map

$$S^K/P^K \to T^K/L^K$$

is injective.

Let ϕ be the geometric Weil form on h_1A defined by an ample divisor D, and let $\psi = T^\phi|X$. Then $\psi \in \mathit{\Pi}^{\text{CM}}(X)$, and it suffices to show that $R^K(\psi)$ is positive-definite (1.4, 1.5). But $R^K(X) = \text{End}(A_{\mathbb{F}})_{\mathbb{Q}}$ and $R^K(\phi)$ is the trace pairing $u, v \mapsto \text{Tr}(u \cdot v^\dagger)$ of the Rosati involution defined by $D_{\mathbb{F}}$, which is positive-definite (Weil 1948, Théorème 38). □

3. The Hodge standard conjecture

Throughout this section, k is an algebraically closed field and $\mathcal{S}$ is a class of smooth projective varieties over k satisfying (0.1).

By a Weil cohomology theory on $\mathcal{S}$, we mean a contravariant functor $V \mapsto H^*(V)$ satisfying the conditions (1)–(4), (6) of Kleiman 1994, §3 (finiteness, Poincaré duality, Künneth formula, cycle map, strong Lefschetz theorem), except that we remember the Tate twists. For example, ℓ-adic étale cohomology, $\ell \neq \text{char}(k)$, is a Weil cohomology theory in this sense (the strong Lefschetz theorem is proved in Deligne 1980).

For $V \in \mathcal{S}$, $A^*_\sim(V)$ denotes the $\mathbb{Q}$-algebra of algebraic classes on V modulo an admissible equivalence relation $\sim$, for example, numerical equivalence (num), or homological equivalence (hom) with respect to a Weil cohomology H.

We say that a Weil cohomology theory is *good* if homological equivalence coincides with numerical equivalence on algebraic cycles with $\mathbb{Q}$-coefficients for all varieties in $\mathcal{S}$.

Let H be a Weil cohomology theory on $\mathcal{S}$. For a connected variety V in $\mathcal{S}$ of dimension n, define $P^r(V)$ to be the subspace of $A^r_{\text{hom}}(V)$ on which L^{n-2r+1} is zero. Let θ^r be the bilinear form

$$(x, y) \mapsto (-1)^r \langle L^{n-2r} x \cdot y\rangle\colon P^r(V) \times P^r(V) \to \mathbb{Q}, \quad r \leq n/2.$$

As originally stated (Grothendieck 1969), the Hodge standard conjecture asserts that these pairings are positive-definite when H is ℓ-adic étale cohomology. Kleiman (1968,

§3) states the conjecture for any Weil cohomology theory. Note that if the Hodge standard conjecture holds for one good Weil cohomology theory, then it holds for all.

For a Weil cohomology theory H, π^r denotes the projection onto H^r and Λ, ${}^c\Lambda$, $*$, p^r denote the maps defined in Kleiman 1968, 1.4 (corrected in Kleiman 1994, §4).

PROPOSITION 3.1. *For all good Weil cohomology theories H on $\mathcal{S}$, the operators Λ, ${}^c\Lambda$, $*$, p^r, and π^r are defined by algebraic cycles that, modulo numerical equivalence, depend only on L (not H).*

PROOF. Let H be a good Weil cohomology theory on $\mathcal{S}$. Then the Lefschetz standard conjecture holds for all $V \in \mathcal{S}$ (Kleiman 1994, 5-1, 4-1(1)), and the proposition can be proved as in ibid., 5.4, (the Hodge standard conjecture is used there only to deduce that numerical equivalence coincides with homological equivalence on $V \times V$). □

When there exists a good Weil cohomology theory on $\mathcal{S}$, we define $\mathbf{Mot}(k;\mathcal{S})$ to be the category of motives with the commutativity constraint modified using the π^r's given by (3.1). It is semisimple (Jannsen 1992), hence Tannakian (Deligne 1990), and it has a natural structure of a Tate triple. Let $V \in \mathcal{S}$ be connected of dimension n, and let Z be a smooth hyperplane section of V. Then $l =_{\text{df}} \Delta_V(Z) \in A^{n+1}_{\text{num}}(V \times V)$ is a morphism

$$l\colon h(V) \to h(V)(1)$$

of degree 2. Define

$$\varphi^r\colon h^r(V) \otimes h^r(V) \to \mathbb{1}(-r)$$

to be the composite

$$h^r(V) \otimes h^r(V) \overset{\text{id}\otimes *}{\to} h^r(V) \otimes h^{2n-r}(V)(n-r) \to h^{2n}(V)(n-r) \cong \mathbb{1}(-r)$$

($*$ as in 3.1). Let $p^r(V)$ be the largest subobject of

$$\operatorname{Ker}(l^{n-2r+1}\colon h^{2r}(V)(r) \to h^{2n-2r+2}(V)(n-r+1))$$

on which $\pi =_{\text{df}} \pi(\mathbf{Mot}(k;\mathcal{S}))$ acts trivially. For any good Weil cohomology theory H on $\mathcal{S}$,

$$\gamma(p^r(V)) \cong P^r(V)$$

where γ is the tensor equivalence $\operatorname{Hom}(\mathbb{1}, -)$ from $\mathbf{Mot}(k;\mathcal{S})^\pi$ to finite-dimensional $\mathbb{Q}$-vector spaces, and there is a pairing

$$\vartheta^r\colon p^r(V) \otimes p^r(V) \to \mathbb{1},$$

also fixed by π, such that $\gamma(\vartheta^r) = \theta^r$.

PROPOSITION 3.2. *Let H be a good Weil cohomology theory on $\mathcal{S}$. The following statements are equivalent:*

(a) *the Hodge standard conjecture holds for H and the varieties in $\mathcal{S}$;*
(b) *there exists a polarization Π on $\mathbf{Mot}(k;\mathcal{S})$ for which the forms φ^r are positive;*
(c) *there exists a polarization Π on $\mathbf{Mot}(k;\mathcal{S})$ for which the forms ϑ^r are positive.*

PROOF. (a)⇒(b): See Saavedra 1972, VI 4.4.

(b)⇒(c): The restriction of $\varphi^{2r} \otimes \mathrm{id}_{\mathbb{1}(2r)}$ to the subobject $p^r(V)$ of $h^{2r}(V)(r)$ is the form ϑ^r.

(c)⇒(a): Let Π be a polarization on $\mathbf{Mot}(k;\mathcal{S})$ for which the forms ϑ^r are positive. The restriction of Π to $\mathbf{Mot}(k;\mathcal{S})^{w(\mathbb{G}_m)}$ is a symmetric polarization, and so there exists an $\mathbb{R}$-valued fibre functor ω on $\mathbf{Mot}(k;\mathcal{S})^{w(\mathbb{G}_m)}$ carrying Π-positive forms to positive-definite symmetric forms (Deligne and Milne 1982, 4.27). The restriction of ω to $\mathbf{Mot}(k;\mathcal{S})^{\pi}_{(\mathbb{R})}$ is (uniquely) isomorphic to γ, and so $\gamma(\vartheta^r)$ is positive-definite. □

THEOREM 3.3. *Let k be an algebraically closed field. If the Hodge conjecture holds for complex abelian varieties of CM-type, then, for all $\ell \neq$ char(k),*

(a) *numerical equivalence coincides with ℓ-adic étale homological equivalence on abelian varieties over k, and*
(b) *the Hodge standard conjecture holds for all abelian varieties over k and the ℓ-adic étale cohomology theory.*

PROOF. *(a) for $k = \mathbb{F}$.* The Hodge conjecture for complex abelian varieties of CM-type implies the Tate conjecture (Milne 1999b, 7.1), which implies (a) (see, for example, Tate 1994, 2.7).

(b) for $k = \mathbb{F}$. For abelian varieties over $\mathbb{Q}^{\mathrm{al}}$, the Betti cohomology theory is good (Lieberman 1968) and the Hodge standard conjecture holds, and so (3.2) there is a polarization Π on $\mathbf{Mot}(\mathbb{Q}^{\mathrm{al}};\mathcal{C})$ for which the forms

$$\varphi^r\colon h^r(A) \otimes h^r(A) \to \mathbb{1}(-r)$$

are positive. Clearly, Π is the canonical polarization Π^{CM} in §2. Let Z be the hyperplane section of A used in the definition of φ^r. Because $R\colon \Pi^{\mathrm{CM}} \mapsto \Pi^{\mathrm{Mot}}$ (Theorem 2.1), the form

$$\varphi^r\colon h^r(A_{\mathbb{F}}) \otimes h^r(A_{\mathbb{F}}) \to \mathbb{1}(-r)$$

defined by the reduction $Z_{\mathbb{F}}$ of Z on $A_{\mathbb{F}}$ is positive for Π^{Mot}. Every polarized abelian variety A over $\mathbb{F}$ lifts (up to isogeny) to a polarized abelian variety of CM-type over $\mathbb{Q}^{\mathrm{al}}$ (Zink 1983, 2.7), and so

$$(A_{\mathbb{F}}, Z_{\mathbb{F}} \text{ modulo numerical equivalence})$$

is arbitrary. Proposition 3.2 now gives (b).

(a,b) for arbitrary k. For an abelian variety A of dimension n over k, consider the commutative diagram:

$$\begin{array}{ccccc}
H^{2r}(A,\mathbb{Q}_\ell(r)) & \times & H^{2n-2r}(A,\mathbb{Q}_\ell(n-r)) & \xrightarrow{\ \cup\ } & \mathbb{Q}_\ell \\
\uparrow cl & & \uparrow L^{n-2r}\circ cl & & \bigcup \\
P^r(A) & \times & P^r(A) & \xrightarrow{\ \theta\ } & \mathbb{Q}
\end{array}$$

There is a similar diagram for a smooth specialization $A_{\mathbb{F}}$ of A to an abelian variety over $\mathbb{F}$. The specialization maps on the cohomology groups are bijective and hence they are injective on the P's. Since the pairings are compatible, this implies the Hodge standard conjecture for A and ℓ-adic étale cohomology. Because the Lefschetz standard conjecture is known for abelian varieties, this in turn implies that numerical equivalence coincides with ℓ-adic homological equivalence for A (Kleiman 1994, 5-4). □

Corollary 3.4. *If the Hodge conjecture holds for complex abelian varieties of CM-type, then, for any algebraically closed field k,* $\mathbf{Mot}(k;\mathcal{A})$ *has a polarization (necessarily unique) for which the forms ϑ^r and φ^r are positive.*

Proof. For $\ell \neq \mathrm{char}(k)$, Theorem 3.3(a) shows that ℓ-adic étale cohomology is good. Now apply (3.3b) and (3.2). □

Remark 3.5. Assume the Hodge conjecture holds for complex abelian varieties of CM-type, and let H be a Weil cohomology theory on $\mathcal{A}$ (over an algebraically closed field k). Because the Lefschetz standard conjecture is known for abelian varieties, if H is not good, then the Hodge standard conjecture fails for H (Kleiman 1994, 5-1). Thus, the Hodge standard conjecture holds for H if and only if H is good.

Remark 3.6. Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$ satisfying conditions (a) and (b) of the proof of (2.1). The preceding arguments can be modified to show that, if the Hodge conjecture holds for all complex abelian varieties with reflex field contained in K, then the conclusions of Theorem 3.3 hold for all abelian varieties over $\mathbb{F}$ whose endomorphism algebra is split by K. In fact, condition (b) is not necessary for this statement because, as Deligne pointed out to me, the results of Milne 1999b, §6, hold without it.

Remark 3.7. Most of the preceding arguments hold with "algebraic cycle" replaced by "Lefschetz cycle" (cf. Milne 1999a, §5). Let A be an abelian variety over k. Recall that, for any Weil cohomology theory, if a Lefschetz class a on A is not homologically equivalent to zero, then there exists a Lefschetz class b on A of complementary dimension such that $\langle a\cdot b\rangle \neq 0$ (ibid. 5.2). Thus, homological equivalence on Lefschetz classses is independent of the Weil cohomology theory, and coincides with numerical equivalence.

Let $D^r(A)$ be the $\mathbb{Q}$-space of Lefschetz classes modulo numerical equivalence on A of codimension r, and let $DP^r(A)$ be the $\mathbb{Q}$-subspace on which L^{n-2r+1} is zero. With the notations of Milne 1999b, the categories $\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})$ and $\mathbf{LMot}(\mathbb{F})$ of Lefschetz motives have canonical polarizations, and the reduction functor $\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{LMot}(\mathbb{F})$ maps one to the other. The same argument as in the proof of Theorem 3.3 shows that the bilinear forms

$$(x,y) \mapsto (-1)^r\langle L^{n-2r}x\cdot y\rangle \colon DP^r(A)\times DP^r(A)\to\mathbb{Q}$$

are positive-definite for $r \leq n/2$ and A an abelian variety over $\mathbb{F}$. In other words, the Lefschetz analogue of the Hodge standard conjecture holds unconditionally for abelian varieties over $\mathbb{F}$. A specialization argument (as in the proof of 3.3) extends the statement to arbitrary k.

Remark 3.8. Recall that a Hodge, Tate, or algebraic class on a variety is said to be *exotic* if it is not Lefschetz. Remark 3.7 shows that the Hodge standard conjecture holds unconditionally for abelian varieties with no exotic algebraic classes. For examples (discovered by Lenstra, Spiess, and Zarhin) of abelian varieties over $\mathbb{F}$ with no exotic Tate classes, and hence no exotic algebraic classes, see Milne 2001, A.7.

Remark 3.9. Grothendieck (1969) stated: "Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry." Should the Hodge conjecture remain inaccessible, even for abelian varieties of CM-type, Theorem 3.3 suggests a possible approach to proving the Hodge standard conjecture for abelian varieties, namely, improve the theory of absolute Hodge classes (Deligne 1982) sufficiently to remove the hypothesis from the theorem.

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