Reviewer: Milne, James [Featured review 13.06.2002.]

Author: Michael Harris and Richard Taylor

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Review text:

The authors make a detailed study of the geometry and cohomology of some Shimura varieties at primes of bad reduction, and as a consequence obtain the first proof of nonabelian class field[1] for nonarchimedean local fields of characteristic zero (the local Langlands conjecture for GL_n).

Such a field K is a finite extension of $\mathbb{Q}_p$. The Weil group W_K of K is the subgroup of the absolute Galois group G_K consisting of the elements that act on the residue field as an integer power of the Frobenius element. Local class field theory for abelian extensions (that is, for GL_1) establishes the existence of a canonical topological isomorphism Art_K from $K^\times$ onto the largest abelian quotient W_K^{ab} of W_K. Initially abelian local class field theory was deduced from global class field theory (Hasse, Schmidt, 1930)[2], which had been proved earlier[3], but Hasse and Chevalley later[4] found purely local proofs. At that time, Art_K could be characterized locally[5], or it could be described as a local component of the global Artin map, but in 1965 Lubin and Tate gave an explicit local description of it.

Once the class field theory of abelian extensions was proved, the search began[6] for a nonabelian class field theory. For several decades it was unclear what form this should take, or even whether[7][8] it existed, but in 1967 Langlands stated his conjectural[9] functoriality principle, which includes a nonabelian class field theory as a special case. For a local field, it asserts that homomorphisms from W_K into $\mathrm{GL}_n(\mathbb{C})$ correspond to certain representations of $\mathrm{GL}_n(K)$. For $n = 1$, the representations of $\mathrm{GL}_1(K) = K^\times$ are just characters, and the correspondence is given by Art_K, but for $n > 1$ the representations of $\mathrm{GL}_n(K)$ are infinite dimensional.[10] The serious study of infinite dimensional representations in the p-adic case had begun only a few years earlier (Mautner, Bruhat, et al.).[11]

On the automorphic side, let $\mathcal{A}_n(K)$ be the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ (those irreducible representations on complex vector spaces for which the stabilizer of each vector is open). On the Galois side, let $\mathcal{G}_n(K)$ be the set of equivalence classes of pairs (r, N) where r is a semisimple representation of W_K on an n-dimensional complex vector space V, trivial on an open subgroup, and N is a nilpotent[12] endomorphism of V such that conjugating N by $r(\sigma)$ ($\sigma \in W_K$) multiplies it by the absolute value of $Art_K^{-1}(\sigma)$. The local Langlands conjecture[13] for K asserts that there is a family of bijections $(\sigma_n)_{n\geq 1}$,

$$\pi \mapsto \sigma_n(\pi)\colon \mathcal{A}_n(K) \to \mathcal{G}_n(K),$$

such that

(1) the determinant of $\sigma_n(\pi)$, viewed as a character of W_K, corresponds under Art_K to the central character of π;

(2) the map σ_n preserves L-factors and ε-factors of pairs of π's (as defined by Jacquet, Piatetskii-Shapiro, and Shalika[14] on the automorphic side, and by Langlands[15] and Deligne[16] on the Galois side);

(3) for $\chi \in \mathcal{A}_1(K)$, $\sigma_n(\pi \otimes (\chi \circ \det)) = \sigma_n(\pi) \otimes \sigma_1(\chi)$;

(4) σ_n commutes with passage to the contragredient, $\pi \mapsto \pi^\vee$.

For each K, Henniart[17] showed there exists at most one such family.

This is probably the most natural statement of nonabelian local class field theory, but Langlands's original conjecture is both more general (GL_n is replaced by an arbitrary reductive group) and less precise (his conditions do not determine the σ_n uniquely).

All admissible representations can be obtained by an inductive process from supercuspidal representations, and Bernstein and Zelevinsky[18] showed that the inductive structure of $\mathcal{A}_n(K)$ is the same as that of $\mathcal{G}_n(K)$. It follows[19] that it suffices to prove that there exists a family of bijections $\sigma_n\colon \mathcal{A}_n^0(K) \to \mathcal{G}_n^0(K)$ satisfying (1–4), where $\mathcal{A}_n^0(K)$ is the set of equivalence classes of supercuspidal representations and $\mathcal{G}_n^0(K)$ is the set of equivalence classes of pairs (r, N) with r irreducible (hence $N = 0$). Henniart[20] showed that there exist bijections $\mathcal{A}_n^0(K) \to \mathcal{G}_n^0(K)$ preserving conductors and satisfying (3) for unramified χ (the numerical local Langlands conjecture), from which it follows by a counting argument[21] that it suffices to show that there exist maps $\sigma_n\colon \mathcal{A}_n^0(K) \to \mathcal{G}_n^0(K)$ that satisfy (1-4) on a subset of $\mathcal{A}_n^0(K)$ which surjects onto $\mathcal{G}_n^0(K)$. This, in essence, is what the authors do.

Let D_g be a division algebra with centre K and invariant $1/g$, and let $\mathcal{A}_{D_g}$ be the set of equivalence classes of irreducible admissible finite dimensional representations of $D_g^\times$. There exists an injection[22] $JL\colon \mathcal{A}_g^0(K) \to \mathcal{A}_{D_g}$

characterized by an equality of characters (Jacquet, Langlands, et al.)[23]. By considering deformations of formal groups with $\mathcal{O}_K$-action of dimension 1, fixed $\mathcal{O}_K$-height g, and (Drinfeld) structure of level m, one obtains a tower $(Y_m)_{m\geq 1}$ of formal schemes, to which étale cohomology attaches $\mathbb{Q}_l^{\mathrm{ac}}$-vector spaces Ψ_g^i of vanishing cycles (Berkovich). A large subgroup of $GL_g(K)\times D_g^\times \times W_K$ acts on Ψ_g^i. For an irreducible admissible representation τ of $D_g^\times$, let $\Psi_g^i(\tau) = \mathrm{Hom}_{\mathcal{O}_{D_g}^\times}(\tau, \Psi_g^i)$. In 1988, Carayol[24] conjectured that

(a) for each irreducible supercuspidal representation π of $GL_g(K)$, there exists a representation $s_g(\pi)\colon W_K \to \mathrm{GL}_g(\mathbb{Q}_l^{\mathrm{ac}})$ such that $\Psi_g^{g-1}(JL(\pi)) = \pi^\vee \otimes s_g(\pi)$ (as a representations of $GL_g(K)\times W_K$);

(b) the maps $\pi \mapsto s_g(\pi)$ satisfy the conditions (1–4) of the local Langlands conjecture (after a change of normalization and an identification of $\mathbb{Q}_l^{\mathrm{ac}}$ with $\mathbb{C}$).

For $g = 1$, this is a restatement of the original theorem of Lubin and Tate. For $g = 2$, it was proved by Deligne[25] ($K = \mathbb{Q}_p$) and Carayol[26]. For g and K arbitrary, it is proved by the authors except that, in (a), Ψ_g^{g-1} is replaced by an alternating sum of the Ψ_g^i — a virtual representation.

Deligne and Carayol obtain their results from a study of elliptic modular curves and Shimura curves at primes of bad reduction, and the authors obtain their generalization by a similar study of certain carefully chosen Shimura varieties. Specifically, they study the Shimura varieties attached to certain division algebras B over CM-fields F, where, for some p-adic prime w, $F_w \approx K$ and $B \otimes F_w$ is a matrix algebra. These are moduli varieties for polarized abelian varieties with endomorphism and level structure, and they have good reduction at w except when p divides the level. At the prime w, the deformation of the abelian varieties with additional structure is controlled by a part of the formal group of the abelian variety that has the same shape as those considered in the last paragraph. The Main Theorem of the monograph expresses the representation of $GL_n(K)\times W_K$ on the l-adic cohomology of the Shimura variety in terms of its representations on certain of the spaces $\Psi_g^i(\tau)$. From this, the authors deduce (a). Moreover, they are able to extend to all primes a theorem of Clozel[27] that attaches a Galois representation $R(\Pi)$ to a global automorphic representation Π and identifies the local component of $R(\Pi)$ at most good primes (weak form of the conjectured global Langlands correspondence). From this, and an automorphic induction theorem of Harris[28], they are able to deduce (b), and hence the local Langlands conjecture.

The proof of the Main Theorem occupies most of the monograph, and can only be described as a tour de force. Among the tools used are the deformation theory of Barsotti-Tate groups and abelian schemes (Serre, Tate, Grothendieck), Drinfeld level structures, the classification of abelian varieties over finite fields (Weil, Tate, Honda), the étale cohomology of formal schemes (Berkovich), and a Lefschetz trace formula for formal schemes (Fujiwara). Fortunately, the authors are careful to explain the results they use.

Beyond the simple statement of the conjecture, the authors prove that the correspondence is realized in a specific space of vanishing cycles, and they prove a compatibility with the global Langlands correspondence given by the Shimura varieties they study. Several months after the authors distributed a preliminary version of their manuscript, Henniart found a much simpler proof of the Langlands local conjecture, but without these supplements [Invent. Math. 139 (2000), no. 2, 439–455; MR2001e:11052]. His proof is also global and makes use of Shimura varieties, but only at primes of good reduction. As the authors note, neither proof ends the story: one may hope for a local proof of the conjecture and for an explicit description of σ_n (as in the abelian case).[29]

In reviewing the above story, one can not help but be struck by how many mathematicians have made essential contributions to it.[30] Takagi was able to find his proof of global abelian class field theory in the isolation imposed on him in Japan by World War I and its aftermath, and his papers, although very difficult, can be read with only the knowledge acquired in a single graduate course. By contrast, it is unlikely that any single mathematician can claim familiarity with the proofs of all statements used in the present monograph.

Finally, I note that the talk of Carayol [Séminaire Bourbaki, Vol. 1998/99. Astérisque No. 266 (2000), Exp. No. 857, 4, 191–243. MR2001i:11136] is an excellent introduction to the proofs both of Harris and Taylor and of Henniart.

Notes

1. The object of class field theory is to describe the extensions of a local or global in terms of objects intrinsically attached to the field itself, or, as Chevalley (1940) put it, "comment un corps possède en soi les éléments de son propre dépassement" .

2. H. Hasse, Die Normenresttheorie relative-Abelscher Zahlkörper als Klassenkörper im Kleinen, J. für Mathematik (Crelle) 162 (1930), 145–154.

F. K. Schmidt, Zur Klassenkörpertheorie im Kleinen, ibid. 155–168.

3. By Furtwängler, Takagi, and Artin, after earlier work of Kronecker, Weber, and Hilbert.

4. In three papers published in 1933 — see P. Roquette, *Class field theory in characteristic p, its origin and development* (2001), pp558–559 for further discussion and references.

5. There is also the theorem of Dwork — see Serre, Corps Locaux, XIII 5.

6. Takagi mentioned it in the talk at the ICM 1920 in which he announced his proof of the main theorems of abelian class field theory to the world:

> En m'arrêtant ici, je me permets d'attirer votre attention sur un problème important de la théorie des nombres algébriques: à savoir, rechercher s'il est possible de définir la classe d'idéaux d'un corps algébrique de telle manière que le corps supérieur relativement normal mais non abélian puisse être caractérisé par le groupe correspondant de classes d'idéaux du corps de fond.

7. A nonabelian global class field theory would provide, in particular, a description of the sets of primes in a global field that split in a finite extension field. For abelian extensions, abelian class field theory says that these sets are determined by congruence conditions, but it also says that there is no such simple description in the nonabelian case (which seems to give a negative answer to the specific problem posed by Takagi).

8. In 1946, Artin speculated that finding the correct statements was the *only* problem: once one knew what they were, it would be possible to deduce them from abelian class field theory (A Century of Mathematics in America, Part II, (Peter Duren, ed.), 1989, p312). Weil relates that, a year later, Artin said that he had lost faith in the existence of a nonabelian class field theory (Weil, A., Œuvres, Vol. III, p457.)

9. In a letter to Weil, and later (to the rest of us) in *Problems in the theory of automorphic forms* (1970). For an engaging introduction to these works, see Casselman, *The L-group* (2001). TEXed versions of Langlands's works, including the above two, can be found at `http://sunsite.ubc.ca/DigitalMathArchive/Langlands/`.

10. Except, of course, for the (quasi)characters.

11. Mautner 1958, Bruhat 1961, Gel'fand and Graev 1963 1966, Shalika 1966, Tanaka 1966, Kirillov 1966, Harish-Chandra ... — see the introduction to Bernshtein (sic) and Zelevinskii, *Representations of the group $GL(n, F)$ where F is a non-archimedean local field*, 1976, for a brief history.

12. H&Z p1 forget this condition.

13. The statement in this form is due to Henniart — see
Henniart, Guy, Le point sur la conjecture de Langlands pour $GL(N)$ sur un corps local. Séminaire de théorie des nombres, Paris 1983–84, 115–131, Progr. Math., 59, Birkhäuser Boston, Boston, MA, 1985.

or, perhaps, an earlier talk referred to as [He 1] in the above talk.

14. Jacquet, H.; Piatetskii-Shapiro, I. I.; Shalika, J. A. Rankin-Selberg convolutions. Amer. J. Math. 105 (1983), no. 2, 367–464.
15. On the functional equation of the Artin L-functions (incomplete preprint) Yale University, (1970).
16. Deligne, P., Les constantes des équations fonctionnelles des fonctions L. Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501–597. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973.
17. Henniart, Guy, Caractérisation de la correspondance de Langlands locale par les facteurs ε de paires. Invent. Math. 113 (1993), no. 2, 339–350.
18. Bernstein, I. N.; Zelevinsky, A. V. Induced representations of reductive $\mathfrak{p}$-adic groups. I. Ann. Sci. École Norm. Sup. (4) 10 (1977), no. 4, 441–472.

Zelevinsky, A. V., Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $GL(n)$. Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 2, 165–210.

The results of B&Z are also used in the proof of H&Z in the supercuspidal case.
19. This is explained in: §10 of Zelevinsky 1980; Rodier's Bourbaki talk (1982), 4.4; Kudla's talk at the Seattle conference on motives (1994), §4; and, most completely, pp251–255 of Harris and Taylor.
20. Henniart, Guy, La conjecture de Langlands locale numérique pour $GL(n)$, Ann. Sci. École Norm. Sup. (4) 21 (1988), no. 4, 497–544.
21. The point is that, on each side, the set of objects with fixed conductor is finite modulo twists by unramified characters.
22. In fact, there is a bijection $\mathcal{A}_n^d \to \mathcal{A}_D$ where $\mathcal{A}_n^d$ is the set of equivalence classes of irreducible admissible representations whose matrix coefficients are integrable modulo the centre.
23. Jacquet and Langlands for $n = 2$ and Rogawski, Kazhdan, Vigneras for $n > 2$.
24. Carayol, H., Nonabelian Lubin-Tate theory. Automorphic forms, Shimura varieties, and L-functions (L. Clozel and J.S. Milne eds), Vol. II (Ann Arbor, MI, 1988), 15–39, Perspect. Math., 11, Academic Press, Boston, MA, 1990.
25. In an eleven-page handwritten letter to Piatetskii-Shapiro dated March 25, 1973, with a one-page typed covering letter dated April, 1973: "In it I claim (except at 2) to prove for the supercuspidal representations what in your notes [Antwerp Conference LNM 349] you prove for the principal (unramified) series. The idea is that

1. room is left for it in your notes only thanks to the supersingular elliptic curves;
2. supersingular elliptic curves correspond to ideal classes in the quaternion algebra ramified at p and ∞;
3. this, by a global argument using Jacquet Langlands §14, forces the outcome."

Apparently (see MR 50 7095), the letter was published: Matematika—Period. Sb. Perevodov Inostran, Statei 18 (1974), 110–122. It would be useful if someone would put it on the web, since it was the starting point for Carayol and H&T.

In fact, Deligne's proof of (b) was completed by J-L. Brylinski (appendix to Carayol 1986).
26. Carayol, Henri, Sur la mauvaise réduction des courbes de Shimura. Compositio Math. 59 (1986), no. 2, 151–230.

Carayol, Henri, Sur les représentations l-adiques associées aux formes modulaires de Hilbert. Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 3, 409–468.

27. Clozel, Laurent, Représentations galoisiennes associées aux représentations automorphes autoduales de $GL(n)$, Inst. Hautes Études Sci. Publ. Math. No. 73 (1991), 97–145.

Labesse, Jean-Pierre, Cohomologie, stabilisation et changement de base. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen. Astérisque No. 257 (1999), vi+161 pp.

28. Base change and automorphic induction are what (conjecturally) correspond on the automorphic side to restriction and induction on the Galois side. They are known only for cyclic extensions (Arthur and Clozel 1989, Henniart and Herb 1995, respectively), and hence for solvable extensions. Harris (1998) remarked that by combining cyclic automorphic induction with the existence of certain Galois representations (Clozel 1991), one can obtain automorphic induction for certain nonGalois subextensions of solvable extensions. This idea plays an important role in the proofs of both Harris and Taylor and of Henniart.

29. See the continuing work of Bushnell, Henniart, and Kutzko, especially,

Bushnell, Colin J.; Henniart, Guy; Davenport-Hasse relations and an explicit Langlands correspondence. II. Twisting conjectures. Colloque International de Théorie des Nombres (Talence, 1999). J. Théor. Nombres Bordeaux 12 (2000), no. 2, 309–347.

30. Despite all the heavy machinery used, it appears to this innocent observer that Harris and Taylor (and Henniart) only just scrape through.