Motivic complexes and special values of zeta functions

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Abstract

Beginning with the conjecture of Artin and Tate in 1966, there has been a series of successively more general conjectures expressing the special values of the zeta function of an algebraic variety over a finite field in terms of other invariants of the variety. In this article, we present the ultimate such conjecture, and provide evidence for it. In particular, we enhance Voevodsky's $\mathbb{Z}[1/p]$-category of étale motivic complexes with a p-integral structure, and show that, for this category, our conjecture follows from the Tate and Beilinson conjectures. As the conjecture is stated in terms of motivic complexes, it (potentially) applies also to algebraic stacks, log varieties, simplicial varieties, etc..

Contents

1   Introduction

Motivating examples

We begin by reviewing two statements from the 1960s concerning the special values of the zeta functions of varieties over finite fields. Our goal has been to find the ultimate generalization of these statements, and to provide persuasive evidence for it.

Recall that the zeta function of an algebraic variety X over a finite field $\mathbb{F}_q$ is the formal power series $Z(X,t) \in \mathbb{Q}[[t]]$ such that

$$\log(Z(X,t)) = \sum_{n>0} \frac{N_n t_n}{n} \qquad (1)$$

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with $N_n = \#(X(\mathbb{F}_{q^n}))$, and that Dwork (1960) proved that $Z(X,t) \in \mathbb{Q}(t)$.

Let X be a smooth projective surface over a finite field $\mathbb{F}_q$. The Néron-Severi group $\mathrm{NS}(X)$ of X is finitely generated, and the Tate conjecture says that its rank ρ is the order of the pole of $Z(X,t)$ at $t = q^{-1}$. Write

$$Z(X,t) = \frac{P_1(X,t)P_3(X,t)}{(1-t)P_2(X,t)(1-q^2t)}, \quad P_i(X,t) \in \mathbb{Z}[t].$$

Then the Artin-Tate conjecture says that the Brauer group of X is finite, and that its order $[\mathrm{Br}(X)]$ satisfies

$$\lim_{t \to q^{-1}} \frac{P_2(X,t)}{(1-qt)^\rho} = \frac{[\mathrm{Br}(X)] \cdot D}{q^{\alpha(X)} \cdot [\mathrm{NS}(X)_{\mathrm{tors}}]^2}$$

where D is the discriminant of the intersection pairing on $\mathrm{NS}(X)$ and

$$\alpha(X) = \chi(X, \mathcal{O}_X) - 1 + \dim(\mathrm{PicVar}(X)).$$

Now consider two abelian varieties A and B over a finite field, and write

$$Z(A,t) = \frac{P_1(A,t)\cdots}{(1-t)P_2(A,t)\cdots}, \quad P_1(A,t) = \prod_i (1-a_it)$$

$$Z(B,t) = \frac{P_1(B,t)\cdots}{(1-t)P_2(B,t)\cdots}, \quad P_1(B,t) = \prod_j (1-b_jt).$$

Weil showed that $\mathrm{Hom}(A,B)$ is a finitely generated $\mathbb{Z}$-module, and the Tate conjecture (proved by Tate in this case) says that

$$\mathrm{rank}(\mathrm{Hom}(A,B)) = \#\{(i,j) \mid a_i = b_j\}.$$

According to [Milne 1968](), the group $\mathrm{Ext}^1(A,B)$ is finite, and its order satisfies

$$\prod_{a_i \neq b_j} \left(1 - \frac{a_i}{b_j}\right) = [\mathrm{Ext}^1(A,B)] \cdot D \cdot q^{-\dim(A)-\dim(B)}$$

where D is the discriminant of the pairing

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,A) \xrightarrow{\circ} \mathrm{End}(A) \xrightarrow{\mathrm{trace}} \mathbb{Z}.$$

Statement of the conjecture

Let k be a perfect field. Throughout the first six sections of the article, $\mathsf{DM}(k)$ is a triangulated category[1] of "motivic complexes" equipped with exact "realization" functors

$$r_\ell : \mathsf{DM}(k) \to \mathsf{D}^b_c(k, \mathbb{Z}_\ell), \text{ all } \ell \neq p,$$
$$r_p : \mathsf{DM}(k) \to \mathsf{D}^b_c(R)$$

where $\mathsf{D}^b_c(k, \mathbb{Z}_\ell)$ is the ℓ-adic triangulated category (Ekedahl; see §4) and $\mathsf{D}^b_c(R)$ is the triangulated category of coherent complexes of graded modules over the Raynaud ring (Ekedahl-Illusie-Raynaud; see §5). There are Tate twists in $\mathsf{DM}(k)$, compatible with the

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[1]In fact, we work throughout with pretriangulated differential graded categories. This aspect will be ignored in the Introduction

realization functors. We require that $\mathsf{DM}(k)$ have an internal Hom, $R\underline{\mathrm{Hom}}(-,-)$. We do not require $\mathsf{DM}(k)$ to have a t-structure. The Ext of two objects M, N in $\mathsf{DM}(k)$ is defined by the usual formula

$$\mathrm{Ext}^j(M, N) = \mathrm{Hom}_{\mathsf{DM}(k)}(M, N[j]).$$

In §7 (resp. §8) we construct a candidate for $\mathsf{DM}(k)$ based on Voevodsky's category of geometric motives (resp. the conjectural theory of rational Tate classes).

Now let k be a finite field with q elements. Using the finiteness of k, we construct (in §2) a canonical complex

$$E(M, N, r): \quad \cdots \to \mathrm{Ext}^{j-1}(M, N(r)) \to \mathrm{Ext}^j(M, N(r)) \to \mathrm{Ext}^{j+1}(M, N(r)) \to \cdots$$

of abelian groups for each pair M, N in $\mathsf{DM}(k)$. Here (r) denotes the Tate twist in $\mathsf{DM}(k)$.

We expect that each object P of $\mathsf{DM}(k)_{\mathbb{Q}}$ has a zeta function $Z(P, t) \in \mathbb{Q}(t)$ compatible with the realization functors (see §3).

Attached to each P in $\mathsf{D}^b_c(R)$, there is a bounded complex $R_1 \otimes^L_R P$ of graded k-vector spaces whose cohomology groups have finite dimension. The Hodge numbers $h^{i,j}(P)$ of P are defined to be the dimensions of the k-vector spaces $H^j(R_1 \otimes^L_R P)^i$ (see §5).

CONJECTURE 1.1. *Let $M, N \in \mathsf{DM}(k)$ and let $r \in \mathbb{Z}$. Let $P = R\underline{\mathrm{Hom}}(M, N)$.*
  (a) *The groups $\mathrm{Ext}^j(M, N(r))$ are finitely generated $\mathbb{Z}$-modules for all j, and the alternating sum of their ranks is zero.*
  (b) *The zeta function $Z(P, t)$ of P has a pole at $t = q^{-r}$ of order*

$$\rho = \sum_j (-1)^{j+1} \cdot j \cdot \mathrm{rank}_{\mathbb{Z}}(\mathrm{Ext}^j(M, N(r))).$$

  (c) *The cohomology groups of the complex $E(M, N, r)$ are finite, and the alternating product $\chi^{\times}(M, N(r))$ of their orders satisfies*

$$\left| \lim_{t \to q^{-r}} Z(P, t)(1 - q^r t)^{\rho} \right| = \chi^{\times}(M, N(r)) \cdot q^{\chi(P, r)}$$

  where

$$\chi(P, r) = \sum_{i, j \ (i \leq r)} (-1)^{i+j} (r - i) h^{i,j}(r_p(P)).$$

Assume that $\mathsf{DM}(k)$ has a tensor structure, and let $\mathbb{1}$ be the identity object for this structure. Then $R\underline{\mathrm{Hom}}(\mathbb{1}, N) \simeq N$. As we now explain, when we take M to be $\mathbb{1}$, the statement of Conjecture 1.1 simplifies, and it doesn't require the existence of internal Homs.

We define the ***absolute cohomology groups*** of $P \in \mathrm{ob}\,\mathsf{DM}(k)$ by

$$H^j_{\mathrm{abs}}(P, r) = \mathrm{Hom}_{\mathsf{DM}(k)}(\mathbb{1}, P[j](r))$$

(cf. Deligne 1994, 3.2). With $(M, N) = (\mathbb{1}, P)$, the complex $E(M, N, r)$ becomes a complex

$$H^{\bullet}_{\mathrm{abs}}(P, r): \quad \cdots \to H^{j-1}_{\mathrm{abs}}(P, r) \to H^j_{\mathrm{abs}}(P, r) \to H^{j+1}_{\mathrm{abs}}(P, r) \to \cdots.$$

and Conjecture 1.1 becomes the following statement.

CONJECTURE 1.2.   (a) *The groups $H^j_{\mathrm{abs}}(P, r)$ are finitely generated $\mathbb{Z}$-modules for all j, and the alternating sum of their ranks is zero.*

(b) *The zeta function $Z(P,t)$ of P has a pole at $t = q^{-r}$ of order*

$$\rho = \sum\nolimits_j (-1)^{j+1} \cdot j \cdot \mathrm{rank}_{\mathbb{Z}} \left(H_{\mathrm{abs}}^j(P,r) \right).$$

(c) *The cohomology groups of the complex $H_{\mathrm{abs}}^\bullet(P,r)$ are finite, and the alternating product $\chi^\times(P,r)$ of their orders satisfies*

$$\left| \lim_{t \to q^{-r}} Z(P,t)(1 - q^r t)^\rho \right| = \chi^\times(P,r) \cdot q^{\chi(P,r)}$$

where

$$\chi(P,r) = \sum_{i,j \ (i \leq r)} (-1)^{i+j}(r-i) \cdot h^{i,j}(r_p P).$$

Examples

In these examples, we take $\mathsf{DM}(k)$ to be the category defined in §7.

1.3. Let X be a smooth projective surface over $k = \mathbb{F}_q$. The terms of the complex $E(\mathbb{1}, hX, 1)$ are finitely generated $\mathbb{Z}$-modules, and all are finite except for E^2 and E^3, which both have rank $\rho = \mathrm{rank}(\mathrm{NS}(X))$. Therefore (a) of (1.1) is true, and (b) is the Tate conjecture for X. Modulo torsion, the map $E^2 \xrightarrow{d} E^3$ can be identified with the map $\mathrm{NS}(X) \to \mathrm{Hom}(\mathrm{NS}(X), \mathbb{Z})$ defined by the intersection pairing, whose cokernel has order D. In this case, (c) of (1.1) essentially becomes the Artin-Tate conjecture.

1.4. Let $M = h_1 A$, $N = h_1 B$, and $P = R\underline{\mathrm{Hom}}(h_1 A, h_1 B)$ where A and B are abelian varieties over $k = \mathbb{F}_q$. Then $Z(P,t) = \prod_{i,j}(1 - \frac{a_i}{b_j}t)$, and Conjecture 1.1 essentially becomes the statement for A and B discussed above.[2]

1.5. Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$, and let $\mathbb{Z}(r)$ be the complex of étale sheaves on X given by Bloch's higher Chow groups (see the survey article Geisser 2005). Define the ***Weil-étale motivic cohomology groups*** of X to be

$$H_{\mathrm{mot}}^j(X,r) = H^j(X_{\mathrm{Wét}}, \mathbb{Z}(r))$$

where Wét denotes the Weil-étale topology (Lichtenbaum 2005). Geisser and Levine (2000, 2001) have shown that $\mathbb{Z}(r)$ satisfies the "Kummer sequence" axiom (Lichtenbaum 1984, (3), p.130; Milne 1988, (A2)$_p$, p.68). Assume that X satisfies the Tate conjecture and that, for some prime l, the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)$ is nil. It then follows from Milne 1986 (and addendum) that Conjecture 1.2 holds with the groups $H_{\mathrm{abs}}^j(X,r)$ replaced by $H_{\mathrm{mot}}^j(X,r)$.

The proof of this has four steps (X is as above).

◇ From the Kummer sequence axiom, we get exact sequences

$$0 \to H_{\mathrm{mot}}^j(X,r)_l \to H^j(X, \mathbb{Z}_l(r)) \to T_l H_{\mathrm{mot}}^{j+1}(X,r) \to 0.$$

Here $M_l = \varprojlim_n M^{(l^n)}$ is the l-adic completion of M, and $T_l M = \varprojlim_n \mathrm{Ker}(M \xrightarrow{l^n} M)$.

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[2]This should be taken with a grain of salt: the Ext's are in different categories.

⬦   A theorem of Gabber shows that $H^j(X, \mathbb{Z}_l(r))$ is torsion-free for almost all l (i.e., for all but possibly finitely many).

⬦   Tate's conjecture for X implies that the map

$$H^j_{\mathrm{mot}}(X, r)' \otimes \mathbb{Z}_l \to H^j(X, \mathbb{Z}_l(r))$$

is an isomorphism. Here M' is the quotient of M by its largest uniquely divisible subgroup. Together with Gabber's theorem, and the local results in Milne 1986, this implies Conjecture 1.2 for the groups $H^j_{\mathrm{mot}}(X, r)'$.

⬦   If the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)$ is nil, then the groups $H^j_{\mathrm{mot}}(X, r)$ are finitely generated, and so Conjecture 1.2 holds for the groups $H^j_{\mathrm{mot}}(X, r)$.

Notes

1.6.   Statement (b) of Conjecture 1.1 should be regarded as the Tate conjecture for motivic complexes. In particular, Conjecture 1.1 presupposes the usual Tate conjecture *if* the category $\mathrm{DM}(k)$ is defined using algebraic classes, but there is no need to do this. We envisage that Deligne's theory of absolute Hodge classes in characteristic zero can be extended to a theory of rational Tate classes in characteristic p for which the Tate conjecture is automatically true (see Milne 2009). Thus, we expect Conjecture 1.1 to be true (suitably interpreted) even if the Tate conjecture proves false (see §8).

1.7.   There appears to be no direct relation between Conjecture 1.1 and the Bloch-Kato conjecture. The latter applies to a motive over a global field (see, for example, Fontaine 1992), whereas our conjecture applies to a complex of motives over a finite field.

Consider, for example, a smooth projective surface X over a finite field k. As noted above, our conjecture for the motive of X essentially becomes the conjecture of Artin and Tate for the surface X. Now let $f : X \to C$ be a morphism from X onto a curve C, and let J be the Jacobian of the generic fibre of f (so J is an abelian variety over the function field $k(C)$ of C). Assume that f has connected geometric fibres and smooth generic fibre. The conjecture of Bloch and Kato for the motive $h_1(J)$ over $k(C)$ is essentially the conjecture of Birch and Swinnerton-Dyer for J over $k(C)$. It is known that the Artin-Tate conjecture for X/k is equivalent to the Birch/Swinnerton-Dyer conjecture for $J/k(C)$. Under some hypotheses on the map f, this was proved directly (Gordon 1979), but the general proof is difficult and indirect: it proceeds by showing that each conjecture is equivalent to the Tate conjecture for X (Milne 1975, Kato and Trihan 2003). In other words, passing between the two conjectures in this case is no easier than deducing the conjectures separately from the Tate conjecture. We expect that a similar statement is true for the Bloch-Kato conjecture and our conjecture in general.

1.8.   For some background and history to these questions, see Milne 2013.

1.9.   Let P be the motivic complex $\mathbb{1}[-j_0]$ over $\mathbb{F}_q$. Then

$$H^j_{\mathrm{abs}}(\bar{P}, r) \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{DM}(\mathbb{F})}(\mathbb{1}, \mathbb{1}(r)[j - j_0])$$

is zero except that $H^{j_0}_{\mathrm{abs}}(\bar{P}, r) = \mathbb{Z}$. From the spectral sequence

$$H^i(\Gamma_0, H^j_{\mathrm{abs}}(\bar{P}, r)) \implies H^{i+j}_{\mathrm{abs}}(P, r)$$

we find that

$$H^{j_0}_{\mathrm{abs}}(P,r) \simeq \mathbb{Z} \simeq H^{j_0+1}(P,r)$$

and the remaining groups are zero. As $Z(P(r),t) = (1-q^r t)^{(-1)^{j_0+1}}$, we see that Conjecture 1.2 is true for P.

For a general $P \in \mathrm{ob}\,\mathsf{DM}(\mathbb{F}_q)$, the Tate and other conjectures predict that there should be a distinguished triangle

$$P_0 \to P \to P_1 \to P_0[1]$$

with P_0 a direct sum of motivic complexes of the form $\mathbb{1}[-j_0]$ and P_1 such that $Z(P_1,t)$ has no pole or zero at $t = q^{-r}$ and $H^j_{\mathrm{abs}}(P_1,r)$ is finite for all j. From this, (a) and (b) of Conjecture 1.2 follow.

1.10. Conjecture 1.2 is a special case of Conjecture 1.1, and so Conjecture 1.1 implies Conjecture 1.2. On the other hand, for M,N in $\mathsf{DM}(k)$, we should have

$$R\,\mathrm{Hom}(M,N) \simeq R\,\mathrm{Hom}(\mathbb{1}, R\underline{\mathrm{Hom}}(M,N)),$$

and so Conjecture 1.2 for $P = R\underline{\mathrm{Hom}}(M,N)$ implies Conjecture 1.1 for M,N. Nevertheless, it is convenient to have both forms of the conjecture.

Evidence for the conjectures

THE LOCAL VERSIONS ARE TRUE

The statement of Conjecture 1.1 uses few properties of the triangulated category $\mathsf{DM}(k)$. In fact, the same arguments lead to similar conjectures for $\mathsf{D}^b_c(k,\mathbb{Z}_\ell)$ and $\mathsf{D}^b_c(R)$. Assume that the Frobenius maps are semisimple. Then the conjecture for $\mathsf{D}^b_c(\mathbb{Z}_\ell)$ is easy to prove (see §4 below). That for $\mathsf{D}^b_c(R)$ is less easy, but is proved in Milne and Ramachandran 2013 (see §5 below).

The realization functors r_ℓ and r_p define maps

$$\mathrm{Hom}_{\mathsf{DM}(k)}(M,N) \to \mathrm{Hom}_{\mathsf{D}^b_c(k,\mathbb{Z}_\ell)}(r_\ell M, r_\ell N)$$

$$\mathrm{Hom}_{\mathsf{DM}(k)}(M,N) \to \mathrm{Hom}_{\mathsf{D}^b_c(R)}(r_p M, r_p N)$$

$(M,N \in \mathrm{ob}(\mathsf{DM}(k)))$, and hence maps

$$r_\ell(M,N)\colon \mathrm{Hom}_{\mathsf{DM}(k)}(M,N) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathsf{D}^b_c(k,\mathbb{Z}_\ell)}(r_\ell M, r_\ell N)$$

$$r_p(M,N)\colon \mathrm{Hom}_{\mathsf{DM}(k)}(M,N) \otimes \mathbb{Z}_p \to \mathrm{Hom}_{\mathsf{D}^b_c(R)}(r_p M, r_p N).$$

When k is finite, we expect the maps $r_l(M,N)$ to be isomorphisms (see below). If so, then our local results imply a statement that is only a little weaker than Conjecture 1.1 (see 6.8).

VOEVODSKY MOTIVIC COMPLEXES

From Example 1.5, it is clear that, in order for Conjecture 1.2 to be true, we must have

$$\mathrm{Hom}_{\mathsf{DM}(k)}(\mathbb{1}, M(X)(r)[j]) \simeq H^j_{\mathrm{mot}}(X,r)$$

for all smooth projective varieties X, where $M(X)$ is the motivic complex of X and $H^j_{\mathrm{mot}}(X,r)$ is the *Weil-étale* motivic cohomology group. This excludes Voevodsky's category $\mathsf{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}}$, which gives the *Zariski* motivic cohomology groups.

Instead, we need to look at $(\mathsf{DM}_{\mathrm{et}}(k))^{\mathrm{opp}}$. This is known to give the *étale* motivic cohomology groups *except for the p-part*; in fact, $\mathsf{DM}_{\mathrm{et}}(k)$ is a $\mathbb{Z}[p^{-1}]$-linear category. In order to obtain a category that is also correct for p-torsion, we define an exact functor $(\mathsf{DM}_{\mathrm{et}}(k))_{\mathbb{Q}}^{\mathrm{opp}} \to \mathsf{D}_c^b(R)_{\mathbb{Q}}$ and form the "fibred product" category

$$\begin{array}{ccc} \mathsf{DM}(k) & \longrightarrow & (\mathsf{DM}_{\mathrm{et}}(k))^{\mathrm{opp}} \\ \downarrow & & \downarrow \\ \mathsf{D}_c^b(R) & \longrightarrow & \mathsf{D}_c^b(R)_{\mathbb{Q}}. \end{array}$$

This category gives the full *étale* motivic cohomology groups, which is correct over $\mathbb{F}$, but over $\mathbb{F}_q$ we want the *Weil-étale* motivic cohomology groups. There are two more-or-less equivalent ways of achieving this: repeat the above construction with Lichtenbaum's Weil-étale topology, or define $\mathsf{DM}_{\mathrm{W\acute{e}t}}(\mathbb{F}_q)$ in terms of $\mathsf{DM}_{\mathrm{et}}(\mathbb{F})$. See §7.

Once $\mathsf{DM}(k)$ has been defined, we can proceed as in Example 1.5 (we use the same notations).

◇ From the rigidity theorem of Suslin and Voevodsky (Voevodsky 2000, 3.3.3) and its p counterpart, we obtain exact sequences

$$0 \to \mathrm{Ext}^j(M, N(r))_l \to \mathrm{Ext}^j(r_l M, r_l N(r)) \to T_l \mathrm{Ext}^j(M, N(r)) \to 0$$

for all l.

◇ Assuming that the category $\mathsf{DM}(k)_{\mathbb{Q}}$ is semisimple, we show that, for almost all l, the group $\mathrm{Ext}^j(r_l M, r_l N(r))$ is torsion-free.

◇ Tate's conjecture for smooth projective varieties implies that the map

$$\mathrm{Ext}^j(M, N)' \otimes \mathbb{Z}_l \to \mathrm{Ext}^j(r_l M, r_l N)$$

is an isomorphism. Together with the preceding statement and our local results (§4,§5), this implies Conjecture 1.2 for the groups $\mathrm{Ext}^j(M, N)'$.

◇ The groups $\mathrm{Ext}^j(M, N)$ contain no nontrivial uniquely divisible subgroups if and only if some realization functor r_l is faithful (in which case, they all are). This is true, for example, if the Beilinson conjecture[3] holds for smooth projective varieties.

Of course, a theorem that assumes the Tate conjecture for *all* smooth projective varieties over $\mathbb{F}_q$ is not of much value, since we may never be able to prove this. Fortunately, our results are more precise (see §§6,7,8).

Motivic complexes for rational Tate classes.

If the category $\mathsf{DM}(k)$ is defined using algebraic cycles, then Conjecture 1.1 requires the Tate conjecture. However, we expect that the theory of rational Tate classes will provide a (unique) extension of Deligne's theory of absolute Hodge cycles to mixed characteristic. Assuming the rationality conjecture (8.1), which is weaker than the Hodge conjecture for CM abelian varieties, we construct in §8 a category $\mathsf{DM}(\mathbb{F}_q)$ of motivic complexes for which the conjectures are automatically true.

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[3]The Beilinson conjecture says that, for smooth projective varieties over a finite field, rational equivalence coincides with numerical equivalence with $\mathbb{Q}$-coefficients.

Notations

We use ℓ for a prime number $\neq p$, and l for a prime number, possibly p. The l-adic absolute value is normalized so that

$$|a|_l^{-1} = (\mathbb{Z}_l : a\mathbb{Z}_l), \quad a \in \mathbb{Z}_l.$$

For an object M of an abelian category, $M^{(n)}$ denotes the cokernel of multiplication by n on M. For an object M of a triangulated category, $M^{(n)}$ denotes the cone of $M \xrightarrow{n} M$.

2   The complex $E(M,N,r)$

Differential graded enhancements

Recall that a **graded category** is an additive category equipped with gradations

$$\mathrm{Hom}(A,B) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(A,B)$$

on the Hom groups that are compatible with composition of morphisms; in particular, $\mathrm{id}_A \in \mathrm{Hom}^0(A,A)$. Such a category is a **differential graded (dg) category** if, in addition, it is equipped with differentials $d : \mathrm{Hom}^n(A,B) \to \mathrm{Hom}^{n+1}(A,B)$ such that $d \circ d = 0$ and

$$d(g \circ f) = dg \circ f + (-1)^{\deg g}(g \circ df)$$

whenever g is homogeneous and $g \circ f$ is defined. The **homotopy category** $\mathrm{Ho}(\mathcal{C})$ of a dg category $\mathcal{C}$ has the same objects a $\mathcal{C}$ but

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A,B) = H^0(\mathrm{Hom}_{\mathcal{C}}^\bullet(A,B)).$$

Let C be a triangulated category. By a **dg enhancement** of C, we mean a dg category $\mathcal{C}$ such that $\mathrm{Ho}(\mathcal{C}) = \mathsf{C}$ and

$$\mathrm{Hom}_{\mathcal{C}}^\bullet(A,B[m]) \simeq \mathrm{Hom}_{\mathcal{C}}^\bullet(A,B)[m] \tag{2}$$

for $A,B \in \mathrm{ob}\,\mathsf{C}$, and

$$\mathrm{Hom}_{\mathcal{C}}(C,\mathrm{Cone}(f)) \simeq \mathrm{Cone}(\mathrm{Hom}_{\mathcal{C}}(C,A) \xrightarrow{f \circ -} \mathrm{Hom}(C,B))$$

for all $C \in \mathrm{ob}\,\mathcal{C}$ and morphism $f : A \to B$ in $\mathcal{C}$ such that $d \circ f = 0$. More generally, we allow a dg enhancement of C to be a pretriangulated dg category $\mathcal{C}$ together with an equivalence to C from the triangulated category associated with $\mathcal{C}$.

When C has a fixed dg enhancement of $\mathcal{C}$, we write

$$R\,\mathrm{Hom}_{\mathsf{C}}^\bullet(A,B) = \mathrm{Hom}_{\mathcal{C}}^\bullet(A,B). \tag{3}$$

Thus, $R\,\mathrm{Hom}_{\mathsf{C}}^\bullet(A,B)$ is a complex such that

$$H^n(R\,\mathrm{Hom}_{\mathsf{C}}^\bullet(A,B)) = \mathrm{Hom}_{\mathsf{C}}(A,B[n]) \overset{\mathrm{def}}{=} \mathrm{Ext}^n(A,B).$$

The cohomology of Γ_0

Let Γ_0 be the free abelian group generated by a single element γ (thus $\Gamma_0 \simeq \mathbb{Z}$), and let $\mathbb{Z}\Gamma_0$ be its group ring. For a Λ-module M, we let M_* denote the corresponding co-induced module. Recall that this consists of the maps from Γ_0 to M, and that $\tau \in \Gamma_0$ acts on $f \in M_*$ according to the rule $(\tau f)(\sigma) = f(\sigma\tau)$. For each Γ_0-module M, there is an exact sequence

$$0 \to M \longrightarrow M_* \xrightarrow{\alpha_\gamma} M_* \to 0, \tag{4}$$

in which the first map sends $m \in M$ to the map $\sigma \mapsto \sigma m$ and the second map sends $f \in M_*$ to $\sigma \mapsto f(\sigma\gamma) - \gamma f(\sigma)$. Let F denote the functor $M \mapsto M^{\Gamma_0} \colon \mathsf{Mod}(\mathbb{Z}\Gamma_0) \to \mathsf{Mod}(\mathbb{Z})$. The class of co-induced $\Lambda\Gamma_0$-modules is F-injective, and so (4) defines isomorphisms

$$RF(M) \simeq F(M_* \xrightarrow{\alpha_\gamma} M_*) \simeq (M \xrightarrow{1-\gamma} M)$$

in $\mathsf{D}^+(\mathbb{Z})$. For the second isomorphism, note that $(M_*)^{\Gamma_0}$ consists of the constant functions $\Gamma_0 \to M$, and that if f is the constant function with value m, then

$$(a_\gamma f)(\sigma) = f(\sigma\gamma) - \gamma f(\sigma) = m - \gamma m, \text{ all } \sigma \in \Gamma_0.$$

For every complex X of $\mathbb{Z}\Gamma_0$-modules, there is an exact sequence

$$0 \to X \to X_* \xrightarrow{\alpha_\gamma} X_* \to 0 \tag{5}$$

of complexes with $X_*^j = (X^j)_*$ for all j. We deduce from (5) isomorphisms

$$RF(X) \simeq s(F(X_* \longrightarrow X_*)) \simeq s(X \xrightarrow{1-\gamma} X)$$

in $\mathsf{D}^+(\mathbb{Z})$ where $X \xrightarrow{1-\gamma} X$ is a double complex with X as both its zeroth and first column and s denotes the associated total complex. In other words,

$$RF(X)[1] \simeq \mathrm{Cone}(1 - \gamma \colon X \to X). \tag{6}$$

From (6), we get a long exact sequence

$$\cdots \to H^{j-1}(X) \xrightarrow{1-\gamma} H^{j-1}(X) \to R^j F(X) \to H^j(X) \xrightarrow{1-\gamma} H^j(X) \to \cdots. \tag{7}$$

Note that
$$R^1 F(\mathbb{Z}) \simeq H^1(\Gamma_0, \mathbb{Z}) \simeq \mathrm{Hom}(\Gamma_0, \mathbb{Z}),$$

which has a canonical element $\theta \colon \gamma \mapsto 1$. We can regard θ as an element of

$$\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}[1]) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathsf{D}^+(\mathbb{Z}\Gamma_0)}(\mathbb{Z}, \mathbb{Z}[1]).$$

Thus, for X in $\mathsf{D}^+(\mathbb{Z}\Gamma_0)$, we obtain maps

$$\theta \colon X \to X[1]$$
$$R\theta \colon RF(X) \to RF(X)[1]. \tag{8}$$

The second map is described explicitly by the following map of double complexes:

$$
\begin{array}{ccccc}
RF(X) & & X & \xrightarrow{\ 1-\gamma\ } & X \\
\downarrow{\scriptstyle R\theta} & & & \downarrow{\scriptstyle \gamma} & \\
RF(X)[1] & X & \xrightarrow{\ 1-\gamma\ } & X & \\
& {\scriptstyle -1} & {\scriptstyle 0} & & {\scriptstyle 1}
\end{array}
$$

For all j, the following diagram commutes

$$
\begin{array}{ccc}
R^j F(X) & \xrightarrow{\ d^j\ } & R^{j+1} F(X) \\
\downarrow & & \uparrow \\
H^j(X) & \xrightarrow{\ \mathrm{id}\ } & H^j(X)
\end{array}
\tag{9}
$$

where $d^j = H^j(R\theta)$ and the vertical maps are those in (7). The sequence

$$
\cdots \to R^{j-1} F(X) \xrightarrow{d^{j-1}} R^j F(X) \xrightarrow{d^j} R^{j+1} F(X) \to \cdots
\tag{10}
$$

is a complex because $R\theta \circ R\theta = 0$.

Construction of the complex $E(M,N,r)$

Let $\bar{k}$ be an algebraic closure of k, and let $\Gamma_0 \subset \mathrm{Gal}(\bar{k}/k)$ be the Weil group of k. We let γ denote the generator $x \mapsto x^q$ of Γ_0.

We require a dg enhancement $\mathcal{DM}(k)$ of $\mathsf{DM}(k)$. In particular, this means that there is a functor

$$
R\operatorname{Hom}\colon \mathsf{DM}(k)^{\mathrm{opp}} \times \mathsf{DM}(k) \to \mathsf{D}^+(\mathbb{Z})
$$

such that

$$
\operatorname{Hom}_{\mathsf{DM}(k)}(M, N[j]) \simeq H^j(R\operatorname{Hom}(M,N))
$$

for $M, N \in \operatorname{ob}\mathsf{DM}(k)$ and $j \in \mathbb{Z}$. We require that $R\operatorname{Hom}$ be related to the internal Hom by

$$
R\operatorname{Hom}(\mathbb{1}, R\underline{\operatorname{Hom}}(M,N)) \simeq R\operatorname{Hom}(M,N), \quad M, N \in \operatorname{ob}\mathsf{DM}(k).
\tag{11}
$$

We require that the functor

$$
R\operatorname{Hom}(\mathbb{1},-)\colon \mathsf{DM}(k) \to \mathsf{D}^+(\mathbb{Z})
$$

factors through $\mathsf{D}^+(\mathbb{Z}\Gamma_0)$; moreover, that this factorization arises from a factorization of $M \rightsquigarrow \operatorname{Hom}_{\mathsf{DM}(k)}(\mathbb{1},M)$ through $M \rightsquigarrow \operatorname{Hom}_{\mathsf{DM}(\bar{k})}(\mathbb{1},\bar{M})$.[4]

From (11), we see that

$$
R\operatorname{Hom}\colon \mathsf{DM}(k)^{\mathrm{opp}} \times \mathsf{DM}(k) \to \mathsf{D}^+(\mathbb{Z})
$$

---

[4]For the motivic complex of a smooth projective variety X, this amounts to requiring the existence of a spectral sequence

$$
H^i(\Gamma_0, H^j_{\mathrm{abs}}(\bar{X},r)) \implies H^{i+j}_{\mathrm{abs}}(X,r)
$$

for each integer r.

factors through $RF \colon \mathsf{D}^+(\mathbb{Z}\Gamma_0) \to \mathsf{D}^+(\mathbb{Z})$:

$$
\begin{array}{ccc}
& \xrightarrow{\;R\operatorname{Hom}(\mathbb{1},-)\;} & \\
\mathsf{DM}(k) \xrightarrow{\hspace{3cm}} \mathsf{D}^+(\mathbb{Z}\Gamma_0) \xrightarrow{\;RF\;} \mathsf{D}^+(\mathbb{Z}) \\
\Big\uparrow{\scriptstyle R\underline{\operatorname{Hom}}(-,-)} \qquad\qquad \xrightarrow{\;R\operatorname{Hom}(-,-)\;} \\
\mathsf{DM}(k)^{\mathrm{opp}} \times \mathsf{DM}(k)
\end{array}
$$

Therefore, for each pair M, N of objects of $\mathsf{DM}(k)$, there is a well-defined object X of $\mathsf{D}(\mathbb{Z}\Gamma_0)$ such that $RFX = R\operatorname{Hom}(M,N)$, and

$$
H^j(RFX) = H^j(R\operatorname{Hom}(M,N)) \simeq \operatorname{Hom}_{\mathsf{DM}(k)}(M, N[j]) \overset{\mathrm{def}}{=} \operatorname{Ext}^j(M,N).
$$

Now (10) becomes a complex

$$
E(M,N,r)\colon \quad \cdots \longrightarrow \operatorname{Ext}^{j-1}(M,N(r)) \xrightarrow{d^{j-1}} \operatorname{Ext}^j(M,N(r)) \xrightarrow{d^j} \operatorname{Ext}^{j+1}(M,N(r)) \longrightarrow \cdots.
$$

3   The zeta function of a motivic complex

Throughout this section, k is a perfect field.

Let P be an object of $\mathsf{D}^b_c(k, \mathbb{Z}_\ell)$. We can think of P as a bounded complex of $\mathbb{Z}_\ell$-modules with finitely generated cohomology, equipped with a continuous action of Γ. For an endomorphism α of P, we define $c_\alpha(t)$ to be the alternating product of the characteristic polynomials of α acting on the $\mathbb{Q}_\ell$-vector spaces $H^i(P)_{\mathbb{Q}_\ell}$:

$$
c_\alpha(t) = \prod_i \det(1 - \alpha t \,|\, H^i(P)_{\mathbb{Q}_\ell})^{(-1)^{i+1}} \in \mathbb{Q}_\ell(t).
$$

Let P be an object of $\mathsf{D}^b_c(R)$ and let α be an endomorphism of P. We define $c_\alpha(t)$ to be the alternating product of the characteristic polynomials of α acting on the isocrystals $H^i(sP)_K$:

$$
c_\alpha(t) = \prod_i \det(1 - \alpha t \,|\, H^i(sP)_K)^{(-1)^{i+1}} \in \mathbb{Q}_p(t)
$$

(cf. Demazure 1972, p.89).

Determinants

Let T be a triangulated category, and let $\mathsf{T}_\approx$ denote the subcategory with the same objects but with only the isomorphisms as morphisms. A ***determinant functor*** on T is a functor $f \colon \mathsf{T}_\approx \to \mathsf{P}$ from T to a Picard category satisfying certain natural conditions (Breuning 2011, 3.1). Every (essentially) small triangulated category admits a universal determinant functor $f \colon \mathsf{T}_\approx \to \mathsf{P}$, which is unique up to a non-unique isomorphism (ibid. §4). The automorphism group of an object X of P is independent of X up to a well-defined isomorphism — we denote it by $\pi_1(\mathsf{P})$. The ***determinant*** $\det(\alpha)$ of an automorphism α of an object P of T is defined to be the element $f(\alpha)$ of $\operatorname{Aut}_{\mathsf{P}}(f(P)) \simeq \pi_1(\mathsf{P})$.

Let $f \colon \mathsf{DM}(k)_{\mathbb{Q}} \to \mathsf{P}$ be the universal determinant functor for the triangulated category $\mathsf{DM}(k)_{\mathbb{Q}}$. Because $\mathsf{DM}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, $\pi_1(\mathsf{P}) \supset \mathbb{Q}^\times$.

PROPOSITION 3.1. *Let $P \in \mathrm{ob}\,\mathsf{DM}(k)_{\mathbb{Q}}$, and let α be an automorphism of $P_{\mathbb{Q}}$. If $\det(1 - \alpha n) \in \mathbb{Q}^{\times}$ for all $n \in \mathbb{Z}$, then there exists a unique $c_{\alpha}(t) \in \mathbb{Q}(t)$ such that $c_{\alpha}(n) = \det(1 - \alpha n)$ for all $n \in \mathbb{Z}$. Moreover, $c_{\alpha}(t) = c_{r_l(\alpha)}(t)$ for all l.*

PROOF. Let $P, P_1, Q, Q_1 \in \mathbb{Q}[t]$ be such that $P(n)/Q(n) = P_1(n)/Q_1(n)$ for all $n \in \mathbb{Z}$. Then $P(t)Q_1(t) - P_1(t)Q(t)$ has infinitely many roots, and so is zero. This proves the uniqueness of $c_{\alpha}(t)$.

Fix an l (possibly p). There is a commutative diagram

$$
\begin{array}{ccc}
\mathsf{DM}(k) & \longrightarrow & \mathsf{D}_c^b(\mathbb{Z}_l) \\
\downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\
\mathsf{P} & \dashrightarrow & \mathsf{P}_l
\end{array}
$$

in which the vertical arrows are universal. Let $c_{r_l(\alpha)}(t) = P(t)/Q(t)$ with $P(t), Q(t) \in \mathbb{Q}_l[t]$. Then $P(n)/Q(n) = \det(1 - \alpha n)$ for all $n \in \mathbb{Z}$. Let $P(t) = \sum c_i t^i$ and $Q(t) = \sum d_j t^j$. Choose distinct rational numbers $n_1, \ldots, n_r$ with r at least $\max(\deg(P), \deg(Q))$. Then $(c_1, c_2, \ldots, d_1, \ldots)$ is the unique solution of the system of linear equations

$$
\sum c_i n_s^i = \det(1 - \alpha n_s) \cdot \sum d_j n_s^j, \quad s = 1, \ldots, r.
$$

As the coefficients of these linear equations lie in $\mathbb{Q}$, their solution does also. $\square$

ASIDE 3.2. The definition of the characteristic polynomial of an endomorphism of a motivic complex in (3.1) follows that in Milne 1994, 2.1, for an endomorphism of a motive, which in turn follows that in Weil 1948, IX, for an endomorphism of an abelian variety.

ASIDE 3.3. What (conjecturally) is the P attached to $\mathsf{DM}(k)$?

Traces

Let P be an object in a tensor category. When P has a dual $(P^{\vee}, \mathrm{ev}, \delta)$, we can define the ***trace*** of an endomorphism α of P to be the composite of

$$
\mathbf{1} \xrightarrow{\delta} P \otimes P^{\vee} \xrightarrow{\text{transpose}} P^{\vee} \otimes P \xrightarrow{\mathrm{id} \otimes \alpha} P^{\vee} \otimes P \xrightarrow{\mathrm{ev}} \mathbf{1}.
$$

It is an element of $\mathrm{End}(\mathbf{1})$.

Now assume that $\mathsf{DM}(k)_{\mathbb{Q}}$ has the structure of a rigid tensor category and that each of the realization functors r_l is a tensor functor. Define $c_{\alpha}(t)$ to be the power series satisfying (1) with $N_n = \mathrm{Tr}(\alpha^n | P)$. Then $c_{\alpha}(t)$ maps to $c_{r_l(\alpha)}(t)$ under $\mathbb{Q}[[t]] \mapsto \mathbb{Q}_l[[t]]$ for all l (including $l = p$). As[5] $\mathbb{Q}(t) = \mathbb{Q}[[t]] \cap \mathbb{Q}_l(t)$, this shows that $c_{\alpha}(t) \in \mathbb{Q}(t)$.

Zeta functions

Now assume that k is finite, with $q = p^a$ elements, and let γ be the generator $x \mapsto x^q$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. We define the zeta function $Z(P, t)$ of an object P of $\mathsf{D}_c^b(k, \mathbb{Z}_{\ell})$ (resp. $\mathsf{D}_c^b(R)$) to be $c_{\gamma}(t)$ (resp. $c_{F^a}(t)$). Let P be an object of $\mathsf{DM}(k)$. We say that an element $Z(P, t)$ of

---

[5]The condition that a power series be a rational function is linear; see Bourbaki, Algèbre, Chap. IV, §4, Ex. 1.

$\mathbb{Q}(t)$ is the ***zeta function*** of P if $Z(P,t) = Z(r_l(P),t)$ for all prime numbers l (including $l = p$) — clearly $Z(P,t)$ is unique if it exists.

Now assume that there is a Frobenius endomorphism π of $\mathrm{id}_{\mathsf{DM}(k)}$ such that $r_\ell(\pi_P)$ acts as γ on $r_\ell(P)$ and $r_p(\pi_P)$ acts as F^a on $r_p(P)$.

PROPOSITION 3.4. *Under each of the following two hypotheses, every object of* $\mathsf{DM}(k)$ *has a zeta function:*

(a) *for all* $P \in \mathrm{ob}\,\mathsf{DM}(k)$ *and* $n \in \mathbb{Z}$, $\det(1 - \pi_P n) \in \mathbb{Q}^\times$;
(b) *the category* $\mathsf{DM}(k)_\mathbb{Q}$ *has a rigid tensor structure, and each of the realization functors* r_l *is a tensor functor.*

PROOF. Immediate consequence of the above discussion.   □

4   The local conjecture at ℓ

4.1. For a scheme X of finite type over a field k and a prime $\ell \neq \mathrm{char}(k)$, we let $\mathsf{D}_c^b(X, \mathbb{Z}_\ell)$ denote the triangulated category of bounded constructible $\mathbb{Z}_\ell$-complexes on X (Ekedahl 1990). This can be constructed as follows. Let $\Lambda = \mathbb{Z}_\ell$ and let $\Lambda_n = \mathbb{Z}_\ell / \ell^n \mathbb{Z}_\ell$. The inverse systems

$$F = (\cdots \leftarrow F_n \leftarrow F_{n+1} \leftarrow \cdots)$$

in which F_n is a sheaf of Λ_n-modules on X form an abelian category $\mathcal{S}(X, \Lambda_\bullet)$, whose derived category we denote by $\mathsf{D}(X, \Lambda_\bullet)$. An inverse system $F = (F_n) \in \mathrm{ob}\,\mathcal{S}(X, \Lambda_\bullet)$ is said to be ***essentially zero*** if, for each n, there exists an $m \geq n$ such that the transition map $F_m \to F_n$ is zero. A complex $K \in \mathrm{ob}\,\mathsf{D}(X, \Lambda_\bullet)$ is ***essentially zero*** if each inverse system $H^i(K)$ is essentially zero. Now $\mathsf{D}_c^b(X, \mathbb{Z}_\ell)$ can defined to be the full subcategory of $\mathsf{D}(X, \Lambda_\bullet)$ consisting of the complexes $M = (M_n)_{n \in \mathbb{N}}$ such that the inverse system

$$\Lambda_1 \otimes_{\Lambda_\bullet}^L M \overset{\mathrm{def}}{=} (\Lambda_1 \otimes_{\Lambda_n}^L M_n)_{n \in \mathbb{N}}$$

of complexes is isomorphic, modulo essentially zero complexes, to the constant inverse system defined by an object of $\mathsf{D}_c^b(X, \Lambda_1)$ (triangulated category of bounded complexes of Λ_1-sheaves on X with constructible cohomology).

4.2. Let $\mathcal{S}(X, \mathbb{Z}_\ell)$ denote the category of sheaves of $\mathbb{Z}_\ell$-modules on X_{et}. The obvious functors

$$\mathcal{S}(X, \Lambda_\bullet) \xrightarrow{\pi_*} \mathcal{S}(X, \mathbb{Z}_\ell) \xrightarrow{\pi^*} \mathcal{S}(X, \Lambda_\bullet)$$

induce functors on the corresponding derived categories

$$\mathsf{D}^+(X, \Lambda_\bullet) \xrightarrow{R\pi_*} \mathsf{D}^+(X, \mathbb{Z}_\ell)_{\mathrm{litt}} \xrightarrow{L\pi^*} \mathsf{D}^+(X, \Lambda_\bullet)$$

whose composite we denote by $M \rightsquigarrow \hat{M}$. The essential image $\mathsf{D}^+(X, \Lambda_\bullet)_{\mathrm{norm}}$ of $L\pi^*$ consists of the complexes M such that $M \simeq \hat{M}$. Let $\mathsf{D}_c^b(X, \Lambda_\bullet)$ denote the full subcategory of $\mathsf{D}^+(X, \Lambda_\bullet)$ consisting of complexes M such that $M \simeq \hat{M}$ and $\Lambda_1 \otimes_{\Lambda_\bullet}^L M$ lies in $\mathsf{D}_c^b(X, \Lambda_1)$. The canonical functor from $\mathsf{D}^+(X, \Lambda_\bullet)_{\mathrm{norm}}$ to Ekedahl's category $\mathsf{D}^+(X, \mathbb{Z}_\ell)$ induces an equivalence of categories

$$\mathsf{D}_c^b(X, \Lambda_\bullet) \to \mathsf{D}_c^b(X, \mathbb{Z}_\ell).$$

See Fargues 2009, 5.15.

4.3. Let k be a field, and let Γ be its absolute Galois group. Let $\mathsf{Mod}(\mathbb{Z}_l\Gamma)$ denote the category of $\mathbb{Z}_l$-modules with a continuous action of Γ, and let $\mathsf{D}(\mathbb{Z}_l\Gamma)$ denote its derived category. Define $\mathsf{D}_c^b(\mathbb{Z}_l\Gamma)$ to be the subcategory of $\mathsf{D}(\mathbb{Z}_l\Gamma)$ of bounded complexes with finitely generated cohomology (as a $\mathbb{Z}_l$-module). It is a triangulated category with t-structure whose heart is the category of continuous representations of Γ on finitely generated $\mathbb{Z}_l$-modules. When $l \neq \mathrm{char}(k)$, the functor sending $(M_n)_n \rightsquigarrow \varprojlim M_n(k^{\mathrm{sep}})$ derives to a functor

$$\alpha: \mathsf{D}_c^b(k, \mathbb{Z}_l) \to \mathsf{D}_c^b(\mathbb{Z}_l\Gamma).$$

4.4. Now let k be a finite field with q elements, and equip $\Gamma \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ with the topological generator $\gamma: x \mapsto x^q$. For M in $\mathsf{D}_c^b(\mathbb{Z}_l\Gamma)$, we write $\bar{M}$ for M as an object of $\mathsf{D}_c^b(\mathbb{Z}_l)$.

Let $M, N \in \mathsf{D}_c^b(\mathbb{Z}_\ell\Gamma)$, and let $P = R\underline{\mathrm{Hom}}(M, N)$. Note that

$$P = R\,\mathrm{Hom}(\bar{M}, \bar{N})$$

regarded as a continuous $\mathbb{Z}_\ell\Gamma$-module. Let

$$f_j: \mathrm{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma \to \mathrm{Ext}^j(\bar{M}, \bar{N}(r))_\Gamma$$

be the map induced by the identity map on $\mathrm{Ext}^j(\bar{M}, \bar{N}(r))$.

Let $[S]$ denote the cardinality of a set S. For a homomorphism $f: M \to N$ of abelian groups, we let $z(f) = [\mathrm{Ker}(f)]/[\mathrm{Coker}(f)]$ when both cardinalities are finite. On applying Lemma 5.1 of Milne and Ramachandran 2013 to the $\mathbb{Z}_\ell\Gamma$-module

$$H^j(P) = \mathrm{Ext}^j(\bar{M}, \bar{N}),$$

we obtain the following statement:

> $z(f_j)$ is defined if and only if the minimum polynomial of γ on $H^j(P(r))_\mathbb{Q}$ does not have q^r as a multiple root, in which case
>
> $$z(f_j) = \left| \prod_{i,\, a_{j,i} \neq q^r} \left(1 - \frac{a_{j,i}}{q^r}\right) \right|,$$
>
> where $(a_{j,i})_i$ is the family of eigenvalues of γ acting on $H^j(P)_\mathbb{Q}$.

4.5. By a $\Lambda_\bullet\Gamma$-module, we mean an inverse system

$$M = (M_0 \leftarrow \cdots \leftarrow M_m \leftarrow M_{m+1} \leftarrow \cdots)$$

with M_m a discrete Γ-module killed by l^m. For example, $\Lambda_\bullet$ is the $\Lambda_\bullet\Gamma$-module $(\mathbb{Z}/l^m\mathbb{Z})_m$ with the trivial action of Γ. Let

$$F: \mathsf{Mod}(\Lambda_\bullet\Gamma) \to \mathsf{Mod}(\mathbb{Z}_l)$$

denote the functor sending $M = (M_m)$ to the $\mathbb{Z}_l$-module $\varprojlim M_m^\Gamma$. If M satisfies the Mittag-Leffler condition, then

$$R^j F(M) \simeq H_{\mathrm{cts}}^j(\Gamma, \varprojlim M_m)$$

(cohomology with respect to continuous cocycles).

Let $\mathsf{D}(\Lambda_\bullet \Gamma)$ denote the derived category of complexes of $\Lambda_\bullet \Gamma$-modules. Then F derives to a functor $RF: \mathsf{D}^+(\Lambda_\bullet \Gamma) \to \mathsf{D}^+(\mathbb{Z}_l)$. For X in $\mathsf{D}(\Lambda_\bullet \Gamma)$,

$$RF(X) \simeq s(\bar{X} \xrightarrow{1-\gamma} \bar{X}) = \mathrm{Cone}(1-\gamma)[-1] \qquad (12)$$

where $\bar{X} = (R\varprojlim)(X)$ (see Milne and Ramachandran 2013, §5). From (12), we get a long exact sequence

$$\cdots \to H^{j-1}(\bar{X}) \xrightarrow{1-\gamma} H^{j-1}(\bar{X}) \to R^j F(X) \to H^j(\bar{X}) \xrightarrow{1-\gamma} H^j(\bar{X}) \to \cdots \qquad (13)$$

The canonical generator γ of Γ defines a canonical element θ_l in $H^1_{\mathrm{cts}}(\Gamma, \mathbb{Z}_l)$, which we can regard as an element of

$$\mathrm{Ext}^1_{\Lambda_\bullet \Gamma}(\Lambda_\bullet, \Lambda_\bullet) \simeq \mathrm{Hom}_{\mathsf{D}(\Lambda_\bullet \Gamma)}(\Lambda_\bullet, \Lambda_\bullet[1]).$$

For each X in $\mathsf{D}^+(\Lambda_\bullet \Gamma)$, θ_l defines morphisms

$$\theta_l: X \to X[1]$$
$$R\theta_l: RF(X) \to RF(X)[1].$$

The second map is described explicitly by the following map of double complexes:

$$
\begin{array}{ccccc}
RF(X) & & \bar{X} & \xrightarrow{1-\gamma} & \bar{X} \\
\downarrow{\scriptstyle R\theta} & & & \downarrow{\scriptstyle \gamma} & \\
RF(X)[1] & & \bar{X} & \xrightarrow{1-\gamma} & \bar{X} \\
& & -1 & 0 & 1
\end{array}
$$

For all j, the following diagram commutes

$$
\begin{array}{ccc}
R^j F(X) & \xrightarrow{d^j} & R^{j+1} F(X) \\
\downarrow & & \uparrow \\
H^j(\bar{X}) & \xrightarrow{\mathrm{id}} & H^j(\bar{X})
\end{array}
\qquad (14)
$$

where $d^j = H^j(R\theta)$ and the vertical maps are those in (13). The sequence

$$\cdots \to R^{j-1} F(X) \xrightarrow{d^{j-1}} R^j F(X) \xrightarrow{d^j} R^{j+1} F(X) \to \cdots \qquad (15)$$

is a complex because $R\theta \circ R\theta = 0$.

4.6. Let $M, N \in \mathsf{D}^b_c(k, \mathbb{Z}_\ell)$. The bifunctor

$$R\,\mathrm{Hom}: \mathsf{D}^b_c(k, \mathbb{Z}_\ell)^{\mathrm{opp}} \times \mathsf{D}^b_c(\mathbb{Z}_\ell \Gamma) \to D^+(\mathbb{Z}_\ell)$$

factors canonically through $RF: \mathsf{D}^+(\mathbb{Z}_\ell \Gamma) \to \mathsf{D}(\mathbb{Z}_\ell)$:

$$
\begin{array}{ccc}
 & \xrightarrow{\ R\mathrm{Hom}(\mathbb{1},-)\ } & \\
\mathsf{D}^b_c(k, \mathbb{Z}_\ell) \longrightarrow & \mathsf{D}^+(\mathbb{Z}_\ell \Gamma) \xrightarrow{\ RF\ } & \mathsf{D}^+(\mathbb{Z}) \\
{\scriptstyle R\underline{\mathrm{Hom}}(-,-)}\uparrow & \nearrow{\scriptstyle R\mathrm{Hom}(-,-)} & \\
\mathsf{D}^b_c(k, \mathbb{Z}_\ell)^{\mathrm{opp}} \times \mathsf{D}^b_c(k, \mathbb{Z}_\ell) & &
\end{array}
$$

Hence

$$R\operatorname{Hom}(M, N(r)) = RF(X)$$

for a well-defined object X in $\mathsf{D}(\mathbb{Z}_\ell \Gamma)$. The sequence (13) gives us short exact sequences

$$0 \to \operatorname{Ext}^{j-1}(\bar{M}, \bar{N}(r))_\Gamma \to \operatorname{Ext}^j(M, N(r)) \to \operatorname{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma \to 0 \qquad (16)$$

in which $(-)^\Gamma$ and $(-)_\Gamma$ denote the kernel and cokernel of $1 - \gamma$. Moreover, (15) becomes a complex $E(M, N, r)$,

$$\cdots \longrightarrow \operatorname{Ext}^{j-1}(M, N(r)) \xrightarrow{d^{j-1}} \operatorname{Ext}^j(M, N(r)) \xrightarrow{d^j} \operatorname{Ext}^{j+1}(M, N(r)) \longrightarrow \cdots.$$

This is the unique complex for which the following diagram commutes,

$$
\begin{array}{ccc}
\operatorname{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma & \xrightarrow{\ f^j\ } & \operatorname{Ext}^j(\bar{M}, \bar{N}(r))_\Gamma \\
\uparrow & & \downarrow \\
\cdots \longrightarrow \operatorname{Ext}^{j-1}(M, N(r)) \xrightarrow{d^{j-1}} \operatorname{Ext}^j(M, N(r)) \xrightarrow{d^j} \operatorname{Ext}^{j+1}(M, N(r)) \longrightarrow \cdots \\
\downarrow & & \uparrow \\
\operatorname{Ext}^{j-1}(\bar{M}, \bar{N}(r))^\Gamma & \xrightarrow{\ f^{j-1}\ } & \operatorname{Ext}^{j-1}(\bar{M}, \bar{N}(r))_\Gamma
\end{array}
$$

$$(17)$$

(the vertical maps are those in (14) and the maps f^j are induced by the identity map).

PROPOSITION 4.7. *Let $M, N \in \mathsf{D}^b_c(k, \mathbb{Z}_\ell)$, and let $P = R\underline{\operatorname{Hom}}(M, N)$. Let $r \in \mathbb{Z}$, and assume that, for all j, the minimum polynomial of γ on $H^j(\alpha P)_{\mathbb{Q}_\ell}$ does not have q^r as a multiple root.*
   (a) *The groups $\operatorname{Ext}^j(M, N(r))$ are finitely generated $\mathbb{Z}_\ell$-modules, and the alternating sum of their ranks is zero.*
   (b) *The zeta function $Z(P, t)$ of P has a pole at $t = q^{-r}$ of order*

$$\rho = \sum (-1)^{j+1} \cdot j \cdot \operatorname{rank}_{\mathbb{Z}_\ell}\left(\operatorname{Ext}^j(M, N(r))\right).$$

   (c) *The cohomology groups of the complex $\operatorname{Ext}^\bullet(M, N(r))$ are finite, and the alternating product $\chi^\times(M, N(r))$ of their orders satisfies*

$$\left| \lim_{t \to q^{-r}} Z(P, t)(1 - q^r t)^\rho \right|_\ell^{-1} = \chi^\times(M, N(r)).$$

PROOF. (a) Note that $\operatorname{Ext}^j(\bar{M}, \bar{N}(r)) = H^j(\alpha P(r))$, which is a finitely generated $\mathbb{Z}_\ell$-module. From (16), we see that

$$\operatorname{rank}(\operatorname{Ext}^j(M, N(r))) = \operatorname{rank}(H^{j-1}(\alpha P(r))_\Gamma) + \operatorname{rank}^j(H^j(\alpha P(r))^\Gamma).$$

The hypothesis on the action of the Frobenius element implies that

$$H^j(\alpha P(r))^\Gamma \otimes \mathbb{Q} \simeq H^j(\alpha P(r))_\Gamma \otimes \mathbb{Q}$$

for all j, and so

$$\sum_j (-1)^j \operatorname{rank}(\operatorname{Ext}^j(M, N(r)))$$
$$= \sum_j (-1)^j \left(\operatorname{rank}(H^{j-1}(\alpha P(r))^\Gamma) + \operatorname{rank}(H^j(\alpha P(r))^\Gamma) \right)$$
$$= 0$$

(b) Let ρ_j be the multiplicity of q^r as an inverse root of P_j. Then

$$\rho_j = \operatorname{rank} H^j(\alpha P(r))^\Gamma = \operatorname{rank} H^j(\alpha P(r))_\Gamma,$$

and so

$$\sum_j (-1)^{j+1} \cdot j \cdot \operatorname{rank}(\operatorname{Ext}^j(M, N(r))) = \sum_j (-1)^{j+1} \cdot j \cdot (\rho_{j-1} + \rho_j)$$
$$= \sum_j (-1)^{j+1} \rho_j$$
$$= -\rho.$$

(c) From Lemma 5.2 of Milne and Ramachandran 2013 applied to the diagram (17), we find that
$$\chi^\times(M, N(r)) = \prod_j z(f^j)^{(-1)^j}.$$

According to (4.4),
$$z(f^j) = \left| \prod_{i,\, a_{j,i} \neq q^r} \left(1 - \frac{a_{j,i}}{q^r} \right) \right|_\ell$$

where $(a_{j,i})_i$ is the family of eigenvalues of γ acting on $H^j(\alpha P(r))_{\mathbb{Q}_\ell}$. Note that

$$\prod_{i,\, a_{j,i} \neq q^r} \left(1 - \frac{a_{j,l}}{q^r} \right) = \lim_{t \to q^{-r}} \frac{P_j(t)}{(1 - q^r t)^{\rho_j}},$$

and so
$$\chi^\times(M, N(r)) = \left| \lim_{t \to q^{-r}} Z(M, N, t) \cdot (1 - q^r t)^\rho \right|_\ell^{-1}. \qquad \square$$

5   The local conjecture at p

This section is a brief review of Milne and Ramachandran 2013. The definitions and results reviewed in (5.1, 5.2, 5.3) are due to Ekedahl, Illusie, and Raynaud.

5.1. Let k be a perfect field, and let W be the ring of Witt vectors over k equipped with its Frobenius automorphism σ. We let K denote the field of fractions of W. Recall that the Raynaud ring is the graded algebra $R = R^0 \oplus R^1 = R^0[d]$ where R^0 is the Dieudonné ring $W_\sigma[F, V]$ and d (of degree 1) satisfies $d^2 = 0$, $FdV = d$, $ad = da$ ($a \in W$).

5.2. The graded R-modules and homomorphisms of degree 0 form an abelian category, whose derived category is denoted by $\mathsf{D}(R)$. There is a well-defined "completion" functor $M \rightsquigarrow \hat{M} \colon \mathsf{D}(R) \to \mathsf{D}(R)$, and an object M of $\mathsf{D}(R)$ is said to be complete if $M \simeq \hat{M}$. A coherent complex of graded R-modules is a complete complex M in $\mathsf{D}(R)$ such that $R_1 \otimes_R^L M$ is bounded with finite-dimensional cohomology. Here $R_1 = R/(VR + dVR)$. The coherent complexes form a full triangulated subcategory $\mathsf{D}_c^b(R)$ of $\mathsf{D}(R)$.

5.3. A complex of graded R-modules is often viewed as a bicomplex $M = M^{\bullet\bullet}$ of R^0-modules in which the first index corresponds to the R-gradation. Let F' act on $M^{i\bullet}$ as $p^i F$ (assuming only nonnegative i's occur). The differentials in the bicomplex commute with F', and so the associated simple complex sM is a complex of $W_\sigma[F']$-modules. When M is coherent, $H^n(sM)_K$ is an F-isocrystal.

We now take $k = \mathbb{F}_q$, $q = p^a$.

5.4. We define the zeta function $Z(M,t)$ of an M in $\mathsf{D}_c^b(R)$ to be the alternating product of the characteristic polynomials of F^a acting on the F-isocrystals $H^n(sM)_K$ (see §3).

5.5. Let Γ denote the Galois group of $\bar{k}/k$ equipped with its generator $x \mapsto x^q$. By a $\Lambda_\bullet\Gamma$-module, we mean an inverse system $M = (M_m)_{m \in \mathbb{N}}$ of discrete Γ-modules M_m killed by p^m. For example, $\Lambda_\bullet$ denotes the $\Lambda_\bullet\Gamma$-module $(\mathbb{Z}/p^m\mathbb{Z})_m$. Let F denote the functor sending a $\Lambda_\bullet\Gamma$-module M to the $\mathbb{Z}_p$-module $\varprojlim M_m^\Gamma$. If M satisfies the Mittag-Leffler condition, then

$$R^j F(M) \simeq H_{\mathrm{cts}}^j(\Gamma, \varprojlim M_m)$$

(cohomology with respect to continuous cocycles). Because Γ has a canonical generator, there is a canonical element θ_p in $H_{\mathrm{cts}}^1(\Gamma, \mathbb{Z}_p)$, which we can regard as an element of

$$\mathrm{Ext}_{\Lambda_\bullet\Gamma}^1(\Lambda_\bullet, \Lambda_\bullet) \simeq \mathrm{Hom}_{\mathsf{D}(\Lambda_\bullet\Gamma)}(\Lambda_\bullet, \Lambda_\bullet[1]).$$

Here $\mathsf{D}(\Lambda_\bullet\Gamma)$ is the derived category of the category of $\Lambda_\bullet\Gamma$-modules. For each X in $\mathsf{D}^+(\Lambda_\bullet\Gamma)$, θ_p defines morphisms

$$\theta_p \colon X \to X[1]$$
$$R\theta_p \colon RF(X) \to RF(X)[1].$$

5.6. The functor

$$R\,\mathrm{Hom} \colon \mathsf{D}_c^b(R)^{\mathrm{opp}} \times \mathsf{D}_c^b(R) \to D(\mathbb{Z}_p)$$

factors canonically through

$$RF \colon \mathsf{D}^+(\Lambda_\bullet\Gamma) \to \mathsf{D}(\mathbb{Z}_p).$$

This means that, for each pair $M, N \in \mathsf{D}_c^b(R)$, there is a well-defined $X \in \mathsf{D}^+(\Lambda_\bullet\Gamma)$ such that

$$R\,\mathrm{Hom}(M, N(r)) = RF(X).$$

Now $R\theta_p$ is a morphism

$$R\,\mathrm{Hom}(M, N(r)) \to R\,\mathrm{Hom}(M, N(r))[1]$$

On setting $\mathrm{Ext}^j(M, N(r)) = H^j(R\,\mathrm{Hom}(M, N(r)))$, we obtain a complex

$$E(M, N, r) \colon \quad \cdots \to \mathrm{Ext}^j(M, N(r)) \xrightarrow{d^j} \mathrm{Ext}^{j+1}(M, N(r)) \to \cdots.$$

PROPOSITION 5.7. *Let $M, N \in \mathsf{D}_c^b(R)$, and let $P = R\underline{\mathrm{Hom}}(M, N)$. Let $r \in \mathbb{Z}$, and assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H^j(sP)_K$.*

(a) *The groups $\mathrm{Ext}^j(M, N(r))$ are finitely generated $\mathbb{Z}_p$-modules, and the alternating sum of their ranks is zero.*

(b) *The zeta function $Z(P, t)$ has a pole at $t = q^{-r}$ of order*

$$\rho = \sum (-1)^{j+1} \cdot j \cdot \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Ext}^j(M, N(r)).$$

(c) *The cohomology groups of the complex $E(M, N, r)$ are finite, and the alternating product $\chi^\times(M, N(r))$ of their orders satisfies*

$$\left| \lim_{t \to q^{-r}} Z(M, N, t) \cdot (1 - q^r t)^\rho \right|_p^{-1} = \chi^\times(M, N(r)) \cdot q^{\chi(M, N, r)}$$

where

$$\chi(M, N, r) = \sum_{i < r} (-1)^i (r - i) \left(\sum_j (-1)^j h^{i,j}(P) \right).$$

6 How to prove the conjectures

This section consists of a series of somewhat unrelated subsections. Throughout, $\mathsf{DM}(k)$ is a triangulated category of motivic complexes over a perfect field k, and r_l is an exact realization functor to $\mathsf{D}_c^b(k, \mathbb{Z}_l)$ $(l \neq p)$ or $\mathsf{D}_c^b(R)$ $(l = p)$. We sometimes write $\mathsf{D}_c^b(k, \mathbb{Z}_p)$ for $\mathsf{D}_c^b(R)$.

It suffices to construct $\mathsf{DM}(k)$ *for k algebraically closed*

Throughout this subsection, all dg categories are pretriangulated. We write $\mathcal{D}_c^b(k, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}_c^b(k, \mathbb{Z}_p)$) for the natural dg enhancement of $\mathsf{D}_c^b(k, \mathbb{Z}_\ell)$ (resp. $\mathsf{D}_c^b(R)$). When $k = \mathbb{F}$, each object of $\mathcal{D}_c^b(k, \mathbb{Z}_l)$ is equipped with a germ of a Frobenius element (cf. Milne 1994, p.422). We assume that the same is true of $\mathcal{DM}(\mathbb{F})$ and that the realization functors send germs to germs.

Suppose that we have constructed a dg category $\mathcal{DM}(\mathbb{F})$ and dg realization functors $r_l: \mathcal{DM}(\mathbb{F}) \to \mathcal{D}_c^b(\mathbb{F}, \mathbb{Z}_l)$ for each l (including $l = p$). In this subsection we construct a dg category $\mathcal{DM}(\mathbb{F}_q)$ and dg functors

$$\mathcal{DM}(\mathbb{F}_q) \to \mathcal{DM}(\mathbb{F}),$$
$$r_l: \mathcal{DM}(\mathbb{F}_q) \to \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l)$$

such that

$$\begin{array}{ccc}
\mathcal{DM}(\mathbb{F}) & \xrightarrow{r_l} & \mathcal{D}_c^b(\mathbb{F}, \mathbb{Z}_l) \\
\uparrow & & \uparrow \\
\mathcal{DM}(\mathbb{F}_q) & \xrightarrow{r_l} & \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l)
\end{array} \tag{18}$$

commutes. Moreover, the composite

$$\mathcal{DM}(\mathbb{F}_q) \to \mathcal{DM}(\mathbb{F}) \xrightarrow{R\mathrm{Hom}(\mathbb{1}, -)} \mathcal{D}(\mathbb{Z})$$

factors canonically through $\mathcal{D}(\mathbb{Z}\Gamma_0)$:

$$
\begin{array}{ccc}
\mathcal{DM}(\mathbb{F}) & \longrightarrow & \mathcal{D}(\mathbb{Z}) \\
\uparrow & & \uparrow \\
\mathcal{DM}(\mathbb{F}_q) & \longrightarrow & \mathcal{D}(\mathbb{Z}\Gamma_0).
\end{array}
\tag{19}
$$

We define $\mathcal{DM}(\mathbb{F}_q)$ to be the category whose objects are pairs (X, π_X) consisting of an object X of $\mathcal{DM}(\mathbb{F})$ and a q-representative of the germ of a Frobenius element on X. A morphism $(X, \pi_X) \to (Y, \pi_Y)$ is a morphism $X \to Y$ sending π_X to π_Y.

Similarly, let $\mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l)'$ be the category whose objects are pairs (X, π_X) consisting of an object X of $\mathcal{D}_c^b(\mathbb{F}, \mathbb{Z}_l)$ and a q-representative of the germ of a Frobenius element on X. The functors

$$(X, \pi_X) \rightsquigarrow X : \mathcal{DM}(\mathbb{F}_q) \to \mathcal{DM}(\mathbb{F})$$

$$(X, \pi_X) \rightsquigarrow (r_l(X), r_l(\pi_X)) : \mathcal{DM}(\mathbb{F}_q) \to \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l)'$$

clearly make the diagram (18) commute. The functor

$$X \rightsquigarrow (\bar{X}, \pi_X) : \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l) \to \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l)'$$

is an equivalence of categories. On choosing a quasi-inverse, we obtain the functors making (18) commute.

An object of $\mathsf{D}(\mathbb{Z}\Gamma_0)$ is just an object of $\mathsf{D}(\mathbb{Z})$ together with an action of γ_0, and so the factorization of $R\operatorname{Hom}(\mathbf{1}, -) : \mathsf{DM}(\mathbb{F}_q) \to \mathsf{D}(\mathbb{Z})$ through $\mathsf{D}(\mathbb{Z}\Gamma_0)$ is obvious.

PROPOSITION 6.1. *If* $r_l : \mathcal{DM}(\mathbb{F}) \to \mathcal{D}_c^b(\mathbb{F}, \mathbb{Z}_l)$ *induces an equivalence of categories*

$$r_l : \mathcal{DM}(\mathbb{F}, \mathbb{Z}/l^n\mathbb{Z}) \to \mathcal{D}_c^b(\mathbb{F}, \mathbb{Z}/l^n\mathbb{Z})$$

for all n, *then* $r_l : \mathcal{DM}(\mathbb{F}_q) \to \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}_l)$ *induces an equivalence of categories*

$$r_l : \mathcal{DM}(\mathbb{F}_q, \mathbb{Z}/l^n\mathbb{Z}) \to \mathcal{D}_c^b(\mathbb{F}_q, \mathbb{Z}/l^n\mathbb{Z})$$

for all n.

PROOF. Omitted.   □

Here $\mathcal{DM}(k, \mathbb{Z}/l^n\mathbb{Z})$ is the subcategory of $\mathcal{DM}(k, \mathbb{Z}_l)$ of objects killed by l^n.

Some folklore

Throughout this subsection, k is finitely generated over the prime field.[6] We write $P \rightsquigarrow \bar{P}$ for base change to the algebraic closure $\bar{k}$ of k. Let $\Gamma = \operatorname{Gal}(\bar{k}/k)$.

Let P be an object of $\mathsf{DM}(k)$. Set

$$
\left\{
\begin{array}{rcl}
H_{\mathrm{abs}}^j(P, r) & = & \operatorname{Hom}_{\mathsf{DM}(k)}(\mathbf{1}, P[j](r)) \\
H^j(P, \mathbb{Z}_l(r)) & = & \operatorname{Hom}_{\mathsf{D}(k, \mathbb{Z}_l)}(\mathbf{1}, r_l(P)[j](r)) \\
H^j(P, (\mathbb{Z}/l^n\mathbb{Z})(r)) & = & \operatorname{Hom}_{\mathsf{D}(k, \mathbb{Z}_l)}(\mathbf{1}, r_l(P^{(l^n)})[j](r)).
\end{array}
\right.
\tag{20}
$$

---

[6]So either k is finite or a finitely generated field extension of $\mathbb{Q}$.

Recall that, for $l = p$, $\mathsf{D}(k, \mathbb{Z}_l) \overset{\text{def}}{=} \mathsf{D}_c^b(R)$ and that, for an object P of a triangulated category, $P^{(l^n)}$ is the cone on $l^n \colon P \to P$.

Let $A^j(P, r)$ denote the image of the canonical map

$$H_{\text{abs}}^j(P, r)_{\mathbb{Q}} \to V_l^j(P, r) \overset{\text{def}}{=} \mathbb{Q}_l \otimes_{\mathbb{Z}_l} H^j(\bar{P}, \mathbb{Z}_l(r)).$$

Let P' be a second object of $\mathsf{DM}(k)$ equipped with a pairing

$$P \otimes^L P' \to \mathbf{1}(-d)$$

for some integer d. For example, $P' = R\underline{\text{Hom}}(P, \mathbf{1}(-d))$. Assume that the induced pairings

$$\langle\,,\,\rangle \colon V_l^j(P, r) \times V_l^{2d-j}(P', d-r) \to \mathbb{Q}_l$$

are nondegenerate, and that $V_l^j(P, r)$ and $V_l^{2d-j}(P', d-r)$ are (noncanonically) isomorphic as Γ-modules. Elements $a \in A^j(P, r)$ and $a' \in A^{2d-j}(P', d-r)$ pair to a rational number $\langle a, a'\rangle$ independent of l. Let

$$N^j(P, r) = \{a \in A^j(P, r) \mid \langle a, a'\rangle = 0 \text{ for all } a' \in A^{2d-j}(P', d-r)\}.$$

There is a canonical map

$$\mathbb{Q}_l \otimes_{\mathbb{Q}} A^j(P, r) \to V_l^j(P, r)^\Gamma. \tag{21}$$

Consider the following statements.

$T_l^j(P, r)$**:** The map (21) is surjective, i.e., $\mathbb{Q}_l A^j(P, r) = V_l^j(P, r)^\Gamma$.

$I_l^j(P, r)$**:** The map (21) is injective.

$E_l^j(P, r)$**:** The vector space $N^j(P, r) = 0$.

$S_l^j(P, r)$**:** The map $V_l^j(P, r)^\Gamma \to V^j(P, r)_\Gamma$ induced by the identity map is bijective.

In the next statement, we abbreviate $T_l^j(P, r)$ to T and $T_l^{2d-j}(P', d-r)$ to T' etc..

THEOREM 6.2. *The following statements are equivalent:*

  (a) $dim_{\mathbb{Q}}(A^j(P, r)/N^j(P, r)) = \dim_{\mathbb{Q}_l} V_l^j(P, r)^\Gamma$;

  (b) $T + E$;

  (c) $T + T' + S$;

  (d) $T + T' + E + E' + I + I' + S + S'$.

Moreover, if k is finite, then these statements are equivalent to:

  (e) *the multiplicity of 1 as a root of the characteristic polynomial of the Frobenius on $V_l^j(P, r)^\Gamma$ is equal to* $\dim_{\mathbb{Q}}(A^j(P, r)/N^j(P, r))$.

PROOF. See Tate 1994, §2, and Milne 2009, §1.   □

We say that the **Tate conjecture holds** for P, j, r, l if the equivalent conditions (a),…,(d) of the theorem hold for P, j, r, l. We say that the **Tate conjecture holds** for P and l if the equivalent conditions of the theorem hold for all quadruples P, j, r, l (fixed P, l).

Local torsion (Gabber's theorem)

Throughout this subsection, k is a separably closed field of characteristic $p \neq 0$ (not necessarily perfect).

CONJECTURE 6.3. *Let $M, N \in \mathrm{ob}\,\mathrm{DM}(k)$, and let $j \in \mathbb{Z}$. The group $\mathrm{Ext}^j(r_l M, r_l N)$ is torsion-free for almost all l.*

Let X be an algebraic variety over k. Then (6.3) applied to $\mathbf{1}$ and the motivic complexes attached to X predicts that the étale cohomology groups $H^j(X_{\mathrm{et}}, \mathbb{Z}_l(r))$ and $H^j_c(X_{\mathrm{et}}, \mathbb{Z}_l(r))$ are torsion-free for almost all $l \neq p$.

6.4. Let X be a smooth projective variety over k. In this case Gabber (1983) shows that $H^j(X_{\mathrm{et}}, \mathbb{Z}_l(r))$ is torsion-free for almost $l \neq p$. A specialization argument shows that it suffices to prove this in the case that $k = \mathbb{F}$. When $k = \mathbb{F}$, Gabber applied Deligne 1980 to obtain the statement.

6.5. Let X be proper and smooth over k. An application of Chow's lemma and de Jong's alteration theorem shows that there exists a morphism $\pi \colon X' \to X$ with X' projective and smooth and π projective, surjective, and generically finite and étale. The composite

$$H^j(X_{\mathrm{et}}, \mathbb{Z}_\ell(r)) \xrightarrow{\pi^*} H^j(X'_{\mathrm{et}}, \mathbb{Z}_\ell(r)) \xrightarrow{\pi_*} H^j(X_{\mathrm{et}}, \mathbb{Z}_\ell(r))$$

is multiplication by $\deg(\pi)$. Hence $H^j(X_{\mathrm{et}}, \mathbb{Z}_\ell(r))$ is torsion-free for almost all ℓ (Suh 2012, 1.4).

We now take $k = \mathbb{F}$.

6.6. Let X be an arbitrary variety over k. Then de Jong's alteration theorem (de Jong 1996) says that there exists an alteration $X' \to X$ of X such that X' is smooth; moreover, X' can be chosen to be the complement of a divisor with strict normal crossings in some smooth projective variety.[7]

Therefore, by the argument in (6.5), we may suppose that U is an open subvariety of a smooth projective variety X with complement a strict normal crossings divisor D. There is then an exact sequence

$$\cdots \to H^i_c(U_{\mathrm{et}}, \mathbb{Z}_\ell) \to H^i(X_{\mathrm{et}}, \mathbb{Z}_\ell) \to H^i(D_{\mathrm{et}}, \mathbb{Z}_\ell) \to H^{i+1}_c(U_{\mathrm{et}}, \mathbb{Z}_\ell) \to \cdots.$$

Induction on the dimension of X allows us to suppose that $H^*(D_{\mathrm{et}}, \mathbb{Z}_\ell)$ is torsion-free for almost all ℓ, and so it remains to show that the cokernel of

$$H^i(X_{\mathrm{et}}, \mathbb{Z}_\ell) \to H^i(D_{\mathrm{et}}, \mathbb{Z}_\ell) \tag{22}$$

is torsion-free for almost all ℓ. This will be true if the map (22) arises from a map $h^i(X) \to h^i(D)$ of motives in a category that becomes semisimple when tensored with $\mathbb{Q}$.

---

[7]An alteration is a dominant proper morphism that preserves the dimension. Let D be a divisor in a variety X, and let $D_i \subset D$, $i \in I$, be its irreducible components (both D and the D_i are closed subvarieties of codimension 1 in X). Then D is a strict normal crossings divisor if (a) every point $s \in D$ is a smooth point on X, (b) for every $J \subset I$, the closed subscheme $D_J = \bigcap_{i \in J} D_i$ is a smooth subvariety of codimension $\#J$ in V.

6.7. We now consider the general statement. Assume that the triangulated category $\mathsf{DM}(k)_{\mathbb{Q}}$ is semisimple (i.e., every distinguished triangle splits). This is certainly expected to be true when $k = \mathbb{F}$ (see, for example, Milne 1994, 2.49). Fix M, and let

$$N' \to N \to N'' \to N'[1]$$

be a distinguished triangle. If (6.3) is true for two of the pairs (M, N'), (M, N), (M, N''), then it is true for all three (cf. the last paragraph). A similar statement is true in the first variable. It follows that if $\mathsf{DM}(k)$ is generated as a triangulated category by the motives of smooth varieties (which is true for the examples in §7[8] and §8), then (6.3) is true for all $M, N \in \mathrm{ob}\,\mathsf{DM}(k)$.

Consequences of $r_l(M, N)$ being an isomorphism

In the remainder of this section, we require that each object of $\mathsf{DM}(k)$ has a zeta function (see §3), and that the realization functors r_l are such that there is a canonical map of complexes

$$r_l E(M, N, r) \to E(r_l M, r_l N, r)$$

for M, N in $\mathsf{DM}(k)$. More specifically, we require that the r_l have dg-enhancements, and that they map the factorization of

$$R\,\mathrm{Hom}\colon \mathsf{DM}(k)^{\mathrm{opp}} \times \mathsf{DM}^+(k) \to \mathsf{D}(\mathbb{Z})$$

through $RF\colon \mathsf{D}(\mathbb{Z}\Gamma_0) \to \mathsf{D}(\mathbb{Z})$ onto the canonical factorizations of

$$R\,\mathrm{Hom}\colon \mathsf{D}(k, \mathbb{Z}_\ell)^{\mathrm{opp}} \times \mathsf{D}^+(k, \mathbb{Z}_\ell) \to \mathsf{D}(\mathbb{Z}_\ell)$$
$$R\,\mathrm{Hom}\colon \mathsf{D}^b_c(R)^{\mathrm{opp}} \times \mathsf{D}^b_c(R) \to \mathsf{D}(\mathbb{Z}_p)$$

through $\mathsf{D}(\mathbb{Z}_l\Gamma)$.

Let $M, N \in \mathrm{ob}\,\mathsf{DM}(k)$. From the realization maps we get maps

$$r_\ell(M, N)\colon \mathrm{Hom}_{\mathsf{DM}(k)}(M, N) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathsf{D}^b_c(\mathbb{Z}_\ell)}(r_\ell M, r_\ell N)$$
$$r_p(M, N)\colon \mathrm{Hom}_{\mathsf{DM}(k)}(M, N) \otimes \mathbb{Z}_p \to \mathrm{Hom}_{\mathsf{D}^b_c(R)}(r_p M, r_p N)$$

We say that an abelian group X is **pseudofinite** if it is torsion and its l-primary component $X(l)$ is finite for all primes l. The order of such a group is the formal product $\prod l^{n(l)}$ where $l^{n(l)}$ is the order of $X(l)$. We say that an abelian group X is pseudofinitely generated if X_{tors} is pseudofinite and X/X_{tors} is finitely generated.

THEOREM 6.8. *Let $M, N \in \mathsf{DM}(k)$, and let $r \in \mathbb{Z}$. Let $P = R\underline{\mathrm{Hom}}(M, N)$. Assume that $r_l(M, N)$ is an isomorphism for all primes l and that the Frobenius endomorphism of P is semisimple.*
   (a) *The groups $\mathrm{Ext}^j(M, N(r))$ are pseudofinitely generated, and the alternating sum of their ranks is zero.*
   (b) *The zeta function $Z(P, t)$ has a zero at $t = q^{-r}$ of order*

$$\rho = \sum (-1)^{j+1} \cdot j \cdot \mathrm{rank}_{\mathbb{Z}}(\mathrm{Ext}^j(M, N(r))).$$

---

[8]See, for example, Bondarko 2011.

(c) *The cohomology groups of the complex* $\text{Ext}^\bullet(M, N(r))$ *are pseudofinite, and the alternating product* $\chi^\times(M, N(r))$ *of their orders satisfies*

$$\left| \lim_{t \to q^{-r}} Z(P, t)(1 - q^r t)^\rho \right| = \chi^\times(M, N(r)) \cdot q^{\chi(P,r)}$$

where

$$\chi(P, r) = \sum_{i, j \ (i \le r)} (-1)^{i+j} (r - i) h^{i,j}(R_p(P)).$$

If, in addition, $\text{DM}(k)_\mathbb{Q}$ *is semisimple, then Conjecture 1.1 holds for M and N.*

PROOF. The statements (a,b,c) are an immediate consequence of the hypotheses and Propositions 4.7 and 5.7. The final statement follows from (6.7).   □

Consequences of rigidity and the Tate conjecture

Let $k = \mathbb{F}_q$. We assume that, for all primes l, the realization functor $r_l \colon \text{DM}(k) \to \text{D}_c^b(k, \mathbb{Z}_l)$ defines an equivalence on the subcategories of objects killed by l^n. For Voevodsky's category $\text{DM}^{\text{eff}}_{-,\text{et}}(k)$ and $l \ne p$, this is the rigidity theorem (Suslin and Voevodsky 1996, §4; Voevodsky 2000, 3.3.3).

THEOREM 6.9. *Let $P \in \text{ob}\,\text{DM}(k)$, and let $r \in \mathbb{Z}$. Assume that the Tate conjecture holds for P, r, and a fixed prime l. Then, for all $i \in \mathbb{Z}$,*
  (a) *the first Ulm subgroup U^i of $H^i_{\text{abs}}(P, r)$ is uniquely divisible by l;*
  (b) *the group $H^i_{\text{abs}}(P, r)' \overset{\text{def}}{=} H^i_{\text{abs}}(P, r)/U^i$ is finitely generated modulo torsion, and its l-primary subgroup is finite;*
  (c) *the map $H^i_{\text{abs}}(P, r)' \otimes \mathbb{Z}_l \to H^i(P, \mathbb{Z}_l(r))$ is an isomorphism for all i.*

The proof will occupy the rest of this subsection.

PRELIMINARIES ON ABELIAN GROUPS

In this subsection, we review some elementary results on abelian groups from the first section of the appendix to Milne 1986. An abelian group N is said to be ***bounded*** if $nN = 0$ for some $n \ge 1$, and a subgroup M of N is ***pure*** if $M \cap mN = mM$ for all $m \ge 1$.

LEMMA 6.10. *(a) Every bounded abelian group is a direct sum of cyclic groups.*
  (b) Every bounded pure subgroup M of an abelian group N is a direct summand of N.

LEMMA 6.11. *Let M be a subgroup of N, and let l^n be a prime power. If $M \cap l^n N = 0$ and M is maximal among the subgroups with this property, then M is a direct summand of N.*

Every abelian group M contains a largest divisible subgroup M_{div}, which is obviously contained in the first Ulm subgroup of M, $U(M) \overset{\text{def}}{=} \bigcap_{n \ge 1} nM$. Note that $U(M/U(M)) = 0$.

PROPOSITION 6.12. *If M/nM is finite for all $n \ge 1$, then $U(M) = M_{\text{div}}$.*

COROLLARY 6.13. *If $TM = 0$ and all quotients M/nM are finite, then $U(M)$ is uniquely divisible (= divisible and torsion-free = a $\mathbb{Q}$-vector space).*

For an abelian group M, we let M_l denote the completion of M with respect to the l-adic topology. The quotient maps $M \to M/l^n M$ induce an isomorphism $M_l \to \varprojlim_n M/l^n M$. The kernel of the map $M \to M_l$ is $\bigcap_n l^n M$.

LEMMA 6.14. *Let N be a torsion-free abelian group. If N/lN is finite, then the l-adic completion of N is a free finitely generated $\mathbb{Z}_l$-module.*

PROPOSITION 6.15. *Let $\phi: M \times N \to \mathbb{Z}$ be a bi-additive pairing of abelian groups whose extension $\phi_l: M_l \times N_l \to \mathbb{Z}_l$ to the l-adic completions has trivial left kernel. If N/lN is finite and $\bigcap_n l^n M = 0$, then M is free and finitely generated.*

PROOF OF THEOREM 6.9

Let $P \in \mathrm{ob}\,\mathrm{DM}(k)$. We use the notations (20). From the factorization

$$\mathrm{DM}(k) \xrightarrow{\hspace{2cm}} \mathrm{D}^+(\mathbb{Z}\Gamma_0) \xrightarrow{\overrightarrow{RF}} \mathrm{D}^+(\mathbb{Z}),$$
$$\overset{R\,\mathrm{Hom}(\mathbb{1},-)}{\overbrace{\hspace{4cm}}}$$

we get a spectral sequence

$$H^i(\Gamma_0, H^j_{\mathrm{abs}}(\bar{P}, r)) \implies H^{i+j}_{\mathrm{abs}}(P, r),$$

and hence exact sequences

$$0 \to H^{i-1}_{\mathrm{abs}}(\bar{P}, r)_{\Gamma_0} \to H^i_{\mathrm{abs}}(P, r) \to H^i_{\mathrm{abs}}(\bar{P}, r)^{\Gamma_0} \to 0.$$

On applying the rigidity theorem to the exact cohomology sequence of the distinguished triangle

$$P \xrightarrow{l^n} P \longrightarrow P^{(l^n)} \longrightarrow P[1]$$

(see the Notations), we get an exact sequence

$$0 \to H^i_{\mathrm{abs}}(P, r)^{(l^n)} \to H^i(P, (\mathbb{Z}/l^n\mathbb{Z})(r)) \to H^{i+1}_{\mathrm{abs}}(P, r)_{l^n} \to 0. \tag{23}$$

Here the subscript l^n denotes the kernel of multiplication by l^n. We deduce that $H^i_{\mathrm{abs}}(P, r)^{(l^n)}$ and $H^{i+1}_{\mathrm{abs}}(P, r)_{l^n}$ are finite for all i and n. On passing to the inverse limit, we get an exact sequence

$$0 \to H^i_{\mathrm{abs}}(P, r)_l \to H^i(P, \mathbb{Z}_l(r)) \to T_l H^{i+1}_{\mathrm{abs}}(P, r) \to 0.$$

We now apply $T^i_l(P, r)$: the map

$$H^i_{\mathrm{abs}}(P, r) \otimes \mathbb{Q}_l \to H^i(\bar{P}, \mathbb{Q}_l(r))^\Gamma$$

is surjective. This implies that the cokernel of the map

$$H^i_{\mathrm{abs}}(P, r) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to H^i(P, \mathbb{Z}_l(r))$$

is torsion. As the map factors through $H^i_{\mathrm{abs}}(P, r)_l$, it follows that $T_l H^{i+1}_{\mathrm{abs}}(P, r) = 0$ and

$$H^i_{\mathrm{abs}}(P, r)_l \simeq H^i(P, \mathbb{Z}_l(r)).$$

Consider the diagram

$$
\begin{array}{ccc}
H^i_{\mathrm{abs}}(P,r)_l & \xrightarrow{\ \simeq\ } & H^i(P,\mathbb{Z}_l(r)) \\
\downarrow{\scriptstyle (d^i)_l} & & \downarrow{\scriptstyle \delta^i} \\
H^{i+1}_{\mathrm{abs}}(P,r)_l & \longrightarrow & H^{i+1}(P,\mathbb{Z}_l(r)).
\end{array}
$$

Here the vertical maps are the differentials in $H^\bullet_{\mathrm{abs}}(P,r)$ (see p.3) and its l-analogue. As δ^i has finite cokernel, so does the bottom arrow, and so $T_l H^{2i+2}_{\mathrm{abs}}(P,r) = 0$. We have now shown that

$$T_l H^i_{\mathrm{abs}}(P,r) = 0 \text{ for all } i$$

and so

$$
\begin{cases}
H^i_{\mathrm{abs}}(P,r)_l \simeq H^i(P,\mathbb{Z}_l(r)) \\
U(H^i_{\mathrm{abs}}(P,r)) \text{ is uniquely } l\text{-divisible}
\end{cases}
\quad \text{for all } i.
$$

We have now proved (a) of the theorem, and we have proved (b) and (c) except that each group $H^i_{\mathrm{abs}}(P,r)'$ has been replaced by its l-adic completion. It remains to prove that $H^i_{\mathrm{abs}}(P,r)'$ is finitely generated for all i (for then $H^i_{\mathrm{abs}}(P,r)_l \simeq H^i_{\mathrm{abs}}(P,r)' \otimes \mathbb{Z}_l$).

The maps $H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))' \to H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))_l$ are injective, and so $H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))'$ is finite unless 1 is an eigenvalue of the Frobenius map on $H^i(\bar{P},\mathbb{Z}_l(r))$ or $H^{i-1}(\bar{P},\mathbb{Z}_l(r))$.

We next show that the group $H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))'$ is finitely generated. There is a commutative diagram

$$
\begin{array}{ccccc}
H^i_{\mathrm{abs}}(P,r)'/\mathrm{tors} & \times & H^{2d-i+1}_{\mathrm{abs}}(P,d-r)'/\mathrm{tors} & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
H^i(P,\mathbb{Z}_l(r))/\mathrm{tors} & \times & H^{2d-i+1}(P,\mathbb{Z}_l(d-r))/\mathrm{tors} & \longrightarrow & \mathbb{Z}_l
\end{array}
$$

to which we wish to apply (6.15). The bottom pairing is nondegenerate, $U(H^i_{\mathrm{abs}}(P,r)') = 0$ (the quotient of a group by its first Ulm group has trivial first Ulm group), and the group $H^{2d-i+1}_{\mathrm{abs}}(P,\mathbb{Z}(d-r))^{(l)}$ is finite. Therefore (6.15) shows that $H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))'/\mathrm{tors}$ is finitely generated. Because $U(H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))') = 0$, the torsion subgroup of $H^i_{\mathrm{abs}}(P,\mathbb{Z}(r))'$ injects into the torsion subgroup of $H^i(P,\mathbb{Z}_l(r))$, which is finite. Hence $H^i_{\mathrm{abs}}(P,r)'$ is finitely generated modulo prime-to-l torsion.

The bijectivity of r_l

Let $k = \mathbb{F}_q$. Let P be an object of $\mathsf{DM}(k)$ such that the Tate conjecture is true for P and all i,r,l. According to Theorem 6.9, the map

$$
\left(H^i_{\mathrm{abs}}(P,r)/U^i \right) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to H^i_{\mathrm{et}}(P,\mathbb{Z}_l(r))
$$

is an isomorphism for all i,r,l. Here U^i is the first Ulm subgroup of $H^i_{\mathrm{abs}}(P,r)$, which is uniquely divisible (hence a $\mathbb{Q}$-vector space). Moreover, $H^i_{\mathrm{abs}}(P,r)' \stackrel{\mathrm{def}}{=} H^i_{\mathrm{abs}}(P,r)/U^i$

is finitely generated and its l-primary subgroup is finite for all l. Therefore the group $\text{Tor}_1^{\mathbb{Z}}(H^i_{\text{abs}}(P,r)', \mathbb{Z}_l)$ is torsion, and so the sequence

$$0 \to U^i \otimes \mathbb{Z}_l \to H^i_{\text{abs}}(P,r) \otimes \mathbb{Z}_l \to H^i_{\text{abs}}(P,r)' \to 0$$

is exact (because $\mathbb{Q} \otimes \mathbb{Z}_l \simeq \mathbb{Q}_l$ is torsion-free). To show that $U^i = 0$, it suffices to show that

$$r_l : (\mathbb{1}, P) : H^i_{\text{abs}}(P,r) \otimes \mathbb{Z}_l \to H^i_{\text{et}}(P, \mathbb{Z}_l(r)) \tag{24}$$

is injective *for a single l*.

When P is the motivic complex with compact support of an algebraic variety X, we write X for P.

LEMMA 6.16. *Suppose that the map*

$$r_l(\mathbb{1}, X) : H^*_{\text{abs}}(X,r) \otimes \mathbb{Z}_l \to H^*(X, \mathbb{Z}_l(r)) \tag{25}$$

is bijective for almost all primes l when X is smooth and projective. Then this is true for all varieties X.

PROOF. From an alteration $\pi : X' \to X$ we get a commutative diagram

$$
\begin{array}{ccccc}
H^*_{\text{abs}}(X,r) \otimes \mathbb{Z}_l & \xrightarrow{\pi^*} & H^*_{\text{abs}}(X',r) \otimes \mathbb{Z}_l & \xrightarrow{\pi_*} & H^*_{\text{abs}}(X,r) \otimes \mathbb{Z}_l \\
\downarrow & & \downarrow & & \downarrow \\
H^*_{\text{et}}(X, \mathbb{Z}_l(r)) & \xrightarrow{\pi^*} & H^*_{\text{et}}(X, \mathbb{Z}_l(r)) & \xrightarrow{\pi_*} & H^*_{\text{et}}(X, \mathbb{Z}_l(r)).
\end{array}
$$

in which both composites $\pi_* \circ \pi^*$ are multiplication by $\deg(\pi)$, and are therefore isomorphisms for almost all primes l. Therefore the statement is true for X if it is true for X'. Now de Jong's alteration theorem (de Jong 1996) allow us to suppose that X is smooth and is the complement of a divisor D with strict normal crossings in a smooth projective variety Y. Induction on the dimension of X allows us to assume the statement for D, and a five-lemma argument using the exact cohomology sequence

$$\cdots \to H^i_c(X) \to H^i(Y) \to H^i(D) \to$$

proves it for X.   □

ASIDE 6.17. Fix a prime $l \neq p$. Gabber's improvement of de Jong's theorem (see Bondarko 2011, 1.2.1) allows one to assume in the above proof that the degree of π is prime to l. Therefore we obtain the following stronger result: if $r_l(\mathbb{1}, X)$ is an isomorphism for all smooth projective varieties, then it is an isomorphism for all varieties (fixed $l \neq p$). Of course, with resolution, this would be true also for $l = p$.

LEMMA 6.18. *Let X be a smooth projective variety over $\mathbb{F}_q$ that satisfies the Tate conjecture. If the ideal of l-homologically trivial correspondences in $\text{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil, then $r_l(\mathbb{1}, X)$ is bijective.*

PROOF. See, for example, the appendix to Milne 1986.   □

ASIDE 6.19. For a smooth projective algebraic variety X whose Chow motive is finite-dimensional, the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil for all prime l (Kimura). It is conjectured (Kimura and O'Sullivan) that the Chow motives of algebraic varieties are always finite-dimensional, and this is known for those in the category generated by the motives of abelian varieties. On the other hand, Beilinson has conjectured that, over finite fields, rational equivalence with $\mathbb{Q}$-coefficients coincides with with numerical equivalence, which implies that the ideal in question is always null (not merely nil).

THEOREM 6.20. *Assume that, for each smooth projective variety X over $\mathbb{F}_q$, the Tate conjecture holds for X and, for some prime l, the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil. If $\mathsf{DM}(k)$ is generated as a triangulated category by the motives of smooth algebraic varieties, then the maps $r_l(M, N)$ are isomorphisms for all l (except possibly p).*

PROOF. The hypotheses imply that $r_l(\mathbb{1}, X)$ is an isomorphism for all l if X is smooth and projective (see 6.18 and the discussion above). Therefore $r_l(\mathbb{1}, X)$ is an isomorphism for all l (except possibly p) if X is smooth (apply 6.16, 6.17). If $r_l(\mathbb{1}, P)$ is an isomorphism for two of the terms in a distinguished triangle, then it is an isomorphism for all three of the terms. Because $\mathsf{DM}(k)$ is generated by the motives of smooth algebraic varieties, we deduce that $r_l(\mathbb{1}, P)$ is an isomorphism all l (except possibly p). This proves the theorem for $(\mathbb{1}, P)$, and it can be deduced for (M, N) by taking $P = R\underline{\mathrm{Hom}}(M, N)$. □

The exception at p in the theorem can be removed under either of the following two hypotheses.

(a) *Resolution of singularities.* This allows us to drop the condition $l \neq p$ in (6.17), and hence in the rest of the proof.

(b) *Numerical equivalence coincides with l-homological equivalence for $l = p$ and at least one other prime l_1 (for smooth projective varieties; $\mathbb{Q}$-coefficients).* Let X be a smooth variety. If $r_l(\mathbb{1}, X)$ is an isomorphism for all $l \neq p$, then the Tate conjecture holds for X and all $l \neq p$. Under our hypothesis on the equivalence relations, Condition (e) of (6.2) holds for l_1 if and only if it holds for p (cf. Katz 1994, p.28). Therefore the Tate conjecture holds for X and p, which in which in turn implies that $r_p(\mathbb{1}, X)$ is an isomorphism (6.9 et seq.). We now know that $r_l(\mathbb{1}, X)$ is an isomorphism for all smooth varieties X and all primes l, and so the proof can be completed as before.

THEOREM 6.21. *Assume that the Tate and Beilinson conjectures hold for all smooth projective varieties over $\mathbb{F}_q$. Then Theorem 6.8 holds for all M, N in $\mathsf{DM}(k)$. If in addition, $\mathsf{DM}(k)_{\mathbb{Q}}$ is semisimple, then Conjecture 1.1 holds for all M, N in $\mathsf{DM}(k)$.*

PROOF. Immediate from the above. □

7   Motivic complexes following Voevodsky

Let k be a perfect field of characteristic $p > 0$, let $W = W(k)$ be the ring of Witt vectors over k, and let $K = W \otimes \mathbb{Q}$ be its fraction field. Let $\bar{k}$ be an algebraic closure of k, and let Γ the Galois group of $\bar{k}$ over k. Recall that an F-isocrystal is a $K_\sigma[F]$-module that is finite-dimensional as a K-vector space and such that F is bijective.

At present, it is not known how to construct (in a natural way, i.e., without using realizations) a triangulated category of motivic complexes k equipped with an integral p-adic cohomological realization functor to $\mathsf{D}_c^b(R)$. We propose constructing such a category as a dg fibred product of Voevodsky's category with the category $\mathsf{D}_c^b(R)$.

In our earlier article Milne and Ramachandran 2004, we constructed the abelian category of integral mixed motives as a fibred product. Our construction here is similar in spirit, but takes place at the level of dg-categories.

There are roughly three steps:
 (a) definition of a dg crystalline/rigid realization (k algebraically closed);
 (b) formation of the fibred product category at the dg level and comparison of the Hom's in the various categories (k algebraically closed);
 (c) passage to a finite base field and the Weil étale cohomology groups.

Various definitions of $\mathsf{DM}_{gm}^{\mathit{eff}}(k, \mathbb{Q})$

Voevodsky's category $\mathsf{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Z})$ of effective étale motives over k has no p-torsion and is $\mathbb{Z}[p^{-1}]$-linear. Thus

$$\mathsf{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Z}) = \mathsf{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Z}[p^{-1}]).$$

With $\mathbb{Q}$-coefficients, there is a canonical equivalence of categories

$$\mathsf{DM}_{\mathrm{Nis}}^{\mathrm{eff}}(k, \mathbb{Q}) \to \mathsf{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Q})$$

between the triangulated categories of effective Voevodsky motives and effective étale motives with $\mathbb{Q}$-coefficients (Mazza et al. 2006, 14.30, p.118). In fact, the various definitions of the triangulated category of motivic complexes all give the same answer for rational coefficients. In particular, the following categories are all canonically equivalent (Déglise 2013):
 ◇ $\mathsf{DM}_{\mathrm{et}}(k, \mathbb{Q})$ of étale motives (Voevodsky)
 ◇ $\mathsf{DM}_{\mathrm{Nis}}(k, \mathbb{Q})$ of Voevodsky sheaves in the Nisnevich motives (Voevodsky)
 ◇ $\mathsf{DA}_{\mathrm{et}}(k, \mathbb{Q})$ via motivic homotopy (Morel, Ayoub)
 ◇ $\mathsf{DM}_{\mathrm{h}}(k, \mathbb{Q})$ of h-topology motives (Voevodsky).

The first two involve presheaves with transfers, but the third one does not. For a definition of $\mathsf{DA}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Q})$, see Ayoub 2013, begining of Section 3, just before 3.1. Ayoub (ibid., B.1) has provided a canonical equivalence

$$\mathsf{DA}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Q}) \to \mathsf{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Q}).$$

For details on the other categories, see Cisinski and Déglise 2012b

The integral versions of these categories, other than the second, are all $\mathbb{Z}[p^{-1}]$-linear, i.e., they do not have any p-torsion. For this reason, we write $\mathsf{DM}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ for the integral version of $\mathsf{DM}_{\mathrm{et}}(k, \mathbb{Q})$. The p-part of the integral Nisnevich category remains mysterious; no connection, as yet, has been established between the Nisnevich category and the de Rham-Witt complex.

Glossary of categories

 ◇ $\mathrm{Ho}(\mathcal{C})$ the homotopy category of a dg-category $\mathcal{C}$. When $\mathcal{C}$ is pretriangulated, $\mathrm{Ho}(\mathcal{C})$ is a triangulated category.

⋄ $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ Voevodsky's triangulated category of effective étale motives over k.

⋄ $\mathsf{D}^b_c(R)$ the triangulated category of coherent complexes of graded modules over the Raynaud ring R (see §5).

⋄ $\mathsf{D}^b_c(K_\sigma[F])$ the bounded derived category of complexes of $K_\sigma[F]$-modules whose cohomology groups are F-isocrystals.

⋄ $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ the natural dg enhancement of $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{=1}])$ (Beilinson and Vologodsky 2008).

⋄ $\mathcal{D}(B)$ the derived dg-category of an exact category B; this is the dg quotient of the dg category $C(B)$ of unbounded complexes by the subcategory of acyclic ones (Drinfeld 2004, Tabuada 2010b).

⋄ $\mathcal{D}^b_c(R)$ the natural dg enhancement of $\mathsf{D}^b_c(R)$.

⋄ $\mathcal{D}^b_c(K_\sigma[F])$ the natural dg enhancement of $\mathsf{D}^b_c(K_\sigma[F])$.

⋄ $\mathcal{DM}(k, \mathbb{Z})$ the dg-category of integral motivic complexes over k (constructed below).

The dg-enhancement of Voevodsky's category

For every $\mathbb{Q}$-algebra A, there is a canonical equivalence of categories

$$\sigma \colon \mathsf{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k, A) \to \mathsf{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, A) \tag{26}$$

between the triangulated categories of effective Voevodsky motives and effective étale motives with coefficients in A (Mazza et al. 2006, 14.30, p.118). Voevodsky's category $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k, \mathbb{Q})$ of effective geometric motives over k admits a crystalline realization functor (homology) (Cisinski and Déglise 2012a). It is a mixed Weil cohomology theory in the terminology of Cisinski and Déglise. When composed with a quasi-inverse of σ, it gives a crystalline realization of $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Q})$.

We want to lift this to a dg-realization, i.e. a dg-quasifunctor

$$\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}]) \to \mathcal{D}(K_\sigma[F])$$

where $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ is the dg-enhancement of $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ constructed in Beilinson and Vologodsky 2008.

We shall freely use Beilinson and Vologodsky 2008 and Vologodsky 2012.

In particular, Vologodsky (2012, Theorem 2, p.384; also the start of Section 2) has provided a very convenient criterion for constructing dg-quasifunctors on $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$. For rational coefficients, i.e., for quasi-functors on $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Q})$, this criterion states[9]

> A dg-realization $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Q}) \to \mathcal{C}$ into a cocomplete compactly generated dg-category $\mathcal{C}$ is just an "ordinary" functor from the category of smooth connected schemes to $\mathcal{C}$ which is $\mathbb{A}^1$-homotopy invariant and satisfies the descent property for Voevodsky's h-topology.

In other words, $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, \mathbb{Q})$ is universal for dg-realizations of motives.

The canonical equivalence σ admits a dg-enhancement: the canonical dg-quasifunctor on the dg-enhancements $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{Nis}}(k, A) \to \mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k, A)$ is a homotopy equivalence for every $\mathbb{Q}$-algebra A (Vologodsky 2012, p.380, following Remark 2.6,).

---

[9]Vologodsky informed us that, by results of Ayoub, the h-topology in the criterion can be replaced by étale topology.

REMARK 7.1. The construction of the category $\mathsf{DM}^{\mathrm{eff}}_{\mathrm{et}}(k,\mathbb{Q})$ is based on $\mathbb{A}^1$-homotopy invariant étale presheaves with transfer on the category of all smooth schemes over k. The category $\mathsf{DA}^{\mathrm{eff}}_{\mathrm{et}}(k,\mathbb{Q})$ in Ayoub's theorem is a variant "sans transfers". The equivalence in Ayoub's theorem provides a variant of Theorem 2.8(c) of Vologodsky 2012. Because of this, one does not have to check for good properties with respect to transfers in constructing dg-realizations.

The dg-realization of rigid homology

We assume that k is algebraically closed. Then the category of F-isocrystals is semisimple, and so every object of $\mathcal{D}(K_\sigma[F])$ is isomorphic to its homology (viewed as a complex with trivial differentials).

LEMMA 7.2. *The category $C(K_\sigma[F])$ of unbounded complexes of $K_\sigma[F]$-modules is a pre-triangulated cocomplete compactly generated dg-category over $\mathbb{Q}_p$.*

PROOF. Apply Beilinson and Vologodsky 2008, Example before 1.5.5, p.1718, which shows that for every abelian category A, the category of complexes $C(\mathsf{A})$ is pretriangulated. □

Our task is to define the dg-quasifunctor

$$\mathrm{Crys} \colon \mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k,\mathbb{Q}) \to \mathcal{D}(K_\sigma[F]).$$

By Vologodsky's criterion and Ayoub's theorem (Ayoub 2013, B.1) it suffices to
  (a) construct a functor from smooth connected schemes over k to $C(K_\sigma[F])$ (hence to $\mathcal{D}(K_\sigma[F])$) that is $\mathbb{A}^1$-homotopy invariant and satisfies étale descent, and
  (b) check that the image lies in the dg subcategory $\mathcal{D}^b_c(K_\sigma[F]))$ of $\mathcal{D}(K_\sigma[F]))$.

N. Tsuzuki has proved proper cohomological descent for rigid cohomology and, together with B. Chiaroletto, étale descent for rigid cohomology (Tsuzuki 2003, Chiarellotto and Tsuzuki 2003). It is known that rigid cohomology is $\mathbb{A}^1$-homotopy invariant.[10]

For a smooth scheme X over k, we define $\mathrm{Crys}(X)$ to be Besser's rigid complex $\mathbb{R}\Gamma(X/K)$ (Besser 2000, 4.9, 4.13), which is a canonical functorial complex of K-vector spaces that computes the rigid cohomology of X. It is compatible with base change $k \to k'$ and is endowed with a Frobenius map (ibid. Proposition 4.21, Corollary 4.22). Therefore $\mathrm{Crys}(X)$ is an object of $C(K_\sigma[F])$. The assignment $X \rightsquigarrow \mathrm{Crys}(X)$ is a functor; it satisfies étale descent and $\mathbb{A}^1$-homotopy invariance because rigid cohomology satisfies étale descent and $\mathbb{A}^1$-homotopy invariance.

In summary, there exists a dg-crystalline realization functor

$$\mathrm{Crys} \colon \mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}]) \to \mathcal{D}(K_\sigma[F]).$$

NOTES

7.3. The functor Crys becomes covariant if we take $\mathrm{Crys}(X)$ to be the dual of Besser's complex.

7.4. On $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}])$ this becomes the Borel-Moore rigid homology functor.

---

[10]It satisfies the Künneth formula, agrees with Monsky-Washnitzer cohomology on smooth affine varieties, and the Monsky-Washnitzer cohomology of affine space is trivial except in degree 0 (see, for example, van der Put 1986).

7.5. As the Besser complex is compatible with base change, $\text{Crys}(X)$ for any smooth variety X over a finite field k is a complex of $K_\sigma[F]$-modules with an action of Γ_0.

7.6. Another method of obtaining Crys is to use the overconvergent site of B. Le Stum (Le Stum 2011).

7.7. There is an alternative construction of Crys. The restriction functor from étale sheaves on smooth schemes over k to étale sheaves on smooth affine schemes over k is an equivalence of categories. Therefore, it suffices to define Crys on smooth affine schemes. Here we can take the Monsky-Washnitzer complex of a smooth affine variety. It is known that the cohomology of this complex computes the rigid cohomology of the variety.

Homotopy fibred products of dg-categories

In this subsection, we review the definition of homotopy fibred products of dg categories (Drinfeld 2004, Section 15, Appendix IV; Tabuada 2010a, Chapter 3; Ben-Bassat and Block 2012, Section 4).

Let $\mathcal{B}$ be a dg category. Given objects x, y of $\mathcal{B}$, we write $\text{Hom}_{\mathcal{B}}^\bullet(x, y)$ for the $\mathbb{Z}$-graded complex of morphisms from x to y. For the homotopy category $\text{Ho}\mathcal{B}$ of $\mathcal{B}$,

$$\text{Hom}_{\text{Ho}\mathcal{B}}(x, y) = H^0(\text{Hom}_{\mathcal{B}}^\bullet(x, y)).$$

Consider a diagram of dg-categories and dg-functors:

$$
\begin{array}{ccc}
 & & \mathcal{B} \\
 & & \downarrow{\scriptstyle G} \\
\mathcal{C} & \xrightarrow{\ L\ } & \mathcal{D}.
\end{array}
$$

The homotopy fibred product $\mathcal{C} \times_{\mathcal{D}} \mathcal{B}$ is a dg-category (Ben-Bassat and Block 2012, Section 4). Its objects are triples

$$x = (M, N, \phi) \qquad M \in \mathcal{B}, \quad N \in \mathcal{C}, \quad \phi \in \text{Hom}_{\mathcal{D}}^0(G(M), L(N))$$

such that ϕ is closed and becomes invertible in $\text{Ho}(D)$. The morphisms of degree i from an object (M_1, N_1, ϕ_1) to an object (M_2, N_2, ϕ_2) are the triples

$$(\mu, \nu, \gamma) \in \text{Hom}_{\mathcal{B}}^i(M_1, M_2) \oplus \text{Hom}_{\mathcal{C}}^i(N_1, N_2) \oplus \text{Hom}_{\mathcal{D}}^{i-1}(G(M_1), L(N_2)),$$

and the differential is

$$d(\mu, \nu, \gamma) = (d\mu, d\nu, d\gamma + \phi_2 G(\mu) - (-1)^i L(\nu)\phi_1).$$

Definition of $\mathcal{DM}(k, \mathbb{Z})$

The full category $\mathcal{DM}_{\text{et}}(k, \mathbb{Q})$ of étale dg-motives is defined to be the localization of $\mathcal{DM}_{\text{et}}^{\text{eff}}(k, \mathbb{Q})$ by the Tate motive (Beilinson and Vologodsky 2008, Section 6.1). Since the Tate motive maps under Crys to an invertible object in $\mathcal{D}_c^b(K_\sigma[F])$, the dg-realization Crys automatically extends to

$$\text{Crys}: \mathcal{DM}_{\text{et}}(k, \mathbb{Q}) \to \mathcal{D}_c^b(K_\sigma[F]).$$

REMARK 7.8. (a) As mentioned earlier, the equivalence σ in (26) has a dg enhancement; this clearly extends to the non-effective categories. So the definitions of Beilinson and Vologodsky 2008 for $\mathcal{DM}(k,\mathbb{Q})$ of Voevodsky motives (Nisnevich topology) can be replaced with the étale version.

(b) (Beilinson and Vologodsky 2008, Section 6.1). Explicitly, an object of $\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Q})$ is represented as $M(a)$, $M \in \mathcal{DM}_{\mathrm{et}}^{\mathrm{eff}}(k,\mathbb{Q})$, $a \in \mathbb{Z}$, and

$$\mathrm{Hom}_{\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Q})}(M(a),N(b)) = \varinjlim_{n} \mathrm{Hom}_{\mathcal{DM}_{\mathrm{et}}^{\mathrm{eff}}(k,\mathbb{Q})}(M(a+n),N(b+n));$$

here the inductive limit is taken as $n \to +\infty$ with $a+n$, $b+n$ nonnegative. Cf. the definition of the category **Crys**(k), just before Lemma 1.7, in Milne and Ramachandran 2004.

We write Crys again for the composite

$$\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}]) \to \mathcal{DM}_{\mathrm{et}}(k,\mathbb{Q}) \xrightarrow{\mathrm{Crys}} \mathcal{D}(K_\sigma[F]).$$

Let $s\colon \mathcal{D}_c^b(R) \to \mathcal{D}_c^b(K_\sigma[F])$ be the dg functor sending a complex of graded R-modules to the associated simple complex tensored with K. We define the ***dg category of motives*** over k to be the fibred product

$$\mathcal{DM}(k,\mathbb{Z}) = \mathcal{DM}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}]) \times_{\mathcal{D}_c^b(K_\sigma[F])} \mathcal{D}_c^b(R).$$

In other words, the following diagram is cartesian:

$$
\begin{array}{ccc}
\mathcal{DM}(k,\mathbb{Z}) & \longrightarrow & \mathcal{D}_c^b(R) \\
\downarrow & & \downarrow{\scriptstyle s} \\
\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}]) & \xrightarrow{\mathrm{Crys}} & \mathcal{D}_c^b(K_\sigma[F])
\end{array}
$$

In the diagram, the vertical arrows are contravariant and covariant respectively, and the horizontal arrows are covariant and contravariant respectively. We also write $\mathcal{DM}(k)$ for $\mathcal{DM}(k,\mathbb{Z})$.

By definition, objects of $\mathcal{DM}(k,\mathbb{Z})$ are triples (M,N,ϕ) where N is an object of $\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}])$, M is an object of $\mathcal{D}(R)$ and $\phi \in \mathcal{D}_c^b(K_\sigma[F])(s(M),\mathrm{Crys}(N))$ is closed and becomes invertible in $\mathrm{Ho}(\mathcal{D}(K_\sigma[F])) = D(K_\sigma[F])$. The homotopy category of $\mathcal{DM}(k,\mathbb{Z})$ is the triangulated category $\mathsf{DM}(k) = \mathsf{DM}(k,\mathbb{Z})$ of integral motivic complexes over k.

ASIDE 7.9. If we take the complex dual to Besser's complex, we will get a covariant rigid realization (Borel-Moore homology, dual to cohomology) and then all the arrows in the above diagram will be covariant. But then the motive defined by a smooth projective variety will be the triple $(DR\Gamma(X,W\Omega_X),hX,-)$ where the first component is the dual of the complex computing the de Rham-Witt cohomology of X, or the triple $(R\Gamma(X,W\Omega_X),DhX,-)$ where we use duality in Voevodsky's category.

Mayer-Vietoris sequence

Fix a commutative ring A and an A-linear dg-category $\mathcal{B}$. The dg-category $\mathcal{B}$-Mod of $\mathcal{B}$-modules is defined to be the category of dg-functors from $\mathcal{B}$ to the dg-category of complexes

of A-modules. Every object y of $\mathcal{B}$ defines a dg-module $x \rightsquigarrow \mathrm{Hom}_{\mathcal{B}}(x,y)$. There is a natural embedding $\mathcal{B} \to \mathcal{B}$-Mod (Yoneda) given by $y \mapsto \mathrm{Hom}_{\mathcal{B}}(-,y)$. Every dg-functor $f: \mathcal{B}' \to \mathcal{B}$ induces a pullback dg-functor $f^*: \mathcal{B}\text{-Mod} \to \mathcal{B}'\text{-Mod}$.

Every y in $\mathcal{B}$ defines the dg-module $\mathrm{Hom}_{\mathcal{B}}(-,y)$ whereas the bi-functor $\mathrm{Hom}_{\mathcal{B}}(-,-)$ defines a canonical bimodule $M_{\mathcal{B}}$. See Section 2 of Tabuada 2013 for more details.

Consider the two bimodules U and V on $\mathcal{DM}(k,\mathbb{Z})$ defined as follows. Say $x = (M,N,\phi)$ and $x' = (M',N',\phi')$ are objects of $\mathcal{DM}(k,\mathbb{Z})$. We define

$$U(x,x') = \mathrm{Hom}_{\mathcal{D}_c^b(K_\sigma[F])}(s(M),\mathrm{Crys}(N'))[1]$$

and

$$V(x,x') = \mathrm{Hom}_{\mathcal{D}_c^b(R)}(M,M') \oplus \mathrm{Hom}_{\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}])}(N,N').$$

The definition of the morphisms in the homotopy fibred dg category provides the exact sequence of $\mathcal{DM}(k,\mathbb{Z})$-bimodules

$$0 \to U \to M_{DZ} \to V \to 0.$$

Applied to any pair x,x' of objects in $\mathcal{DM}(k,\mathbb{Z})$, we get a short exact sequence of $\mathbb{Z}$-graded complexes

$$0 \to U(x,x') \to \mathrm{Hom}_{\mathcal{DM}(k,\mathbb{Z})}(x,x') \to V(x,x') \to 0$$

and the associated (Mayer-Vietoris) long exact sequence

$$\cdots \to \mathrm{Ext}_{\mathcal{DM}(k)}^i(x,x') \to \mathrm{Ext}_{\mathcal{D}_c^b(R)}^i(M,M) \oplus \mathrm{Ext}_{\mathcal{DM}_{\mathrm{et}}(k,\mathbb{Z}[p^{-1}])}^i(N,N') \to$$
$$\to \mathrm{Ext}_{\mathcal{D}_c^b(K_\sigma[F])}^i(sM,\mathrm{Crys}(N')) \to \mathrm{Ext}_{\mathcal{DM}(k,\mathbb{Z})}^{i+1}(x,x') \to \cdots.$$

Note: the homotopy fibre-product is designed to give the Mayer-Vietoris sequence!

Properties of DM(k)

We now develop some of the properties of DM(k).

TRIANGULATED STRUCTURE

All the dg-categories involved in the construction of $\mathcal{DM}(k,\mathbb{Z})$ are dg-enhancements of triangulated categories, and are therefore pretriangulated. Recall that a dg-category $\mathcal{C}$ is (strongly) pretriangulated if

◇   for each object A of $\mathcal{C}$ and $m \in \mathbb{Z}$, there exists an object, denoted $A[m]$, representing the functor $C \rightsquigarrow \mathrm{Hom}^m(C,A))$, so

$$\mathrm{Hom}(C,A[m]) = \mathrm{Hom}^m(C,A), \text{ all } C \in \mathrm{ob}(\mathsf{C});$$

◇   for each morphism $f: A \to B$ in $\mathcal{C}$ with $d \circ f = 0$, there exists an object, denoted $\mathrm{Cone}(f)$, representing the functor sending each $C \in \mathrm{ob}(\mathsf{C})$ to the cone on

$$\mathrm{Hom}(C,A) \xrightarrow{f \circ -} \mathrm{Hom}(C,B).$$

If C is strongly pre-triangulated, then Ho(C) has a translation functor, namely, $A \rightsquigarrow A[1]$, and a class of distinguished triangles, namely, those isomorphic to one of the form

$$A \xrightarrow{f} B \to \mathrm{Cone}(f) \to A[1].$$

With this structure, Ho(C) becomes a triangulated category (Beilinson and Vologodsky 2008, 1.5.4).

For any object $x = (N, N, \phi)$ of $\mathcal{DM}(k, \mathbb{Z})$ and any integer n, the object $x[n]$ representing the shift exists, and is given by $(M[n], N[n], \phi_n)$. Similarly, one can check that the cone of any map $f : x \to x'$ is representable. Therefore $\mathsf{DM}(k)$ is a triangulated category

MOTIVES OF SMOOTH VARIETIES

The category $\mathsf{DM}(k)$ contains an identity object $\mathbf{1} = (W, N(\mathrm{Spec}\, k), \mathrm{id})$ where $N(\mathrm{Spec}\, k)$, the motive of a point, is the identity object of $\mathcal{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Z}[p^{-1}])$ (Beilinson and Vologodsky 2008, §2.2, §2.3). The object W is the identity object of $\mathsf{D}_c^b(R)$, and $s(W) = K$ considered as a complex in degree zero with $F = \sigma$. This is $\mathrm{Crys}(N(\mathrm{Spec}(k))$.

For any smooth proper variety X, consider the object

$$(R\Gamma(X, W\Omega), N(X), \phi) \in \mathcal{DM}(k, \mathbb{Z})$$

where the first term is a suitable complex computing the de Rham-Witt cohomology of X, the second term is the motive of X, and the third term is the canonical isomorphism between de Rham-Witt cohomology (tensored with K) and rigid cohomology (Berthelot 1986). The resulting object h_X in $\mathcal{DM}(k, \mathbb{Z})$ is well defined because any two suitable complexes computing the de Rham-Witt cohomology are quasi-isomorphic.

For any smooth variety X, similarly consider the object $(C(X), N(X), \phi)$ where $N(X)$ is the motive of X (see above) and $C(X)$ is the object of $\mathsf{D}_c^b(R)$ attached to X in §6 of Milne and Ramachandran 2013. The map ϕ comes from the canonical isomorphism between logarithmic de Rham-Witt cohomology and rigid cohomology (Nakkajima and Shiho 2008, Nakkajima 2012).

TENSOR STRUCTURE AND INTERNAL HOM'S

We expect that the homotopy categories $\mathsf{DM}(k)$ and $\mathsf{DM}^{\mathrm{eff}}(k)$ are tensor triangulated categories.

We sketch the proof for $\mathsf{DM}^{\mathrm{eff}}(k)$. There is a homotopy tensor structure on $\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Z}[p^{-1}])$ (Beilinson and Vologodsky 2008, Section 2.2, 2.3; Vologodsky 2012, line before Lemma 2.3). There are clearly also homotopy tensor structures on $\mathcal{D}_c^b(R)$ and $\mathcal{D}_c^b(K_\sigma[F])$. The functors s and Crys are compatible with the tensor structure (up to homotopy) — for Crys this is a consequence of the Künneth property for rigid cohomology. This suffices to endow $\mathsf{DM}^{\mathrm{eff}}(k)$ with the structure of a tensor triangulated category.

More precisely, the cartesian product of schemes induces a tensor product structure on $\mathcal{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Z}[p^{-1}])$ (Vologodsky 2012, line before Lemma 2.3). The category is generated by the motives $N(X)$ of smooth varieties. Thus, the homotopy tensor structure on $\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Z})$, which defines the tensor product structure on $\mathsf{DM}^{\mathrm{eff}}(k)$ is determined by $h(X) \otimes h(Y) \simeq h(Y) \otimes h(X)$.

Similar comments apply to the internal Hom. The internal Hom of $N(X)$ and $N(Y)$ in $\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Z}[p^{-1}])$ is defined by the following equality for all smooth varieties Z,

$$\mathrm{Hom}(N(Z), R\underline{\mathrm{Hom}}(N(X), N(Y))) = \mathrm{Hom}(N(Z \times X), N(Y))$$

([Beilinson and Vologodsky 2008](#), Section 2.2).

On the other hand, the existence of a homotopy tensor structure on $\mathcal{DM}(k, \mathbb{Z}[p^{-1}])$ does not seem to be known (ibid., remark at the end of Section 6.1). Therefore, it does not seem to be known that $\mathsf{DM}(k)$ has a natural tensor structure.

<small>TATE TWIST</small>

The Tate object $R(1)$ in $\mathcal{DM}_{\mathrm{et}}^{\mathrm{eff}}(k, \mathbb{Z}[p^{-1}])$ is determined by the property

$$N(\operatorname{Spec} k) \oplus R(1)[1] = N(\mathbb{G}_m)$$

where $N(X)$ is the motive of the smooth variety X ([Beilinson and Vologodsky 2008](#), Section 2.2). The Tate twist is given by $F(1) = F \otimes R(1)$. Recall that the rigid cohomology groups of $\mathbb{G}_m$ are

$$H_{\mathrm{rig}}^0(\mathbb{G}_m) = K(0), \quad H_{\mathrm{rig}}^1(\mathbb{G}_m) = K(1)$$

where $K(m)$ is the F-isocrystal K with $F = p^m \sigma$. Thus $\mathrm{Crys}(R(1)) = K(1)$.

The Tate object $E(1)$ in $\mathsf{DM}(k)$ is the triple

$$E(1) = (W(1), R(1), \phi)$$

where $W(1)$ is the Tate twist of the identity object of $\mathsf{D}_c^b(R)$ ([Milne and Ramachandran 2005](#), Section 2, or [Milne and Ramachandran 2013](#), 1.6) and ϕ is the natural isomorphism between $sW(1) = K(1)$ and $\mathrm{Crys}(R(1))$. The Tate twist on $\mathsf{DM}(k)$ is given by $M(1) = M \otimes E(1)$ for M an object of $\mathsf{DM}(k)$.

<small>REALIZATION FUNCTORS</small>

There are realization functors r_l on $\mathsf{DM}(k)$ for all primes l.

The dg functor

$$pr_2 \colon \mathcal{DM}(k, \mathbb{Z}) \to \mathcal{DM}(k, \mathbb{Z}[p^{-1}]), \quad (M, N, \phi) \rightsquigarrow N,$$

induces a functor $pr_2 \colon \mathsf{DM}(k) \to \mathsf{DM}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ which is clearly an exact functor between these triangulated categories. For each prime $l \neq p$, there is an étale realization functor

$$r_l' \colon \mathsf{DM}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}]) \to \mathsf{D}_c^b(k, \mathbb{Z}_l)$$

which is an exact functor of tensor triangulated categories ([Ayoub 2013](#)). We define the l-adic realization r_l to be the composite

$$r_l' \circ pr_2 \colon \mathsf{DM}(k) \to \mathsf{D}_c^b(k, \mathbb{Z}_l).$$

As r_l' is an exact functor of triangulated categories, so also is r_l.

The dg functor

$$pr_1 \colon \mathcal{DM}(k, \mathbb{Z}) \to \mathcal{D}_c^b(R), \quad (M, N, \phi) \rightsquigarrow M,$$

induces a functor $\mathsf{DM}(k) \to \mathsf{D}_c^b(R)$ on the associated triangulated categories. This is the realization functor r_p.

RIGIDITY

PROPOSITION 7.10. *For all l, including $l = p$, the realization functor r_l defines an equivalence on the subcategories of objects killed by l^n.*

PROOF. This follows from the Mayer-Vietoris sequence and Voevodsky 2000, 3.3.3.   □

We say that an object X of a triangulated category C is *n-torsion* ($n \in \mathbb{Z}$) if the abelian group $\mathrm{Hom}(X, Y)$ is killed by n for all objects Y of C. We say that X is *l-power torsion* for a prime l if every element of $\mathrm{Hom}(X, Y)$ is killed by a power of l, all Y in C.

PROPOSITION 7.11. *For any $x = (M, N, \phi)$ and $x' = (M', N', \phi')$ in $\mathcal{DM}(k, \mathbb{Z})$, and every positive integer n, the map*

$$\mathrm{Hom}_{\mathsf{DM}(k)}(x, x') \otimes \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Hom}_{\mathsf{D}_c^b(R)}(M, M') \otimes \mathbb{Z}/p^n\mathbb{Z}$$

induced by r_p is an isomorphism.

PROOF. Omitted.   □

Motivic complexes over $\mathbb{F}_q$

We need a category $\mathsf{DM}(\mathbb{F}_q)$ such that the functor

$$R\,\mathrm{Hom}(\mathbb{1}, -) \colon \mathsf{DM}(\mathbb{F}_q) \to \mathsf{D}(\mathbb{Z})$$

factors through $\mathsf{D}(\mathbb{Z}\varGamma_0)$. We can either proceed as above, but with the Weil-étale topology for the étale topology or, more simply, as in §6.

Applications to algebraic varieties

To give an object of $\mathcal{DM}(k)$ amounts to giving objects of $\mathcal{DM}_{\mathrm{et}}(k, \mathbb{Z}[p^{-1}])$ and $\mathcal{D}_c^b(R)$ together with an isomorphism between their realizations in $\mathcal{D}_c^b(K_\sigma[F])$. In the final section of Milne and Ramachandran 2013, we explained how to attach an object of $\mathsf{D}_c^b(R)$ to an arbitrary variety, a variety with log structure, a Deligne-Mumford stack, etc.. All of the statements there carry over *mutatis mutandis* to the present situation.

7.12. Choudhury (2012) attaches an object in Voevodsky's category $\mathsf{DM}^{\mathrm{eff}}(k, \mathbb{Q})$ to a smooth Deligne-Mumford stack.

7.13. Voevodsky 2010 attaches an object in his category $\mathsf{DM}_-^{\mathrm{eff}}(k)$ to a smooth simplicial scheme over a field k.

8   Motivic complexes for rational Tate classes

In this section, we sketch the construction of a category $\mathsf{DM}(k)$ for which the "Tate conjecture" is automatically true. The construction requires only the rationality conjecture of Milne 2009, which is much weaker than the Tate conjecture.

Rational Tate classes

In this subsection, we review part of Milne 2009. Let $\mathbb{Q}^{\mathrm{al}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Fix a p-adic prime w of $\mathbb{Q}^{\mathrm{al}}$, and let $\mathbb{F}$ be its residue field. Then $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$. We assume that the reader is familiar with the theory of absolute Hodge classes (Deligne 1982).

RATIONALITY CONJECTURE 8.1. Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction to an abelian variety A_0 over $\mathbb{F}$, and let $d = \dim(A)$. An absolute Hodge class γ of codimension i on A defines (by specialization) classes $\gamma_l \in H^{2i}(A_0, \mathbb{Q}_l(i))$ for all $l \neq p$ and $\gamma_p \in H^{2i}_{\mathrm{crys}}(A_0/W)(i)_{\mathbb{Q}}$. Let $D_1, \ldots, D_{d-i}$ be divisors on A_0, and let $\delta_1(l), \ldots, \delta_{d-i}(l)$ denote their l-cohomology classes. The conjecture says that

$$\gamma_l \cdot \delta_1(l) \cdot \cdots \cdot \delta_{d-i}(l),$$

which a priori lies in $\mathbb{Q}_l$ or $\left(\mathbb{Q}^{\mathrm{al}}\right)_w$, is a rational number independent of l.

Now let $\mathcal{S}$ be a class of smooth projective algebraic varieties over $\mathbb{F}$ that is closed under passage to a connected component and under the formation of finite products and disjoint unions. We assume that $\mathcal{S}$ contains the class $\mathcal{S}_0$ of all abelian varieties over $\mathbb{F}$, and that the Frobenius elements of the varieties in $\mathcal{S}$ act semisimply on cohomology. For an X in $\mathcal{S}$, we let $H^{2i}_{\mathbb{A}}(X)(i)$ denote the restricted product of the cohomology groups $H^{2i}(X, \mathbb{Q}_l(i))$ for $l \neq p$ with $H^{2i}_{\mathrm{crys}}(X/W)_{\mathbb{Q}}$.

DEFINITION 8.2. A family $(\mathcal{R}^*(X))_{X \in \mathcal{S}}$ with each $\mathcal{R}^*(X)$ a graded $\mathbb{Q}$-subalgebra of $H^{2*}_{\mathbb{A}}(X)(*)$ is a ***good theory of rational Tate classes on*** $\mathcal{S}$ if it satisfies the following conditions:

(R1) for every regular map $f : X \to Y$ of varieties in $\mathcal{S}$, f^* maps $\mathcal{R}^*(Y)$ into $\mathcal{R}^*(X)$ and f_* maps $\mathcal{R}^*(X)$ into $\mathcal{R}^*(Y)$;

(R2) for every X in $\mathcal{S}$, $\mathcal{R}^1(X)$ contains the divisor classes;

(R3) for all CM abelian varieties A over $\mathbb{Q}^{\mathrm{al}}$, the absolute Hodge classes on A map to elements of $\mathcal{R}^*(A_0)$ under the specialization map;

(R4) For all varieties X in $\mathcal{S}$, the $\mathbb{Q}$-algebra $\mathcal{R}^*(X)$ is of finite degree, and the l-primary components of every element of $\mathcal{R}^*(X)$ are l-adic Tate classes.

Recall that an abelian variety A is CM if its endomorphism algebra $\mathrm{End}(A)_{\mathbb{Q}}$ contains an étale subalgebra of degree $2 \dim A$ over $\mathbb{Q}$, and that such an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ has good reduction at all the primes of $\mathbb{Q}^{\mathrm{al}}$. For the space $\mathcal{T}^i_l(X)$ of Tate classes in $H^{2i}(X, \mathbb{Q}_\ell(i))$ or $H^{2i}_{\mathrm{crys}}(X/W)(i)_{\mathbb{Q}}$, see ibid. pp.112–113.

THEOREM 8.3. *(a) There exists at most one good theory of rational Tate classes on $\mathcal{S}$.*

(b) There exists a good theory of rational Tate classes on $\mathcal{S}_0$ if the rationality conjecture holds.

(c) The Tate conjecture holds for every good theory of rational Tate classes, i.e., the maps $\mathcal{R}^i(X) \otimes \mathbb{Q}_l \to \mathcal{T}^i_l(X)$ induced by the projection maps are isomorphisms for all X, i, and l.

PROOF. Ibid. Theorem 3.3; Theorem 4.5; Theorem 3.2.   □

The category $\mathsf{DM}(k)$

We assume that there exists a good theory of rational Tate classes for some class $\mathcal{S}$ as in the last subsection. Grothendieck's construction now gives us a Tannakian $\mathbb{Q}$-linear category $\mathsf{Mot}(k, \mathbb{Q})$ of motives, and we define $\mathsf{DM}(k, \mathbb{Q})$ to be its derived category.

Let $\mathbb{Z}^p = \prod_{l \neq p} \mathbb{Z}_l$. It is generally believed that the constructions of the ℓ-adic triangulated category $\mathsf{D}_c^b(X, \mathbb{Z}_\ell)$ (Deligne 1980, Ekedahl 1990, Bhatt and Scholze 2013) will generalize to give an adic category $\mathsf{D}_c^b(X, \mathbb{Z}^p)$, but as far as we know, no proof has been written out. Instead, we give an ad hoc construction of $\mathsf{D}_c^b(k, \mathbb{Z}^p)$. Let Γ be the absolute Galois group of k. Let $\Lambda = \mathbb{Z}^p$ and let $\Lambda_m = \mathbb{Z}/m\mathbb{Z}$ when $(m, p) = 1$. The inverse systems $M = (M_m)_m$ in which M_m is a continuous $\Lambda_m \Gamma$-module form an abelian category whose derived category we denote by $\mathsf{D}(k, \Lambda_\bullet)$. As in (4.2), there is an obvious "completion" functor $M \rightsquigarrow \hat{M} : \mathsf{D}(k, \Lambda_\bullet) \to \mathsf{D}(k, \Lambda_\bullet)$. We define $\mathsf{D}_c^b(k, \mathbb{Z}^p)$ to be the full subcategory of $\mathsf{D}(k, \Lambda_\bullet)$ consisting of complexes M such that $M \simeq \hat{M}$ and $(\mathbb{Z}/\ell\mathbb{Z}) \otimes_{\Lambda_\bullet}^L M$ is a bounded complex with bounded finite-dimensional cohomology for all primes $\ell \neq p$. The functor $(M_n)_n \rightsquigarrow \varprojlim M_n(k^{\mathrm{sep}})$ defines a functor

$$\alpha : \mathsf{D}_c^b(k, \mathbb{Z}^p) \to \mathsf{D}(\mathbb{Z}^p \Gamma).$$

It follows from the next lemma that the cohomology groups of αM are finitely presented $\mathbb{Z}^p$-modules.

LEMMA 8.4. *The following conditions on a $\mathbb{Z}^p$- module are equivalent.*
  (a) M *is of finite presentation.*
  (b) *The $\mathbb{Z}_\ell$-module $\mathbb{Z}_\ell \otimes_{\mathbb{Z}^p} M$ is finitely generated for all ℓ, the natural map $M \to \prod_{\ell \neq p} (\mathbb{Z}_\ell \otimes_{\mathbb{Z}^p} M)$ is an isomorphism, and $\dim_{\mathbb{F}_\ell}(M/\ell M)$ is bounded (independently of ℓ).*
  (c) *The natural map $M \to \varprojlim_{(m,p)} M/mM$ is an isomorphism, and $\dim_{\mathbb{F}_\ell}(M/\ell M)$ is bounded (independently of ℓ).*

PROOF. Elementary exercise.   □

There is an exact functor of triangulated categories $\mathsf{DM}(k, \mathbb{Q}) \to \mathsf{D}_c^b(\mathbb{Z}^p)_{\mathbb{Q}} \times \mathsf{D}_c^b(R)_{\mathbb{Q}}$, and we define $\mathsf{DM}(k)$ to be the universal object fitting into a diagram

$$
\begin{array}{ccc}
\mathsf{DM}(k) & \longrightarrow & \mathsf{DM}(k, \mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathsf{D}_c^b(\mathbb{Z}^p) \times \mathsf{D}_c^b(R) & \longrightarrow & \mathsf{D}_c^b(\mathbb{Z}^p)_{\mathbb{Q}} \times \mathsf{D}_c^b(R)_{\mathbb{Q}}.
\end{array}
$$

More precisely, we form the homotopy fibred product of the natural dg enhancements of the categories, and then pass to the associated triangulated category (see §7). Conjecturally $\mathsf{DM}(k)$ is independent of the choice of $\mathcal{S}$ containing $\mathcal{S}_0$.

For this category, the "Tate conjecture" holds automatically, and so we can apply the results of §6.

F<small>INAL NOTE</small>

Contrary to our earlier claim, (1.1) is not in fact the ultimate conjecture. The zeta function of an Artin stack is also defined, but it is a power series in t (not a rational function). To accomodate Artin stacks, one will need to state a conjecture for a category of *unbounded* motivic complexes, but we (the authors, and perhaps also the reader) are already exhausted.

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