Descent for algebraic schemes

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Abstract

This is an elementary exposition of the basic descent theorems for algebraic schemes over fields (Grothendieck, Weil,...).

Contents

Consider fields $k \subset \Omega$. An algebraic scheme V over k defines a scheme V_Ω over Ω by extension of the base field. Descent theory provides answers to the following question: what additional structure do we need to place on an algebraic scheme over Ω, or a morphism of algebraic schemes over Ω, in order to ensure that it comes from k? We are most interested in the case that Ω is algebraically closed and k is perfect.

In this article, we shall make free use of the axiom of choice (usually in the form of Zorn's lemma).

This is a revised version of Chapter 16 of my notes *Algebraic Geometry*. I've posted it on the arXiv in order to have a convenient reference.

0 Preliminaries from algebraic geometry

Let k be a field. An algebraic scheme over k is a separated scheme of finite type over $\operatorname{Spec} k$. It is integral if it is reduced and irreducible, and it is an algebraic variety if it is geometrically reduced. An affine k-algebra is a finitely generated k-algebra A such that $K \otimes_k A$ is reduced for all fields K containing k (it suffices to check this for an algebraic closure of k). A regular map of algebraic schemes (or varieties) over k is a k-morphism. For an affine algebraic scheme V over k, $k[V] = \mathcal{O}_V(V)$ (so $V = \operatorname{Spec} k[V]$), and for an integral algebraic scheme V over k, $k(V)$ is the field of rational functions on V (the local ring at the generic point of V).

0.1. *In an algebraic scheme V over k, the intersection of any two open affine subsets is again an open affine subset.*

Let U and U' be open affine subsets of V. Then $U \cap U'$ is certainly open, and the diagonal map $U \cap U' \hookrightarrow U \times U'$ is a closed immersion because it is the pullback of the diagonal $\Delta_V \hookrightarrow V \times V$,

$$\begin{tikzcd} U \cap U' \arrow[r] \arrow[d] & U \times U' \arrow[d] \\ \Delta_V \arrow[r] & V \times V. \end{tikzcd}$$

Now $U \cap U'$ is an affine scheme because it is a closed subscheme of an affine scheme.

0.2. *Let V be an algebraic variety over k. If k is separably closed, then $V(k)$ is dense in $|V|$ (for the Zariski topology).*

We may assume that V is irreducible. Then $k(V)$ admits a separating transcendence basis over k. This means that V is birationally equivalent to a hypersurface

$$f(X_1, \dots, X_{d+1}) = 0, \quad d = \dim V,$$

where f has the property that $\partial f / \partial X_{d+1} \neq 0$. This implies that the closed points P such that $k(P)$ is separable over k form a dense subset of $|V|$. In particular, $V(k)$ is dense in $|V|$ when k is separably closed.

0.3. *Let V be a quasi-projective scheme over an infinite field. Every finite set of closed points of V is contained in an open affine subset.*

Embed V as a subscheme of $\mathbb{P}^n$. Let $\bar{V}$ be the closure of V in $\mathbb{P}^n$, and let $Z = \bar{V} \setminus V$ be the boundary. For each $P \in S$, there exists a homogeneous polynomial $F_P \in I(Z)$ such that $F_P(P) \neq 0$. We may suppose that the F_P have the same degree. Because k is infinite, some linear combination F of the F_P has the property that, for all $P \in S$, $F(P) \neq 0$. Then $\bar{V} \cap D(F)$ is an open affine subset of V containing S.

0.4. *Let $A \subset B$ be rings with B integral over A. Let $\mathfrak{p}$ be a prime ideal of A. Then there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} = \mathfrak{q} \cap A$. If $\mathfrak{q}' \supset \mathfrak{q}$ is a second such prime ideal, then $\mathfrak{q}' = \mathfrak{q}$.*

See, for example, 7.3 and 7.5 of my notes *A Primer of Commutative Algebra.*

0.5 (CHEVALLEY'S THEOREM). *Let $\phi\colon W \to V$ be a dominant morphism of irreducible algebraic schemes over k. Then $\phi(W)$ contains a dense open subset of V.*

See, for example, Theorem 15.8 of my notes *A Primer of Commutative Algebra.*

0.6. *Let A and B be k-algebras. Assume that k is algebraically closed and A is finitely generated over k.*

(a) *If A and B are reduced. so also is $A \otimes_k B$.*

(b) *If A and B are integral domains, so also is $A \otimes_k B$.*

Let $\alpha \in A \otimes_k B$. Then $\alpha = \sum_{i=1}^{n} a_i \otimes b_i$, some $a_i \in A$, $b_i \in B$. If one of the b_j is a k-linear combination of the remaining b_i, say, $b_n = \sum_{i=1}^{n-1} c_i b_i$, $c_i \in k$, then, using the bilinearity of $\otimes$, we find that

$$\alpha = \sum_{i=1}^{n-1} a_i \otimes b_i + \sum_{i=1}^{n-1} c_i a_n \otimes b_i = \sum_{i=1}^{n-1} (a_i + c_i a_n) \otimes b_i.$$

Thus we can suppose that in the original expression of α, the b_i are linearly independent over k.

Now assume A and B to be reduced, and suppose that α is nilpotent. Let $\mathfrak{m}$ be a maximal ideal of A. From $a \mapsto \bar{a}\colon A \to A/\mathfrak{m} = k$ we obtain homomorphisms

$$a \otimes b \mapsto \bar{a} \otimes b \mapsto \bar{a}b\colon A \otimes_k B \to k \otimes_k B \xrightarrow{\simeq} B.$$

The image $\sum \bar{a}_i b_i$ of α under this homomorphism is a nilpotent element of B, and hence is zero (because B is reduced). As the b_i are linearly independent over k, this means that the $\bar{a}_i$ are all zero. Thus, the a_i lie in all maximal ideals $\mathfrak{m}$ of A, and so are zero (because A is reduced). Hence $\alpha = 0$, and we have shown that $A \otimes_k B$ is reduced.

Now assume that A and B are integral domains, and let $\alpha, \alpha' \in A \otimes_k B$ be such that $\alpha\alpha' = 0$. As before, we can write $\alpha = \sum a_i \otimes b_i$ and $\alpha' = \sum a'_i \otimes b'_i$ with the sets $\{b_1, b_2, ...\}$ and $\{b'_1, b'_2, ...\}$ each linearly independent over k. For each maximal ideal $\mathfrak{m}$ of A, we know that $(\sum \bar{a}_i b_i)(\sum \bar{a}'_i b'_i) = 0$ in B, and so either $(\sum \bar{a}_i b_i) = 0$ or $(\sum \bar{a}'_i b'_i) = 0$. Thus either all the $a_i \in \mathfrak{m}$ or all the $a'_i \in \mathfrak{m}$. This shows that

$$\operatorname{spm}(A) = V(a_1, \dots, a_m) \cup V(a'_1, \dots, a'_n).$$

As $\operatorname{spm}(A)$ is irreducible, it follows that $\operatorname{spm}(A)$ equals either $V(a_1, \dots, a_m)$ or $V(a'_1, \dots, a'_n)$. In the first case $\alpha = 0$, and in the second $\alpha' = 0$.

REMARK 0.7. If K and L are fields containing k, then $K \otimes_k L$ need by not be reduced, and if it is reduced, then it need not be an integral domain. For example, let $K = k[\alpha]$, where $\alpha^p = a \in k$, but $\alpha \notin k$. Then K is a field, but $K \otimes_k K$ contains the nilpotent element $\alpha \otimes 1 - 1 \otimes \alpha$. On the other hand, if K is a separable extension of k and L is a Galois closure of K/k, then

$$K \otimes_k L \simeq \prod\nolimits_{\sigma\colon K \to L} L_\sigma,$$

where L_σ is a copy of L. Thus, 0.6 may fail if k is not algebraically closed. However, if A and B are finitely generated reduced k-algbras and k is perfect, then $A \otimes_k B$ is reduced.

NOTATION 0.8. Let V be an algebraic scheme over a field F. For a homomorphism of fields $i\colon F \to L$, we sometimes write iV for

$$V_L \stackrel{\text{def}}{=} V \times_{\operatorname{Spec} F} \operatorname{Spec} L.$$

For example, if V is embedded in affine space, then we get iV by applying i to the coefficients of the polynomials defining V. If $\sigma \in \operatorname{Aut}(L/iF)$, so $\sigma \circ i = i$, then $(\sigma \circ i) V \simeq iV$. We often view this as an equality $\sigma V_L = V_L$.

A morphism $\varphi\colon V \to W$ defines a morphism $\varphi_L\colon V_L \to W_L$, which we sometimes denote $i\varphi\colon iV \to iW$. Note that $(i\varphi)(iZ) = i(\varphi(Z))$ for any algebraic subscheme Z of V. For schemes embedded in affine space, $i\varphi$ is obtained from φ by applying i to the coefficients of the polynomials defining φ.

1 Models

Let $\Omega \supset k$ be fields, and let V be an algebraic scheme over Ω. A ***model*** of V over k is an algebraic scheme V_0 over k together with an isomorphism $\varphi\colon V \to V_{0\Omega}$. An algebraic scheme over Ω need not have a model over k, and when it does it typically will have many nonisomorphic models.[1]

Let V be an affine algebraic variety over Ω. An embedding $V \hookrightarrow \mathbb{A}^n_\Omega$ defines a model of V over k if $I(V)$ is generated by polynomials in $k[X_1,\ldots,X_n]$, because then $I_0 \stackrel{\text{def}}{=} I(V) \cap k[X_1,\ldots,X_n]$ is a radical ideal, $k[X_1,\ldots,X_n]/I_0$ is an affine k-algebra, and $V(I_0) \subset \mathbb{A}^n_k$ is a model of V. Moreover, every model of V arises in this way from an embedding in affine space, because every model of an affine algebraic variety is affine. However, different embeddings in affine space will usually give rise to different models. Similar remarks apply to projective varieties.

Note that the condition that $I(V)$ be generated by polynomials in $k[X_1,\ldots,X_n]$ is stronger than asking that V be the zero set of some polynomials in $k[X_1,\ldots,X_n]$. For example, let $V = V(X + Y + \alpha)$, where α is an element of Ω such that $\alpha^p \in k$ but $\alpha \notin k$. Then V is the zero set of the polynomial $X^p + Y^p + \alpha^p$, which has coefficients in k, but $I(V) = (X + Y + \alpha)$ is not generated by polynomials in $k[X, Y]$.

2 Fixed fields

Let $\Omega \supset k$ be fields, and let Γ be the group $\operatorname{Aut}(\Omega/k)$ of automomorphisms of Ω (as an abstract field) fixing the elements of k. Define the ***fixed field*** Ω^Γ of Γ to be

$$\{a \in \Omega \mid \sigma a = a \text{ for all } \sigma \in \Gamma\}.$$

PROPOSITION 2.1. *The fixed field of Γ equals k in each of the following two cases:*

(a) *Ω is a Galois extension of k (possibly infinite);*

(b) *Ω is an algebraically closed field and k is perfect.*

[1] For example, an elliptic curve E over $\mathbb{C}$ has a model over a number field if and only if its j-invariant $j(E)$ is an algebraic number. If $Y^2Z = X^3 + aXZ^2 + bZ^3$ is one model of E over a number field k (meaning, $a, b \in k$), then $Y^2Z = X^3 + ac^2XZ^2 + bc^3Z^3$ is a second, which is isomorphic to the first only if c is a square in k.

PROOF. (a) See for example, Milne 2022, 7.9.

(b) See for example, Milne 2022, 9.29. □

REMARK 2.2. (a) The proof of Proposition 2.1 requires the axiom of choice. For example, without the axiom of choice, every function $\mathbb{C} \to \mathbb{C}$ is measurable, hence continuous, but the only continuous automorphisms of $\mathbb{C}$ are complex conjugation and the identity map. Therefore, without the axiom of choice, $\mathbb{C}^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q})} = \mathbb{R}$.

(b) Suppose that Ω is algebraically closed and k is not perfect. Then k has characteristic $p \neq 0$ and Ω contains an element α such that $\alpha \notin k$ but $\alpha^p = a \in k$. As α is the unique root of $X^p - a$, every automorphism of Ω fixing k also fixes α, and so $\Omega^\Gamma \neq k$.

The ***perfect closure*** of k in Ω is the subfield

$$k^{p^{-\infty}} = \{\alpha \in \Omega \mid \alpha^{p^n} \in k \text{ for some } n\}.$$

The field $k^{p^{-\infty}}$ is purely inseparable over k. When Ω is algebraically closed, it is the smallest perfect subfield of Ω containing k.

COROLLARY 2.3. *If Ω is separably closed, then $\Omega^{\mathrm{Aut}(\Omega/k)}$ is a purely inseparable algebraic extension of k. In particular, $\Omega^{\mathrm{Aut}(\Omega/k)} = k$ if k is perfect.*

PROOF. When k has characteristic zero, $\Omega^\Gamma = k$, and there is nothing to prove. Thus, we may suppose that k has characteristic $p \neq 0$. Choose an algebraic closure Ω^{al} of Ω, and let $k^{p^{-\infty}}$ be the perfect closure of k in Ω^{al}. As Ω^{al} is purely inseparable over Ω, every element σ of $\mathrm{Aut}(\Omega/k)$ extends uniquely to an automorphism of Ω^{al}: if $\alpha \in \Omega^{\mathrm{al}}$, then $\alpha^{p^n} \in \Omega$ for some n, and so an extension of σ to Ω^{al} must send α to the unique root of $X^{p^n} - \sigma(\alpha^{p^n})$ in Ω^{al}. The action of $\mathrm{Aut}(\Omega/k)$ on Ω^{al} identifies it with $\mathrm{Aut}(\Omega^{\mathrm{al}}/k^{p^{-\infty}})$. According to (b) of the proposition, $(\Omega^{\mathrm{al}})^\Gamma = k^{p^{-\infty}}$, and so

$$k^{p^{-\infty}} \supset \Omega^\Gamma \supset k.$$ □

3 Descending subspaces of vector spaces

Let $\Omega \supset k$ be fields, and let V be a k-subspace of an Ω-vector space $V(\Omega)$ such that the map

$$c \otimes v \mapsto cv\colon \Omega \otimes_k V \to V(\Omega) \tag{1}$$

is an isomorphism. This means that $\Omega V = V(\Omega)$ and that k-linearly independent sets in V are Ω-linearly independent. Such k-spaces V are the k-spans of Ω-bases of $V(\Omega)$.

LEMMA 3.1. *Let W be an Ω-subspace of $V(\Omega)$. There exists at most one k-subspace W_0 of V such that $\Omega \otimes_k W_0$ maps isomorphically onto W under (1). The subspace W_0 exists if and only if V contains a set spanning W, in which case $W_0 = V \cap W$.*

PROOF. If W_0 exists, then $\Omega W_0 = W$, so it contains a set spanning W. Conversely, if V contains a set spanning W, then any k-basis for $V \cap W$ is an Ω-basis for W, and so $c \otimes w \mapsto cw\colon \Omega \otimes_k (V \cap W) \to W$ is an isomorphism. No proper k-subspace of $V \cap W$ can have this property. □

EXAMPLE 3.2. Consider the fields $\mathbb{C} \supset \mathbb{Q}$, and let $V = \mathbb{Q}^2$ and $V(\Omega) = \mathbb{C}^2$. If W is the $\mathbb{C}$-subspace $\{(x, y) \in \mathbb{C}^2 \mid y = \sqrt{2}x\}$ of $V(\Omega)$, then $W \cap V = 0$, and no W_0 exists.

Now assume that k is the fixed field of $\Gamma \overset{\text{def}}{=} \operatorname{Aut}(\Omega/k)$, and let Γ act on $\Omega \otimes_k V$ through its action on Ω,

$$\sigma(\sum c_i \otimes v_i) = \sum \sigma c_i \otimes v_i, \quad \sigma \in \Gamma, \quad c_i \in \Omega, \quad v_i \in V. \tag{2}$$

There is a unique action of Γ on $V(\Omega)$ fixing the elements of V and such that each $\sigma \in \Gamma$ acts σ-linearly,

$$\sigma(cv) = \sigma(c)\sigma(v) \text{ all } \sigma \in \Gamma, c \in \Omega, v \in V(\Omega). \tag{3}$$

LEMMA 3.3. *We have* $V = V(\Omega)^{\Gamma}$.

PROOF. Let $(e_i)_{i \in I}$ be a k-basis for V. Then $(1 \otimes e_i)_{i \in I}$ is an Ω-basis for $\Omega \otimes_k V$, and $\sigma \in \Gamma$ acts on $v = \sum c_i \otimes e_i$ according to the rule (2). Thus, v is fixed by Γ if and only if each c_i is fixed by Γ and so lies in k. □

LEMMA 3.4. *Let W be a Ω-subspace of $V(\Omega)$ stable under the action of Γ. If $W \neq 0$, then $W^{\Gamma} \neq 0$.*

PROOF. As $V(\Omega) = \Omega V$, every nonzero element w of W can be expressed in the form

$$w = c_1 v_1 + \cdots + c_n v_n, \quad c_i \in \Omega \smallsetminus \{0\}, \quad v_i \in V, \quad n \geq 1.$$

Let w be a nonzero element of W for which n takes its smallest value. After scaling, we may suppose that $c_1 = 1$. For $\sigma \in \Gamma$, the element

$$\sigma w - w = (\sigma c_2 - c_2)e_2 + \cdots + (\sigma c_n - c_n)e_n$$

lies in W and has at most $n-1$ nonzero coefficients, and so is zero. Thus, $w \in W^{\Gamma}$. □

PROPOSITION 3.5. *A subspace W of $V(\Omega)$ is of the form $W = \Omega W_0$ for some k-subspace W_0 of V if and only if it is stable under the action of Γ, in which case $W_0 = V \cap W = W^{\Gamma}$.*

PROOF. Certainly, if $W = \Omega W_0$, then it is stable under Γ (and $W = \Omega(W \cap V)$). Conversely, assume that W is stable under Γ, and let W' be a complement to $W \cap V$ in V, so that

$$V = (W \cap V) \oplus W'.$$

Then

$$(W \cap \Omega W')^{\Gamma} = W^{\Gamma} \cap \left(\Omega W'\right)^{\Gamma} = (W \cap V) \cap W' = 0,$$

and so, by Lemma 3.4,

$$W \cap \Omega W' = 0. \tag{4}$$

As $W \supset \Omega(W \cap V)$ and

$$V(\Omega) = \Omega(W \cap V) \oplus \Omega W',$$

this implies that $W = \Omega(W \cap V)$: write an element w of W as $w = w_1 + w_2$ with $w_1 \in \Omega(W \cap V)$ and $w_2 \in \Omega W'$; then $w_2 = w - w_1 \in W \cap \Omega W'$, and so it is 0. □

4 Descending subschemes of algebraic schemes

Let $\Omega \supset k$ be fields.

PROPOSITION 4.1. *Let V be an algebraic scheme over k, and let W be a closed subscheme of V_Ω. There exists at most one closed subscheme W_0 of V such that $W_{0\Omega} = W$ (as a subscheme of V_Ω).*

PROOF. If $V = \operatorname{Spec} A$ and I is an ideal in $\Omega \otimes_k A$, then there is at most one ideal I_0 in A such that $\Omega \otimes_k I_0$ maps isomorphically onto I under $c \otimes a \mapsto ca$. Moreover, the ideal I_0 exists if and only if A contains a set of generators for the ideal I, in which case $I_0 = I \cap A$ (see 3.1). To prove the general case, cover V with open affines. □

PROPOSITION 4.2. *Let V and W be algebraic schemes over k, and let $\varphi\colon V_\Omega \to W_\Omega$ be a morphism over Ω. There exists at most one morphism $\varphi_0\colon V \to W$ such that $\varphi_{0\Omega} = \varphi$.*

PROOF. As W is separated, the graph Γ_φ of φ is closed in $V \times W$, and so we can apply 4.1. □

Now assume that k is perfect and Ω is separably closed. Then k is the fixed field of $\Gamma = \operatorname{Aut}(\Omega/k)$.

For any algebraic variety V over Ω, $V(\Omega)$ is Zariski dense in V (see 0.2). It follows that two regular maps $V \rightrightarrows W$ of algebraic varieties coincide if they agree on $V(\Omega)$.

For any algebraic scheme V over k, Γ acts on $V(\Omega)$. For example, if V is embedded in $\mathbb{A}^n$ or $\mathbb{P}^n$ over k, then Γ acts on the coordinates of a point. If $V = \operatorname{Spec} A$, then

$$V(\Omega) = \operatorname{Hom}(A, \Omega) \qquad (k\text{-algebra homomorphisms}),$$

and Γ acts through its action on Ω.

PROPOSITION 4.3. *Let V be an algebraic scheme over k, and let W be a reduced closed subscheme of V_Ω. There exists a closed subscheme W_0 of V such that $W = W_{0\Omega}$ if and only if $W(\Omega)$ is stable under the action of Γ on $V(\Omega)$.*

PROOF. Certainly, the condition is necessary. For the converse, suppose first that V is affine, and let $I(W)$ be the ideal in $\Omega[V_\Omega]$ corresponding to W. Note that $\Omega[V_\Omega] = \Omega \otimes_k k[V]$. Because $W(\Omega)$ is stable under Γ, so also is $I(W)$, and Proposition 3.5 shows that $I(W)$ is spanned by $I_0 \overset{\text{def}}{=} I(W) \cap k[V]$. The closed subscheme W_0 of V corresponding to I_0 has the property that $W = W_{0\Omega}$.

To deduce the general case, cover V with open affines $V = \bigcup V_i$. Then $W_i \overset{\text{def}}{=} V_{i\Omega} \cap W$ is such that $W_i(\Gamma)$ is stable under Γ, and so it arises from a closed subscheme W_{i0} of V_i; a similar statement holds for $W_{ij} \overset{\text{def}}{=} W_i \cap W_j$. Define W_0 to be the scheme obtained by patching the W_{i0} along the open subschemes W_{ij0}. □

COROLLARY 4.4. *Let V and W be algebraic varieties over k, and let $f\colon V_\Omega \to W_\Omega$ be a regular map. If $f(\Omega)\colon V(\Omega) \to W(\Omega)$ commutes with the actions of Γ, then f arises from a (unique) regular map $V \to W$ over k.*

PROOF. Apply Proposition 4.3 to the graph of f, $\Gamma_f \subset (V \times W)_\Omega$. □

COROLLARY 4.5. *The functor*

$$V \rightsquigarrow (V_\Omega, \text{ action of } \Gamma \text{ on } V(\Omega)) \tag{5}$$

from algebraic varieties over k to algebraic varieties over Ω equipped with an action of Γ on their Ω-points is fully faithful.

PROOF. Restatement of 4.4. □

In particular, an algebraic variety V over k is uniquely determined up to a unique isomorphism by the algebraic variety V_Ω equipped with the action of Γ on $V(\Omega)$.

In Theorems 11.5 and 11.6 below, we obtain sufficient conditions for a pair to lie in the essential image of the functor (5).

5 Galois descent of vector spaces

Let Γ be a group acting on a field Ω, and let k be a subfield of Ω^Γ. By a ***semilinear action*** of Γ on an Ω-vector space V we mean a homomorphism $\rho\colon \Gamma \to \mathrm{Aut}_{k\text{-linear}}(V)$ such that, for all $\sigma \in \Gamma$, $\rho(\sigma)$ acts σ-linearly on V,

$$\rho(\sigma)(cv) = \sigma(c)v, \quad c \in \Omega, \quad v \in V.$$

For example, if V is a k-vector space, then $\sigma(c \otimes v) = \sigma c \otimes v$ is a semilinear action of Γ on $\Omega \otimes_k V$.

LEMMA 5.1. *Let S be the standard $M_n(k)$-module (i.e., $S = k^n$ with $M_n(k)$ acting by left multiplication). The functor $V \rightsquigarrow S \otimes_k V$ from k-vector spaces to left $M_n(k)$-modules is an equivalence of categories.*

PROOF. Let V and W be k-vector spaces. The choice of bases $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ for V and W identifies $\mathrm{Hom}_k(V,W)$ with the set of matrices $(a_{ji})_{(j,i)\in J\times I}$, $a_{ji} \in k$, such that, for a fixed i, all but finitely many a_{ji} are zero. Because S is a simple $M_n(k)$-module and $\mathrm{End}_{M_n(k)}(S) = k$, the set $\mathrm{Hom}_{M_n(k)}(S \otimes_k V, S \otimes_k W)$ has the same description, and so the functor $V \rightsquigarrow S \otimes_k V$ from k-modules to left $M_n(k)$-modules is fully faithful.

The functor $V \rightsquigarrow S \otimes_k V$ sends a vector space V with basis $(e_i)_{i\in I}$ to a direct sum of copies of S indexed by I. Therefore, to show that the functor is essentially surjective, it suffices to prove that every left $M_n(k)$-module is a direct sum of copies of S.

We first prove this for $M_n(k)$ regarded as a left $M_n(k)$-module. For $1 \le i \le n$, let $L(i)$ be the set of matrices in $M_n(k)$ whose entries are zero except for those in the ith column. Then $L(i)$ is a left ideal in $M_n(k)$, and $L(i)$ is isomorphic to S as an $M_n(k)$-module. Hence,

$$M_n(k) = \bigoplus_i L(i) \simeq S^n \qquad \text{(as a left } M_n(k)\text{-module).}$$

We now prove it for an arbitrary (nonzero) left $M_n(k)$-module M. The choice of a set of generators for M realizes it as a quotient of a sum of copies of $M_n(k)$, and so M is a sum of copies of S. It remains to show that the sum can be made direct. Let I be the set of submodules of M isomorphic to S, and let Ξ be the set of subsets J of I such that the sum $N(J) \stackrel{\text{def}}{=} \sum_{N\in J} N$ is direct, i.e., such that for any $N_0 \in J$ and finite subset J_0

of J not containing N_0, $N_0 \cap \sum_{N \in J_0} N = 0$. If $J_1 \subset J_2 \subset \ldots$ is a chain of sets in Ξ, then $\bigcup J_i \in \Xi$, and so Zorn's lemma implies that Ξ has maximal elements. For any maximal J, $M = N(J)$ because otherwise, there exists an element S' of I not contained in $N(J)$; because S' is simple, $S' \cap N(J) = 0$, and it follows that $J \cup \{S'\} \in \Xi$, contradicting the maximality of J. □

ASIDE 5.2. The above argument proves the following statement: let A be a ring (not necessarily commutative) and S a simple left A-module; if ${}_AA$ is a sum of submodules isomorphic to S, then every left A-module is a direct sum of submodules isomorphic to S.

ASIDE 5.3. Let A and B be rings (not necessarily commutative), and let S be A-B-bimodule (this means that A acts on S on the left, B acts on S on the right, and the actions commute). When the functor $M \rightsquigarrow S \otimes_B M : \mathsf{Mod}_B \to \mathsf{Mod}_A$ is an equivalence of categories, A and B are said to be ***Morita equivalent through*** S. In this terminology, the lemma says that $M_n(k)$ and k are Morita equivalent through S.

PROPOSITION 5.4. *Let Ω be a finite Galois extension of k with Galois group Γ. The functor $V \rightsquigarrow (\Omega \otimes_k V, *)$ from k-vector spaces to Ω-vector spaces endowed with a semilinear action of Γ is an equivalence of categories.*

PROOF. Let $\Omega[\Gamma]$ be the Ω-vector space with basis $\{\sigma \in \Gamma\}$, and make $\Omega[\Gamma]$ into a k-algebra by setting

$$\Big(\sum_{\sigma\in\Gamma} a_\sigma \sigma\Big)\Big(\sum_{\tau\in\Gamma} b_\tau \tau\Big) = \sum_{\sigma,\tau}(a_\sigma \cdot \sigma b_\tau)\sigma\tau.$$

Then $\Omega[\Gamma]$ acts k-linearly on Ω by the rule

$$\Big(\sum\nolimits_{\sigma\in\Gamma} a_\sigma\sigma\Big)c = \sum\nolimits_{\sigma\in\Gamma} a_\sigma(\sigma c),$$

and Dedekind's theorem on the independence of characters (Milne 2022, 5.14) implies that the homomorphism

$$\Omega[\Gamma] \to \operatorname{End}_k(\Omega)$$

defined by this action is injective. By counting dimensions over k, one sees that it is an isomorphism. Therefore, Lemma 5.1 shows that $\Omega[\Gamma]$ and k are Morita equivalent through Ω, i.e., the functor $V \mapsto \Omega \otimes_k V$ from k-vector spaces to left $\Omega[\Gamma]$-modules is an equivalence of categories. This is precisely the statement of the lemma. □

When Ω is an infinite Galois extension of k, we endow Γ with the Krull topology, and we say that a semilinear action of Γ on an Ω-vector space V is ***continuous*** if every element of V is fixed by an open subgroup of Γ, i.e., if

$$V = \bigcup\nolimits_\Delta V^\Delta \qquad \text{(union over the open subgroups } \Delta \text{ of } \Gamma).$$

For example, the action of Γ on Ω is continuous, and it follows that, for any k-vector space V, the action of Γ on $\Omega \otimes_k V$ is continuous.

PROPOSITION 5.5. *Let Ω be a Galois extension of k (possibly infinite) with Galois group Γ. For any Ω-vector space V equipped with a continuous semilinear action of Γ, the map*

$$\sum c_i \otimes v_i \mapsto \sum c_i v_i : \Omega \otimes_k V^\Gamma \to V$$

is an isomorphism.

PROOF. Suppose first that Γ is finite. According to Proposition 5.4, there is a subspace W of V such that $\Omega \otimes_k W \simeq V$. Moreover, $W = V^\Gamma$ by 3.3, and so $\Omega \otimes_k V^\Gamma \simeq V$.

When Γ is infinite, the finite case shows that $\Omega \otimes_k (V^\Delta)^{\Gamma/\Delta} \simeq V^\Delta$ for every open normal subgroup Δ of Γ. Now pass to the direct limit over Δ, recalling that tensor products commute with direct limits. □

PROPOSITION 5.6. *The functor*

$$W \rightsquigarrow (\Omega \otimes_k W, *)$$

from k-vector spaces to Ω-vector spaces equipped with a continuous semilinear action of Γ is an equivalence of categories, with quasi-inverse $(V, *) \rightsquigarrow V^\Gamma$.

PROOF. We have constructed natural isomorphisms $W \simeq (\Omega \otimes_k W)^\Gamma$ (see 3.3) and $\Omega \otimes_k V^\Gamma \simeq V$ (see 5.5). □

6 Descent data

Let $\Omega \supset k$ be fields, and let $\Gamma = \mathrm{Aut}(\Omega/k)$. An Ω/k-***descent system*** on an algebraic scheme V over Ω is a family $(\varphi_\sigma)_{\sigma\in\Gamma}$ of isomorphisms $\varphi_\sigma : \sigma V \to V$ satisfying the cocycle condition,

$$\varphi_\sigma \circ (\sigma\varphi_\tau) = \varphi_{\sigma\tau} \text{ for all } \sigma, \tau \in \Gamma,$$

$$\begin{tikzcd} \sigma\tau V \arrow[r, "\sigma\varphi_\tau"'] \arrow[rr, bend left, "\varphi_{\sigma\tau}"] & \sigma V \arrow[r, "\varphi_\sigma"'] & V. \end{tikzcd}$$

A model (V_0, φ) of V over a subfield K of Ω containing k ***splits*** the descent system if $\varphi_\sigma = \varphi^{-1} \circ \sigma\varphi$ for all σ fixing K,

$$\begin{tikzcd} \sigma V \arrow[r, "\sigma\varphi"'] \arrow[rr, bend left, "\varphi_\sigma"] & \sigma(V_{0\Omega}) = V_{0\Omega} & V \arrow[l, "\varphi"] \end{tikzcd}$$

A descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ is said to be ***continuous*** if it is split by some model over a subfield K of Ω that is *finitely generated* over k. A ***descent datum*** is a continuous descent system. A descent datum is ***effective*** if it is split by some model over k. In a given situation, we say that ***descent is effective*** if every descent datum is effective.

Let V_0 be an algebraic scheme over k, and let $V = V_{0\Omega}$. For $\sigma \in \Gamma$, let φ_σ denote the canonical isomorphism $\sigma V \to V$. Then $(\varphi_\sigma)_{\sigma\in\Gamma}$ is an Ω/k-descent datum, split by V_0.

Let $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$ be an Ω/k-descent system on an algebraic scheme V over Ω, and let Ω^{sep} be a separable closure of Ω. The restriction map $\mathrm{Aut}(\Omega^{\mathrm{sep}}/k) \to \mathrm{Aut}(\Omega/k)$ is surjective, and we can extend $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$ to an Ω^{sep}/k-descent system $(\varphi'_\sigma)_{\sigma\in\mathrm{Aut}(\Omega^{\mathrm{sep}}/k)}$ on $V_{\Omega^{\mathrm{sep}}}$ by setting $\varphi'_\sigma = (\varphi_{\sigma|\Omega})_{\Omega^{\mathrm{sep}}}$. A model of V over a subfield K of Ω splits $(\varphi_\sigma)_\sigma$ if and only if it splits $(\varphi'_\sigma)_\sigma$. This observation sometimes allows us to assume that Ω is separably closed.

PROPOSITION 6.1. *Assume that $k = \Omega^{\mathrm{Aut}(\Omega/k)}$, and let $(\varphi_\sigma)_{\sigma\in\Gamma}$ and $(\varphi'_\sigma)_{\sigma\in\Gamma}$ be Ω/k-descent data on algebraic varieties V and V' over Ω. If (V_0, φ) and (V'_0, φ') are models over*

k splitting $(\varphi_\sigma)_{\sigma\in\Gamma}$ and $(\varphi'_\sigma)_{\sigma\in\Gamma}$ respectively, then to give a regular map $\alpha_0\colon V_0 \to V_0$ is the same as giving a regular map $\alpha\colon V \to V'$ such that diagrams

$$\begin{tikzcd} \sigma V \arrow[r, "\varphi_\sigma"] \arrow[d, "\sigma\alpha"] & V \arrow[d, "\alpha"] \\ \sigma V' \arrow[r, "\varphi'_\sigma"] & V' \end{tikzcd} \tag{6}$$

commute for all $\sigma \in \Gamma$.

PROOF. Given α_0, define α to make the right hand square in

$$\begin{tikzcd} \sigma V \arrow[r, "\sigma\varphi"] \arrow[d, "\sigma\alpha"] & V_{0\Omega} \arrow[d, "\alpha_{0\Omega}"] & V \arrow[l, "\varphi"'] \arrow[d, "\alpha"] \\ \sigma V' \arrow[r, "\sigma\varphi'"] & V'_{0\Omega} & V' \arrow[l, "\varphi'"] \end{tikzcd}$$

commute. The left hand square is obtained from the right hand square by applying σ, and so it also commutes. The outer square is (6).

In proving the converse, we may assume that Ω is separably closed. Given α, use φ and φ' to transfer α to a regular map $\alpha'\colon V_{0\Omega} \to V'_{0\Omega}$. Then the hypothesis implies that α' commutes with the actions of Γ on $V_0(\Omega)$ and $V'_0(\Omega)$, and so is defined over k (4.4). □

COROLLARY 6.2. *Assume that $k = \Omega^{\mathrm{Aut}(\Omega/k)}$. Let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be a descent datum on a variety V over Ω, and let (V_0, φ) be a model over k splitting $(\varphi_\sigma)_{\sigma\in\Gamma}$. Let W be an algebraic scheme over k. To give a regular map $W \to V_0$ (resp. $V_0 \to W$) is the same as giving a regular map $\alpha\colon W_\Omega \to V$ (resp. $\alpha\colon V \to W_\Omega$) compatible with the descent datum, i.e., such that $\varphi_\sigma \circ \sigma\alpha = \alpha$ (resp. $\alpha \circ \varphi_\sigma = \sigma\alpha$).*

PROOF. This is the special case of the proposition in which W_Ω is endowed with its canonical descent datum. □

REMARK 6.3. Proposition 6.1 shows that the functor taking an algebraic variety V over k to V_Ω endowed with its canonical descent datum,

$$\{\text{varieties over } k\} \to \{\text{varieties over } \Omega + \Omega/k\text{-descent datum}\}$$

is fully faithful. We are interested in determining when it is essentially surjective.

Let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be an Ω/k-descent system on V. For a subscheme W of V, we set

$$^{\sigma}W = \varphi_\sigma(\sigma W).$$

Then the following diagram commutes,

$$\begin{tikzcd} \sigma V \arrow[r, "\varphi_\sigma", "\simeq"'] & V \\ \sigma W \arrow[u, hook] \arrow[r, "\varphi_\sigma|\sigma W", "\simeq"'] & {}^{\sigma}W. \arrow[u, hook] \end{tikzcd} \tag{7}$$

LEMMA 6.4. *The following hold.*

(a) *For all $\sigma, \tau \in \Gamma$ and $W \subset V$, ${}^{\sigma}({}^{\tau}W) = {}^{\sigma\tau}W$.*

(b) *Suppose that $(\varphi_\sigma)_{\sigma\in\Gamma}$ is split by a model (V_0, φ) of V over k_0, and let W be an algebraic subscheme of V. If $W = \varphi^{-1}(W_{0\Omega})$ for some algebraic subscheme W_0 of V_0, then ${}^{\sigma}W = W$ for all $\sigma \in \Gamma$; the converse is true if $\Omega^{\Gamma} = k$.*

PROOF. (a) By definition

$$^{\sigma}({}^{\tau}W) = \varphi_\sigma(\sigma(\varphi_\tau(\tau W)) = (\varphi_\sigma \circ \sigma\varphi_\tau)(\sigma\tau W) = \varphi_{\sigma\tau}(\sigma\tau W) = {}^{\sigma\tau}W.$$

In the second equality, we used that $(\sigma\varphi)(\sigma W) = \sigma(\varphi W)$.

(b) Let $W = \varphi^{-1}(W_{0\Omega})$. By hypothesis $\varphi_\sigma = \varphi^{-1}\circ\sigma\varphi$, and so

$$^{\sigma}W = (\varphi^{-1}\circ\sigma\varphi)(\sigma W) = \varphi^{-1}(\sigma(\varphi W)) = \varphi^{-1}(\sigma W_{0\Omega}) = \varphi^{-1}(W_{0\Omega}) = W.$$

Conversely, suppose ${}^{\sigma}W = W$ for all $\sigma \in \Gamma$. Then

$$\varphi(W) = \varphi({}^{\sigma}W) = (\sigma\varphi)(\sigma W) = \sigma(\varphi(W)).$$

Therefore, $\varphi(W)$ is stable under the action of Γ on $V_{0\Omega}$, and so is defined over k (see 4.3). □

For a descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on V and a regular function f on an open subset U of V, we define ${}^{\sigma}f$ to be the function $(\sigma f)\circ\varphi_\sigma^{-1}$ on ${}^{\sigma}U$, so that ${}^{\sigma}f({}^{\sigma}P) = \sigma(f(P))$ for all $P \in U$. Then ${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f$, and so this defines an action of Γ on the regular functions.

The ***Krull topology*** on Γ is that for which the subgroups of Γ fixing a subfield of Ω finitely generated over k form a basis of open neighbourhoods of 1 (see, for example, Milne 2022, Chapter 7). An action of Γ on an Ω-vector space V is ***continuous*** if

$$V = \bigcup_{\Delta} V^{\Delta} \qquad \text{(union over the open subgroups } \Delta \text{ of } \Gamma).$$

For a subfield L of Ω containing k, let $\Delta_L = \mathrm{Aut}(\Omega/L)$.

PROPOSITION 6.5. *Assume that Ω is separably closed. An Ω/k-descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on an affine algebraic scheme V is continuous if and only if the action of Γ on $\Omega[V]$ is continuous.*

PROOF. If $(\varphi_\sigma)_{\sigma\in\Gamma}$ is continuous, it is split by a model of V over some subfield K of Ω finitely generated over k. By definition, Δ_K is open, and $\Omega[V]^{\Delta_K}$ contains a set $\{f_1, \ldots, f_n\}$ of generators for $\Omega[V]$ as an Ω-algebra. Now $\Omega[V] = \bigcup L[f_1, \ldots, f_n]$, where L runs over the subfields of Ω containing K and finitely generated over k. As $L[f_1, \ldots, f_n] \subset \Omega[V]^{\Delta_L}$, this shows that $\Omega[V] = \bigcup \Omega[V]^{\Delta_L}$.

Conversely, if the action of Γ on $\Omega[V]$ is continuous, then for some subfield L of Ω finitely generated over k, $\Omega[V]^{\Delta_L}$ will contain a set of generators $f_1, \ldots, f_n$ for $\Omega[V]$ as an Ω-algebra. According to 2.3, Ω^{Δ_L} is a purely inseparable algebraic extension of L, and so, after possibly replacing L with a finite extension, we may suppose that the embedding $V \hookrightarrow \mathbb{A}^n$ defined by the f_i determine a model of V over L. This model splits $(\varphi_\sigma)_{\sigma\in\Gamma}$, which is therefore continuous. □

PROPOSITION 6.6. *A descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on an algebraic scheme V over Ω is continuous if there exists a finite set S of points in $V(\Omega)$ such that*

(a) *the only automorphism of V fixing all $P \in S$ is the identity, and*

(b) *there exists a subfield K of Ω finitely generated over k such that ${}^{\sigma}P = P$ for all $\sigma \in \Gamma$ fixing K.*

PROOF. Let (V_0, φ) be a model of V over a subfield K of Ω finitely generated over k. After possibly replacing K by a larger finitely generated field, we may suppose (i) that ${}^{\sigma}P = P$ for all $\sigma \in \Gamma$ fixing K and all $P \in S$ (because of (b)) and (ii) that $\varphi(P) \in V_0(K)$ for all $P \in S$ (because S is finite). Then, for $P \in S$ and every σ fixing K,

$$\varphi_\sigma(\sigma P) \overset{\text{def}}{=} {}^{\sigma}P \overset{\text{(i)}}{=} P$$

$$(\sigma\varphi)(\sigma P) = \sigma(\varphi P) \overset{\text{(ii)}}{=} \varphi P,$$

and so both φ_σ and $\varphi^{-1} \circ \sigma\varphi$ are isomorphisms $\sigma V \to V$ sending σP to P. Therefore, φ_σ and $\varphi^{-1} \circ \sigma\varphi$ differ by an automorphism of V fixing the $P \in S$, and so are equal. This says that (V_0, φ) splits $(\varphi_\sigma)_{\sigma \in \Gamma}$. □

PROPOSITION 6.7. *Let V be an algebraic scheme over Ω such that the only automorphism of V is the identity map. If V has a model over k, then every Ω/k-descent datum on V is effective. More precisely, every Ω/k-descent datum on V is split by the model.*

PROOF. If $(\varphi_\sigma)_\sigma$ is a descent datum on V and (V, φ) is a model of V over k, then φ_σ and $\varphi^{-1} \circ \sigma\varphi$ are both isomorphisms $\sigma V \to V$, hence differ by an automorphism of V. Thus $\varphi_\sigma = \varphi^{-1} \circ \sigma\varphi$. □

Of course, in Proposition 6.6, S does not have to be a finite set of points. The proposition will hold with S any additional structure on V that rigidifies V (i.e., such that $\mathrm{Aut}(V, S) = 1$) and is such that (V, S) has a model over a finitely generated extension of k.

7 Galois descent of algebraic schemes

In this section, Ω is a Galois extension of k with Galois group Γ.

THEOREM 7.1. *A descent datum $(\varphi_\sigma)_{\sigma \in \Gamma}$ on an algebraic scheme V is effective if V is covered by open affines U with the property that ${}^{\sigma}U = U$ for all $\sigma \in \Gamma$.*

PROOF. Assume first that V is affine, and let $A = k[V]$. A descent datum $(\varphi_\sigma)_{\sigma \in \Gamma}$ defines a continuous action of Γ on A (see 6.5). From 5.5, we know that the map

$$c \otimes a \mapsto ca : \Omega \otimes_k A^\Gamma \to A \tag{8}$$

is an isomorphism. Let $V_0 = \operatorname{Spec} A^\Gamma$, and let φ be the isomorphism $V \to V_{0\Omega}$ defined by (8). Then (V_0, φ) splits the descent datum.

Next note that if ${}^{\sigma}U = U$ for all $\sigma \in \Gamma$, then a descent datum on V restricts to a descent datum on U (see the diagram (7)).

In the general case, we can write V as a finite union of open affines U_i such that ${}^{\sigma}U_i = U_i$ for all $\sigma \in \Gamma$. Then V is the algebraic scheme over Ω obtained by patching the U_i by means of the maps

$$U_i \hookleftarrow U_i \cap U_j \hookrightarrow U_j. \tag{9}$$

Each intersection $U_i \cap U_j$ is again affine (0.1), and so the system (9) descends to k. The algebraic scheme over k obtained by patching the descended system is a model of V over k splitting the descent datum. □

COROLLARY 7.2. *Let Ω^{sep} be a separable closure of Ω. If every finite set of points of $V(\Omega^{\mathrm{sep}})$ is contained in an open affine algebraic subscheme of $V_{\Omega^{\mathrm{sep}}}$, then every descent datum on V is effective.*

PROOF. As we noted in the paragraph before 6.1, an Ω/k-descent datum for V extends in a natural way to an Ω^{sep}/k-descent datum for $V_{\Omega^{\mathrm{sep}}}$, and if a model (V_0, φ) over k splits the second descent datum, then it also splits the first. Thus, we may suppose that Ω is separably closed.

Let $(\varphi_\sigma)_{\sigma\in\Gamma}$ be a descent datum on V, and let U be an open subscheme of V. By definition, (φ_σ) is split by a model (V_1, φ) of V over some finite extension k_1 of k. After possibly replacing k_1 with a larger finite extension, which we may suppose to be Galois over k, we have that there exists an open subscheme U_1 of V_1 such that $\varphi(U) = U_{1\Omega}$. Now 6.4b shows that ${}^\sigma U$ depends only on the coset $\sigma\Delta$, where $\Delta = \mathrm{Gal}(\Omega/k_1)$. In particular, $\{{}^\sigma U \mid \sigma \in \Gamma\}$ is finite, and so the scheme $U' \overset{\mathrm{def}}{=} \bigcap_{\sigma\in\Gamma} {}^\sigma U$ is open in V. Note that (see 6.4a)

$$ {}^\tau U' = {}^\tau\Big(\bigcap_{\sigma\in\Gamma} {}^\sigma U\Big) = \Big(\bigcap_{\sigma\in\Gamma} {}^{\tau\sigma} U\Big) = U' $$

for all $\tau \in \Gamma$.

Let P be a closed point of V. Because $\{{}^\sigma P \mid \sigma \in \Gamma\}$ is finite, it is contained in an open affine U of V. Now $U' \overset{\mathrm{def}}{=} \bigcap_{\sigma\in\Gamma} {}^\sigma U$ is an open affine in V containing P and such that ${}^\sigma U' = U'$ for all $\sigma \in \Gamma$. It follows that the scheme V satisfies the hypothesis of Theorem 7.1. □

COROLLARY 7.3. *Descent is effective in each of the following two cases:*

(a) *V is quasi-projective, or*

(b) *an affine algebraic group G acts transitively on V.*

PROOF. (a) Apply 0.3.

(b) We may assume Ω to be separably closed. Let S be a finite set of points of $V(\Omega)$, and let U be an open affine in V. For each $P \in S$, there is a nonempty open algebraic subscheme G_P of G such that $G_P \cdot P \subset U$. Because Ω is separably closed, there exists a $g \in (\bigcap_{P\in S} G_P \cdot P)(\Omega)$ (by 0.2; the separable points are dense in an algebraic scheme). Now $g^{-1}U$ is an open affine containing S. □

8 Application: Weil restriction

Let K/k be a finite extension of fields, and let V be an algebraic scheme over K. A pair (V_*, φ) consisting of an algebraic scheme V_* over k and a regular map $\varphi\colon V_{*K} \to V$ is called the K/k-***Weil restriction*** of V if it has the following universal property: for any algebraic scheme T over k and regular map $\varphi'\colon T_K \to V$, there exists a unique regular map $\psi\colon T \to V$ (of k-scheme) such that $\varphi \circ \psi_K = \varphi'$, i.e.,

$$\text{given}\quad \begin{tikzcd} T_K \arrow[dr, "\varphi'"] & \\ V_{*K} \arrow[r, "\varphi"] & V \end{tikzcd} \quad\text{there exists a unique}\quad \begin{tikzcd} T \arrow[d, "\psi"] \\ V_* \end{tikzcd} \quad\text{such that}\quad \begin{tikzcd} T_K \arrow[d, "\psi_K"] \arrow[dr, "\varphi'"] & \\ V_{*K} \arrow[r, "\varphi"] & V \end{tikzcd} \quad\text{commutes.}$$

In other words, (V_*, φ) is the K/k-Weil restriction of V if φ defines an isomorphism

$$\psi \mapsto \varphi \circ \psi_K\colon \ \mathrm{Mor}_k(T, V_*) \to \mathrm{Mor}_K(T_K, V)$$

(natural in the k-algebraic scheme T); in particular,

$$V_*(A) \simeq V(K \otimes_k A)$$

(natural in the affine k-algebra A). If it exists, the K/k-Weil restriction of V is uniquely determined by its universal property (up to a unique isomorphism).

When (V_*, φ) is the K/k-Weil restriction of V, the algebraic scheme V_* is said to have been obtained from V ***by (Weil) restriction of scalars*** or ***by restriction of the base field***.

PROPOSITION 8.1. *If V is a quasi-projective variety and K/k is separable, then the K/k-Weil restriction of V exists.*

PROOF. Let Ω be a Galois extension of k large enough to contain all conjugates of K, i.e., such that $\Omega \otimes_k K \simeq \prod_{\tau\colon K \to \Omega} \tau K$. Let $V' = \prod \tau V$ — this is an algebraic scheme over Ω. For $\sigma \in \mathrm{Gal}(\Omega/k)$, define $\varphi_\sigma\colon \sigma V' \to V'$ to be the regular map that acts on the factor $\sigma(\tau V)$ as the canonical isomorphism $\sigma(\tau V) \simeq (\sigma\tau)V$. Then $(\varphi_\sigma)_{\sigma \in \mathrm{Gal}(\Omega/k)}$ is a descent datum, and so defines a model (V_*, φ_*) of V' over k.

Choose a $\tau_0\colon K \to \Omega$. The projection map $V' \to \tau_0 V$ is invariant under the action of $\mathrm{Gal}(\Omega/\tau_0 K)$, and so defines a regular map $(V_*)_{\tau_0 K} \to \tau_0 V$ (4.4), and hence a regular map $\varphi\colon V_{*K} \to V$. It is easy to check that this has the correct universal property. □

9 Specialization

Let U be an integral algebraic scheme over k, and let $\varphi\colon V \to U$ be a dominant map. The ***generic fibre*** of φ is a regular map $\varphi_K\colon V_K \to \operatorname{Spec} K$, where $K = k(U)$. For example, if V and U are affine, then φ is Spec of an injective homomorphism of rings $f\colon A \to B$, and φ_K is Spec of $K \simeq A \otimes_k K \to B \otimes_k K$, where K is the field of fractions of the integral domain A.

Let K be a field finitely generated over k, and let V_K be an algebraic scheme over K. For any integral algebraic scheme U over k with $k(U) = K$, there exists a dominant map $\varphi\colon V \to U$ with generic fibre $V_K \to \operatorname{Spec} K$. For example, if $U = \operatorname{Spec}(A)$, where A is a finitely generated k-subalgebra of K, we only have to invert the coefficients of some set of polynomials defining V_K. Let P be a closed point in the image of φ. Then the fibre of V over P is an algebraic scheme $V(P)$ over $k(P)$, called the ***specialization*** of V at P. If k is algebraically closed, then $k(P) = k$.

10 Rigid descent

PROPOSITION 10.1. *Let V and W be algebraic schemes over an algebraically closed field k. If V and W become isomorphic over some field containing k, then they are already isomorphic over k.*

PROOF. The hypothesis implies that, for some field K finitely generated over k, there exists an isomorphism $\varphi\colon V_K \to W_K$. Let U be an integral algebraic scheme over k such that $k(U) = K$. After possibly replacing U with an open subscheme, we may extend φ to an isomorphism $\varphi_U\colon U \times V \to U \times W$. The fibre of φ_U at any closed point of U is an isomorphism $V \to W$. □

EXAMPLE 10.2. Let $\Omega \supset k$ be algebraically closed fields, and let E be an elliptic curve over Ω. There exists a model of E over a subfield K of Ω if and only if $j(E) \in K$. Therefore, if there exist models of E over subfields K_1, K_2 of Ω such that $K_1 \cap K_2 = k$, then E has a model over k. We now prove a similar statement for an arbitrary algebraic scheme over Ω.

Let $\Omega \supset k$ be fields. Subfields K_1 and K_2 of Ω containing k are said to be ***linearly disjoint*** over k if the homomorphism

$$\sum a_i \otimes b_i \mapsto \sum a_i b_i\colon K_1 \otimes_k K_2 \to K_1 \cdot K_2 \subset \Omega$$

is injective.

PROPOSITION 10.3. *Let $\Omega \supset k$ be algebraically closed fields, and let V be an algebraic scheme over Ω. If there exist models of V over subfields K_1, K_2 of Ω finitely generated over k and linearly disjoint over k, then there exists a model of V over k.*

PROOF. The model of V over K_1 extends to a model over an integral affine algebraic scheme U_1 with $k(U_1) = K_1$, i.e., there exists a surjective map $V_1 \to U_1$ of k-schemes whose generic fibre is the model of V over K_1. A similar statement applies to the model over K_2. Because K_1 and K_2 are linearly disjoint, $K_1 \otimes_k K_2$ is an integral domain with field of fractions $k(U_1 \times U_2)$. From the map $V_1 \to U_1$, we get a map $V_1 \times U_2 \to U_1 \times U_2$, and similarly for V_2.

Assume initially that $V_1 \times U_2$ and $U_1 \times V_2$ are isomorphic over $U_1 \times U_2$, so that we have a commutative diagram,

$$\begin{tikzcd}
V_1 \arrow[d] & V_1 \times U_2 \arrow[l, "\mathrm{pr}_1"'] \arrow[rr, "\approx"] \arrow[dr] & & U_1 \times V_2 \arrow[r, "\mathrm{pr}_2"] \arrow[dl] & V_2 \arrow[d] \\
U_1 & & U_1 \times U_2 \arrow[ll] \arrow[rr] & & U_2.
\end{tikzcd}$$

Let P be a closed point of U_1. When we pull back the central triangle to the algebraic subscheme $P \times U_2$ of $U_1 \times U_2$, we get the diagram at left below. Note that $P \simeq \operatorname{Spec} k$ (because k is algebraically closed) and $P \times U_2 \simeq U_2$.

$$\begin{tikzcd}
V_1(P) \times U_2 \arrow[rr, "\approx"] \arrow[dr] & & P \times V_2 \arrow[dl] \\
& P \times U_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
V_1(P)_{K_2} \arrow[rr, "\approx"] \arrow[dr] & & V_{2K_2} \arrow[dl] \\
& \operatorname{Spec}(K_2). &
\end{tikzcd}$$

The generic fibre of this diagram is the diagram at right. Here $V_1(P)_{K_2}$ is the algebraic scheme over K_2 obtained from $V_1(P)$ by extension of scalars $k \to K_2$. As V_{2K_2} is a model V over K_2, it follows that $V_1(P)$ is a model of V over k.

We now prove the general case. The schemes $(V_1 \times U_2)_{k(U_1 \times U_2)}$ and $(U_1 \times V_2)_{k(U_1 \times U_2)}$ become isomorphic over some finite field extension L of $k(U_1 \times U_2)$. Let $\bar{U}$ be the normalization[2] of $U_1 \times U_2$ in L, and let U be a dense open subset of $\bar{U}$ such that some isomorphism of $(V_1 \times U_2)_L$ with $(U_1 \times V_2)_L$ extends to an isomorphism over U. Then 0.4 shows that $\bar{U} \to U_1 \times U_2$ is surjective, and so the image U' of U in $U_1 \times U_2$ contains a nonempty (hence dense) open subset of $U_1 \times U_2$ (see 0.5). In particular, U' contains a subset $P \times U_2'$ with U_2' a nonempty open subset of U_2. Now the previous argument gives us schemes $V_1(P)_{K_2}$ and V_{2K_2} over K_2 that become isomorphic over $k(U'')$, where U'' is the inverse image of $P \times U_2'$ in $\bar{U}$. As $k(U'')$ is a finite extension of K_2, this again shows that $V_1(P)$ is a model of V over k. □

PROPOSITION 10.4. *Let Ω be algebraically closed of infinite transcendence degree over k, and assume that k is algebraically closed in Ω. For any $K \subset \Omega$ finitely generated over k, there exists a $\sigma \in \mathrm{Aut}(\Omega/k)$ such that K and σK are linearly disjoint over k.*

PROOF. Let $a_1, \ldots, a_n$ be a transcendence basis for K/k, and extend it to a transcendence basis $a_1, \ldots, a_n, b_1, \ldots, b_n, \ldots$ of Ω/k. Let σ be any permutation of the transcendence basis such that $\sigma(a_i) = b_i$ for all i. Then σ defines a k-automorphism of $k(a_1, \ldots a_n, b_1, \ldots, b_n, \ldots)$, which we extend to an automorphism of Ω.

Let $K_1 = k(a_1, \ldots, a_n)$. Then $\sigma K_1 = k(b_1, \ldots, b_n)$, and certainly K_1 and σK_1 are linearly disjoint. Note that $K_1 \otimes_k \sigma K_1 \subset K \otimes_k \sigma K$ are integral domains (by 0.6) and that $K \otimes_k \sigma K$ is integral over $K_1 \otimes_k \sigma K_1$. The kernel of $K \otimes_k \sigma K \to K \cdot \sigma K$ is a prime ideal $\mathfrak{q}$ such that

$$\mathfrak{q} \cap (K_1 \otimes_k \sigma K_1) = 0 = \{0\} \cap (K_1 \otimes_k \sigma K_1),$$

and so $\mathfrak{q} = 0$ (by 0.4). □

PROPOSITION 10.5. *Let $\Omega \supset k$ be algebraically closed fields, and let V be an algebraic scheme over Ω. If V is isomorphic to σV for every $\sigma \in \mathrm{Aut}(\Omega/k)$, then V has a model over k.*

PROOF. After replacing Ω with a larger algebraically closed field, we may suppose that it has infinite transcendence degree over k. There exists a model (V_0, φ) of V over a subfield K of Ω finitely generated over k. According to Proposition 10.4, there exists a $\sigma \in \mathrm{Aut}(\Omega/k)$ such that K and σK are linearly disjoint. Now

$$(\sigma V_0, (\sigma V_0)_\Omega = \sigma(V_{0\Omega}) \xrightarrow{\sigma\varphi} \sigma V \approx V)$$

is a model of V over σK, and so we can apply Proposition 10.3. □

In the next two theorems, $\Omega \supset k$ is an algebraically closed field containing a perfect field (so $k = \Omega^\Gamma$, $\Gamma = \mathrm{Aut}(\Omega/k)$).

THEOREM 10.6. *Let V be a quasi-projective scheme over Ω, and let $(\varphi_\sigma)_{\sigma \in \Gamma}$ be an Ω/k-descent system for V. If the only automorphism of V is the identity map, then V has a model over k splitting $(\varphi_\sigma)_\sigma$.*

[2] If $U_1 \times U_2 = \mathrm{Spec}\, C$, then $\bar{U} = \mathrm{Spec}\, \bar{C}$, where $\bar{C}$ is the integral closure of C in L.

PROOF. According to Proposition 10.5, V has a model (V_0, φ) over the algebraic closure k^{al} of k in Ω, which (see 6.7) splits $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k^{\mathrm{al}})}$.

Now $\varphi'_\sigma \overset{\mathrm{def}}{=} \varphi^{-1}\circ\varphi_\sigma\circ\sigma\varphi$ is stable under $\mathrm{Aut}(\Omega/k^{\mathrm{al}})$, and hence is defined over k^{al} (4.4). Moreover, φ'_σ depends only on the restriction of σ to k^{al}, and $(\varphi'_\sigma)_{\sigma\in\mathrm{Gal}(k^{\mathrm{al}}/k)}$ is a descent system for V_0. It is continuous by Proposition 6.6, and so V_0 has a model (V_{00}, φ') over k splitting $(\varphi'_\sigma)_{\sigma\in\mathrm{Gal}(k^{\mathrm{al}}/k)}$. Now $(V_{00}, \varphi\circ\varphi'_\Omega)$ splits $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$. □

We now consider pairs (V, S), where V is an algebraic scheme over Ω and $S = (P_i)_{1\le i\le n}$ is a family of closed points on V. A morphism $(V, (P_i)_{1\le i\le n}) \to (W, (Q_i)_{1\le i\le n})$ is a regular map $\varphi\colon V \to W$ such that $\varphi(P_i) = Q_i$ for all i.

THEOREM 10.7. *Let V be a quasi-projective scheme over Ω, and let $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$ be a descent system for V. Let $S = (P_i)_{1\le i\le n}$ be a finite set of points of V such that*

(a) *the only automorphism of V fixing each P_i is the identity map, and*
(b) *there exists a subfield K of Ω finitely generated over k such that ${}^\sigma P = P$ for all $\sigma \in \Gamma$ fixing K.*

Then V has a model over k splitting (φ_σ).

PROOF. The preceding propositions hold with V replaced by (V, S) (with the same proofs), and so the proof of Theorem 10.6 applies. □

EXAMPLE 10.8. Theorem 10.7 sometimes allows us to construct objects over subfields of $\mathbb{C}$ by working entirely over $\mathbb{C}$. We illustrate this with the Jacobian variety of a complete smooth curve. For such a curve C over $\mathbb{C}$, the complex torus

$$J(C)(\mathbb{C}) = \Gamma(C, \Omega^1)^\vee / H_1(C, \mathbb{Z}).$$

has a unique structure of a projective algebraic variety (hence of an abelian variety). Let $P \in C(\mathbb{C})$. The Abel–Jacobi map $Q \mapsto [Q - P]\colon C(\mathbb{C}) \to J(C)(\mathbb{C})$ arises from a (unique) regular map $f^P\colon C \to J(C)$. This has the following universal property:[3] for any regular map $f\colon C \to A$ from C to an abelian variety sending P to 0, there is a unique homomorphism $\phi\colon J \to A$ such that $\phi\circ f^P = f$.

Now let C be a complete smooth curve over a subfield k of $\mathbb{C}$. From C, we get a curve $\bar{C}$ over $\mathbb{C}$ and a descent datum $(\varphi_\sigma)_{\sigma\in\mathrm{Aut}(\mathbb{C}/k)}$. Let $J(\bar{C})$ denote the Jacobian variety of $\bar{C}$, and let $f\colon \bar{C} \to J(\bar{C})$ be the Abel–Jacobi map defined by some point in $C(\mathbb{C})$. For each $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$, there is a unique isomorphism $\phi_\sigma\colon \sigma J \to J$ such that

$$\begin{tikzcd} \sigma\bar{C} \arrow[r, "\varphi_\sigma"] \arrow[d, "\sigma f"] & \bar{C} \arrow[d, "f"] \\ \sigma J \arrow[r, "\phi_\sigma"] & J \end{tikzcd}$$

commutes up to a translation (apply the universality of (J, f) to get ϕ_σ^{-1} and $\sigma\phi_\sigma$). The family $(\phi_\sigma)_\sigma$ is a descent system for J, and if we take S to be the set of points of order 3 on $J(C)$, then the conditions of the theorem are satisfied. For (a), this is proved, for example, in Milne 1986, 17.5. For (b), we may suppose (after possibly extending k) that

[3] It suffices to check this in the complex-analytic category.

$C(k)$ is nonempty, say, $P \in C(k)$. When we set $f = f^P$, the above diagram commutes exactly, and $f({}^\sigma Q) = {}^\sigma f(Q)$. According to the Jacobi inversion theorem, the map

$$\sum m_i Q_i \mapsto \sum m_i f(Q_i) \colon \operatorname{Div}^0(\bar{C}) \to J(\bar{C})(\mathbb{C})$$

is surjective. Now K can be taken to be any finitely generated field such that the subgroup of $J(\bar{C})(\mathbb{C})$ generated $\{f(Q) \mid Q \in C(K)\}$ contains all elements of order 3.

We can therefore define the Jacobian of C over k to be the model of $J(\bar{C})$ over k splitting $(\phi_\sigma)_\sigma$.

ASIDE 10.9. The Theorem 10.7 is Corollary 1.2 of Milne 1999, where it was used to show that the conjecture of Langlands on the conjugation of Shimura varieties (a statement about Shimura varieties over $\mathbb{C}$) implies the existence of canonical models (Shimura's conjecture). There it was deduced from Weil's theorems (see below). The present more elementary proof was suggested by Wolfart's elementary proof of the 'obvious' part of Belyi's theorem (Wolfart 1997; see also Derome 2003).

11 Restatement in terms of group actions

In this subsection, $\Omega \supset k$ are fields with k perfect and Ω algebraically closed (so $k = \Omega^\Gamma$, $\Gamma = \mathrm{Gal}(\Omega/k)$). Recall that for any algebraic variety V over k, there is a natural action of Γ on $V(\Omega)$. In this subsection, we describe the essential image of the functor

$$\{\text{quasi-projective varieties over } k\} \to \{\text{quasi-projective varieties over } \Omega + \text{action of } \Gamma\}.$$

In other words, we determine which pairs $(V, *)$, with V a quasi-projective variety over Ω and $*$ an action of Γ on $V(\Omega)$,

$$(\sigma, P) \mapsto \sigma * P \colon \Gamma \times V(\Omega) \to V(\Omega),$$

arise from an algebraic variety over k. There are two obvious necessary conditions for this.

REGULARITY CONDITION

Obviously, the action should recognize that $V(\Omega)$ is not just a set, but rather the set of points of an algebraic variety. For $\sigma \in \Gamma$, let σV be the algebraic variety obtained by applying σ to the coefficients of the equations defining V, and for $P \in V(\Omega)$ let σP be the point on σV obtained by applying σ to the coordinates of P.

DEFINITION 11.1. We say that the action $*$ is ***regular*** if the map

$$\sigma P \mapsto \sigma * P \colon (\sigma V)(\Omega) \to V(\Omega)$$

is a regular isomorphism for all σ.

A priori, this is only a map of sets. The condition requires that it be induced by a regular map $\varphi_\sigma \colon \sigma V \to V$. If $V = V_{0\Omega}$ for some algebraic variety V_0 defined over k, then $\sigma V = V$, and φ_σ is the identity map, and so the condition is clearly necessary.

When V is affine, $V = \operatorname{Spec} A$, then $*$ is regular if and only if it induces an action

$$(\sigma * f)(\sigma * P) = \sigma(f(P))$$

of Γ on A by semilinear automorphisms.

REMARK 11.2. The maps φ_σ satisfy the cocycle condition $\varphi_\sigma \circ \sigma\varphi_\tau = \varphi_{\sigma\tau}$. In particular, $\varphi_\sigma \circ \sigma\varphi_{\sigma^{-1}} = \mathrm{id}$, and so if $*$ is regular, then each φ_σ is an isomorphism, and the family $(\varphi_\sigma)_{\sigma\in\Gamma}$ is a descent system. Conversely, if $(\varphi_\sigma)_{\sigma\in\Gamma}$ is a descent system, then

$$\sigma * P = \varphi_\sigma(\sigma P)$$

defines a regular action of Γ on $V(\Omega)$. Note that if $* \leftrightarrow (\varphi_\sigma)$, then $\sigma * P = {}^\sigma P$.

CONTINUITY CONDITION

DEFINITION 11.3. We say that the action $*$ is ***continuous*** if there exists a subfield L of Ω finitely generated over k and a model V_0 of V over L such that the action of $\Gamma(\Omega/L)$ is that defined by V_0.

For an affine algebraic variety V, an action of Γ on V gives an action of Γ on $\Omega[V]$, and one action is continuous if and only if the other is.

Continuity is obviously necessary. It is easy to write down regular actions that fail it, and hence do not arise from varieties over k.

EXAMPLE 11.4. The following are examples of actions that fail the continuity condition (the second two are regular).

(a) Let $V = \mathbb{A}^1$ and let $*$ be the trivial action.

(b) Let $\Omega/k = \mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$, and let N be a normal subgroup of finite index in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ that is not open,[4] i.e., that fixes no extension of $\mathbb{Q}$ of finite degree. Let V be the zero-dimensional algebraic variety over $\mathbb{Q}^{\mathrm{al}}$ with $V(\mathbb{Q}^{\mathrm{al}}) = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})/N$ equipped with its natural action.

(c) Let k be a finite extension of $\mathbb{Q}_p$, and let $V = \mathbb{A}^1$. The homomorphism $k^\times \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ can be used to twist the natural action of Γ on $V(\Omega)$.

RESTATEMENT OF THE MAIN THEOREMS

Recall that $\Omega \supset k$ are fields with k perfect and Ω algebraically closed (so $k = \Omega^\Gamma$, $\Gamma = \mathrm{Gal}(\Omega/k)$).

THEOREM 11.5. *Let V be a quasi-projective algebraic variety over Ω, and let $*$ be a regular action of Γ on $V(\Omega)$. Let $S = (P_i)_{1\le i\le n}$ be a finite set of points of V such that*

(a) *the only automorphism of V fixing each P_i is the identity map, and*

(b) *there exists a subfield K of Ω finitely generated over k such that $\sigma * P = P$ for all $\sigma \in \Gamma$ fixing K.*

Then $*$ arises from a model of V over k.

PROOF. This a restatement of Theorem 10.7. □

THEOREM 11.6. *Let V be a quasi-projective algebraic variety over Ω with an action $*$ of Γ. If $*$ is regular and continuous, then $*$ arises from a model of V over k in each of the following cases:*

(a) *Ω is algebraic over k, or*

[4]For a proof that such subgroups exist, see, for example, Milne 2022, 7.29.

(b) *Ω is has infinite transcendence degree over k.*

PROOF. (a) Restatement of 7.1, 7.3

(b) Restatement of 13.3 below (which depends on Weil's theorem 13.2). □

The condition "quasi-projective" is necessary, because otherwise the action may not stabilize enough open affine subsets to cover V. In fact, an example shows that if V is not quasi-projective, then V_0 need not exist, unless it is allowed to be an algebraic space in the sense of Artin (see, for example, p. 131 of Dieudonné, J., Fondements de la Géométrie Algébrique Moderne, Presse de l'Université de Montréal, 1964).

12 Faithfully flat descent

Recall that a homomorphism $f: A \to B$ of rings is flat if the functor "extension of scalars" $M \rightsquigarrow B \otimes_A M$ is exact. It is ***faithfully flat*** if a sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-modules is exact if and only if

$$0 \to B \otimes_A M' \to B \otimes_A M \to B \otimes_A M'' \to 0$$

is exact. For a field k, a homomorphism $k \to A$ is always flat (because exact sequences of k-vector spaces are split-exact), and it is faithfully flat if $A \neq 0$.

The next theorem and its proof are quintessential Grothendieck.

THEOREM 12.1. *If $f: A \to B$ is faithfully flat, then the sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{d^0} B^{\otimes 2} \to \cdots \to B^{\otimes r} \xrightarrow{d^{r-1}} B^{\otimes r+1} \to \cdots$$

is exact, where

$$\begin{aligned} B^{\otimes r} &= B \otimes_A B \otimes_A \cdots \otimes_A B \qquad (r \text{ times}) \\ d^{r-1} &= \textstyle\sum (-1)^i e_i \\ e_i(b_0 \otimes \cdots \otimes b_{r-1}) &= b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}. \end{aligned}$$

PROOF. It is easily checked that $d^r \circ d^{r-1} = 0$. We assume first that f admits a section, i.e., that there is a homomorphism $g: B \to A$ such that $g \circ f = 1$, and we construct a contracting homotopy $k_r: B^{\otimes r+2} \to B^{\otimes r+1}$. Define

$$k_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0) b_1 \otimes \cdots \otimes b_{r+1}, \qquad r \geq -1.$$

It is easily checked that

$$k_{r+1} \circ d^{r+1} + d^r \circ k_r = 1, \quad r \geq -1,$$

and this shows that the sequence is exact.

Now let A' be an A-algebra. Let $B' = A' \otimes_A B$ and let $f' = 1 \otimes f: A' \to B'$. The sequence corresponding to f' is obtained from the sequence for f by tensoring with A' (because $B^{\otimes r} \otimes A' \cong B'^{\otimes r}$ etc.). Thus, if A' is a faithfully flat A-algebra, it suffices to prove the theorem for f'. Take $A' = B$, and then $b \overset{f}{\mapsto} b \otimes 1: B \to B \otimes_A B$ has a section, namely, $g(b \otimes b') = bb'$, and so the sequence is exact. □

THEOREM 12.2. *If $f\colon A \to B$ is faithfully flat and M is an A-module, then the sequence*

$$0 \to M \xrightarrow{1\otimes f} M\otimes_A B \xrightarrow{1\otimes d^0} M\otimes_A B^{\otimes 2} \to \cdots \to M\otimes_B B^{\otimes r} \xrightarrow{1\otimes d^{r-1}} B^{\otimes r+1} \to \cdots$$

is exact.

PROOF. As in the above proof, one may assume that f has a section, and use it to construct a contracting homotopy. □

REMARK 12.3. Let $f\colon A \to B$ be a faithfully flat homomorphism, and let M be an A-module. Write M' for the B-module $f_*M = B\otimes_A M$. The module $e_{0*}M' = (B\otimes_A B)\otimes_B M'$ may be identified with $B\otimes_A M'$, where $B\otimes_A B$ acts by $(b_1\otimes b_2)(b\otimes m) = b_1 b\otimes b_2 m$, and $e_{1*}M'$ may be identified with $M'\otimes_A B$, where $B\otimes_A B$ acts by $(b_1\otimes b_2)(m\otimes b) = b_1 m\otimes b_2 b$. There is a canonical isomorphism $\phi\colon e_{1*}M' \to e_{0*}M'$ arising from

$$e_{1*}M' = (e_1 f)_*M = (e_0 f)_*M = e_{0*}M';$$

explicitly, it is the map

$$(b\otimes m)\otimes b' \mapsto b\otimes(b'\otimes m)\colon M'\otimes_A B \to B\otimes_A M'.$$

Moreover, M can be recovered from the pair (M',ϕ) because

$$M = \{m\in M' \mid 1\otimes m = \phi(m\otimes 1)\}.$$

Conversely, every pair (M',ϕ) satisfying certain obvious conditions does arise in this way from an A-module. Given $\phi\colon M'\otimes_A B \to B\otimes_A M'$, define

$$\begin{aligned}
\phi_1 &\colon B\otimes_A M'\otimes_A B \to B\otimes_A B\otimes_A M'\\
\phi_2 &\colon M'\otimes_A B\otimes_A B \to B\otimes_A B\otimes_A M',\\
\phi_3 &\colon M'\otimes_A B\otimes_A B \to B\otimes_A M'\otimes_A B
\end{aligned}$$

by tensoring ϕ with id_B in the first, second, and third positions respectively. Then a pair (M',ϕ) arises from an A-module M as above if and only if $\phi_2 = \phi_1\circ\phi_3$. The necessity is easy to check. For the sufficiency, define

$$M = \{m\in M' \mid 1\otimes m = \phi(m\otimes 1)\}.$$

There is a canonical map $b\otimes m \mapsto bm\colon B\otimes_A M \to M'$, and it suffices to show that this is an isomorphism (and that the map arising from M is ϕ). Consider the diagram

$$\begin{tikzcd}
M'\otimes_A B \arrow[r, shift left, "\alpha\otimes\mathrm{id}_B"] \arrow[r, shift right, "\beta\otimes\mathrm{id}_B"'] \arrow[d, "\phi"] & B\otimes_A M'\otimes_A B \arrow[d, "\phi_1"] \\
B\otimes_A M' \arrow[r, shift left, "e_0\otimes\mathrm{id}_{M'}"] \arrow[r, shift right, "e_1\otimes\mathrm{id}_{M'}"'] & B\otimes_A B\otimes_A M'
\end{tikzcd}$$

in which $\alpha(m) = 1\otimes m$ and $\beta(m) = \phi(m)\otimes 1$. As the diagram commutes with either the upper of the lower horizontal maps (for the lower maps, this uses the relation $\phi_2 = \phi_1\circ\phi_3$), ϕ induces an isomorphism on the kernels. But, by definition of M, the kernel of the pair $(\alpha\otimes 1, \beta\otimes 1)$ is $M\otimes_A B$, and, according to (12.2), the kernel of the pair $(e_0\otimes 1, e_1\otimes 1)$ is M'. This completes the proof.

THEOREM 12.4. *Let $f\colon A \to B$ be a faithfully flat homomorphism. Let R be a B-algebra and $\phi\colon R \otimes_A B \to B \otimes_A R$ a homomorphism of B-algebras. There exists an A-algebra R_0 and an isomorphism $\varphi\colon B \otimes_A R_0 \to R$ such that $\phi = (\mathrm{id}_B \otimes \varphi)\circ(\varphi \otimes \mathrm{id}_B)^{-1}$ if and only if (with the above notation)*

$$\phi_2 = \phi_1 \circ \phi_3.$$

Moreover, when this is so, the pair (R, φ) is unique up to a unique isomorphism, and R_0 is finitely generated if R is finitely generated.

PROOF. When M is a B-module, we proved this in 12.3. The same argument applies to algebras. □

A morphism $p\colon W \to V$ of schemes is ***faithfully flat*** if it is surjective on the underlying sets and $\mathcal{O}_{\varphi(P)} \to \mathcal{O}_P$ is flat for all $P \in W$.

THEOREM 12.5. *Let $p\colon W \to V$ be a faithfully flat map of schemes. Let U be a scheme quasi-projective over W and $\phi\colon \mathrm{pr}_1^* U \to \mathrm{pr}_2^* U$ an isomorphism of $W \times_V W$-schemes. There exists a scheme U_0 over V and an isomorphism $\varphi_0\colon p^*U_0 \to U$ such that $\phi = \mathrm{pr}_2^*(\varphi_0)\circ \mathrm{pr}_1^*(\varphi_0)^{-1}$ if and only if*

$$\mathrm{pr}_{31}^*(\phi) = \mathrm{pr}_{32}^*(\phi)\circ \mathrm{pr}_{21}^*(\phi).$$

Moreover, when this is so, the pair (V, φ_0) is unique up to a unique isomorphism, and V_0 is quasi-projective.

Here pr_1 and pr_2 denote the projections $W \times_V W \to W$ and pr_{ji} denotes the projection $W \times_V W \times_V W \to W \times_V W$ such that $p_{ji}(w_1, w_2, w_3) = (w_j, w_i)$.

PROOF. When U, V, and W are affine, this becomes the statement 12.4. We omit the proof of the extension to the general case. □

EXAMPLE 12.6. Let Γ be a finite group, viewed as an algebraic group over k of dimension 0. Let V be an algebraic scheme over k. A scheme ***Galois over*** V ***with Galois group*** Γ is a finite morphism $W \to V$ of k-schemes together with a morphism $W \times \Gamma \to W$ such that

(a) for all k-algebras R, $W(R) \times \Gamma(R) \to W(R)$ is an action of the group $\Gamma(R)$ on the set $W(R)$ in the usual sense, and the map $W(R) \to V(R)$ is compatible with the action of $\Gamma(R)$ on $W(R)$ and its trivial action on $V(R)$, and

(b) the morphism $(w, \sigma) \mapsto (w, w\sigma)\colon W \times \Gamma \to W \times_V W$ is an isomorphism.

Then there is a commutative diagram

$$\begin{tikzcd}
V \arrow[d, equal] & W \arrow[l] \arrow[d, equal] & W \times \Gamma \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\simeq"] & W \times \Gamma \times \Gamma \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, "\simeq"] \\
V & W \arrow[l] & W \times_V W \arrow[l, shift left] \arrow[l, shift right] & W \times_V W \times_V W \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

in which the vertical isomorphisms are

$$(w, \sigma) \mapsto (w, w\sigma)$$
$$(w, \sigma_1, \sigma_2) \mapsto (w, w\sigma_1, w\sigma_1\sigma_2).$$

Therefore, in this case, Theorem 12.5 says that to give an algebraic scheme affine over V is the same as giving an algebraic scheme affine over W together with an action of Γ on it compatible with that on W. When we take W and V to be the spectra of fields, then this becomes the affine case of Theorem 7.1.

Noncommutative rings

DEFINITION 12.7. Let $f\colon A \to B$ be a homomorphism of rings, not necessarily commutative, such that B is a faithfully flat as a left A-module. A ***descent datum*** on a right B-module M is a homomorphism of right B-modules $\rho_M\colon M \to M \otimes_A B$ such that the two composed maps

$$M \xrightarrow{\rho_M} M \otimes_A B \underset{m\otimes b \mapsto m \otimes 1_B \otimes b}{\overset{\rho_M \otimes B}{\rightrightarrows}} M \otimes_A B \otimes_A B$$

are equal and the map

$$M \xrightarrow{\rho_Y} M \otimes_A B \xrightarrow{m\otimes b \mapsto mb} M$$

equals the identity map.

With the obvious notion of morphism, the pairs (M, ρ) consisting of a right B-module and a descent datum form a category $\mathsf{Desc}(B/A)$.

THEOREM 12.8 (FAITHFULLY FLAT DESCENT). *The functor*

$$\Phi\colon \mathsf{Mod}_A \to \mathsf{Desc}(B/A), \quad M \rightsquigarrow (M \otimes_A B, \rho_M), \quad \rho_M(m \otimes b) = m \otimes 1 \otimes b$$

is an equivalence of categories.

This follows from the next more precise statement.

LEMMA 12.9. *Let (N, ρ_N) be a right B-module equipped with a descent datum. Then*

$$N' \overset{\text{def}}{=} \{y \in N \mid \rho_N(n) = n \otimes 1\}$$

is an A-submodule of N such that

$$N' \otimes_A B \simeq N.$$

PROOF. This follows from the comonadicity theorem in category theory. See, for example, Deligne 1990, Proposition 4.4. □

When the rings are commutative, it is possible to show that descent data in the above sense correspond to descent data in the commutative sense. This gives a different approach to faithfully flat descent for commutative rings, which, however, is not simpler than the direct approach (12.3).

13 Weil's descent theorems

In this section, Ω is an algebraically closed field containing the field k. We let k^{sep} denote the separable closure of k in Ω. The next statement is essentially Theorem 3 of Weil 1956.

THEOREM 13.1. *Let K be a finite separable extension of k, and let I be the set of k-homomorphisms $K \to \Omega$. Let V be a quasi-projective algebraic scheme over K, and for each pair (σ, τ) of elements of I, let $\phi_{\tau,\sigma}$ be an isomorphism $\sigma V \to \tau V$ (of algebraic schemes over Ω). Then there exists an algebraic scheme V_0 over k and an isomorphism $\phi\colon V_{0K} \to V$ such that $\phi_{\tau,\sigma} = \tau\phi\circ(\sigma\phi)^{-1}$ for all $\sigma, \tau \in I$ if and only if the $\phi_{\tau,\sigma}$ are defined over k^{sep} and satisfy the following conditions,*

(a) *$\phi_{\tau,\rho} = \phi_{\tau,\sigma}\circ\phi_{\sigma,\rho}$ for all $\rho, \sigma, \tau \in I$;*

(b) *$\phi_{\tau\omega,\sigma\omega} = \omega\phi_{\tau,\sigma}$ for all $\sigma, \tau \in I$ and all k_0-automorphisms ω of k_0^{al} over k_0.*

Moreover, when this is so, the pair (V_0, ϕ) is unique up to isomorphism over k_0, and V_0 is quasi-projective or quasi-affine if V is.

PROOF. The conditions are obviously necessary. For the converse, fix an embedding $i\colon K \to k^{\mathrm{sep}}$. Then the isomorphisms $\phi_{\sigma,\tau}$ define a descent datum on iV, and Corollary 7.3 provides us with a pair (V_0, ϕ) satisfying the required conditions (and (V_0, ϕ) is unique up to a *unique* isomorphism over k_0). □

An extension K of a field k is said to be ***regular*** if it is finitely generated, admits a separating transcendence basis, and k is algebraically closed in K. These are precisely the fields that arise as the field of rational functions on a geometrically irreducible algebraic variety over k.

Let k be a field, and let $k(t)$, $t = (t_1, \ldots, t_n)$, be a regular extension of k (in Weil's terminology, t is a *generic point* of an algebraic variety over k). By $k(t')$ we shall mean a field isomorphic to $k(t)$ by $t \mapsto t'$, and we write $k(t, t')$ for the field of fractions of $k(t) \otimes_k k(t')$.[5] When V_t is an algebraic scheme over $k(t)$, we shall write $V_{t'}$ for the algebraic scheme over $k(t')$ obtained from V_t by base change with respect to $t \mapsto t'\colon k(t) \to k(t')$. Similarly, if f_t denotes a regular map of schemes over $k(t)$, then $f_{t'}$ denotes the regular map over $k(t')$ obtained by base change. Similarly, $k(t'')$ is a second field isomorphic to $k(t)$ by $t \mapsto t''$ and $k(t, t', t'')$ is the field of fractions of $k(t) \otimes_k k(t') \otimes_k k(t'')$.

The next statement is essentially Theorem 6 and Theorem 7 of Weil 1956.

THEOREM 13.2. *With the above notation, let V_t be a quasi-projective scheme over $k(t)$, and, for each pair (t, t'), let $\phi_{t',t}$ be an isomorphism $V_t \to V_{t'}$ defined over $k(t, t')$. Then there exists an algebraic scheme V defined over k and an isomorphism $\phi_t\colon V_{k(t)} \to V_t$ (of schemes over $k(t)$) such that $\phi_{t',t} = \phi_{t'}\circ\phi_t^{-1}$ if and only if $\phi_{t',t}$ satisfies the following condition,*

$$\phi_{t'',t} = \phi_{t'',t'}\circ\phi_{t',t} \quad \text{(isomorphism of schemes over } k(t, t', t'')\text{).}$$

Moreover, when this is so, the pair (V, ϕ_t) is unique up to an isomorphism over k, and V is quasi-projective or quasi-affine if V is.

[5] If $k(t)$ and $k(t')$ are linearly disjoint subfields of Ω, then $k(t, t')$ is the subfield of Ω generated over k by t and t'.

PROOF. The condition is obviously necessary. Assume initially that V_t is affine, and let $R_r = \mathcal{O}(V_t)$. From $\phi_{t,t'}$ we get a commutative diagram

$$\begin{array}{ccc} R_t \otimes_{k(t)} k(t,t') & \xleftarrow{\mathcal{O}(\phi_{t',t})} & R_{t'} \otimes_{k(t')} k(t,t') \\ \uparrow & & \uparrow \\ R_t \otimes_k k(t') & \xleftarrow{\mathcal{O}(\phi)} & R_{t'} \otimes_k k(t). \end{array}$$

On replacing t' with t, we get a homomorphism $\mathcal{O}(\phi)\colon R_t \otimes_k k(t) \to k(t) \otimes_k R_t$ satisfying the condition $\mathcal{O}(\phi)_2 = \mathcal{O}(\phi)_1 \circ \mathcal{O}(\phi)_3$ of Theorem 12.2. Thus, there exists a k-algebra R and an isomorphism $\mathcal{O}(\varphi)\colon k(t) \otimes_k R \to R_t$ such that

$$\mathcal{O}(\phi) = (\mathrm{id}_B \otimes \mathcal{O}(\varphi)) \circ (\mathcal{O}(\varphi) \otimes \mathrm{id}_B)^{-1}.$$

Now $(\mathrm{Spec}(R), \varphi)$ is the requred pair.

In the general case, there is a commutative diagram

$$\begin{array}{ccc} V_t \times_{\mathrm{Spec}\, k(t)} \mathrm{Spec}\, k(t,t') & \xrightarrow{\phi_{t',t}} & V_{t'} \times_{\mathrm{Spec}\, k(t')} \mathrm{Spec}\, k(t,t') \\ \downarrow & & \downarrow \\ V_t \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k(t') & \xrightarrow{\phi} & V_{t'} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k(t). \end{array}$$

This case follows from Theorem 12.5. □

THEOREM 13.3. *Let Ω be an algebraically closed field of infinite transcendence degree over a perfect field k. Then descent is effective for quasi-projective schemes over Ω.*

PROOF. Let $(\varphi_\sigma)_\sigma$ be a descent datum on an algebraic scheme V over Ω. Because $(\varphi_\sigma)_\sigma$ is continuous, it is split by a model of V over some subfield K of Ω finitely generated over k. Let k' be the algebraic closure of k in K; then k' is a finite extension of k and K is a regular extension of k. Write $K = k(t)$, and let (V_t, φ') be a model of V over $k(t)$ splitting (φ_σ). According to Lemma 10.4, there exists a $\sigma \in \mathrm{Aut}(\Omega/k)$ such that $k(t') \overset{\text{def}}{=} \sigma k(t)$ and $k(t)$ are linearly disjoint over k. The isomorphism

$$V_{t\Omega} \xrightarrow{\varphi'} V \xrightarrow{\varphi_\sigma^{-1}} \sigma V \xrightarrow{(\sigma\varphi')^{-1}} V_{t',\Omega}$$

is defined over $k(t,t')$ and satisfies the conditions of Theorem 13.2. Therefore, there exists a model (W, φ) of V over k' splitting $(\varphi_\sigma)_{\sigma \in \mathrm{Aut}(\Omega/k(t))}$.

For $\sigma, \tau \in \mathrm{Aut}(\Omega/k)$, let $\varphi_{\tau,\sigma}$ be the composite of the isomorphisms

$$\sigma W \xrightarrow{\sigma\varphi} \sigma V \xrightarrow{\varphi_\sigma} V \xrightarrow{\varphi_\tau^{-1}} \tau V \xrightarrow{\tau\varphi} \tau W.$$

Then $\varphi_{\tau,\sigma}$ is defined over the algebraic closure of k in Ω and satisfies the conditions of Theorem 13.1, which gives a model of W over k splitting $(\varphi_\sigma)_{\sigma \in \mathrm{Aut}(\Omega/k)}$. □

NOTES. Weil 1956 is the first important paper in descent theory. Its results were not superseded by the results of Grothendieck. As noted the statements of Theorems 13.1 and 13.2 are from Weil's paper. Their proofs are probably also Weil's. Theorem 13.3 is Theorem 1.1. of Milne 1999.

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