Hodge classes on abelian varieties

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Abstract

We prove, following Deligne and André, that the Hodge classes on abelian varieties of CM-type can be expressed in terms of divisor classes and split Weil classes, and we describe some consequences. In particular, we show that Grothendieck's standard conjecture of Lefschetz type implies the Hodge conjecture for abelian varieties (Abdulali, André, ...). No new results, but the proofs are shorter.

Contents

1 Review of abelian varieties of CM-type

1.1. A complex abelian variety is said to be of CM-type if $\mathrm{End}^0(A)$ contains a CM-algebra[1] E such that $H^1(A,\mathbb{Q})$ is free of rank 1 as an E-module. Let $S = \mathrm{Hom}(E,\mathbb{C})$, and let $H^1(A) = H^1(A,\mathbb{C})$. Then

$$H^1(A) \simeq H^1(A,\mathbb{Q}) \otimes \mathbb{C} = \bigoplus_{s\in S} H^1(A)_s, \quad H^1(A)_s \overset{\text{def}}{=} H^1(A) \otimes_{E,s} \mathbb{C}.$$

Here $H^1(A)_s$ is the (one-dimensional) subspace of $H^1(A)$ on which E acts through s. We have

$$H^{1,0}(A) = \bigoplus_{s\in\Phi} H^1(A)_s, \quad H^{0,1}(A) = \bigoplus_{s\in\bar{\Phi}} H^1(A)_s,$$

where Φ is a CM-type on E, i.e., a subset of S such that $S = \Phi \sqcup \bar{\Phi}$. Every pair (E,Φ) consisting of a CM-algebra E and a CM-type Φ on E arises in this way from an abelian variety. Sometimes we identify a CM-type with its characteristic function $\phi\colon S \to \{0,1\}$.

1.2. Let A be a complex abelian variety of CM-type, and let E be a CM-subalgebra of $\mathrm{End}^0(A)$ such that $H^1(A,\mathbb{Q})$ is a free E-module of rank 1. Let K be a Galois extension of

[1] That is, a product of CM-fields.

$\mathbb{Q}$ in $\mathbb{C}$ splitting[2] E, and let $S = \mathrm{Hom}(E, K)$. We regard the CM-type Φ of A as a subset of S. Let $H^r(A) = H^r(A, \mathbb{Q}) \otimes_{\mathbb{Q}} K$. Then

$$H^1(A) = \bigoplus\nolimits_{s \in S} H^1(A)_s,$$

where $H^1(A)_s \stackrel{\text{def}}{=} H^1(A, \mathbb{Q}) \otimes_{E,s} K$ is the (one-dimensional) K-subspace of $H^1(A)$ on which E acts through s.

(a) We have

$$H^r(A) \simeq \bigwedge\nolimits_K^r H^1(A) = \bigoplus\nolimits_{\Delta} H^r(A)_\Delta \qquad (K\text{-vector spaces}),$$

where Δ runs over the subsets of S of size $|\Delta| = r$ and $H^r(A)_\Delta \stackrel{\text{def}}{=} \bigotimes_{s \in \Delta} H^1(A)_s$ is the (one-dimensional) subspace on which $a \in E$ acts as $\prod_{s \in \Delta} s(a)$.

(b) Let $H^{1,0} = \bigoplus_{s \in \Phi} H^1(A)_s$ and $H^{0,1} = \bigoplus_{s \in \bar{\Phi}} H^1(A)_s$. Then

$$H^{p,q} \stackrel{\text{def}}{=} \bigwedge\nolimits^p H^{1,0} \otimes \bigwedge\nolimits^q H^{0,1} = \bigoplus\nolimits_\Delta H^{p+q}(A)_\Delta \qquad (K\text{-vector spaces}),$$

where Δ runs over the subsets of S with $|\Delta \cap \Phi| = p$ and $|\Delta \cap \bar{\Phi}| = q$.

(c) (Pohlmann 1968, Theorem 1.) Let $B^p = H^{2p}(A, \mathbb{Q}) \cap H^{p,p}$ ($\mathbb{Q}$-vector space of Hodge classes of degree p on A). Then

$$B^p \otimes K = \bigoplus\nolimits_\Delta H^{2p}(A)_\Delta,$$

where Δ runs over the subsets of S with

$$|(t \circ \Delta) \cap \Phi| = p = |(t \circ \Delta) \cap \bar{\Phi}| \text{ for all } t \in \mathrm{Gal}(K/\mathbb{Q}). \tag{1}$$

2 Review of Weil classes

2.1. Let A be a complex abelian variety and ν a homomorphism from a CM-field E into $\mathrm{End}^0(A)$. The pair (A, ν) is said to be of Weil type if $H^{1,0}(A)$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module. In this case, $d \stackrel{\text{def}}{=} \dim_E H^1(A, \mathbb{Q})$ is even and the subspace $W_E(A) \stackrel{\text{def}}{=} \bigwedge_E^d H^1(A, \mathbb{Q})$ of $H^d(A, \mathbb{Q})$ consists of Hodge classes (Deligne 1982, 4.4). When E has degree 2 over $\mathbb{Q}$, these Hodge classes were studied by Weil (1977), and for this reason are called Weil classes. A polarization of (A, ν) is a polarization λ of A whose Rosati involution stabilizes $\nu(E)$ and acts on it as complex conjugation. The Riemann form of such a polarization can be written

$$(x, y) \longmapsto \mathrm{Tr}_{E/\mathbb{Q}}(f\phi(x, y))$$

for some totally imaginary element f of E and E-hermitian form ϕ on $H_1(A, \mathbb{Q})$. If λ can be chosen so that ϕ is split (i.e., admits a totally isotropic subspace of dimension $d/2$), then (A, ν) is said to be of split Weil type. A pair (A, ν) of Weil type is split if and only if

$$\mathrm{disc}(\phi) \equiv (-1)^{\dim(A)/[E:\,\mathbb{Q}]} \quad \text{modulo } \mathrm{Nm}(E^\times).$$

2.2. (Deligne 1982, §5.) Let E be a CM-field, let $\phi_1, \ldots, \phi_{2p}$ be CM-types on E, and let $A = \prod_i A_i$, where A_i is an abelian variety of CM-type (E, ϕ_i). If $\sum_i \phi_i(s) = p$ for all

[2] That is, such that $E \otimes_{\mathbb{Q}} F$ is isomorphic to a product of copies of F.

$s \in T \stackrel{\text{def}}{=} \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})$, then A, equipped with the diagonal action of E, is of split Weil type. Let $I = \{1, \dots, 2p\}$ and $H^r(A) = H^r(A, \mathbb{Q}^{\mathrm{al}})$. In this case, there is a diagram

$$\begin{array}{ccccc} W_E(A) \otimes \mathbb{Q}^{\mathrm{al}} \stackrel{\text{def}}{=} \left(\bigwedge_E^{2p} H^1(A, \mathbb{Q})\right) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{al}} & \hookrightarrow & \left(\bigwedge_{\mathbb{Q}}^{2p} H^1(A, \mathbb{Q})\right) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{al}} = H^{2p}(A) \\ \Vert & & \Vert \\ \bigoplus_{t \in T} \left(\bigotimes_{i \in I} H^1(A_i)_t\right) & \longhookrightarrow & \bigoplus_{\substack{J \subset I \times T \\ |J| = 2p}} \left(\bigotimes_{(i,t) \in J} H^1(A_i)_t\right) \end{array}$$

3 Theorem 1.

THEOREM 1 (ANDRÉ 1992). *Let A be a complex abelian variety of CM-type. There exist abelian varieties A_Δ of split Weil type and homomorphisms $f_\Delta \colon A \to A_\Delta$ such that every Hodge class t on A can be written as a sum $t = \sum f_\Delta^*(t_\Delta)$ with t_Δ a Weil class on A_Δ.*

PROOF. Let A be of CM-type and let $p \in \mathbb{N}$. After replacing A with an isogenous variety, we may suppose that it is a product of simple abelian varieties A_i (not necessarily distinct). Let $E = \prod_i \mathrm{End}^0(A_i)$. Then E is a CM-algebra, and A is of CM-type (E, ϕ) for some CM-type ϕ on E. Let K be a CM subfield of $\mathbb{C}$, Galois over $\mathbb{Q}$, splitting the centre of $\mathrm{End}^0(A)$. Then K splits E, and we let $S = \mathrm{Hom}(E, K)$. We shall show that Theorem 1 holds with each A_Δ of split Weil type relative to K. Let $T = \mathrm{Hom}(K, \mathbb{Q}^{\mathrm{al}})$, where $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. As $K \subset \mathbb{Q}^{\mathrm{al}}$, we can identify T with $\mathrm{Gal}(K/\mathbb{Q})$.

Fix a subset Δ of S satisfying (1). For $s \in \Delta$, let[3] $A_s = A \otimes_{E,s} K$. Then A_s is an abelian variety of CM type (K, ϕ_s), where $\phi_s(t) = \phi(t \circ s)$ for $t \in T$. Because Δ satisfies (1),

$$\sum\nolimits_{s \in \Delta} \phi_s(t) \stackrel{\text{def}}{=} \sum\nolimits_{s \in \Delta} \phi(t \circ s) = p, \text{ all } t \in T,$$

and so we can apply 2.2: the abelian variety $A_\Delta \stackrel{\text{def}}{=} \prod_{s \in \Delta} A_s$ equipped with the diagonal action of K is of split Weil type. There is a homomorphism $f_\Delta \colon A \to A_\Delta$ such that

$$f_{\Delta *} \colon H_1(A, \mathbb{Q}) \to H_1(A_\Delta, \mathbb{Q}) \simeq H_1(A, \mathbb{Q}) \otimes_E K^\Delta$$

is $x \longmapsto x \otimes 1$. Here K^Δ is a product of copies of K indexed by Δ. The map $f_\Delta^* \colon H^1(A_\Delta, \mathbb{Q}) \to H^1(A, \mathbb{Q})$ is the E-linear dual of $f_{\Delta *}$.

Note that A_Δ has complex multiplication by K^Δ. According to 1.2(a),

$$H^{2p}(A_\Delta) \stackrel{\text{def}}{=} H^{2p}(A_\Delta, \mathbb{Q}^{\mathrm{al}}) = \bigoplus\nolimits_J H^{2p}(A_\Delta)_J, \quad H^{2p}(A_\Delta)_J \stackrel{\text{def}}{=} \bigotimes\nolimits_{(s,t) \in J} H^1(A_s)_t,$$

where J runs over the subsets of $\Delta \times T$ of size $2p$. Let $W_K(A_\Delta) \subset H^{2p}(A_\Delta, \mathbb{Q})$ be the space of Weil classes on A_Δ. Then $W_K(A_\Delta) \otimes \mathbb{Q}^{\mathrm{al}} = \bigoplus_{t \in T} H^{2p}(A_\Delta)_{\Delta \times \{t\}}$. Note that $a \in E$ acts on $H^{2p}(A_\Delta)_{\Delta \times \{t\}} \stackrel{\text{def}}{=} \bigotimes_{s \in \Delta} H^1(A_s)_t$ as multiplication by $\prod_{s \in \Delta} (t \circ s)(a)$. Therefore, $f_\Delta^* \otimes 1 \colon H^{2p}(A_\Delta) \to H^{2p}(A)$ maps $H^{2p}(A_\Delta)_{\Delta \times \{t\}}$ into $H^{2p}(A)_{t \circ \Delta} \subset B^p(A) \otimes \mathbb{Q}^{\mathrm{al}}$.

In summary: for every subset Δ of S satisfying (1), we have a homomorphism $f_\Delta \colon A \to A_\Delta$ from A into an abelian variety A_Δ of split Weil type relative to K; moreover, $f_\Delta^*(W_K(A_\Delta)) \otimes \mathbb{Q}^{\mathrm{al}}$ is contained in $B^p(A) \otimes \mathbb{Q}^{\mathrm{al}}$ and contains $H^{2p}(A)_\Delta$. As the subspaces

[3] Better, let $A_s = A \otimes_{\mathcal{O}_E, s} \mathcal{O}_K$.

$H^{2p}(A)_\Delta$ span $B^p \otimes \mathbb{Q}^{\mathrm{al}}$ (see 1.2(c)), this implies that the subspaces $f_\Delta^*(W_K(A_\Delta))$ span B^p.[4]

4 Deligne's original version of Theorem 1.

Let E be a CM-field Galois over $\mathbb{Q}$, and let $\mathcal{S}$ be the set of CM-types on E. For each $\Phi \in \mathcal{S}$ choose an abelian variety A_Φ of CM-type (E, Φ), and let $A_{\mathcal{S}} = \prod_{\Phi \in \mathcal{S}} A_\Phi$. Define G^H (resp. G^W) to be the algebraic subgroup of $\mathrm{GL}_{H^1(A_{\mathcal{S}}, \mathbb{Q})}$ fixing all Hodge classes (resp. divisor classes and split Weil classes) on all products of powers of the A_Φ, $\Phi \in \mathcal{S}$.

THEOREM 2. *The algebraic groups G^H and G^W are equal.*

PROOF. As divisor classes and Weil classes are Hodge classes, certainly $G^H \subset G^W$. On the other hand, the graphs of homomorphisms of abelian varieties are in the $\mathbb{Q}$-algebra generated by divisor classes (Milne 1999, 5.6), and so, with the notation of Theorem 1, G^W fixes the elements of $f_\Delta^*(W_F(A_\Delta))$. As these span the Hodge classes, we deduce that $G^W \subset G^H$.

REMARK 1. Theorem 2 is Deligne's original theorem (1982, §5) except that, instead of requiring G^W to fix all divisor classes, he requires it to fix certain specific homomorphisms.

5 Applications

5.1. Call a rational cohomology class c on a smooth projective complex variety X *accessible* if it belongs to the smallest family of rational cohomology classes such that,

(a) the cohomology class of every algebraic cycle is accessible;
(b) the pull-back by a map of varieties of an accessible class is accessible;
(c) if $(X_s)_{s\in S}$ is an algebraic family of smooth projective varieties with S connected and smooth and $(t_s)_{s\in S}$ is a family of rational classes (i.e., a global section of $R^r f_* \mathbb{Q}$ ...) such that t_s is accessible for one s, then t_s is accessible for all s.

Accessible classes are automatically Hodge, even absolutely Hodge (Deligne 1982, §§2,3).

THEOREM 3. *For abelian varieties, every Hodge class is accessible.*

PROOF. This is proved in Deligne 1982 (see its Introduction) except that the statement there includes an extra "tannakian" condition on the accessible classes (ibid., p. 10, (c)). However, this condition is used only in the proof that Hodge classes on CM abelian varieties are accessible. Conditions (a) and (c) of 5.1 imply that split Weil classes are accessible (ibid., 4.8), and so this follows from Theorem 1 (using 5.1(b)).

REMARK 2. In particular, we see that the Hodge conjecture holds for abelian varieties if algebraic classes satisfy the variational Hodge conjecture (i.e., condition 5.1(c)).

[4] Let W and W' be subspaces of a k-vector space V, and let K be a field containing k. If $W \otimes_k K \subset W' \otimes_k K$, then $W \subset W'$. Indeed,

$$W \not\subset W' \iff \frac{W + W'}{W'} \neq 0 \iff \frac{W \otimes K + W' \otimes K}{W \otimes K} \neq 0 \iff W \otimes K \not\subset W' \otimes K.$$

REMARK 3. For Theorem 3, it suffices to assume that 5.1(c) holds for families of abelian varieties over a *complete smooth curve* S. Indeed, (c) is used in the proof of the theorem only for families of abelian varieties $(A_s)_{s\in\mathbb{S}}$ with additional structure over a locally symmetric variety S. More precisely, there is a semisimple algebraic group G over $\mathbb{Q}$, a bounded symmetric domain X on which $G(\mathbb{R})$ acts transitively with finite kernel, and a congruence subgroup $\Gamma \subset G(\mathbb{Q})$ such that $S(\mathbb{C}) = \Gamma\backslash X$ (Deligne 1982, proofs of 4.8, 6.1). For $s \in S(\mathbb{C})$, the points s' of the orbit $G(\mathbb{Q})\cdot s$ are dense in S and each abelian variety $A_{s'}$ is isogenous to A_s. The boundary of S in its minimal (Baily-Borel) compactification has codimension ≥ 2. After Bertini, for any pair of points $s_1, s_2 \in S(\mathbb{C})$, we can find a smooth linear section of S meeting both orbits $G(\mathbb{Q})\cdot s_1$ and $G(\mathbb{Q})\cdot s_2$ but not meeting the boundary. This proves what we want. Cf. André 1996, p. 32.

Let X be an algebraic variety of dimension d, and let $L\colon H^*(X,\mathbb{Q}) \to H^{*+2}(X,\mathbb{Q})$ be the Lefschetz operator defined by a hyperplane section of X. The strong Lefschetz theorem says that $L^{d-i}\colon H^i(X,\mathbb{Q}) \to H^{2d-i}(X,\mathbb{Q})$ is an isomorphism for all $i \leq d$. Let $\mathit{a}H^{2i}(X,\mathbb{Q})$ denote the $\mathbb{Q}$-subspace of $H^{2i}(X,\mathbb{Q})$ spanned by the algebraic classes. Then L^{d-2i} induces an injective map $L^{d-2i}\colon \mathit{a}H^{2i}(X,\mathbb{Q}) \to \mathit{a}H^{2d-2i}(X,\mathbb{Q})$. The standard conjecture of Lefschetz type asserts that this map is surjective if $2i \leq d$. It is known to be true for abelian varieties.

PROPOSITION 1 (ABDULALI 1994, p. 1122). *Let $f\colon A \to S$ be an abelian scheme over a smooth complete complex variety S. Assume that the Lefschetz standard conjecture holds for A. Let t be a global section of the sheaf $R^{2r}f_*\mathbb{Q}(r)$; if $t_s \in H^{2r}(A_s,\mathbb{Q}(r))$ is algebraic for one $s \in S(\mathbb{C})$, then it is algebraic for all s.*

PROOF. For $n \in \mathbb{N}$, let θ_n denote the endomorphism of A/S acting as multiplication by n on the fibres. By a standard argument (Kleiman 1968, p. 374), θ_n^* acts as n^j on $R^jf_*\mathbb{Q}$. As θ_n^* commutes with the differentials d_2 of the Leray spectral sequence $H^i(S,R^jf_*\mathbb{Q}) \Rightarrow H^{i+j}(A,\mathbb{Q})$, we see that the spectral sequence degenerates at the E_2-term and

$$H^r(A,\mathbb{Q}) \simeq \bigoplus\nolimits_{i+j=r} H^i(S,R^jf_*\mathbb{Q})$$

with $H^i(S,R^jf_*\mathbb{Q})$ the subspace of $H^{i+j}(A,\mathbb{Q})$ on which θ_n acts as n^j. Let $s \in S(\mathbb{C})$ and $\pi = \pi_1(S,s)$. The inclusion $j_s\colon A_s \hookrightarrow A$ induces an isomorphism $j_s^*\colon H^0(S,R^{2r}f_*\mathbb{Q}) \hookrightarrow H^{2r}(A_s,\mathbb{Q})^\pi$ preserving algebraic classes, and so

$$\dim \mathit{a}H^0(S,R^{2r}f_*\mathbb{Q}) \leq \dim \mathit{a}H^{2r}(A_s,\mathbb{Q})^\pi. \tag{2}$$

Similarly, the Gysin map $j_{s*}\colon H^{2d-2r}(A_s,\mathbb{Q}) \to H^{2d-2r+2m}(A,\mathbb{Q})$, where $m = \dim(S)$ and $d = \dim(A/S)$, induces a map $H^{2d-2r}(A_s,\mathbb{Q})^\pi \to H^{2m}(S,R^{2d-2r}f_*\mathbb{Q})$ preserving algebraic classes, and so

$$\dim \mathit{a}H^{2d-2r}(A_s,\mathbb{Q})^\pi \leq \dim \mathit{a}H^{2m}(S,R^{2d-2r}f_*\mathbb{Q}). \tag{3}$$

Because the Lefschetz standard conjecture holds for A_s,

$$\dim \mathit{a}H^{2r}(A_s,\mathbb{Q})^\pi = \dim \mathit{a}H^{2d-2r}(A_s,\mathbb{Q})^\pi. \tag{4}$$

Hence,

$$\begin{aligned}\dim \mathit{a}H^0(S,R^{2r}f_*\mathbb{Q}) &\overset{(2)}{\leq} \dim \mathit{a}H^{2r}(A_s,\mathbb{Q})^\pi \overset{(4)}{=} \dim \mathit{a}H^{2d-2r}(A_s,\mathbb{Q})^\pi\\ &\overset{(3)}{\leq} \dim \mathit{a}H^{2m}(S,R^{2d-2r}f_*\mathbb{Q}).\end{aligned}$$

The Lefschetz standard conjecture for A implies that

$$\dim \mathcal{A}H^0(S, R^{2r} f_*\mathbb{Q}) = \dim \mathcal{A}H^{2m}(S, R^{2d-2r} f_*\mathbb{Q}),$$

and so the inequalities are equalities. Thus

$$\mathcal{A}H^{2r}(A_s, \mathbb{Q})^{\pi} = \mathcal{A}H^0(S, R^{2r} f_*\mathbb{Q}),$$

which is independent of s.

THEOREM 4 (ABDULALI, ANDRÉ). *The Lefschetz standard conjecture for algebraic varieties over $\mathbb{C}$ implies the Hodge conjecture for abelian varieties.*

PROOF. By Theorem 3, it suffices to show that algebraic classes are accessible. They obviously satisfy conditions (a) and (b) of 5.1, and it suffices to check (c) with S a complete smooth curve (Remark 3). This Proposition 1 does.

REMARK 4. Proposition 1 applies also to absolute Hodge classes and motivated classes. As these satisfy the Lefschetz standard conjecture, we deduce that Hodge classes on abelian varieties are absolutely Hodge and motivated.

REMARK 5. Let H_B denote the Betti cohomology theory and $\mathsf{Mot}_H(\mathbb{C})$ the category of motives over $\mathbb{C}$ for homological equivalence generated by the algebraic varieties for which the Künneth projectors are algebraic. If the Betti fibre functor ω_B on $\mathsf{Mot}_H(\mathbb{C})$ is conservative, then the Lefschetz standard conjecture holds for the varieties in question.

Indeed, as $L^{d-2i}\colon H_B^{2i}(X)(i) \to H_B^{2d-2i}(X,\mathbb{Q})(d-i)$ is an isomorphism (strong Lefschetz theorem), so also is $l^{d-2i}\colon h^{2i}(X)(i) \to h^{2d-2i}(X)(d-i)$ by our assumption on ω_B. When we apply the functor $\mathrm{Hom}(\mathbb{1}, -)$ to this last isomorphism, it becomes $L^{d-2i}\colon \mathcal{A}H_B^{2i}(X)(i) \to \mathcal{A}H_B^{2d-2i}(X)(d-i)$, which is therefore an isomorphism. Thus the standard conjecture $A(X,L)$ is true, and $A(X \times X, L \otimes 1 + 1 \otimes L)$ implies $B(X)$.

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Notes

For the convenience of the reader, we include proofs of some of the statements used in the proof of the theorems. Subsection 5.1 is from Pohlmann 1968; 5.3 and 5.4 are extracted from Deligne 1982.

5.1. *Proof of 1.2(c).*

Let $G = \mathrm{Gal}(K/\mathbb{Q})$. For each $s \in S$, let e_s be a nonzero element of $H^1(A)_s$; then $(e_s)_{s\in S}$ is a basis for $H^1(A)$. The e_s can be chosen so that $te_s = e_{t\circ s}$ for $t \in \mathrm{Gal}(K/\mathbb{Q})$.[5] Once we fix an ordering of S, we can set $e_\Delta = \bigwedge_{s\in\Delta} e_s$ for any subset Δ of S.

Let Δ be a subset of S satisfying (1); we shall show that $e_\Delta \in B^p \otimes K$. Let $\{u_1, \dots, u_n\}$ be a basis for K over $\mathbb{Q}$, and let $f_i = \sum_{t\in G} tu_i \cdot te_\Delta$ for $i = 1, \dots, n$. As $te_\Delta = \pm e_{t\Delta}$ and Δ satisfies (1), $f_i \in H^{p,p}$. As it is fixed by all $\sigma \in G$, we see that $f_i \in B^p$. Now $\det(t(u_i))_{t,i} \neq 0$, and so we can solve the linear equations $f_i = \sum_{t\in G} tu_i \cdot te_\Delta$ to find that $te_\Delta \in B^p \otimes K$. In particular, $e_\Delta \in B^p \otimes K$.

Let $f \in B^p$, and write $f = \sum_{|\Delta|=2p} c_\Delta \cdot e_\Delta$ with $c_\Delta \in K$. As $tf = f \in H^{p,p}$, we have $|t\Delta \cap \Phi| = p$ for all $t \in G$ and all Δ with $c_\Delta \neq 0$. Therefore Δ satisfies (1), and so f is an K-linear combination of e_Δ with Δ satisfying (1).

5.2. *Some linear algebra; the diagram in 2.2.*

Let Q be a field, E an étale Q-algebra, and V a free E-module of finite rank. Let $T = \mathrm{Hom}_Q(E, Q^{\mathrm{al}})$ and set $V \otimes_Q Q^{\mathrm{al}} = \bigoplus_{t\in T} V_t$ with $V_t = V \otimes_{E,t} Q^{\mathrm{al}}$. Let $n \in \mathbb{N}$. There is a unique inclusion $\bigwedge_E^n V \hookrightarrow \bigwedge_Q^n V$ that, when tensored with Q^{al}, becomes

$$\begin{tikzcd}
\left(\bigwedge_E^n V\right) \otimes_Q Q^{\mathrm{al}} \arrow[r, hook] \arrow[d, equal] & \left(\bigwedge_Q^n V\right) \otimes_Q Q^{\mathrm{al}} \arrow[d, equal] \\
\bigoplus_{t\in T} \left(\bigwedge_{Q^{\mathrm{al}}}^n V_t\right) \arrow[r, hook, "\text{obvious}", "\text{inclusion}"'] & \bigoplus_{\sum n_t = n} \left(\bigotimes_{t\in T} \bigwedge_{Q^{\mathrm{al}}}^{n_t} V_t\right).
\end{tikzcd}$$

Now suppose that $V = \bigoplus_{i\in I} V_i$ with V_i free of rank 1 as an E-module, and let $n = |I|$. Then $V_i \otimes_Q Q^{\mathrm{al}} = \bigoplus_{t\in T} V_{i,t}$ with $V_{i,t}$ a Q^{al}-vector space of dimension 1, and the diagram becomes

$$\begin{tikzcd}
\left(\bigwedge_E^n V\right) \otimes_Q Q^{\mathrm{al}} \arrow[r, hook] \arrow[d, equal] & \left(\bigwedge_Q^n V\right) \otimes_Q Q^{\mathrm{al}} \arrow[d, equal] \\
\bigoplus_{t\in T} \left(\bigotimes_{i\in I} V_{i,t}\right) \arrow[r, hook, "\text{obvious}", "\text{inclusion}"'] & \bigoplus_{\substack{J\subset I\times T \\ |J|=n}} \left(\bigotimes_{(i,t)\in J} V_{i,t}\right).
\end{tikzcd}$$

The equality at right follows directly from the decomposition $V \otimes_Q Q^{\mathrm{al}} = \bigoplus_{(i,t)\in I\times T} V_{i,t}$, $\dim V_{i,t} = 1$.

[5] The choice of a nonzero element of $H_1(A, \mathbb{Q})$ allows us to identify $H_1(A, \mathbb{Q})$ with E, $H^1(A, \mathbb{Q})$ with $\mathrm{Hom}_{\mathbb{Q}\text{-linear}}(E, \mathbb{Q})$, and $H^1(A)$ with $\mathrm{Hom}_{\mathbb{Q}\text{-linear}}(E, F)$. According to Dedekind's theorem on the independence of characters, the set S is an F-basis for this last space.

Let E be a CM-field, let $\phi_1, \ldots, \phi_{2p}$ be CM-types on E, and let $A = \prod_i A_i$, where A_i is an abelian variety of CM-type (E, ϕ_i). Let $I = \{1, \ldots, 2p\}$ and $T = \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})$. Assume that $\sum_{i \in I} \varphi_i = p$. Set $H^r(A) = H^r(A, \mathbb{Q}^{\mathrm{al}})$. The last diagram becomes the diagram in 2.2.

5.3. *Review of hermitian forms*

Recall that a CM-field is a number field E such that, for every embedding $E \hookrightarrow \mathbb{C}$, complex conjugation induces a nontrivial automorphism $e \longmapsto \bar{e}$ on E that is independent of the embedding. The fixed field of the automorphism is then a totally real field F over which E has degree two.

A bi-additive form

$$\phi \colon V \times V \to E$$

on a vector space V over a CM-field E is Hermitian if

$$\phi(ev, w) = e\phi(v, w), \quad \phi(v, w) = \overline{\phi(w, v)}, \quad \text{all } e \in E, v, w \in V.$$

For an embedding $\tau \colon F \hookrightarrow \mathbb{R}$ we obtain a Hermitian form ϕ_τ in the usual sense on the vector space $V_\tau = V \otimes_{F,\tau} \mathbb{R}$, and we let a_τ and b_τ denote the dimensions of the maximal subspaces of V_τ on which ϕ_τ is positive definite and negative definite respectively. If $d = \dim V$, then ϕ defines a Hermitian form on $\bigwedge^d V$ that, relative to some basis vector, is of the form $(x, y) \longmapsto fx\bar{y}$. The element f is in F, and is independent of the choice of the basis vector up to multiplication by an element of $\mathrm{Nm}_{E/F} E^\times$. It is called the discriminant of ϕ. Let $(v_1, \ldots, v_d)$ be an orthogonal basis for ϕ, and let $\phi(v_i, v_i) = c_i$; then a_τ is the number of i for which $\tau c_i > 0$, b_τ the number of i for which $\tau c_i < 0$, and $f = \prod c_i \pmod{\mathrm{Nm}_{E/F} E^\times}$. If ϕ is nondegenerate, then $f \in F^\times / \mathrm{Nm}\, E^\times$, and

$$a_\tau + b_\tau = d, \quad \mathrm{sign}(\tau f) = (-1)^{b_\tau}, \text{all } \tau. \tag{5}$$

PROPOSITION 2. *Suppose given nonnegative integers $(a_\tau, b_\tau)_{\tau \colon F \hookrightarrow \mathbb{C}}$ and an element $f \in F^\times / \mathrm{Nm}\, E^\times$ satisfying (5). Then there exists a non-degenerate Hermitian form ϕ on an E-vector space V of dimension d with invariants (a_τ, b_τ) and f; moreover, (V, ϕ) is unique up to isomorphism.*

PROOF. The result is due to Landherr (1936).[6]

COROLLARY 1. *Let (V, ϕ) be a non-degenerate Hermitian space, and let $d = \dim V$. The following conditions are equivalent:*

(a) *$a_\tau = b_\tau$ for all τ and* $\mathrm{disc}(f) = (-1)^{d/2}$;

(b) *there exists a totally isotropic subspace of V of dimension $d/2$ (i.e., ϕ is split).*

PROOF. Let W be a totally isotropic subspace of V of dimension $d/2$. The map $v \longmapsto \phi(-, v) \colon V \to W^\vee$ induces an antilinear isomorphism $V/W \to W^\vee$. Thus, a basis $e_1, \ldots, e_{d/2}$ of W can be extended to a basis $\{e_i\}$ of V such that

$$\phi(e_i, e_{\frac{d}{2}+i}) = 1, \quad 1 \le i \le d/2,$$
$$\phi(e_i, e_j) = 0, \quad j \neq i \pm d/2.$$

[6] Landherr, Walther, Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper. Abh. Math. Semin. Hamb. Univ. 11, 245–248 (1936).

It is now easy to check that (V,ϕ) satisfies (a). Conversely, (E^d,ϕ), where

$$\phi((a_i),(b_i)) = \sum_{1\le i\le d/2} a_i\bar{b}_{\frac{d}{2}+i} + a_{\frac{d}{2}+i}\bar{b}_i,$$

is, up to isomorphism, the only Hermitian space satisfying (a), and it also satisfies (b).

5.4. *Proof of 2.2.*

Let E be a CM field and $\phi_1,\dots,\phi_d$ CM-types on E such that $\sum_i \phi_i$ is the constant function. Let $A = \bigoplus_{i=1}^d A_i$, where A_i is of CM-type (E,ϕ_i). Then E acts diagonally on A and $H^1(A,\mathbb{Q}) = \bigoplus_{i=1}^d H^1(A_i,\mathbb{Q})$ has dimension d over E. For each i, there is an E-linear isomorphism

$$H^1(A_i,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \to \bigoplus\nolimits_{s\in S} H^1(A_i)_s,$$

where $H^1(A_i)_s$ is the (one-dimensional) $\mathbb{C}$-vector space on which E acts through s. To say that A_i is of CM-type ϕ_i means that

$$H^{1,0}(A_i) = \bigoplus\nolimits_{\phi_i(s)=1} H^1(A_i)_s.$$

Now $H^{1,0}(A)_s$ has dimension $\sum_i \phi_i(s)$, which, being constant, implies that $H^{1,0}$ is a free $E\otimes\mathbb{C}$-module. Therefore $W_E(A) \stackrel{\text{def}}{=} \bigwedge_E^d H^1(A,\mathbb{Q}) \subset H^d(A,\mathbb{Q})$ consists of Hodge classes.

For each i, choose a polarization θ_i for A_i whose Rosati involution stabilizes E, and let ψ_i be the corresponding Riemann form. For any totally positive elements f_i in F (the maximal totally real subfield of E) $\theta = \bigoplus_i f_i\theta_i$ is a polarization for A. Choose $v_i \neq 0$, $v_i \in H_1(A_i,\mathbb{Q})$; then $\{v_i\}$ is a basis for $H_1(A_i,\mathbb{Q})$ over E. There exist $\zeta_i \in E^\times$ such that $\bar{\zeta}_i = -\zeta_i$ and $\psi_i(xv_i,yv_i) = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta_i x\bar{y})$ for all $x,y\in E$. Thus ϕ_i, where $\phi_i(xv_i,yv_i) = \frac{\zeta_i}{\zeta_1}x\bar{y}$, is an E-Hermitian form on $H_1(A_i,\mathbb{Q})$ such that $\psi_i(v,w) = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta_1\phi_i(v,w))$. The E-Hermitian form on $H_1(A,\mathbb{Q})$

$$\phi(\sum x_iv_i,\sum y_iv_i) = \sum_i f_i\phi_i(x_iv_i,y_iv_i)$$

has the property that $\psi(v,w) \stackrel{\text{def}}{=} \mathrm{Tr}_{E/\mathbb{Q}}(\zeta_1\phi(v,w))$ and is the Riemann form of θ. The discriminant of ϕ is $\prod_i f_i(\frac{\zeta_i}{\zeta_1})$. On the other hand, if $s\in S$ restricts to τ on F, then

$$\mathrm{sign}(\tau\mathrm{disc}(\phi)) = (-1)^{b_s} = (-1)^{d/2}.$$

Thus,

$$\mathrm{disc}(\phi) = (-1)^{d/2} f$$

for some totally positive element f of F. After replacing one f_i with f_i/f, we have that $\mathrm{disc}(\phi) = (-1)^{d/2}$, and therefore ϕ is split.

5.5. *The degeneration of the Leray spectral sequence*

THEOREM 5. *(Blanchard, Deligne) If $f: X \to S$ is smooth projective, then the Leray spectral sequence,*

$$H^p(S,R^qf_*\mathbb{Q}) \Rightarrow H^{p+q}(X,\mathbb{Q}),$$

degenerates at E_2.

PROOF. The relative Lefschetz operator $L = c_1(\mathcal{L}) \cup \cdot$ acts on the whole spectral sequence, and induces a Lefschetz decomposition

$$R^q f_* \mathbb{Q} = \bigoplus_r L^r (R^{q-2r} f_* \mathbb{Q})_{\mathrm{prim}}.$$

It suffices to prove that $d_2 \alpha = 0$ for $\alpha \in H^p(S, (R^q f_* \mathbb{Q})_{\mathrm{prim}})$. In the diagram,

$$\begin{array}{ccc}
H^p(S, (R^q f_* \mathbb{Q})_{\mathrm{prim}}) & \xrightarrow{d_2} & H^{p+2}(S, R^{q-1} f_* \mathbb{Q}) \\
{\scriptstyle 0}\big\downarrow{\scriptstyle L^{n-q+1}} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle L^{n-q+1}} \\
H^p(S, R^{2n-q+2}\mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(S, R^{2n-q+1}\mathbb{Q}),
\end{array}$$

the map at left is zero because L^{n-q+1} is zero on $(R^q f_* \mathbb{Q})_{\mathrm{prim}}$ and the map at right is an isomorphism because $L^{n-q+1} : R^{q-1} f_* \mathbb{Q} \to R^{2n-q+1} f_* \mathbb{Q}$ is an isomorphism. Hence $d_2 \alpha = 0$.

Grothendieck conjectured the degeneration of the Leray spectral sequence by consideration of weights. Blanchard proved the result (1956) when the base is simply connected, and Deligne (1968)[7] proved it in general. See also Griffiths and Harris, 1978, p. 466.

Now consider an abelian scheme $f : A \to S$. In this case, we get a diagram

$$\begin{array}{ccc}
H^p(S, R^q f_* \mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(S, R^{q-1} f_* \mathbb{Q}) \\
{\scriptstyle n^q}\big\downarrow{\scriptstyle \theta_n} & & {\scriptstyle n^{q-1}}\big\downarrow{\scriptstyle \theta_n} \\
H^p(S, R^q f_* \mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(S, R^{q-1} f_* \mathbb{Q}),
\end{array}$$

which proves our claims in the proof of Proposition 1.

5.6. *Proof of Proposition 1*

Abdulali (1994) states his results for Kuga fibre varieties, but notes that they hold more generally. In his original manuscript, he refers to Kuga 1964[8], Theorem II-3-12, p.~94, for the decomposition of the cohomology of A, and to Kuga 1966[9], 1.3.5, p. 17, for a proof that j_s^* induces an isomorphism of $H^0(S, R^{2r} f_* \mathbb{Q})$ with $H^{2r}(A_s, \mathbb{Q})^{\pi}$.

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[7]Deligne, P. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. Inst. Hautes Études Sci. Publ. Math. No. 35 1968 259–278.

[8]Kuga, Michio, Fiber varieties over a symmetric space whose fibers are abelian varieties, Vol. I, Lecture Notes, Univ. Chicago, 1964 (reprinted by Higher Education Press, 2018).

[9]Kuga, Michio. Fibred variety over symmetric space whose fibres are abelian varieties. 1966 Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965) pp. 72–81 Nippon Hyoronsha, Tokyo