

Abelian motives and Shimura varieties in nonzero characteristic

J.S. Milne

November 2, 2025, v2.2

For me, [the Hodge conjecture] is part of the story of motives, and it is not crucial whether it is true or false. If it is true, that's very good, and it solves a large part of the problem of constructing motives in a reasonable way. If one can find another purely algebraic notion of cycles for which the analogue of the Hodge conjecture holds, this will serve the same purpose, and I would be as happy as if the Hodge conjecture were proved. For me it is motives, not Hodge, that is crucial.

Deligne, May 2013.¹

Abstract

This article represents my attempt to construct a theory of Shimura varieties as simple and elegant as that in Grothendieck's "paradis motiviques", but without assuming the Hodge, Tate, or standard conjectures.

Deligne's theorem (1982), that Hodge classes on abelian varieties are absolutely Hodge, allows us to construct a category of abelian motives $\text{Mot}(k)$ over any field k of characteristic zero. This is a tannakian category over \mathbb{Q} with most of the properties anticipated for Grothendieck's category of abelian motives, and it equals Grothendieck's category if the Hodge conjecture is true for abelian varieties. Deligne's theorem makes it possible to realize Shimura varieties of abelian type with rational weight as moduli varieties (Milne 1994b), which greatly simplifies the theory of Shimura varieties in characteristic zero.

The goal of this article is to extend the theory to characteristic p .

We study elliptic modular curves by realizing them as moduli curves for elliptic curves. This works, not only in characteristic zero, but also in mixed characteristic and characteristic p . Some Shimura curves cannot be realized as moduli curves, but a trick of Shimura allows us to deduce their properties from those that can.

In this article, we suggest an approach that makes the theory of Shimura varieties of abelian type as simple, at least conceptually, as that of Shimura curves.

Much of the work on Shimura varieties over the last thirty years has been devoted to constructing the theory that would follow from a good notion of motives, one incorporating the Hodge, Tate, and standard conjectures. These conjectures are

¹Interview on the award of the Abel prize, Eur. Math. Soc. Newsl. No. 89 (2013), 15–23.

believed to be beyond reach, and may not even be correct as stated.² I argue in this article that there exists a theory of motives, accessible to proof, weaker than Grothendieck's, but with many of the same consequences.

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Introduction

To know a scheme is to know its functor of points, and many properties of a scheme can be seen in its functor of points.

Raynaud 2014, describing Grothendieck's view.

In his Corvallis article (1979), Deligne introduced the notion of a connected Shimura variety and its canonical model, and showed that a Shimura variety has a canonical model if and only if its associated connected Shimura variety has a canonical model. In particular, if two Shimura varieties have isomorphic associated connected Shimura varieties, and one has a canonical model, then both do. Starting from the Shimura varieties that are moduli varieties for abelian varieties, he was able to prove in this way the existence of canonical models for a large class of Shimura varieties, now said to be of abelian type. His proof is a tour de force. It does not give a description of the canonical model but only a characterization of it in terms of reciprocity laws at the special points.

Later it was realized (Milne 1994b) that the Shimura varieties of abelian type with rational weight are exactly the moduli varieties of abelian motives with additional structure. This allows us to prove the existence of canonical models for these Shimura varieties by a simple descent argument and it describes the canonical model as a moduli variety. The theory can be extended to varieties with nonrational weight by applying a “trick” of Shimura.

The approach in the last paragraph applies only to Shimura varieties in characteristic zero because the abelian motives are defined using Deligne's theory of absolute Hodge classes, which works only in characteristic zero.

²While we all hope that Grothendieck's motivic paradise exists, it may be too optimistic to expect that it can be constructed using actual algebraic classes.

In this article, I outline a program to extend Deligne’s theory of absolute Hodge classes to characteristic p , thereby obtaining a good theory of abelian motives in mixed characteristic. Once completed, this will make possible similar simplifications in the theory of Shimura varieties in mixed characteristic.

Throughout, \mathbb{Q}^{al} is an algebraic closure of \mathbb{Q} and w a prime of \mathbb{Q}^{al} lying over p . The residue field at w is an algebraic closure \mathbb{F} of \mathbb{F}_p . For a variety X , we write $H_{\mathbb{A}}^r(X)$ for the restricted product of the étale cohomology groups $H_{\text{ét}}^r(X, \mathbb{Q}_\ell)$ with the de Rham cohomology group (characteristic 0) or crystalline cohomology group (characteristic p).

Statement of the conjectures

In this subsection, we fix a collection \mathfrak{s} of abelian varieties over \mathbb{Q}^{al} with good reduction at w , including the CM abelian varieties. Every A in \mathfrak{s} specializes to an abelian variety A_0 over \mathbb{F} .

CONJECTURE (A). *Let $A \in \mathfrak{s}$ and let γ be an absolute Hodge class on A . For all Lefschetz classes δ on A_0 of complementary codimension, $\langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q}$.*

In more detail, an absolute Hodge class on A is an element γ of $H_{\mathbb{A}}^{2*}(A)(*)$, and its specialization γ_0 is an element of $H_{\mathbb{A}}^{2*}(A_0)(*)$. If $\delta_1, \dots, \delta_r$, where $r = \dim(\gamma)$, are divisor classes on A_0 , then

$$\langle \gamma_0 \cdot \delta_1 \cdots \delta_r \rangle \in H_{\mathbb{A}}^{2d}(A_0)(d) \simeq \mathbb{A}_f^p \times \mathbb{Q}_w^{\text{al}}, \quad d = \dim A.$$

The conjecture says that it lies in $\mathbb{Q} \subset \mathbb{A}_f^p \times \mathbb{Q}_w^{\text{al}}$.

CONJECTURE (C). *There exists a unique family of a graded \mathbb{Q} -subalgebras $\mathcal{R}^*(A)$ of $H_{\mathbb{A}}^{2*}(A)(*)$, indexed by the abelian varieties over \mathbb{F} , whose elements we call rational Tate classes, satisfying the following conditions:*

- (R1) *for any morphism f of abelian varieties over \mathbb{F} , f^* and f_* map rational Tate classes to rational Tate classes;*
- (R2) *divisor classes are rational Tate classes;*
- (R3) *absolute Hodge classes on abelian varieties in \mathfrak{s} specialize to rational Tate classes;*
- (R4) *the inclusion $\mathcal{R}^*(A) \hookrightarrow H_{\mathbb{A}}^{2*}(A)(*)$ induces an injection*

$$\mathcal{R}^*(A) \otimes \mathbb{A}_f \rightarrow H_{\mathbb{A}}^{2*}(A)(*).$$

CONJECTURE (D). *There exists a tannakian category $\text{Mot}(\mathbb{F})$ over \mathbb{Q} and exact tensor functors $\text{LMot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}; \mathbb{A}_f)$ and $\text{Mot}^{\mathfrak{s}}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ such that*

$$\begin{array}{ccccc} \text{LMot}^{\mathfrak{s}}(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}^{\mathfrak{s}}(\mathbb{Q}^{\text{al}}) & & \\ \downarrow R & & \downarrow R & \searrow \xi_f & \\ \text{LMot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}; \mathbb{A}_f) \\ & & \searrow \xi_f & & \end{array}$$

commutes and

$$\text{Mot}(\mathbb{F})_{(\mathbb{A}_f)} \longrightarrow \text{Mot}(\mathbb{F}; \mathbb{A}_f)$$

is faithful.

Here $\text{LMot}^\delta(\mathbb{Q}^{\text{al}})$ and $\text{LMot}(\mathbb{F})$ are categories of abelian motives defined using Lefschetz classes, $\text{Mot}^\delta(\mathbb{Q}^{\text{al}})$ is the category of abelian motives defined using absolute Hodge classes, and $\text{Mot}(\mathbb{F}; \mathbb{A}_f)$ is the category of abelian motives defined using Tate classes. See later for precise definitions.

Equivalence of the conjectures

Conjecture C obviously implies Conjecture A because γ_0 and δ are both rational Tate classes, and in §4 we prove a converse statement (4.7):

Assume that Conjecture A holds for CM abelian varieties over \mathbb{Q}^{al} ; let $\mathcal{T}^*(A)$ denote the \mathbb{A}_f -algebra of Tate classes on an abelian variety A over \mathbb{F} ; then there exists a unique family $\mathcal{R}^*(A)$ of \mathbb{Q} -structures on the $\mathcal{T}^*(A)$ satisfying (R1) and (R2) and such that absolute Hodge classes on CM abelian varieties over \mathbb{Q}^{al} specialize to elements of \mathcal{R}^* .

Given Conjecture C, we define $\text{Mot}(\mathbb{F})$ to be the category of motives based on the abelian varieties over \mathbb{F} using the rational Tate classes as correspondences. Conversely, we can recover the family $(\mathcal{R}^*(A))$ from $\text{Mot}(\mathbb{F})$ by setting

$$\mathcal{R}^r(A) = \text{Hom}_{\text{Mot}(\mathbb{F})}(\mathbf{1}, h^{2r}(A)(r)).$$

Summary

When we assume that Conjecture A holds for CM abelian varieties over \mathbb{Q}^{al} , we get a well-defined tannakian category $\text{Mot}(\mathbb{F})$ of (abelian) motives over \mathbb{Q} extending Deligne's category of abelian motives to characteristic p ; moreover, for all prime numbers l , the functor

$$\text{Mot}(\mathbb{F})_{(\mathbb{Q}_l)} \rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_l)$$

defined by the standard l -adic realization is an equivalence of tannakian categories. Here $\text{Mot}(\mathbb{F}; \mathbb{Q}_l)$ is the category of motives defined using l -adic Tate classes.

To prove Conjecture A for CM abelian varieties, it suffices to prove it for split Weil classes on CM abelian varieties (1.39, 2.5). This, of course, would follow from knowing that the classes are algebraic. Concerning this, Markman (2025) writes:

We will present in this paper a general strategy for proving the algebraicity of the Weil classes on abelian varieties of split Weil type but implement it fully only for dimension ≤ 6 and K imaginary quadratic.³

Certainly, Conjecture A is implied by the Hodge conjecture, but it should be more accessible to proof.

³Once Markman has implemented his strategy for all classes on abelian varieties of split Weil type, the Hodge conjecture will be known for all CM abelian varieties (Deligne 1982, André 1992), Grothendieck's standard conjectures will be known for all abelian varieties (Milne 2002), the Tate conjecture will be known for abelian varieties over finite fields (Milne 1999c), and it will be possible to implement the strategy outlined in this article in a strong form.

Proof of the uniqueness of rational Tate classes

For an abelian variety A over \mathbb{F} and a prime number l , we let $\mathcal{T}_l^*(A)$ denote the \mathbb{Q}_l -algebra of Tate classes on A ,

$$\mathcal{T}_\ell^r(A) \stackrel{\text{def}}{=} \bigcup_{A_1/k_1} H_{\text{ét}}^{2r}(A, \mathbb{Q}_\ell(r))^{\text{Gal}(\mathbb{F}/k_1)} \quad \mathcal{T}_p^r(A) \stackrel{\text{def}}{=} \bigcup_{A_1/k_1} H_p^{2r}(A_1)(r)^F,$$

where A_1/k_1 runs over the models of A over finite subfields k_1 of \mathbb{F} . We let $\mathcal{T}^*(A)$ denote the restricted product of the $\mathcal{T}_l^*(A)$ (an \mathbb{A}_f -algebra).

Tate conjectured that the algebraic cycles on A modulo numerical equivalence provide a \mathbb{Q} -structure on $\mathcal{T}^*(A)$. Here we prove this for rational Tate classes (assuming they exist).

THEOREM 0.1. *Let $(\mathcal{R}^*(A))_{A \in \mathfrak{A}}$ be a family of rational Tate classes, as in Conjecture C. Then, for all $A \in \mathfrak{A}$, the map*

$$\mathcal{R}^*(A) \otimes \mathbb{A}_f \rightarrow H_{\mathbb{A}}^{2*}(A)(*)$$

has image $\mathcal{T}^(A)$, i.e., $\mathcal{R}^*(A)$ is a \mathbb{Q} -structure on $\mathcal{T}^*(A)$.*

PROOF. This can be proved by the argument in used in [Milne 1999c](#) to show that if the Hodge conjecture holds for CM abelian varieties then the Tate conjecture holds for abelian varieties over \mathbb{F} . See [Milne 2009](#), Theorem 2.2. \square

THEOREM 0.2. *Let $(\mathcal{R}_1^*(A))_{A \in \mathfrak{A}}$ and $(\mathcal{R}_2^*(A))_{A \in \mathfrak{A}}$ be two families as in the statement of Conjecture C. Then $\mathcal{R}_1^*(A) = \mathcal{R}_2^*(A)$ as \mathbb{Q} -subalgebras of $H_{\mathbb{A}}^{2*}(A)(*)$.*

PROOF. If $\mathcal{R}_1^*(A) \subset \mathcal{R}_2^*(A)$, then they are equal because they are both \mathbb{Q} -structures on $\mathcal{T}^*(A)$. The family $(\mathcal{R}_1^*(A) \cap \mathcal{R}_2^*(A))_{A \in \mathfrak{A}}$ obviously satisfies the conditions (R₁), (R₂), (R₃), and (R₄), and so equals both $\mathcal{R}_1^*(A)$ and $\mathcal{R}_2^*(A)$. \square

ASIDE 0.3. For an abelian variety A over \mathbb{F} , let $\mathcal{A}^*(A)$ denote the \mathbb{Q} -subalgebra of $H_{\mathbb{A}}^{2*}(A)(*)$ of algebraic classes. The family $(\mathcal{A}^*(A))$ satisfies (R1), (R2), and (R4). If it satisfies (R3), i.e., absolute Hodge classes specialize to algebraic classes, then the Tate conjecture holds for abelian varieties over \mathbb{F} .

Let $\text{Mot}(\mathbb{F}; \mathbb{A}_f)$ denote the category based on the abelian varieties over \mathbb{F} , using the \mathbb{A}_f -algebras of Tate classes as correspondences.

THEOREM 0.4. *Assume Conjecture D. The canonical functor $\text{Mot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}; \mathbb{A}_f)$ induces an equivalence of categories*

$$\text{Mot}(\mathbb{F})_{(\mathbb{A}_f)} \rightarrow \text{Mot}(\mathbb{F}; \mathbb{A}_f).$$

In other words, $\text{Mot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}; \mathbb{A}_f)$ is a \mathbb{Q} -structure on the \mathbb{A}_f -linear tannakian category $\text{Mot}(\mathbb{F}; \mathbb{A}_f)$.

PROOF. Restatement of Theorem 0.1. \square

Consequences of the conjectures

We list some consequences of the conjectures. For more details, see §6 and later sections. We add question marks as a reminder that these statements depend on conjectures.

0.5 (?). Deligne (2006) notes that the following would be a “particularly interesting corollary of the Hodge conjecture”:

Let A be an abelian variety over \mathbb{F} . Lift A in two different ways to characteristic 0, to complex abelian varieties A_1 and A_2 defined over \mathbb{C} . Pick Hodge classes γ_1 and γ_2 on A_1 and A_2 of complementary dimension. Interpreting γ_1 and γ_2 as ℓ -adic cohomology classes, one can define the intersection number κ of the reductions of γ_1 and γ_2 over \mathbb{F} . Is κ a rational number independent of ℓ ?

The answer is yes, because the reductions of γ_1 and γ_2 are both rational Tate classes on the abelian variety A .

0.6 (?). Let $l \neq p$. For a certain countable subfield Q of \mathbb{Q}_l , André defines a Q -linear tannakian category of motives $\text{Mot}(\mathbb{F}; Q)$ such that $\text{Mot}(\mathbb{F}; Q)_{(\mathbb{Q}_l)} \simeq \text{Mot}(\mathbb{F}; \mathbb{Q}_l)$. The category $\text{Mot}(\mathbb{F})$ is a \mathbb{Q} -structure on $\text{Mot}(\mathbb{F}; Q)$, that is,

$$\text{Mot}(\mathbb{F})_{(Q)} \simeq \text{Mot}(\mathbb{F}; Q).$$

Thus, rational Tate classes become motivated over the field Q .

0.7 (?). The category $\text{Mot}(\mathbb{F})$ and the functor $R : \text{Mot}^{\text{al}}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ have the properties predicted by the Hodge, Tate, and standard conjectures. In particular, $\text{Mot}(\mathbb{F})$ is a tannakian category over \mathbb{Q} with fundamental group the Weil-number protorus P , and there is a unique polarization on $\text{Mot}(\mathbb{F})$ for which R becomes a functor of polarized tannakian categories.

0.8 (?). Define $\text{Mot}(\mathbb{F}_q)$ to be the tannakian category over \mathbb{Q} whose objects are pairs (M, π_M) with M an object of $\text{Mot}(\mathbb{F})$ and π_M a (Frobenius) endomorphism compatible with the action of P on M . Then $\text{Mot}(\mathbb{F}_q)$ is a tannakian category over \mathbb{Q} with the properties predicted by the Tate and standard conjectures.

0.9 (?). Let M be an abelian motive over a number field $L \subset \mathbb{Q}^{\text{al}}$ (in the sense of absolute Hodge classes), and let G_M denote the Mumford–Tate group of the Hodge structure $\omega_B(M)$. Let $\ell \neq p$ be a prime number. After possibly replacing L with a finite extension, we obtain a representation $\rho : \text{Gal}(\mathbb{Q}^{\text{al}}/L) \rightarrow G_M(\mathbb{Q}_\ell)$. Assume that M has good reduction at the primes of L lying over p (i.e., satisfies the Néron condition). On applying ρ to the Frobenius elements at such primes, we get a conjugacy class in $G_M(\mathbb{Q}_\ell)$, and hence an element $\gamma_\ell \in (c\ell G_M)(\mathbb{Q}_\ell)$. This element lies in $(c\ell G_M)(\mathbb{Q})$ and is independent of ℓ .

0.10. We can extend the notion of an abelian motive to certain schemes, for example:

- ◊ Let S be a connected smooth scheme of finite type over a field k of characteristic zero, and let η be the generic point of S . An abelian motive over S is an abelian motive over $\kappa(\eta)$ such that the action of $\pi_1(\eta, \bar{\eta})$ on $\omega_f(M)$ factors through $\pi_1(S, \bar{\eta})$.

- ◊ Let S be a local scheme with closed point s such that $\kappa(s) \subset \mathbb{F}$. An abelian motive over S is a triple (M, N, φ) , where M is an abelian motive over $\kappa(s)$, N is a p -shtuka (in the sense of Scholze), and φ is an isomorphism from the p -shtuka of M to N_s .

0.11. As noted earlier, over \mathbb{C} the Shimura varieties of abelian type with rational weight are exactly the moduli schemes of abelian motives with additional structure. This description extends to subfields of \mathbb{C} , and Conjecture A allows us to extend it to mixed characteristic. From a “universal” abelian motive over a Shimura variety, we can construct the standard principal bundle, and hence a theory of automorphic vector bundles. To understand Shimura varieties of abelian type without rational weight, we apply Shimura’s trick.

0.12. We say that an abelian motive M over \mathbb{Q}^{al} satisfies the Néron condition if, for some model M_1 of M over a number field $K \subset \mathbb{Q}^{\text{al}}$, the action of $\pi_1(\text{Spec } K)$ on $\omega_\ell(M_1)$ factors through $\pi_1(\text{Spec }(\mathcal{O}_{K,w}))$ (fundamental groups with respect to the base point $\text{Spec}(\mathbb{Q}^{\text{al}})$). The functor $R : \text{Mot}^j(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ should extend to the category $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ of abelian motives over \mathbb{Q} satisfying the Néron condition.

0.13. CAVEAT. The conjectures do not imply that algebraic classes on abelian varieties over \mathbb{F} are rational Tate classes. Indeed, this would imply Grothendieck’s standard conjecture of Hodge type for abelian varieties. Nor do they imply the second of Deligne’s “particularly interesting corollaries” (the intersection number of the reduction of γ_1 with an algebraic cycle on A is rational).

Positive results

0.14. We have proved that the family $(\mathcal{R}^*(A))$ is unique if it exists (see above).

0.15. A commutative diagram of tannakian categories

$$\begin{array}{ccccc}
 \text{LCM}(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}^j(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}(\mathbb{Q}^{\text{al}}; \mathbb{Q}_\ell) \\
 \downarrow R & & \downarrow R & & \downarrow R \\
 \text{LMot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}; \mathbb{Q}_\ell)
 \end{array}$$

would give rise to a commutative diagram of bands

$$\begin{array}{ccccc}
 T & \longleftarrow & G & \longrightarrow & G' \\
 \uparrow & & \uparrow u & & \uparrow \\
 L & \longleftarrow & P & \longrightarrow & P_{\mathbb{Q}_l}
 \end{array}$$

We prove (unconditionally) that such a diagram exists, and that u is independent of l (Theorem 1.18). As a consequence, we find that an abelian motive over a number field with good reduction (suitably defined) defines a compatible system of l -adic representations.⁴

0.16. A necessary condition that the morphism of bands $u : P \rightarrow G$ arise from a morphism of tannakian categories $\text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ is that the (conjectural) class of $\text{Mot}(\mathbb{F})$ in $H^2(\mathbb{Q}, P)$ map to the trivial class in $H^2(\mathbb{Q}, G)$. For this, see §10.

⁴The absence of a proof of this last assertion has been cited by Deligne as one indication that his theory of abelian motives is solely a characteristic zero theory.

A variational approach to proving Conjecture A

An obvious approach to proving Conjecture A is to mimic Deligne's proof (1982) that all Hodge classes on abelian varieties are absolutely Hodge. Recall that this has four main steps.

Step 0: Behaviour of absolute Hodge classes in families:

Let $f : X \rightarrow S$ be a smooth proper map of complex varieties with S smooth and connected, and let γ be a global section of $R^{2n}f_*\mathbb{Q}(n)$. If γ_s is a Hodge class (resp. absolute Hodge class) for one $s \in S(\mathbb{C})$, then it is a Hodge class (resp. absolute Hodge class) for every $s \in S(\mathbb{C})$.

Step 1: Split Weil classes on abelian varieties are absolutely Hodge.

Let (A_1, ν_1, λ_1) be a split Weil triple relative to a CM-algebra E , and let $f : A \rightarrow S$ be the (universal) Weil family containing (A_1, ν_1, λ_1) . Because the Weil family is split, it contains the triple $(A_2, \nu_2, \lambda_2) = (B \otimes \mathcal{O}_E, \dots)$, where B can be any abelian variety of dimension $\dim(A_1)/[E : \mathbb{Q}]$. The Weil classes on such a triple (A_2, ν_2, λ_2) are algebraic, in particular, absolutely Hodge. Now apply Step 0.

Step 2: Use Step 1 to show that Hodge classes on CM abelian varieties are absolutely Hodge.

Since pullbacks of absolute Hodge classes are absolute Hodge, this follows from Theorem 1.39: every Hodge class on a CM abelian variety is a sum of pullbacks of split Weil classes on CM abelian varieties.

Step 3: Use Steps 0 and 2 to prove that all Hodge classes on abelian varieties are absolutely Hodge.

Let A_1 be an abelian variety over \mathbb{C} and γ_1 a Hodge class on A_1 . Choose a polarization λ_1 of A_1 . There exists a family of polarized abelian varieties $f : A \rightarrow S$ containing (A_1, λ_1) and such that γ_1 extends to a global section of $R^{2n}f_*\mathbb{A}(n)$ (Deligne 1982, 6.1). Moreover, S is a Shimura variety, and so for some $s \in S(\mathbb{C})$ in the same connected component as (A_1, λ_1) , A_s is of CM-type. Now apply Steps 0 and 2.

We try to mimic the above argument to prove Conjecture A

Step 0: In §3, we prove some variational theorems, for example:

Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme of finite type over an algebraically closed field k , and let δ be a global section of $R^{2n}f_*\mathbb{A}(n)$ such that δ is fixed by the Lefschetz group $L(A_{\bar{\eta}})$. If δ_s is Lefschetz for one closed $s \in S$, then it is Lefschetz for every closed $s \in S$ (Theorem 3.29).

Step 1: Conjecture A holds for split Weil classes.

This is unknown and is the crux.

Let (A_1, ν_1, λ_1) be a split Weil triple relative to a CM-algebra E , and let $f : A \rightarrow S$ be the Weil family containing (A_1, ν_1, λ_1) — it is defined over \mathbb{Q}^{al} . If $(A_1, \nu_1, \lambda_1)_0$ lifts (up to isogeny) to a triple in the family whose Weil classes satisfy Conjecture A, then the same is true for (A_1, ν_1, λ_1) .

Let δ_1 be a Lefschetz class on $(A_1)_0$ of dimension $d = \dim A_1/[E : \mathbb{Q}]$, and let (A_2, ν_2, λ_2) be a member of the family of the form $B \otimes \mathcal{O}_E$. There exists a complete smooth curve C in S_0 such that $(A_1, \nu_1, \lambda_1)_0$ and $(A_2, \nu_2, \lambda_2)_0$ lie in the pullback $f_C : A_C \rightarrow C$ of $f_0 : A_0 \rightarrow S_0$. If C can be chosen so that δ_1 extends to a global section δ of $R^{2d'} f_{C*} \mathbb{A}(d')$, $d + d' = \dim A_1$, fixed by $L(A_{\bar{\eta}})$, then, for all Weil classes γ on A/S ,

$$\langle (\gamma_1)_0 \cdot \delta_1 \rangle = \langle (\gamma_2)_0 \cdot \delta_2 \rangle$$

lies in \mathbb{Q} because γ_2 is algebraic and δ_2 is Lefschetz (Step 0).

Step 2: Use Step 1 to show that Conjecture A holds for CM abelian varieties.

This again follows from Theorem 1.39 because pullbacks of Hodge classes satisfying Conjecture A satisfy Conjecture A (this uses that push-forwards of Lefschetz classes on abelian varieties are Lefschetz; see 2.5).

Step 3: Use Step 2 to show that Conjecture A holds for all abelian varieties over \mathbb{Q}^{al} with good reduction at w .

This is largely known. Let A_1 be an abelian variety over \mathbb{Q}^{al} with good reduction at w , and let γ_1 be a Hodge class on A_1 . Choose a polarization λ_1 of A_1 . There exists a universal family of polarized abelian varieties $f : A \rightarrow S$ over \mathbb{Q}^{al} containing (A_1, λ_1) in which γ_1 extends to a global section of $R^{2n} f_* \mathbb{A}(n)$. Moreover, S is a Shimura variety. Under some hypotheses on S , the pair $(A_1, \lambda_1)_0$ lifts (up to isogeny) to a CM pair (A_2, λ_2) in the family in the family (Kisin, Vasiu, ...). Now $\langle (\gamma_1)_0 \cdot \delta \rangle = \langle (\gamma_2)_0 \cdot \delta \rangle$, which lies in \mathbb{Q} by Step 2.⁵

For an inductive approach to proving Conjecture A, see Theorem 2.18. For an alternative variational approach, see 3.29.

Outline of the article

After collecting various preliminaries in §1, we discuss in §§2,3 possible proofs of Conjecture A, and a variant Conjecture B, for CM abelian varieties. As outlined above, one approach is to mimic the argument in Deligne 1982.

In §4, we show that Conjecture A for CM abelian varieties implies Conjecture C for CM abelian varieties. In particular, it implies that we have the notion of a rational Tate class on an abelian variety over \mathbb{F} and that Hodge classes on CM abelian varieties specialize to rational Tate classes. In §5 we investigate how to extend this last property to more general abelian varieties. This comes down to a question of the existence of CM-lifts of abelian varieties in families.

In §6, we describe the motivic paradise that results from Conjecture A. In particular, we show how (given Conjecture A), it is possible to construct tannakian categories of abelian motives over various schemes. This allows us in the remaining sections to realize

⁵Thus, to extend the reduction map from $\text{CM}(\mathbb{Q}^{\text{al}})$ to $\text{Mot}^w(\mathbb{Q}^{\text{al}})$, we need to know that points on Shimura varieties over \mathbb{F} lift to (up to isogeny) to CM-points. There is a partial converse: given $R : \text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$, it is possible to show that every point in $\text{Sh}(\mathbb{F})$ lifts to a CM point (at least when the derived group is simply connected...). See Langlands and Rapoport 1987, 5.4; Milne 1992, 4.8.

Shimura varieties of abelian type with rational weight as moduli varieties, even in mixed characteristic (still in progress).

Notation

Throughout,

$$l = 2, 3, \dots, p, \dots, \text{ or } \infty,$$

and ℓ is a prime number $\neq p$,

$$\ell \neq p, \infty.$$

We sometimes let $\mathbb{Q}_\infty = \mathbb{R}$.

The symbol \mathbb{Q}^{al} denotes an algebraic closure of \mathbb{Q} , w denotes a prime of \mathbb{Q}^{al} lying over p , and \mathbb{F} is the residue field at w . We let ι or $z \mapsto \bar{z}$ denote complex conjugation on \mathbb{C} and its subfields.

Let R be a \mathbb{Q} -algebra. By a \mathbb{Q} -structure on an R -module M , we mean a \mathbb{Q} -subspace V such that the map $r \otimes v \mapsto rv : R \otimes_{\mathbb{Q}} V \rightarrow M$ is an isomorphism.

By a Hodge class on an abelian variety A over a field of characteristic zero, we mean an absolute Hodge class in the sense of [Deligne 1982](#). We write $\mathcal{B}^*(A)$ for the \mathbb{Q} -algebra of Hodge classes on A and $\mathcal{D}^*(A)$ for the Lefschetz classes (elements of the \mathbb{Q} -algebra generated by divisor classes).

We often regard abelian varieties as objects in the category with homomorphisms $\text{Hom}^0(A, B)$ (in which isogenies become isomorphisms).

For a complete smooth variety over a perfect field k of characteristic p , we let $H_p^r(X)$ denote the crystalline cohomology group $H^r(X/B)$, where B is the field of fractions of the ring of Witt vectors of k . It is an F -isocrystal over k .

For a complete smooth variety X over an algebraically closed field of characteristic zero,

$$H_{\mathbb{A}}^*(X) \stackrel{\text{def}}{=} \left(\varprojlim_m H_{\text{ét}}^*(X, \mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \times H_{\text{dR}}^*(X),$$

and for a complete smooth variety X_0 over \mathbb{F} ,

$$H_{\mathbb{A}}^*(X) \stackrel{\text{def}}{=} \left(\varprojlim_{p \nmid m} H_{\text{ét}}^*(X, \mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \times H_p^*(X) \otimes_{B(\mathbb{F})} \mathbb{Q}_w^{\text{al}},$$

where \mathbb{Q}_w^{al} is the completion of \mathbb{Q}^{al} at w . If X has good reduction at w to X_0 , then

$$\begin{aligned} H_{\text{ét}}^*(X, \mathbb{Z}/m\mathbb{Z}) &\simeq H_{\text{ét}}^*(X_0, \mathbb{Z}/m\mathbb{Z}) \text{ for all } m \text{ not divisible by } p, \\ H_{\text{dR}}^*(X) \otimes_{\mathbb{Q}^{\text{al}}} \mathbb{Q}_w^{\text{al}} &\simeq H_p^*(X_0) \otimes_{B(\mathbb{F})} \mathbb{Q}_w^{\text{al}}, \end{aligned}$$

and so there is a canonical specialization map $H_{\mathbb{A}}^i(X) \rightarrow H_{\mathbb{A}}^i(X_0)$.

For a connected normal scheme S , we let η denote the generic point of S and $\bar{\eta}$ a geometric generic point (if $\eta = \text{Spec}(K)$, then $\bar{\eta} = \text{Spec}(K^{\text{sep}})$).

The quotient of an affine group scheme G by its action on itself by inner automorphisms is denoted by $\text{cl}G$.

Commutative diagrams of categories and functors are required to commute “on the nose” — the maps of objects and arrows defined by two paths are required to coincide.

$X = Y$ means that X equals Y ;

$X \simeq Y$ means that X is isomorphic to Y with a specific isomorphism (often only implicitly described);

$X \approx Y$ means that X is isomorphic to Y .

1 Preliminaries

We collect various preliminaries, partly to fix notation.

Tannakian categories

1.1. We assume that the reader is familiar with tannakian categories. Let \mathbb{T} be a tannakian category over a field k . Recall that the fundamental group of \mathbb{T} is the group scheme $\pi(\mathbb{T})$ in \mathbb{T} (better, $\text{Ind } \mathbb{T}$) such that $\omega(\pi(\mathbb{T})) = \text{Aut}^{\otimes}(\omega)$ for every fibre functor ω . If $\pi(\mathbb{T})$ is commutative, then it is an affine group scheme over k in the usual sense.

We briefly review the theory of quotients of tannakian categories (Milne 2007).

1.2. Let k be a field. An exact tensor functor $q : \mathbb{T} \rightarrow \mathbb{Q}$ of tannakian categories over k is said to be a quotient functor if every object of \mathbb{Q} is a subquotient of an object in the image of q . Then the full subcategory \mathbb{T}^q of \mathbb{T} consisting of the objects that become trivial⁶ in \mathbb{Q} is a tannakian subcategory of \mathbb{T} , and $X \rightsquigarrow \text{Hom}(\mathbf{1}, qX)$ is a k -valued fibre functor ω^q on \mathbb{T}^q . In particular \mathbb{T}^q is neutral. For X, Y in \mathbb{T} ,

$$\text{Hom}(qX, qY) \simeq \omega^q(\mathcal{H}om(X, Y)^H), \quad (1)$$

where H is the subgroup of $\pi(\mathbb{T})$ such that $\mathbb{T}^q = \mathbb{T}^H$.

Conversely, every k -valued fibre functor ω_0 on a tannakian subcategory \mathbb{S} of \mathbb{T} arises (as above) from a well-defined quotient functor $\mathbb{T} \rightarrow \mathbb{T}/\omega_0$. For example, when \mathbb{T} is semisimple, we can take \mathbb{T}/ω_0 to be the pseudo-abelian hull of the category with one object qX for each object X of \mathbb{T} and whose morphisms are given by (1).

1.3. Let $q : \mathbb{T} \rightarrow \mathbb{Q}$ be a quotient functor. Fix a unit object $\mathbf{1}$ in \mathbb{T} , and let ω^q denote the fibre functor $\text{Hom}(\mathbf{1}, q(-))$ on \mathbb{T}^q . A fibre functor ω on \mathbb{Q} defines a fibre functor $\omega \circ q$ on \mathbb{T} , and the unique isomorphism of fibre functors $\text{Hom}(\mathbf{1}, -) \rightarrow \omega|_{\mathbb{Q}^{\pi(\mathbb{Q})}}$ determines an isomorphism $\omega^q \rightarrow (\omega \circ q)|_{\mathbb{T}^q}$. Conversely, a pair consisting of a fibre functor ω' on \mathbb{T} and an isomorphism $\omega^q \rightarrow \omega'|_{\mathbb{T}^q}$ arises from a unique fibre functor on \mathbb{Q} .

CM abelian varieties

1.4. Let A be an abelian variety over an algebraically closed field k . The reduced degree⁷ of the \mathbb{Q} -algebra $\text{End}^0(A)$ is $\leq 2 \dim A$; when equality holds the abelian variety is said to be CM. An isotypic abelian variety is CM if and only if $\text{End}^0(A)$ contains a field of degree $2 \dim A$ over \mathbb{Q} , and an arbitrary abelian variety is CM if and only if each isotypic isogeny factor of it is. Equivalent conditions:

- (a) the \mathbb{Q} -algebra $\text{End}^0(A)$ contains an étale subalgebra of degree $2 \dim A$ over \mathbb{Q} ;
- (b) for a Weil cohomology $X \rightsquigarrow H^*(X)$ with coefficient field Q , the centralizer of $\text{End}^0(A)$ in $\text{End}_Q(H^1(A))$ is commutative (in which case it equals $C(A) \otimes_{\mathbb{Q}} Q$, where $C(A)$ is the centre of $\text{End}^0(A)$);
- (c) (characteristic zero) the Mumford-Tate group of A is commutative (hence a torus);
- (d) (characteristic $p \neq 0$) A is isogenous to an abelian variety defined over \mathbb{F} (theorems of Tate and Grothendieck).

⁶That is, isomorphic to a direct sum of copies of $\mathbf{1}$

⁷The reduced degree of a simple Q -algebra R with centre C is $[C : Q][R : C]^{1/2}$.

Abelian motives

DEFINITION

To define a category of motives, we need a collection of smooth projective varieties over a field k and a theory of correspondences, i.e., a graded Q -algebra $\mathcal{C}^*(X)$ for each variety X in the collection together with, for each morphism $\phi : X \rightarrow Y$, pull-back and push-forward maps⁸

$$\begin{aligned}\phi^* &: \mathcal{C}^*(Y) \rightarrow \mathcal{C}^*(X) \\ \phi_* &: \mathcal{C}^*(X) \rightarrow \mathcal{C}^{*+\dim Y-\dim X}(Y)\end{aligned}$$

satisfying certain obvious axioms. Here Q is usually a field. The corresponding category of motives $M(k)$ is defined as follows: an object is a triple (X, e, m) , where X is a variety in the collection, e is an idempotent in the Q -algebra $\mathcal{C}^{\dim X}(X \times X)$, and $m \in \mathbb{Z}$; morphisms are defined by

$$\text{Hom}((X, e, m), (Y, f, n)) = f \cdot \mathcal{C}^{n-m+\dim X}(X \times Y) \cdot e.$$

The category $M(k)$ has a tensor product structure

$$(X, e, m) \otimes (Y, f, n) = (X \times Y, e \otimes f, m + n)$$

with unit object

$$\mathbf{1} \stackrel{\text{def}}{=} (\text{Spec } k, \text{id}, 0).$$

In all the cases in this article, the Künneth components of the diagonal lie in \mathcal{C}^* , and we use them to modify the commutativity constraint. In good cases, $M(k)$ is a tannakian category over the Q .

Let k be an algebraically closed field of characteristic zero. We define the category $\text{Mot}(k)$ of *abelian motives* over k to be the category of motives based on the abelian varieties over k using the (absolute) Hodge classes as correspondences.

CM MOTIVES

Let k be an algebraically closed field of characteristic zero. We define the category $\text{CM}(k)$ of *CM motives* over k to be the category of motives based on the CM abelian varieties over k using the (absolute) Hodge classes as correspondences.

1.5. Let K be a CM subfield of \mathbb{Q}^{al} , finite and Galois over \mathbb{Q} . The Serre torus S^K is the quotient of $(\mathbb{G}_m)_{K/\mathbb{Q}}$ such that

$$X^*(S^K) = \{f : \text{Gal}(K/\mathbb{Q}) \rightarrow \mathbb{Z} \mid f(\sigma) + f(\iota\sigma) = \text{constant}\}.$$

The constant value $f(\sigma) + f(\iota\sigma)$ is called the weight of f . For $K' \supset K$, there is a norm map $S^{K'} \rightarrow S^K$, and the Serre protorus is

$$S \stackrel{\text{def}}{=} \varprojlim S^K.$$

⁸In the formulas, we are assuming, for simplicity, that the varieties have pure dimension.

1.6. The category of CM motives over k is a tannakian category over \mathbb{Q} . The choice of an embedding $k \hookrightarrow \mathbb{C}$ defines a (Betti) fibre functor ω_B on $\text{CM}(k)$, and

$$S \simeq \mathcal{A}ut^{\otimes}(\omega_B).$$

In particular, we see that the functor (extension of the base field),

$$\text{CM}(k) \rightarrow \text{CM}(\mathbb{C})$$

is an equivalence of tensor categories.

ABELIAN MOTIVES OVER THE COMPLEX NUMBERS

Let ω_B denote the Betti fibre functor on $\text{Mot}(\mathbb{C})$, and let $G = \mathcal{A}ut^{\otimes}(\omega_B)$. The functor

$$(\omega_B)_{(\mathbb{R})} : \text{Mot}(\mathbb{C})_{(\mathbb{R})} \rightarrow \text{Vec}_{\mathbb{R}}$$

factors canonically through the category $\text{Hdg}_{\mathbb{R}}$ of polarizable real Hodge structures, and so defines a homomorphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$, where $\mathbb{S} \stackrel{\text{def}}{=} (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$ is the Deligne torus. We wish to describe the pair (G, h) . Let

$$\bar{h} : \mathbb{S} \rightarrow G_{\mathbb{R}}^{\text{ad}}$$

denote the composite of h with the quotient map $G_{\mathbb{R}} \rightarrow G_{\mathbb{R}}^{\text{ad}}$.

Consider the following conditions on a homomorphism $h : \mathbb{S} \rightarrow H$ of connected algebraic groups over \mathbb{R} :

SV1: the Hodge structure on the Lie algebra of H defined by $\text{Ad} \circ h$ is of type

$$\{(1, -1), (0, 0), (-1, 1)\};$$

SV2: $\text{inn}(h(i))$ is a Cartan involution of H^{ad} .

THEOREM 1.7. *Let (G, h) be the pair attached to $(\text{Mot}(\mathbb{C}), \omega_B)$ as above.*

- (a) *The quotient of G by its derived group is the Serre protorus (S, h) .*
- (b) *Let H be a semisimple algebraic group over \mathbb{Q} and $\bar{h} : \mathbb{S}/\mathbb{G}_m \rightarrow H_{\mathbb{R}}^{\text{ad}}$ a homomorphism generating H^{ad} and satisfying the conditions SV1,2. The pair (H, \bar{h}) is a quotient of $(G^{\text{der}}, \bar{h})$ if and only if there exists an isogeny $H' \rightarrow H$ with H' a product of almost-simple groups H'_i over \mathbb{Q} such that either*
 - i) H'_i *is simply connected of type A, B, C, or $D^{\mathbb{R}}$, or*
 - ii) H'_i *is of type D_n^{H} ($n \geq 5$) and equals $\text{Res}_{F/\mathbb{Q}} H_0$ for H_0 the double covering of an adjoint group that is a form of $\text{SO}(2n)$.*

The homomorphism $G \rightarrow S$ is defined by the exact tensor functor $\text{CM}(\mathbb{C}) \hookrightarrow \text{Mot}(\mathbb{C})$.

PROOF. See [Milne 1994b](#), 1.27. □

ABELIAN MOTIVES OVER \mathbb{Q}^{al}

Take \mathbb{Q}^{al} to be the algebraic closure of \mathbb{Q} in \mathbb{C} , and let ω_B be the Betti fibre functor defined by the inclusion $\mathbb{Q}^{\text{al}} \hookrightarrow \mathbb{C}$.

THEOREM 1.8. *Let (G, h) be the pair attached to $(\text{Mot}(\mathbb{Q}^{\text{al}}), \omega_B)$. The quotient of G by its derived group is S , and the algebraic quotients of $(G^{\text{der}}, \bar{h})$ have the same description as in Theorem 1.7.*

PROOF. (a) Almost by definition, the motivic Galois group of $\text{CM}(\mathbb{C})$ is the Serre group S . The functor $A \rightsquigarrow A_{\mathbb{C}}$ defines an equivalence of categories $\text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{CM}(\mathbb{C})$, and so the motivic Galois group of $\text{CM}(\mathbb{Q}^{\text{al}})$ is also S .

(b) Let H be a semisimple algebraic group over \mathbb{Q} and \bar{h} a homomorphism $\mathbb{S}/\mathbb{G}_m \rightarrow H_{\mathbb{R}}^{\text{ad}}$ generating H^{ad} and satisfying the conditions (SV1,2). Let $H \rightarrow \text{GL}_V$ be a symplectic representation of (H, \bar{h}) . Recall (Milne 2013, 10.9), that this means that there exists a commutative diagram

$$\begin{array}{ccccc} H & & & & \\ \downarrow & \searrow & & & \\ (H^{\text{ad}}, \bar{h}) & \longleftarrow & (G, h) & \xrightarrow{\rho} & (G(\psi), D(\psi)) \end{array}$$

in which ψ is a nondegenerate alternating form on V , G is a reductive group (over \mathbb{Q}), and h is a homomorphism $\mathbb{S} \rightarrow G_{\mathbb{R}}$; the homomorphism $H \rightarrow G$ is required to have image G^{der} . In particular, $G^{\text{ad}} \simeq H^{\text{ad}}$.

Consider the map of Shimura varieties $\text{Sh}(G, X) \rightarrow \text{Sh}(H^{\text{ad}}, X^{\text{ad}})$ corresponding to the map $(G, h) \rightarrow (H^{\text{ad}}, \bar{h})$. This is defined over \mathbb{Q}^{al} , and induces a finite surjective map

$$\text{Sh}_K(G, X)^{\circ} \rightarrow \text{Sh}_{K^{\text{ad}}}(H^{\text{ad}}, X^{\text{ad}})^{\circ}.$$

Since H^{ad} is a quotient of G , there is an abelian motive (+ structure) corresponding to a point of the second connected Shimura variety, which lifts to the first Shimura variety. \square

CAUTION 1.9. The theorem characterizes the algebraic quotients of G^{der} , where G is the motivic Galois group of $\text{Mot}(\mathbb{Q}^{\text{al}})$. The algebraic quotients of H^{der} , where H is the motivic Galois group of $\text{Mot}(\mathbb{C})$ have exactly the same description, but the two groups are not isomorphic. For example, H^{der} has an uncountably product of copies of SL_2 as a quotient, but G^{der} has only a countable product as a quotient.

ABELIAN MOTIVES WITH GOOD REDUCTION.

We say that an abelian motive over \mathbb{Q}^{al} has *good reduction* at w if some model of it over a subfield of \mathbb{Q}^{al} satisfies the Néron condition. We let $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ denote the full subcategory of $\text{Mot}(\mathbb{Q}^{\text{al}})$ whose objects have good reduction at w . Note that $\text{CM}(\mathbb{Q}^{\text{al}}) \subset \text{Mot}^w(\mathbb{Q}^{\text{al}})$.

THEOREM 1.10. *Let (G, h) be the pair attached to $(\text{Mot}^w(\mathbb{Q}^{\text{al}}), \omega_B)$. The quotient of G by its derived group is S , and the algebraic quotients of $(G^{\text{der}}, \bar{h})$ have the same description as in Theorem 1.7.*

PROOF. Omitted for the moment. \square

In particular, G^{der} is not simply connected unless we exclude abelian varieties of type D^{H} in the definition of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$.

ASIDE 1.11. It will be important to extend everything in this article to abelian varieties over \mathbb{Q}^{al} with (stable) bad reduction at w .

The Weil-number protorus

1.12. Let $q \in p^{\mathbb{N}}$. An algebraic number is a Weil q -number of weight m if

- (a) for every homomorphism $\rho : \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $\rho(\pi) \cdot \overline{\rho(\pi)} = q^m$, and
- (b) for some $n > 0$, $q^n \pi$ is an algebraic integer.

The set of Weil q -numbers in \mathbb{Q}^{al} is denoted by $W(q)$. It is a subgroup of the multiplicative group of \mathbb{Q}^{al} , stable under the action of $\Gamma \stackrel{\text{def}}{=} \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$. We let $P(q)$ denote the affine group scheme of multiplicative type with

$$X^*(P(q)) = W(q).$$

There is a universal element $\pi_{\text{univ}} \in P(q)(\mathbb{Q})$ such that $\chi(\pi_{\text{univ}}) = \pi$ if χ is the character of $P(q)$ corresponding to π — it is the element corresponding to the inclusion map under

$$P(q)(\mathbb{Q}) \simeq \text{Hom}(W(q), \mathbb{Q}^{\text{al}\times})^{\text{Gal } \mathbb{Q}^{\text{al}}/\mathbb{Q}}.$$

1.13. If $n|n'$, then $\pi \mapsto \pi^{n'/n}$ is a homomorphism $W(p^n) \rightarrow W(p^{n'})$. Define

$$W(p^\infty) = \varinjlim W(p^n).$$

There is an action of $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $W(p^\infty)$, and the protorus P over \mathbb{Q} such that

$$X^*(P) = W(p^\infty)$$

is called the Weil-number protorus.

The Shimura–Taniyama homomorphism

1.14. Again let K be a CM subfield of \mathbb{Q}^{al} , finite and Galois over \mathbb{Q} . After possibly enlarging K , we may suppose that t is not in the decomposition group at w_K . Let \mathfrak{p} be the prime ideal of \mathcal{O}_K corresponding to w_K . For some h , \mathfrak{p}^h will be principal, say, $\mathfrak{p}^h = (a)$. Let $\alpha = a^{2n}$, where $n = (U(K) : U(K_+))$. Then, for $f \in X^*(S^K)$, $f(\alpha)$ is independent of the choice of a , and it is a Weil $p^{2nf(\mathfrak{p}/p)}$ -number of weight equal to the weight of f . The map $f \mapsto f(\alpha) : X^*(S^K) \rightarrow W^K(p^\infty)$ is a surjective homomorphism (Milne 2001, A.8), and so corresponds to an injective homomorphism $\rho^K : P^K \rightarrow S^K$. On passing to the direct limit over the $K \subset \mathbb{Q}^{\text{al}}$, we obtain an injective homomorphism

$$\rho : P \rightarrow S,$$

called the Shimura–Taniyama homomorphism.

1.15. The category $\text{CM}(\mathbb{Q}^{\text{al}})$ of CM motives over \mathbb{Q}^{al} is the subcategory $\text{Mot}(\mathbb{Q}^{\text{al}})$ generated by the abelian varieties of CM-type. For any embedding $\mathbb{Q}^{\text{al}} \hookrightarrow \mathbb{C}$, the functor $\text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{CM}(\mathbb{C})$ is an equivalence of categories, and so the fundamental group of $\text{CM}(\mathbb{Q}^{\text{al}})$ is the Serre group. All CM abelian varieties over \mathbb{Q}^{al} have good reduction at w (because they satisfy the Néron condition).

1.16. The Shimura–Taniyama homomorphism $P \rightarrow S$ (see 1.14) has a geometric description. Let A be a CM abelian variety over a subfield k of \mathbb{Q}^{al} . After possibly enlarging k , we may suppose that A has good reduction at w and that some set of generators for the Hodge classes on $A_{\mathbb{Q}^{\text{al}}}$ and its powers is defined over k . Under these assumptions, there is an endomorphism F of A reducing mod p to the Frobenius endomorphism of A_0 . Moreover, F lies in the Mumford–Tate group of A , and so defines a homomorphism $P \rightarrow \text{MT}(A)$. For varying A , these homomorphism define a homomorphism $P \rightarrow \varprojlim \text{MT}(A) = S$. The theory of complex multiplication (Shimura, Taniyama, Weil) shows that this agrees with the Shimura–Taniyama homomorphism ρ defined earlier.

1.17. The Galois groupoid attached to the category $\text{CM}(\mathbb{Q})$ and its Betti fibre functor is called the Taniyama group. There is an explicit description of it, due to Deligne and Langlands. See Milne 1990, §6.

The Galois representations defined by abelian motives

A morphism $\text{Mot}^{\text{al}}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ as in Conjecture D gives rise to a morphism $P \rightarrow G$ of affine bands. Using a recent result of Kisin and Zhou (2025a, 2025b) we show (unconditionally) that such a morphism exists.

In this subsection, \mathbb{Q}^{al} is the algebraic closure of \mathbb{Q} in \mathbb{C} . We define the Mumford–Tate group of an abelian motive M (or abelian variety A) over \mathbb{Q}^{al} to be the Mumford–Tate group of the rational Hodge structure $\omega_B(M)$ (or $\omega_B(h^1 A)$).

An abelian motive over \mathbb{Q}^{al} has *visibly good reduction* at w if it is of the form (A, e, m) with A an abelian variety with good reduction at w . Let $\text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})$ denote the category of abelian motives over \mathbb{Q}^{al} with visibly good reduction at w , and let

$$G = \text{Aut}^{\otimes}(\omega_B \mid \text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})).$$

Let $\ell \neq p$. Let $\text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}}; \mathbb{Q}_{\ell})$ and $\text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell})$ denote the categories of abelian motives defined using the ℓ -adic Tate classes as correspondences. They are \mathbb{Q}_{ℓ} -linear tannakian categories with canonical \mathbb{Q}_{ℓ} -valued fibre functors ω_{ℓ} , and we let

$$\begin{aligned} G_{\ell} &= \text{Aut}^{\otimes}(\omega_{\ell} \mid \text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}}; \mathbb{Q}_{\ell})) \\ P_{\ell} &= \text{Aut}^{\otimes}(\omega_{\ell} \mid \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell})). \end{aligned}$$

From the reduction functor

$$\text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}}; \mathbb{Q}_{\ell}) \rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell}),$$

we get a morphism $u_{\ell} : P_{\ell} \rightarrow G_{\ell}$ of affine bands over \mathbb{Q}_{ℓ} .

Suppose that there exists a commutative diagram of tannakian categories

$$\begin{array}{ccc} \text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}}; \mathbb{Q}_{\ell}) \\ \downarrow & & \downarrow \\ \text{Mot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell}), \end{array}$$

such that

$$\text{Mot}(\mathbb{F})_{(\mathbb{Q}_{\ell})} \rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell})$$

is a tensor equivalence. From such a diagram, we get a commutative diagram of bands

$$\begin{array}{ccc} G & \longrightarrow & G_\ell \\ \uparrow u & & \uparrow u_\ell \\ P & \longrightarrow & P_\ell \end{array} \quad (*)$$

such that

$$P_{\mathbb{Q}^\ell} \rightarrow P_\ell$$

is an isomorphism. Here P is the Weil-number protorus.

Using the main result of [Kisin and Zhou 2025a](#), we show that the diagram (*) exists unconditionally.

THEOREM 1.18. *There exists a unique morphism of bands*

$$u : P \rightarrow G$$

over \mathbb{Q} such that (*) commutes. It is independent of $\ell \neq p$.

PROOF. To give a morphism $P \rightarrow G$ of bands amounts to giving a conjugacy class of homomorphisms $P_{\mathbb{Q}^{\text{al}}} \rightarrow G_{\mathbb{Q}^{\text{al}}}$ stable under the action of $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$.

Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w , and let G_A be its Mumford–Tate group. From a model of A over a number field $L \subset \mathbb{Q}^{\text{al}}$, we get a Frobenius element γ_ℓ in $(\text{cl}G_A)(\mathbb{Q}_\ell)$ (possibly after extending L). According to the main theorem of [Kisin and Zhou 2025a](#), this arises from an element $\gamma \in (\text{cl}G_A)(\mathbb{Q})$ that is independent of ℓ . Let $\mathbb{F}_q \subset \mathbb{F}$ be the residue field at $w|L$. For each lift $\tilde{\gamma}$ of γ to $G_A(\mathbb{Q}^{\text{al}})$, there is a unique homomorphism $P(q)_{\mathbb{Q}^{\text{al}}} \rightarrow (G_A)_{\mathbb{Q}^{\text{al}}}$ of affine group schemes sending the universal element $\pi_{\text{univ}} \in P(q)(\mathbb{Q})$ to γ (apply [Lemma 1.19](#) if this is not obvious). As we vary the lift $\tilde{\gamma}$ of γ , we get a conjugacy class of homomorphisms $P(q)_{\mathbb{Q}^{\text{al}}} \rightarrow (G_A)_{\mathbb{Q}^{\text{al}}}$ stable under $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$, which we regard as a morphism of affine bands $P(q) \rightarrow G_A$. Now pass to the inverse limit over larger and larger abelian varieties,

$$A_1 \subset A_2 \subset \cdots,$$

and use that

$$P = \varprojlim_q P(q) \quad \text{and} \quad G = \varprojlim_A G_A,$$

to obtain the required morphism of bands $P \rightarrow G$. □

LEMMA 1.19. *Let k be a field and R a k -algebra. Let X and Y be algebraic k -schemes with X reduced and Y separated, and let $\Sigma \subset X(k)$ be Zariski dense in $|X|$. A morphism of R -schemes $\phi : X_R \rightarrow Y_R$ arises from a morphism of k -schemes if and only if $\phi(\Sigma) \subset Y(k)$.*

PROOF. The necessity is obvious. Let $S = \text{Spec}(k)$ and $T = \text{Spec}(R)$. For the sufficiency, we have to show that $\text{pr}_1^*(\phi) = \text{pr}_2^*(\phi)$, where pr_1 and pr_2 are the projections $T \times_S T \rightarrow T$. Because X is reduced, Σ is schematically dense in X , and so its inverse image Σ' in $X \times_S (T \times_S T)$ is schematically dense. As Y is separated and $\text{pr}_1^*(\phi)$ and $\text{pr}_2^*(\phi)$ agree on Σ' , they must be equal. □

APPLICATION 1.20. Let M be an abelian motive over a number field $L \subset \mathbb{Q}^{\text{al}}$ with good reduction at $w|L$, and let $G_M \stackrel{\text{def}}{=} \text{Aut}^{\otimes}(\omega_B | \langle M_{\mathbb{Q}^{\text{al}}} \rangle^{\otimes})$ be its Mumford–Tate group. Let $\ell \neq p$ be a prime number. After possibly replacing L with a finite extension, we obtain a representation $\rho: \text{Gal}(\mathbb{Q}^{\text{al}}/L) \rightarrow G_M(\mathbb{Q}_{\ell})$, and hence a Frobenius element $\gamma_{\ell} \in (clG_M)(\mathbb{Q}_{\ell})$. Assume that M has visibly good reduction at $w|L$, i.e., that $M = (A, e, m)$, where A is an abelian variety over L with good reduction at $w|L$. In the proof of Theorem 1.18, we constructed a morphism of bands $P(q) \rightarrow G_A$. As G_M is a quotient of G_A , this gives us a morphism of bands $P(q) \rightarrow G_M$, and hence a morphism of schemes $P(q) \rightarrow clG_M$. The image of $\pi_{\text{univ}} \in P(q)(\mathbb{Q})$ in $(clG_M)(\mathbb{Q})$ maps to $\gamma_{\ell} \in (clG_M)(\mathbb{Q}_{\ell})$ and is independent of ℓ . Thus, we obtain a version of the main theorem of Kisin and Zhou 2025a for abelian motives. (See also 1.23 and Theorem 6.9 below.)

NOTES

1.21. Of course, it would be better to *deduce* Theorem 1.18 from the existence of a diagram of tannakian categories.

1.22. Note that we proved Theorem 1.18 by deducing a statement about a *cofinal* collection of algebraic quotients of G from the theorem of Kisin and Zhou, and then we proved 1.20 by applying Theorem 1.18 to *all* algebraic quotients of G .

1.23. By applying the results of Kisin and Zhou 2025b, it is possible to extend some of the above results to all abelian motives with good reduction at w , and even include p . Specifically, let M be an abelian motive over a number field $L \subset \mathbb{Q}^{\text{al}}$ with good reduction at w , i.e., satisfying the Néron condition. Let $\ell \neq p$. After possibly extending L , we get a Galois representation with values in $G_M(\mathbb{Q}_{\ell})$, which comes via the quotient map $G_A \rightarrow G_M$ from a Weil–Deligne representation with values in $G_A(\mathbb{Q}_{\ell})$. On applying the main theorem *ibid.*, we get an element $\gamma \in (clG_M)(\mathbb{Q})$, which, as rise to a morphism of bands $P(q) \rightarrow G_M$. On passing to the inverse limits, we get the required morphism $u: P \rightarrow G$.

1.24. For CM abelian varieties, this is all much easier, because the Frobenius endomorphism on A_0 lifts to A . See 1.16.

1.25. From the morphism of affine bands $P \rightarrow G$ we get an action of P on the objects of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$, and hence we can define $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$. Constructing a commutative diagram

$$\begin{array}{ccc} \text{Mot}^w(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}^w(\mathbb{Q}^{\text{al}}; \mathbb{Q}_{\ell}) \\ \downarrow R & & \downarrow R \\ \text{Mot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell}), \end{array}$$

amounts to showing that the \mathbb{Q}_{ℓ} -valued fibre functor on $\text{Mot}^w(\mathbb{Q}^{\text{al}}; \mathbb{Q}_{\ell})^P$ defined by the right-hand arrow becomes \mathbb{Q} -valued when restricted to $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$ (see 1.2). In other words, for each object M of $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$ we need a canonical \mathbb{Q} -structure

$$\text{“Hom}_{\text{Mot}(\mathbb{F})}(\mathbf{1}, R(M))\text{”}$$

on $\text{Hom}_{\text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell})}(\mathbf{1}, R(M))$.

1.26. We have a well-defined morphism of affine bands $u : P \rightarrow G$. Does this arise from a morphism of tannakian categories $\text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ making the diagram in (*) commute up to isomorphism? This is question in cohomology. For what can be said about it, see the discussion preceding Theorem 10.1 below.

Similarly, one can ask whether there exists a \mathbb{Q} -valued functor ω_0 on $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$ such that $\omega_0 \otimes \mathbb{Q}_l \approx \omega_l$ for all l ? Again, this is a question in cohomology. For what can be said about it, see Theorem 10.12 below.

1.27. For related results, see [Noot 2009, 2013](#); [Laskar 2014](#); [Commelin 2019](#); [Kisin and Zhou 2025a,b](#)

Lefschetz motives

1.28. For an adequate equivalence relation \sim and a smooth projective variety X over a field k , we let $\mathcal{D}_{\sim}^*(X)$ denote the \mathbb{Q} -subalgebra of $\text{CH}_{\sim}^*(X)$ generated by the divisor classes. The elements of $\mathcal{D}_{\sim}^*(X)$ are called *Lefschetz classes* on X (for the relation \sim).

1.29. Let k be an algebraically closed field. For abelian varieties, the equivalence relation on Lefschetz classes defined by any Weil cohomology theory coincides with that defined by numerical equivalence ([Milne 1999b](#), 5.3). In particular, it is independent of the Weil cohomology theory, and so we set

$$\mathcal{D}^*(A) = \mathcal{D}_{\text{hom}}^*(A) = \mathcal{D}_{\text{num}}^*(A)$$

when A is an abelian variety. If $f : A \rightarrow B$ is a morphism of abelian varieties (as varieties) then f^* and f_* preserve Lefschetz classes (ibid. 5.5). This allows us to define $\text{LMot}(k)$ to be the category based on the abelian varieties over k using the Lefschetz classes as correspondences. It is a semisimple tannakian category over \mathbb{Q} through which the Weil cohomologies factor.

1.30. Let H be a Weil cohomology theory with coefficient field Q , and let ω_H be the fibre functor on $\text{LMot}(k)$ it defines. For A an abelian variety, $\langle hA \rangle^{\otimes}$ denotes the tannakian subcategory of $\text{LMot}(k)$ generated by hA . Define the *Lefschetz group* of A by

$$L(A) = \text{Aut}^{\otimes}(\omega_H | \langle hA \rangle^{\otimes}).$$

It is an algebraic group over Q .

1.31. Let $C(A)$ be the centralizer of $\text{End}^0(A)$ in $\text{End}(H^1(A))$. The Rosati involution \dagger of any ample divisor on A preserves $C(A)$, and its action on $C(A)$ is independent of the ample divisor. Now $L(A)$ is the algebraic group over Q such that

$$L(A)(Q) = \{\gamma \in C(A) \mid \gamma^{\dagger} \gamma \in Q^{\times}\}.$$

For a CM abelian variety A , $C(A) = C_0(A) \otimes_{\mathbb{Q}} Q$, where $C_0(A)$ is the centre of $\text{End}^0(A)$. In this case, we let $L(A)$ denote the algebraic group over \mathbb{Q} such that

$$L(A)(\mathbb{Q}) = \{\gamma \in C_0(A) \mid \gamma^{\dagger} \gamma \in \mathbb{Q}^{\times}\}.$$

Recall (1.4) that all abelian varieties A over \mathbb{F} are CM, and that $C_0(A) = \mathbb{Q}\{\pi_A\}$.

1.32. The inclusion $\mathcal{D}^*(A) \rightarrow H^{2*}(A)(*)$ induces an isomorphism

$$\mathcal{D}^*(A) \otimes_{\mathbb{Q}} \mathbb{Q} \rightarrow H^{2*}(A)(*)^{L(A)},$$

i.e., $\mathcal{D}^*(A)$ is a \mathbb{Q} -structure on $H^{2*}(A)(*)^{L(A)}$ (ibid. 4.5, 5.3). An element of $H^{2*}(A)(*)$ is said to be *Lefschetz* if it is in the image of $\mathcal{D}^*(A) \rightarrow H^{2*}(A)(*)$ and *weakly Lefschetz* if it is in the image of $\mathcal{D}^*(A) \otimes_{\mathbb{Q}} \mathbb{Q} \rightarrow H^{2*}(A)(*)$. Thus, an element of $H^{2*}(A)(*)$ is Lefschetz if it is in the \mathbb{Q} -algebra generated by the divisor classes and weakly Lefschetz if it is fixed by $L(A)$.

Similarly, an element of $H_{\mathbb{A}}^{2*}(A)(*)$ is *Lefschetz* (resp. *weakly Lefschetz*) if it is in the image of $\mathcal{D}^*(A) \rightarrow H_{\mathbb{A}}^{2*}(A)(*)$ (resp. $\mathcal{D}^*(A) \otimes_{\mathbb{Q}} \mathbb{A} \rightarrow H_{\mathbb{A}}^{2*}(A)(*)$).

QUESTION 1.33. Let E be a CM field and \mathbb{F} an algebraic closure of \mathbb{F}_p . Does there exist a simple abelian variety A over \mathbb{F} such that $\text{End}^0(A)$ has centre E ?

A positive answer would allow us to describe the fundamental group of $\text{LMot}(\mathbb{F})$.⁹

ASIDE 1.34. Let X be a smooth projective variety of dimension d over an algebraically closed field. A hyperplane section of $X \subset \mathbb{P}^n$ defines an isomorphism

$$L^{d-2r} : H^{2r}(X, \mathbb{Q}_{\ell}(r)) \rightarrow H^{2d-2r}(X, \mathbb{Q}_{\ell}(d-r))$$

for $r \leq d/2$ (strong Lefschetz theorem). In analogy with the standard conjecture of Lefschetz type, one can ask whether

$$L^{d-2r} : \mathcal{D}^r(X) \rightarrow \mathcal{D}^{d-r}(X)$$

is an isomorphism for $r < d/2$. Apart from abelian varieties, for which it is proved in [Milne 1999b](#), this is known for only a few special varieties, for example, toric varieties, full flag varieties, and products of such varieties, and it fails already for blow-ups of such varieties and for partial flag varieties.

NOTES. The original source of the above theory is [Milne 1999b,c](#). For a more recent exposition, see [Kahn 2024](#).

Weil classes

In this subsection, $H^r(A) = H^r(A, \mathbb{C}) \simeq H^r(A, \mathbb{Q}) \otimes \mathbb{C}$.

1.35. ([Deligne 1982](#), §5.) Let A be a complex abelian variety and ν a homomorphism from a CM-algebra E into $\text{End}^0(A)$. If $H^{1,0}(A)$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$ -module, then $\dim_E H^1(A, \mathbb{Q})$ is even, equal to $2d$ say, and the subspace $W_E(A) \stackrel{\text{def}}{=} \bigwedge_E^{2d} H^1(A, \mathbb{Q})$ of $H^{2d}(A, \mathbb{Q})$ consists of Hodge classes ([Deligne 1982](#), 4.4). We say that (A, ν) is of *Weil type*.¹⁰ When

⁹In arXiv:2505.09589, the following weaker statement is proved (Theorem 1.9): Let E be a CM field. There exists a prime number p and a simple abelian variety A over \mathbb{F}_p^{al} such that $\text{End}^0(A)$ has centre E .

¹⁰Tate, letter to Serre 02.02.65. “I wrote Weil about Mumford’s example of an abelian variety of dimension 4 of CM-type with a rational cycle of type 2, 2 which is not generated by divisors. He remarked that it is a special case of a slightly more generic example, namely, a 4-dimensional family of examples, and then said

“As you and Mumford seem to believe Hodge’s conjecture, it is now up to you to exhibit algebraic cycles corresponding to these abnormal classes. As I incline to disbelieve it, I shall rather attempt to show that there is no such cycle; but I have no idea how this could be done....” Cordially, A. Weil.

Well, I don’t think there is much danger that I will be able to find the algebraic cycle!. Let us move on to formal groups.

$[E : \mathbb{Q}] = 2$, these classes were studied by Weil (1977), and for this reason are called *Weil classes*. A polarization of (A, ν) is a polarization λ of A whose Rosati involution stabilizes $\nu(E)$ and acts on it as complex conjugation. We then call (A, ν, λ) a *Weil triple*. The Riemann form of such a polarization can be written

$$(x, y) \mapsto \text{Tr}_{E/\mathbb{Q}}(f\phi(x, y))$$

for some totally imaginary element f of E and E -hermitian form ϕ on $H_1(A, \mathbb{Q})$. If λ can be chosen so that ϕ is split (i.e., admits a totally isotropic subspace of dimension d), then (A, ν, λ) is said to be of *split Weil type*.

EXAMPLE 1.36. Let E be a CM-algebra over \mathbb{Q} , and let $A = B \otimes \mathcal{O}_E$, where B is an abelian variety of dimension d . With the obvious action ν of E , this is of Weil type. The Weil classes on it are algebraic (Deligne 1982, 4.5).

1.37. (Deligne 1982, §5.) Let E be a CM-field, let ϕ_1, \dots, ϕ_{2p} be CM-types on E , and let A_i be a complex abelian variety of CM-type (E, ϕ_i) . If $\sum_i \phi_i(s) = p$ for all $s \in T \stackrel{\text{def}}{=} \text{Hom}(E, \mathbb{Q}^{\text{al}})$, then $A \stackrel{\text{def}}{=} \prod_i A_i$, equipped with the diagonal action of E , is of split Weil type. Let $I = \{1, \dots, 2p\}$. Then A has complex multiplication by the CM-algebra E^I , and $\text{Hom}(E^I, \mathbb{C}) = I \times T$. When tensored with \mathbb{C} , the inclusion $W_E(A) \hookrightarrow H^{2p}(A, \mathbb{Q})$ becomes,

$$\begin{array}{ccc} W_E(A) \otimes \mathbb{C} & \hookrightarrow & H^{2p}(A) \\ \parallel & & \parallel \\ \bigoplus_{t \in T} H^{2p}(A)_{I \times \{t\}} & \hookrightarrow & \bigoplus_{\substack{J \subset I \times T \\ |J|=2p}} H^{2p}(A)_J. \end{array}$$

EXAMPLE 1.38. Let (A, ν, λ) be a Weil triple relative to the CM field Q , and let

$$\begin{aligned} \text{SU}(\phi) &= \{a \in \text{SL}_Q(V(A)) \mid \phi(ax, ay) = \phi(x, y)\} \\ U(\phi) &= \{a \in \text{GL}_Q(V(A)) \mid \phi(ax, ay) = \phi(x, y)\} \\ \text{GU}(\phi) &= \{a \in \text{SL}_Q(V(A)) \mid \phi(ax, ay) = \mu(a)\phi(x, y), \mu(a) \in Q^\times\} \end{aligned}$$

(unitary similitudes). When (A, ν, λ) is general, there is an exact commutative diagram

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{SU}(\phi) & \longrightarrow & \text{GU}(\phi) & \longrightarrow & Q^\times \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \text{MT}(A) & \longrightarrow & L(A) & \longrightarrow & Q^\times \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & & \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

It follows that for a general A ,

- (a) the Weil classes are Hodge classes but not Lefschetz classes;
- (b) if the Weil classes are algebraic, then the Hodge conjecture holds for A and its powers.

THEOREM 1.39 (ANDRÉ 1992). *Let A be a complex abelian variety of CM-type. There exist CM abelian varieties A_Δ of split Weil type and homomorphisms $f_\Delta : A \rightarrow A_\Delta$ such that every Hodge class t on A can be written as a sum $t = \sum f_\Delta^*(t_\Delta)$ with t_Δ a split Weil class on A_Δ .*

PROOF. Let $p \in \mathbb{N}$. After replacing A with an isogenous variety, we may suppose that it is a product of simple abelian varieties A_i (not necessarily distinct). Let $E = \prod_i \text{End}^0(A_i)$. Then E is a CM-algebra, and A is of CM-type (E, ϕ) for some CM-type ϕ on E . Let K be a CM subfield of \mathbb{Q}^{al} , finite and Galois over \mathbb{Q} , splitting the centre of $\text{End}^0(A)$, and let $S = \text{Hom}(E, K)$. Then

$$e \otimes c \leftrightarrow (se \cdot c)_s : E \otimes K \simeq \prod_{s \in S} K_s,$$

where K_s denotes the E -algebra (K, s) . Let $T = \text{Gal}(K/\mathbb{Q})$.

Let $H^i(A) = H^i(A_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} K$. Then

$$H^{2p}(A) = \bigoplus_{\Delta} H^{2p}(A)_{\Delta},$$

where Δ runs over the subsets Δ of S of order $2p$, and

$$\mathcal{B}^p \otimes_{\mathbb{Q}} K = \bigoplus_{\Delta} H^{2p}(A)_{\Delta},$$

where Δ runs over the subsets satisfying

$$|t\Delta \cap \Phi| = p \quad \text{all } t \in \text{Gal}(K/\mathbb{Q}). \quad (2)$$

Let $K_{\Delta} = \prod_{s \in \Delta} K_s$, and let $A_{\Delta} = A \otimes_{\mathbb{Q}} K_{\Delta}$. Then A_{Δ} is of split Weil type relative to the diagonal action of K . Let $f_{\Delta} : A \rightarrow A_{\Delta}$ be the homomorphism such that

$$H_1(f_{\Delta}) : H_1(A, \mathbb{Q}) \rightarrow H_1(A_{\Delta}, \mathbb{Q}) \simeq H_1(A, \mathbb{Q}) \otimes_E K_{\Delta}$$

is the obvious inclusion. There is a diagram

$$\begin{array}{ccc} W_K(A_{\Delta}) \otimes \mathbb{Q}^{\text{al}} & \hookrightarrow & H^{2p}(A_{\Delta}) \\ \parallel & & \parallel \\ \bigoplus_{t \in T} H^{2p}(A_{\Delta})_{\Delta \times \{t\}} & \hookrightarrow & \bigoplus_{\substack{J \subset \Delta \times T \\ |J|=2p}} H^{2p}(A_{\Delta})_J, \end{array}$$

where $H^{2p}(A_{\Delta})_J$ is the 1-dimensional subspace of $H^{2p}(A_{\Delta})$ on which $a \in K_{\Delta}$ acts as $\prod_{j \in J} j(a)$. Note that $a \in E$ acts on $H^{2p}(A_{\Delta})_{\Delta \times \{t\}}$ as multiplication by $\prod_{s \in \Delta} (t \circ s)(a)$, and so $f_{\Delta}^* : H^{2p}(A_{\Delta}) \rightarrow H^{2p}(A)$ maps $H^{2p}(A_{\Delta})_{\Delta \times \{t\}}$ onto $H^{2p}(A)_{t \circ \Delta}$. Therefore,

$$H^{2p}(A)_{\Delta} \subset f_{\Delta}^*(W_K(A_{\Delta})) \otimes_{\mathbb{Q}} K \subset \mathcal{B}^p(A) \otimes_{\mathbb{Q}} K.$$

As the subspaces $H^{2p}(A)_\Delta$ for Δ satisfying (2) span $\mathcal{B}^p(A) \otimes \mathbb{C}$, this shows that

$$\sum_{\Delta \text{ satisfies (2)}} f_\Delta^*(W_K(A_\Delta)) \otimes_{\mathbb{Q}} K = \mathcal{B}^p(A) \otimes_{\mathbb{Q}} K.$$

As $f_\Delta^*(W_K(A_\Delta))$ and $\mathcal{B}^p(A)$ are both \mathbb{Q} -subspaces of $H^{2p}(A, \mathbb{Q})$, it follows (from 1.40) that

$$\sum_{\Delta \text{ satisfies (2)}} f_\Delta^*(W_K(A_\Delta)) = \mathcal{B}^p(A).$$

See [Milne 2020b](#) for more details. □

LEMMA 1.40. *Let W and W' be subspaces of a k -vector space V , and let R be a k -algebra. If $W \otimes_k R \subset W' \otimes_k R$ (inside $V \otimes R$), then $W \subset W'$.*

PROOF. Because k is a field, the lattice of subspaces of V is preserved when we tensor with R . □

Weil families

1.41. Let (A_1, ν_1, λ_1) be a split Weil triple relative to a CM field E , and let $f : A \rightarrow S$ (a polarized abelian scheme over S with an action of E) be the universal Weil family containing (A_1, ν_1, λ_1) ([Deligne 1982](#), 4.8). Here S is a smooth variety over \mathbb{C} and the fibres of f are split Weil triples (including (A_1, ν_1, λ_1)). There is a local subsystem $W_E(A/S)$ of $R^{2d} f_* \mathbb{Q}(d)$ such that $W_E(A/S)_s = W_E(A_s)$ for all $s \in S(\mathbb{C})$. Here $d = \dim(A)/[E : \mathbb{Q}]$.

1.42. Let B be an abelian variety over \mathbb{C} of dimension d . Then $B \otimes \mathcal{O}_E$ with its action of E and a suitable polarization is a Weil triple (A_2, ν_2, λ_2) . Because (A_1, ν_1, λ_1) is split, (A_2, ν_2, λ_2) lies in the universal family containing (A_1, ν_1, λ_1) .

1.43. The variety S has a unique model over \mathbb{Q}^{al} with the property that every CM-point $s \in S(\mathbb{C})$ lies in $S(\mathbb{Q}^{\text{al}})$. This follows from the general theory of Shimura varieties; or from the general theory of locally symmetric varieties (Faltings; [Peters 2017](#)); or (best) from descent theory ([Milne 1999a](#), 2.3) using that S is a moduli variety over \mathbb{C} and that the moduli problem is defined over \mathbb{Q}^{al} . The morphism f is also defined over \mathbb{Q}^{al} , and we will now simply write $f : A \rightarrow S$ for the family over \mathbb{Q}^{al} . There is a \mathbb{Q} -local subsystem $W_E(A/S)$ of $R^{2d} f_* \mathbb{A}(d)$ such that $W_E(A/S)_s = W_E(A_s)$ for all $s \in S(\mathbb{Q}^{\text{al}})$.

1.44. Now assume that (A_1, ν_1, λ_1) and B are defined over \mathbb{Q}^{al} . Let s_1 and s_2 be the points of $S(\mathbb{Q}^{\text{al}})$ corresponding to (A_1, ν_1, λ_1) and (A_2, ν_2, λ_2) . We assume that A_1 and B both have good reduction at w . We also assume that s_1 and s_2 lie in the same connected component of S , and we replace S with that connected component.

1.45. For technical reasons, we now assume that E contains an imaginary quadratic field in which the prime p splits.

1.46. The family $f : A \rightarrow S$ (without the action of E) defines a morphism from S into a moduli variety M over \mathbb{Q}^{al} for polarized abelian varieties with certain level structures. Let \mathcal{M} be the corresponding moduli scheme over \mathcal{O}_w and \mathcal{M}^* its minimal compactification ([Chai and Faltings 1990](#)). Let S^* be the closure of S in \mathcal{M}^* . The complement of $\mathcal{S}_{\mathbb{F}}^* \cap \mathcal{M}_{\mathbb{F}}$ in $\mathcal{S}_{\mathbb{F}}^*$ has codimension at least two ([André 2006b](#), 2.4.2).

1.47. Recall that s_1 and s_2 are points in $S(\mathbb{Q}^{\text{al}})$ such that $A_{s_1} = A_1$ and $A_{s_2} = B \otimes \mathcal{O}_E$. As A_1 and B have good reduction, the points extend to points \mathfrak{s}_1 and \mathfrak{s}_2 of $\mathcal{S}^* \cap \mathcal{M}$. Let $\bar{\mathcal{S}}$ denote the blow-up of \mathcal{S}^* centred at the closed subscheme defined by the image of \mathfrak{s}_1 and \mathfrak{s}_2 , and let \mathcal{S} be the open subscheme obtained by removing the strict transform of the boundary $\mathcal{S}^* \setminus (\mathcal{S}^* \cap \mathcal{M})$. It follows from 1.46 that $\mathcal{S}_{\mathbb{F}}$ is connected, and that any sufficiently general linear section of relative dimension $\dim(S) - 1$ in a projective embedding $\bar{\mathcal{S}} \hookrightarrow \mathbb{P}_{\mathcal{O}_w}^N$ is a projective flat \mathcal{O}_w -curve \mathcal{C} contained in \mathcal{S} with smooth geometrically connected generic fibre (André 2.5.1). Consider $(\mathcal{A}|\mathcal{C})_{\mathbb{F}} \rightarrow \mathcal{C}_{\mathbb{F}}$.

1.48. After replacing $\mathcal{C}_{\mathbb{F}}$ by its normalization C and pulling back $(\mathcal{A}|\mathcal{C})_{\mathbb{F}}$, we are in the following situation:

- ◊ C is a complete smooth curve over \mathbb{F} ;
- ◊ $\bar{f} : \bar{A} \rightarrow C$ is a polarized abelian scheme over C with a compatible action of E ;
- ◊ there is a \mathbb{Q} -local system $W_E(\bar{A}/C)$ over C that is a \mathbb{Q} -structure on $\bigwedge_{E \otimes \mathbb{A}}^{2d} R^1 \bar{f}_*(\mathbb{A})(d)$;
- ◊ there are points $1, 2 \in C(\mathbb{F})$ such that $(\bar{A}, \bar{\lambda}, \bar{\nu})_1 \approx (A_1, \lambda_1, \nu_1)_0$ and $(\bar{A}, \bar{\lambda}, \bar{\nu})_2 \simeq (A_2, \lambda_2, \nu_2)_0$.

The realization categories

We construct categories $R_l(\mathbb{F})$ for $l = 2, 3, 5, \dots, p, \dots, \infty$. Each is a tannkian category over \mathbb{Q}_l with commutative fundamental group P_l . There is a canonical homomorphism $P_l \rightarrow P_{\mathbb{Q}_l}$ which we can use to modify the category so that its fundamental group is $P_{\mathbb{Q}_l}$ — we denote the new category by $V_l(\mathbb{F})$.¹¹ Under the Tate and standard conjectures, there are (realization) functors $\eta_l : \text{Mot}_{\text{num}}(\mathbb{F}) \rightarrow R_l(\mathbb{F})$ on Grothendieck's category of numerical motives inducing equivalences of \mathbb{Q}_l -linear tensor categories

$$\text{Mot}_{\text{num}}(\mathbb{F})_{(\mathbb{Q}_l)} \xrightarrow{\sim} V_l(\mathbb{F}).$$

THE REALIZATION CATEGORY AT $\ell \neq p, \infty$.

1.49. Let $R_{\ell}(\mathbb{F}_{p^m})$ denote the category of finite-dimensional \mathbb{Q}_{ℓ} -vector spaces equipped with a continuous semisimple action of $\Gamma_m \stackrel{\text{def}}{=} \text{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$. It is a tannakian category over \mathbb{Q}_{ℓ} with the forgetful functor ω as fibre functor. The affine group scheme $T_m \stackrel{\text{def}}{=} \text{Aut}^{\otimes}(\omega)$ is the algebraic hull of Γ_m over \mathbb{Q}_{ℓ} , and $R_{\ell}(\mathbb{F}_{p^m}) \simeq \text{Rep}_{\mathbb{Q}_{\ell}}(T_m)$. In particular, T_m is commutative, and it is of multiplicative type because $R_{\ell}(\mathbb{F}_{p^m})$ is semisimple. On extending scalars to $\mathbb{Q}_{\ell}^{\text{al}}$, we see that $\text{Rep}_{\mathbb{Q}_{\ell}}(T_m)_{(\mathbb{Q}_{\ell}^{\text{al}})} = \text{Rep}_{\mathbb{Q}_{\ell}^{\text{al}}}(T_m)$ is the category of continuous semisimple representations of $\text{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$ on finite-dimensional $\mathbb{Q}_{\ell}^{\text{al}}$ -vector spaces. One shows easily that this is the category of diagonalizable representations of $\text{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$ on finite-dimensional $\mathbb{Q}_{\ell}^{\text{al}}$ -vector spaces such that the eigenvalues of the Frobenius element in $\text{Gal}(\mathbb{F}/\mathbb{F}_{p^m})$ are ℓ -adic units. The simple representations are one-dimensional, parametrized by the units in $\mathcal{O}_{\mathbb{Q}_{\ell}^{\text{al}}}$. Therefore

$$X^*(T_m) \simeq \mathcal{O}_{\mathbb{Q}_{\ell}^{\text{al}}}^{\times}.$$

¹¹An object $V_l(\mathbb{F})$ is object of $R_l(\mathbb{F})$ together with an action of $P_{\mathbb{Q}_l}$ compatible with the action of P_l .

The map on characters corresponding to $R_\ell(\mathbb{F}_{p^m}) \rightarrow R_\ell(\mathbb{F}_{p^{m'}})$ is $a \mapsto a^{m'/m}$. Let $T = T_1$. There is an exact sequence

$$1 \rightarrow T^\circ \rightarrow T \rightarrow \Gamma_{\mathbb{Q}_\ell} \rightarrow 1,$$

where $\Gamma_{\mathbb{Q}_\ell}$ is the profinite \mathbb{Q}_ℓ -group defined by $\text{Gal}(\mathbb{F}/\mathbb{F}_p)$.

1.50. Let $R_\ell(\mathbb{F}) = \varinjlim R_\ell(\mathbb{F}_{p^m})$. This is a tannakian category over \mathbb{Q}_ℓ with a canonical \mathbb{Q}_ℓ -valued fibre functor ω , namely, the forgetful functor.

THE CRYSTALLINE REALIZATION CATEGORY.

1.51. When (M, F) is an isocrystal over \mathbb{F}_q , $q = p^n$, we let $\pi_M = F^n$. It is an endomorphism of M regarded as a vector space over the field of fractions of $W(\mathbb{F}_q)$.

1.52. The following conditions on an isocrystal (M, F) over \mathbb{F}_q are equivalent:

- (a) (M, F) is semisimple, i.e., it is a direct sum of simple isocrystals over \mathbb{F}_q ;
- (b) $\text{End}(M, F)$ is semisimple;
- (c) π_M is a semisimple endomorphism of M .

When these conditions hold, the centre of $\text{End}(M, F)$ is $\mathbb{Q}_p[\pi_M]$. See, for example, [Milne 1994a](#), 2.10.

1.53. Let $R_p(\mathbb{F}_q)$ be the category of semisimple F -isocrystals over \mathbb{F}_q . When V is an object of weight 0,

$$V^F \stackrel{\text{def}}{=} \{v \in V \mid Fv = v\}$$

is a \mathbb{Q}_p -structure on V . The functor $V \rightsquigarrow V^F$ is a \mathbb{Q}_p -valued fibre functor on the tannakian subcategory of isocrystals of weight 0

1.54. Let $R_p(\mathbb{F}) = \varinjlim R_p(\mathbb{F}_q)$. Then $R_p(\mathbb{F})$ is a semisimple tannakian category over \mathbb{Q}_p .

CAUTION 1.55. The canonical functor

$$\varinjlim \text{Isoc}(\mathbb{F}_q) \rightarrow \text{Isoc}(\mathbb{F})$$

is faithful and essentially surjective, but not full. For example, if A_1 and A_2 are ordinary elliptic curves over \mathbb{F}_q with different j -invariants, and Λ_1 and Λ_2 are their Dieudonné isocrystals, then

$$\begin{cases} \text{Hom}_{\varinjlim \text{Isoc}(\mathbb{F}_q)}(\Lambda_1, \Lambda_2) = \text{Hom}(A_{1\mathbb{F}}, A_{2\mathbb{F}}) = 0, \text{ but} \\ \text{Hom}_{\text{Isoc}(\mathbb{F})}(\Lambda_1, \Lambda_2) \approx \mathbb{Q}_p \oplus \mathbb{Q}_p. \end{cases}$$

THE REALIZATION CATEGORY AT INFINITY.

1.56. Let R_∞ be the category of pairs (V, F) consisting of a \mathbb{Z} -graded finite-dimensional complex vector space $V = \bigoplus_{m \in \mathbb{Z}} V^m$ and an ι -semilinear endomorphism F such that $F^2 = (-1)^m$ on V^m . With the obvious tensor structure, R_∞ becomes a tannakian category over \mathbb{R} with fundamental group \mathbb{G}_m . The objects fixed by \mathbb{G}_m are those of weight zero. If (V, F) is of weight zero, then

$$V^F \stackrel{\text{def}}{=} \{v \in V \mid Fv = v\}$$

is an \mathbb{R} -structure on V . The functor $V \rightsquigarrow V^F$ is an \mathbb{R} -valued fibre functor on $R_\infty^{\mathbb{G}_m}$.

The local realizations

In this subsection, we construct, for each prime l (including p and ∞), an exact tensor functor ξ_l from $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ to the \mathbb{Q}_l -linear tannakian category $R_l(\mathbb{F})$,

$$\xi_l : \text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow R_l(\mathbb{F}).$$

The main goal of this article is to construct a universal factorization of these functors through a \mathbb{Q} -linear tannakian category,

$$\text{Mot}^w(\mathbb{Q}^{\text{al}}) \xrightarrow[R]{} \text{Mot}(\mathbb{F}) \xrightarrow[\eta_l]{} R_l(\mathbb{F}).$$

ξ_l

Each of the functors ξ_l defines a fibre functor¹²

$$\omega_l : \text{Mot}^w(\mathbb{Q}^{\text{al}})^P \rightarrow \text{Vec}(\mathbb{Q}_l),$$

and, to achieve our goal, we need to define a fibre functor

$$\omega_0 : \text{Mot}^w(\mathbb{Q}^{\text{al}})^P \rightarrow \text{Vec}(\mathbb{Q})$$

that is a \mathbb{Q} -structure on the restricted product of the ω_l (see §5).

THE LOCAL REALIZATION AT ℓ .

1.57. For each $\ell \neq p, \infty$, we let ξ_ℓ and ω_ℓ denote the functors on $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ defined by ℓ -adic étale cohomology.

THE LOCAL REALIZATION AT p .

1.58. The map

$$(A, e, m) \mapsto e \cdot H_{\text{crys}}^*(A_0)(m)$$

extends to an exact tensor functor

$$\xi_p : \text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow R_p(\mathbb{F}).$$

Let x_p denote the homomorphism $\mathbb{G} \rightarrow G_{\mathbb{Q}_p}$ defined by ξ_p . We obtain a \mathbb{Q}_p -valued fibre functor ω_p on $\text{Mot}^w(\mathbb{Q}^{\text{al}})^{\mathbb{G}}$ as follows:

$$\text{Mot}^w(\mathbb{Q}^{\text{al}})^{\mathbb{G}} \xrightarrow[\xi_p]{} R_p^{\mathbb{G}} \xrightarrow[V \rightsquigarrow V^F]{} \text{Vec}(\mathbb{Q}_p).$$

ω_p

¹²See 1.18 for the action of P on $\text{Mot}^w(\mathbb{Q}^{\text{al}})$.

THE LOCAL REALIZATION AT ∞ .

1.59. Let (V, h) be a real Hodge structure, and let C act on V as $h(i)$. Then the square of the operator $v \mapsto C\bar{v}$ acts as $(-1)^m$ on V^m . Therefore, $\mathbb{C} \otimes_{\mathbb{R}} V$ endowed with its weight gradation and this operator is an object of \mathbb{R}_{∞} . We let

$$\xi_{\infty} : \text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \mathbb{R}_{\infty}, \quad X \rightsquigarrow (\omega_B(X)_{\mathbb{R}}, C),$$

denote the functor sending X to the object of \mathbb{R}_{∞} defined by the real Hodge structure $\omega_B(X)_{\mathbb{R}}$. Then ξ_{∞} is an exact tensor functor, and the cocharacter $x_{\infty} : \mathbb{G}_m \rightarrow \mathbb{G}_{\mathbb{R}}$ it defines is equal to $w_{\mathbb{R}}$. We obtain an \mathbb{R} -valued fibre functor ω_{∞} on $\text{Mot}^w(\mathbb{Q}^{\text{al}})^{\mathbb{G}_m}$ as follows:

$$\text{Mot}^w(\mathbb{Q}^{\text{al}})^{\mathbb{G}_m} \begin{array}{c} \xrightarrow{\omega_{\infty}} \\ \xrightarrow{\xi_{\infty}} \mathbb{R}_{\infty}^{\mathbb{G}_m} \xrightarrow{V \rightsquigarrow V^F} \text{Vec}(\mathbb{R}). \end{array}$$

The proper-smooth base change theorem

Let S be a connected normal scheme and $f : X \rightarrow S$ a smooth proper morphism. Then $R^r f_* \mathbb{Q}_{\ell}$ is a locally constant sheaf. Let

$$M = (R^r f_* \mathbb{Q}_{\ell})_{\bar{\eta}} \simeq H^r(X_{\bar{\eta}}, \mathbb{Q}_{\ell}).$$

Then

$$H^0(S, R^r f_* \mathbb{Q}_{\ell}) = M^{\pi_1(S)},$$

and, for any closed point s of S ,

$$(R^r f_* \mathbb{Q}_{\ell})_s = H^r(X_s, \mathbb{Q}_{\ell}) = M^{\pi_1(s)}.$$

The Leray spectral sequence

THEOREM 1.60 (BLANCHARD, DELIGNE). *If $f : X \rightarrow S$ is smooth projective morphism of smooth varieties over \mathbb{C} , then the Leray spectral sequence,*

$$H^p(S, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q}),$$

degenerates at E_2 .

PROOF. The relative Lefschetz operator $L = c_1(\mathcal{L}) \cup \cdot$ acts on the whole spectral sequence, and induces a Lefschetz decomposition

$$R^q f_* \mathbb{Q} = \bigoplus_r L^r (R^{q-2r} f_* \mathbb{Q})_{\text{prim}}.$$

It suffices to prove that $d_2 \alpha = 0$ for $\alpha \in H^p(S, (R^q f_* \mathbb{Q})_{\text{prim}})$. In the diagram,

$$\begin{array}{ccc} H^p(S, (R^q f_* \mathbb{Q})_{\text{prim}}) & \xrightarrow{d_2} & H^{p+2}(S, R^{q-1} f_* \mathbb{Q}) \\ 0 \downarrow L^{n-q+1} & & \simeq \downarrow L^{n-q+1} \\ H^p(S, R^{2n-q+2} \mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(S, R^{2n-q+1} \mathbb{Q}), \end{array}$$

the map at left is zero because L^{n-q+1} is zero on $(R^q f_* \mathbb{Q})_{\text{prim}}$ and the map at right is an isomorphism because $L^{n-q+1} : R^{q-1} f_* \mathbb{Q} \rightarrow R^{2n-q+1} f_* \mathbb{Q}$ is an isomorphism. Hence $d_2 \alpha = 0$. \square

Grothendieck conjectured the degeneration of the Leray spectral sequence by consideration of weights. [Blanchard 1956](#) proved the result when the base is simply connected, and [Deligne 1968](#) proved it in general. See also [Griffiths and Harris 1978](#), p. 466.

Now consider an abelian scheme $f : A \rightarrow S$. For $n \in \mathbb{N}$, let θ_n denote the endomorphism of A/S acting as multiplication by n on the fibres. By a standard argument ([Kleiman 1968](#), p. 374), θ_n^* acts as n^j on $R^j f_* \mathbb{Q}$. As θ_n^* commutes with the differentials d_2 of the Leray spectral sequence $H^i(S, R^j f_* \mathbb{Q}) \Rightarrow H^{i+j}(A, \mathbb{Q})$,

$$\begin{array}{ccc} H^p(S, R^q f_* \mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(S, R^{q-1} f_* \mathbb{Q}) \\ n^q \downarrow \theta_n & & n^{q-1} \downarrow \theta_n \\ H^p(S, R^q f_* \mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(S, R^{q-1} f_* \mathbb{Q}), \end{array}$$

we see that the spectral sequence degenerates at the E_2 -term and

$$H^{2r}(A, \mathbb{Q}) \simeq \bigoplus_{i+j=2r} H^i(S, R^j f_* \mathbb{Q})$$

with $H^i(S, R^j f_* \mathbb{Q})$ the subspace of $H^{2r}(A, \mathbb{Q})$ on which θ_n acts as n^j . As θ_n^* preserves algebraic classes, this induces a decomposition

$$aH^{2r}(A, \mathbb{Q}) \simeq \bigoplus_{i+j=2r} aH^i(S, R^j f_* \mathbb{Q})$$

of the subspaces of algebraic classes.

THEOREM 1.61 ([DELIGNE 1971](#), 4.1.1). *Let $f : X \rightarrow S$ be a smooth proper morphism of smooth varieties over \mathbb{C} .*

(a) *The Leray spectral sequence*

$$H^r(S, R^s f_* \mathbb{Q}) \Rightarrow H^{r+s}(X, \mathbb{Q})$$

degenerates at E_2 ; in particular, the edge morphism

$$H^n(X, \mathbb{Q}) \rightarrow \Gamma(S, R^n f_* \mathbb{Q})$$

is surjective.

(b) *If \bar{X} is a smooth compactification of X with $\bar{X} \setminus X$ a union of smooth divisors with normal crossings, then the canonical morphism*

$$H^n(\bar{X}, \mathbb{Q}) \rightarrow H^0(S, R^n f_* \mathbb{Q})$$

is surjective.

(c) *Let $(R^n f_* \mathbb{Q})^0$ be the largest constant local subsystem of $R^n \pi_* \mathbb{Q}$ (so $(R^n f_* \mathbb{Q})_s^0 = \Gamma(S, R^n f_* \mathbb{Q})$ for all $s \in S(\mathbb{C})$). For each $s \in S$, $(R^n f_* \mathbb{Q})_s^0$ is a Hodge substructure of $(R^n f_* \mathbb{Q})_s = H^n(X_s, \mathbb{Q})$, and the induced Hodge structure on $\Gamma(S, R^n f_* \mathbb{Q})$ is independent of s .*

In particular, the map

$$H^n(\bar{X}, \mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q})$$

has image $(R^n f_ \mathbb{Q})_s^0$, and its kernel is independent of s .*

Part (b) follows from (a) and the theory of weights. There is an ℓ -adic variant of Theorem 1.61.

THEOREM 1.62. *Let S be a smooth connected scheme over an algebraically closed field k , let $f : X \rightarrow S$ be a smooth projective morphism, and let \bar{X} be a smooth projective compactification of X . For all n , the canonical map*

$$H^n(\bar{X}, \mathbb{Q}_\ell) \rightarrow H^0(S, R^n f_* \mathbb{Q}_\ell)$$

is surjective.

PROOF. When k has characteristic zero, this follows from the case $k = \mathbb{C}$. When $k = \mathbb{F}$, the same argument as in the case $k = \mathbb{C}$ applies when one takes weights in the sense of Deligne 1980. Otherwise, in characteristic p , it can be proved by a specialization argument (see André 2006b, 1.1.1). \square

ASIDE 1.63. Let $f : X \rightarrow S$ be a smooth projective morphism of smooth algebraic varieties over \mathbb{C} . Then the Leray spectral sequence degenerates at E_2 , so

$$H^r(X, \mathbb{Q}) \approx \bigoplus_i H^i(S, R^{r-i} f_* \mathbb{Q}).$$

Moreover, $H^r(X, \mathbb{Q})$ is equipped with a mixed Hodge structure. Each summand $H^i(S, R^{r-i} f_* \mathbb{Q})$ is equipped with a pure Hodge structure if S is complete, but not in general otherwise.

2 Conjecture A

In this section, we state the rationality conjecture (Conjecture A), and we suggest possible proofs of it.

Throughout, \mathbb{Q}^{al} is the algebraic closure of \mathbb{Q} in \mathbb{C} , w is a prime of \mathbb{Q}^{al} lying over p , and \mathbb{F} is the residue field at p .

Statement of the conjecture

DEFINITION 2.1. Let X be a smooth projective variety over \mathbb{Q}^{al} with good reduction at w to a variety X_0 over \mathbb{F} . An absolute Hodge class γ on X is w -rational if $\langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q}$ for all Lefschetz classes δ on X_0 of complementary dimension.

Note that algebraic classes are w -rational.

CONJECTURE (A). *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w . All (absolute) Hodge classes on A are w -rational.*¹³

In more detail, a Hodge class on A is an element γ of $H_{\mathbb{A}}^{2*}(A)(*)$, and its specialization γ_0 is an element of $H_{\mathbb{A}}^{2*}(A_0)(*)$. If $\delta_1, \dots, \delta_r$, where $r = \dim(\gamma)$, are divisor classes on A_0 , then

$$\langle \gamma_0 \cdot \delta_1 \cdots \delta_r \rangle \in H_{\mathbb{A}}^{2d}(A_0)(d) \simeq \mathbb{A}_f^P \times \mathbb{Q}_w^{\text{al}}, \quad d = \dim A.$$

The conjecture says that it lies in $\mathbb{Q} \subset \mathbb{A}_f^P \times \mathbb{Q}_w^{\text{al}}$.

¹³The conjecture should also be stated, *mutatis mutandis*, for abelian varieties over \mathbb{Q}^{al} with bad (semistable) reduction.

2.2. If γ is algebraic, then γ_0 is algebraic, and so Conjecture A holds for γ . Thus, Conjecture A holds for A if the Hodge conjecture holds for A , for example, if A has no exotic Hodge classes.

PROPOSITION 2.3. *If $\text{End}(A) \simeq \text{End}(A_0)$, then Conjecture A holds for A and its powers.*

PROOF. The hypothesis implies that $L(A) \simeq L(A_0)$. It follows that, for all n , the specialization map $\mathcal{D}^*(A^n) \rightarrow \mathcal{D}^*(A_0^n)$ becomes an isomorphism when tensored with \mathbb{Q}_ℓ (any $\ell \neq p$),

$$\begin{array}{ccc} \mathcal{D}^*(A^n) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell & \xrightarrow{\simeq} & H^{2*}(A^n, \mathbb{Q}_\ell(*))^{L(A)} \\ \downarrow & & \downarrow \simeq \\ \mathcal{D}^*(A_0^n) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell & \xrightarrow{\simeq} & H^{2*}(A_0^n, \mathbb{Q}_\ell(*))^{L(A_0)}. \end{array}$$

Therefore, it is an isomorphism, i.e., every Lefschetz class δ on A_0^n lifts uniquely to a Lefschetz class δ' on A^n . If γ is a Hodge class on A^n and δ is a Lefschetz class on A_0 of complementary dimension, then

$$\langle \gamma_0 \cup \delta \rangle = \langle \gamma \cup \delta' \rangle \in \mathbb{Q}. \quad \square$$

The hypothesis in 2.3 holds if A is CM and A_0 is a product of simple ordinary abelian varieties, no two of which are isogenous.

PROPOSITION 2.4. *Let X and Y be smooth projective varieties over \mathbb{Q}^{al} with good reduction at w , and let $f : X \rightarrow Y$ be a morphism. If γ is w -rational on X , then $f_*\gamma$ is w -rational on Y .*

PROOF. Let δ be a Lefschetz class on Y of complementary dimension to $f_*\gamma$. Then

$$\langle (f_*\gamma)_0 \cdot \delta \rangle = \langle f_{0*}\gamma_0 \cdot \delta \rangle = \langle f_{0*}(\gamma_0 \cdot f_0^*\delta) \rangle = \langle \gamma_0 \cdot f_0^*\delta \rangle \in \mathbb{Q}$$

because $f_0^*\delta$ is Lefschetz. □

PROPOSITION 2.5. *Let A and B be abelian varieties over \mathbb{Q}^{al} with good reduction at w , and let $f : A \rightarrow B$ be a morphism. If γ is w -rational on B , then $f^*\gamma$ is w -rational on A .*

PROOF. Let δ be a Lefschetz class on A_0 of complementary dimension to $f^*\gamma$. Then

$$\langle (f^*\gamma)_0 \cdot \delta \rangle = \langle f_0^*\gamma_0 \cdot \delta \rangle = \langle f_{0*}(f_0^*\gamma_0 \cdot \delta) \rangle = \langle \gamma_0 \cdot f_{0*}\delta \rangle \in \mathbb{Q}$$

because $f_{0*}\delta$ is Lefschetz (Milne 1999c, 5.5). □

Let X be a smooth projective variety of dimension n and L a Lefschetz operator. The following is one form of the Lefschetz standard conjecture:

$A(X, L)$: The map $L^{n-2r} : A^r(X) \rightarrow A^{n-r}(X)$ is an isomorphism for all $r \leq n/2$.

Equivalently, a cohomology class γ lies in $A^r(X)$ if $L^{n-2r}\gamma$ lies in $A^{n-r}(X)$.

PROPOSITION 2.6. *Assume that X has good reduction at w . If $A(X_0, L_0)$ holds for Lefschetz classes on X_0 , then $A(X, L)$ holds for w -rational classes.*

PROOF. To see this, let γ be a Hodge class on X of codimension r such that $L^{n-2r}\gamma$ is w -rational, and let δ be a Lefschetz class on X_0 of codimension $n - r$. Then $\delta = L_0^{n-2r}\delta'$ for some Lefschetz class δ' of codimension r on X_0 , and

$$\langle \gamma_0 \cdot \delta \rangle = \langle \gamma \cdot L_0^{n-2r} \delta' \rangle = \langle (L^{n-2r}\gamma)_0 \cdot \delta' \rangle \in \mathbb{Q}.$$

In particular, we see that $A(X, L)$ holds for w -rational classes when X is an abelian variety. Therefore, $*$ preserves w -Lefschetz classes on an abelian variety, the Hodge standard conjecture holds, and conjecture $D(X)$ holds. \square

COROLLARY 2.7. *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w . The Lefschetz standard conjecture holds for w -rational classes on A .*

PROOF. The Lefschetz standard conjecture holds for Lefschetz classes on abelian varieties. \square

Nifty abelian varieties

We let $\text{MT}(A)$ denote the Mumford–Tate group of an abelian variety, and $\text{SMT}(A)$ its special Mumford–Tate group.

DEFINITION 2.8. Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w . We say that A is *nifty* if $\text{MT}(A) \cdot L(A_0) = L(A)$; equivalently, $\text{SMT}(A) \cdot S(A_0) = S(A)$.

As $\text{SMT}(A) = \text{SMT}(A^r)$, $S(A_0) = S(A_0^r)$, and $S(A) = S(A^r)$ for all $r \geq 1$, A^r is nifty if A is.

EXAMPLE 2.9. An abelian variety A is nifty if $\text{MT}(A) = L(A)$, i.e., if all Hodge classes on A and its powers are Lefschetz. There is a large literature listing abelian varieties satisfying this condition.

EXAMPLE 2.10. If $\text{End}^0(A) = \text{End}^0(A_0)$, then A is nifty. This only happens when A is CM.

PROPOSITION 2.11. *Nifty abelian varieties satisfy Conjecture A.*

PROOF. Let A be a nifty abelian variety over \mathbb{Q}^{al} , and let $\langle A \rangle_H$ and $\langle A \rangle_L$ denote the tannakian subcategories of $\text{Mot}(\mathbb{Q}^{\text{al}})$ and $\text{LMot}(\mathbb{Q}^{\text{al}})$ generated by A . To simplify, we assume that A is CM (so that the groups involved are commutative). Let P denote the kernel of the homomorphism $\text{MT}(A) \rightarrow L(A)/L(A_0)$, and consider the diagrams

$$\begin{array}{ccc} \langle A \rangle_H^P \longleftarrow \langle A \rangle_L^{L(A_0)} & & \text{MT}(A)/P \longrightarrow L(A)/L(A_0) \\ \downarrow & & \downarrow \\ \langle A \rangle_H \longleftarrow \langle A \rangle_L & & \text{MT}(A) \longleftarrow L(A) \\ \vdots & & \updownarrow \\ \langle A_0 \rangle \longleftarrow \langle A_0 \rangle_L & & P \longleftarrow L(A_0) \end{array}$$

of tannakian categories and their fundamental groups. The category $\langle A \rangle_L$ is a quotient of $\langle A \rangle_L$, and we let ω denote the fibre functor on $\langle A \rangle_L^{L(A_0)}$ corresponding to it. Because the

homomorphism $\mathrm{MT}(A)/P \rightarrow L(A)/L(A_0)$ is an isomorphism, the functor $\langle A \rangle_L^{L(A_0)} \rightarrow \langle A \rangle_H^P$ is an equivalence of tensor categories, and so ω defines a fibre functor on $\langle A \rangle_H^P$, which we again denote by ω . Define $\langle A_0 \rangle$ to be the quotient of $\langle A \rangle_H/\omega$ of $\langle A \rangle_H$.

Let γ be a Hodge class on A and δ a Lefschetz class on A_0 of complementary dimension. Then it is obvious from the diagram that $(\gamma_{\mathbb{A}})_0 \cdot \delta_{\mathbb{A}} \in \mathbb{Q}$ because the intersection takes place inside the \mathbb{Q} -algebra

$$\mathrm{Hom}(\mathbf{1}, hA_0),$$

where hA_0 is the object of $\langle A_0 \rangle \stackrel{\mathrm{def}}{=} \langle A \rangle_H/\omega$ defined by A_0 . \square

Weil families

DEFINITION 2.12. Let (A, ν, λ) be a Weil triple over \mathbb{Q}^{al} .

- (a) We say (A, λ, ν) has *good reduction* at w if A has good reduction at w (then λ specializes to a polarization on λ_0).
- (b) We say that (A, λ, ν) is CM if A is CM.

DEFINITION 2.13. Let (A_1, λ_1, ν_1) and (A_2, λ_2, ν_2) be two Weil triples over $k \subset \mathbb{C}$ relative to the CM-algebra E . We say that (A_1, λ_1, ν_1) and (A_2, λ_2, ν_2) *lie in the same Weil family* if there exists an E -linear isomorphism

$$H_1(A_1, \mathbb{Q}) \rightarrow H_1(A_2, \mathbb{Q})$$

under which the Riemann forms of λ_1 and λ_2 correspond up to an element of \mathbb{Q}^\times .

In fact, they do then lie in the same Weil family (see [Deligne 1982](#), proof of Theorem 4.8).

LEMMA 2.14. *Let (A_1, λ_1, ν_1) and (A_2, λ_2, ν_2) be Weil triples over \mathbb{Q}^{al} having good reduction to isogenous triples over \mathbb{F} . If (A_1, λ_1, ν_1) and (A_2, λ_2, ν_2) lie in the same Weil family, then the E -vector spaces of Weil classes on A and A' specialize to the same E -subspace of*

$$H_{\mathbb{A}}^{2d}((A_1)_0)(d) \simeq H_{\mathbb{A}}^{2d}((A_2)_0)(d).$$

PROOF. Let $f : A \rightarrow S$ be the universal family over \mathbb{C} containing (A_1, λ_1, ν_1) and (A_2, λ_2, ν_2) (see [Deligne 1982](#), 4.8). Then f has a unique model over \mathbb{Q}^{al} , also denoted $f : A \rightarrow S$, such that the special points of $S(\mathbb{C})$ lie in $S(\mathbb{Q}^{\mathrm{al}})$. The Weil classes form a local \mathbb{Q} -subsystem $W_E(A/S)$ of $R^{2d}f_*\mathbb{A}(d)$, where $d = \dim A/[E : \mathbb{Q}]$. By assumption, the points in $S(\mathbb{Q}^{\mathrm{al}})$ corresponding to (A_1, λ_1, ν_1) and (A_2, λ_2, ν_2) specialize to the same point in $S(\mathbb{F})$ (or a translate), from which the statement follows. \square

An inductive approach to proving Conjecture A

We suggest an approach to proving Conjecture A for CM abelian varieties by induction on the codimension of γ .

PROPOSITION 2.15. *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w , and let $r \in \mathbb{N}$. If all Hodge classes of codimension r on A are w -rational, then the same is true of the Hodge classes of dimension r on A .*

PROOF. Let $g = \dim A$. Let γ be a Hodge class of dimension r on A , and let δ be a Lefschetz class on A_0 of codimension r . Let λ be a polarization of A , and let ξ be the corresponding ample divisor.

Suppose first that $2r \leq g$. In this case, there is an isomorphism

$$\xi^{g-2r} : H_{\mathbb{A}}^{2r}(A)(r) \rightarrow H_{\mathbb{A}}^{2g-2r}(A)(g-r).$$

As the Lefschetz standard conjecture holds for Hodge classes, we can write $\gamma = \xi^{g-2r} \cdot \gamma'$, where γ' is a Hodge class of codimension r on A . Now

$$\langle \gamma_0 \cdot \delta \rangle = \langle (\xi^{g-2r} \cdot \gamma')_0 \cdot \delta \rangle = \langle \gamma'_0 \cdot (\xi_0^{g-2r} \cdot \delta) \rangle,$$

which lies in \mathbb{Q} by the hypothesis.

When $2r > g$, we can replace the Lefschetz operator $L : x \mapsto \xi \cdot x$ in the argument with its quasi-inverse Λ (Kleiman 1994, §4). As Λ is a Lefschetz class (Milne 1999b, 5.9), the same argument applies. \square

DEFINITION 2.16. Let (A, ν, λ) be a CM Weil triple over \mathbb{Q}^{al} . We say that a divisor d on A_0 is *liftable* if there exists a CM Weil triple (A_1, ν_1, λ_1) in the same Weil family as (A, ν, λ) and a divisor d_1 on A_1 such that $(A_1, \nu_1, \lambda_1, d_1)_0$ is isogenous to $(A_0, \nu_0, \lambda_0, d)$. We say that a Lefschetz class δ of codimension r on A_0 is *weakly liftable* if it is in the \mathbb{Q} span of the classes $d_1 \cdots d_r$ with at least one of the d_i liftable.

QUESTION 2.17. Let (A, ν, λ) be a CM Weil triple over \mathbb{Q}^{al} . Is every Lefschetz class of codimension $\dim(A)/[E : \mathbb{Q}]$ on A_0 weakly liftable?

THEOREM 2.18. *If Question 2.17 has a positive answer, then Conjecture A is true for all CM abelian varieties.*

PROOF. We have to prove that every Hodge class γ on a CM abelian variety A is w -rational. Hodge classes of codimension ≤ 1 are w -rational because they are algebraic. We prove the theorem by induction on the codimension of γ . Let $\text{codim}(\gamma) = r > 1$, and assume that every Hodge class of $\text{codim} < r$ on a CM abelian variety is w -rational.

After Theorem 1.39 and 2.5, we may suppose that γ is a Weil class on a CM Weil triple (A, ν, λ) , still of codimension r , so $r = \dim A/[E : \mathbb{Q}]$.

Let $g = \dim A$ and let δ be a Lefschetz class of codimension $g - r$ on A_0 . We have to show that $\langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q}$.

Let ξ be the divisor class on A attached to λ . The isomorphism (strong Lefschetz)

$$\xi_0^{g-2r} : H_{\mathbb{A}}^{2r}(A_0)(r) \rightarrow H_{\mathbb{A}}^{2g-2r}(A_0)(g-r)$$

induces an isomorphism

$$\mathcal{D}^r(A_0) \rightarrow \mathcal{D}^{g-r}(A_0)$$

on Lefschetz classes (Milne 1999b, 5.9). Therefore,

$$\delta = \xi_0^{g-2r} \cdot \delta'$$

with δ' a Lefschetz class of codimension r on A_0 .

Since we are assuming that Question 2.17 has a positive answer, we may suppose that $\delta' = d \cdot \delta''$ with d a liftable divisor on A_0 . This means that there exists a CM Weil

triple (A_1, ν_1, λ_1) in the same Weil family as (A, ν, λ) and a divisor class d_1 on A_1 such that $(A_0, \nu_0, \lambda_0, d)$ is isogenous to $(A_1, \nu_1, \lambda_1, d_1)_0$. According to 2.14, there exists a Weil class γ_1 on A_1 such that $\gamma_0 = (\gamma_1)_0$. Now

$$\langle \gamma_0 \cdot \delta \rangle = \langle \gamma_0 \cdot \xi_0^{g-2r} \cdot \delta' \rangle = \langle \gamma_0 \cdot \xi_0^{g-2r} \cdot d \cdot \delta'' \rangle = \langle (\gamma_1 \cdot \xi_1^{g-2r} \cdot d_1)_0 \cdot \delta'' \rangle.$$

This lies in \mathbb{Q} because $\gamma_1 \cdot \xi_1^{g-2r} \cdot d_1$ is a Hodge class of codimension

$$r + g - 2r + 1 = g - (r - 1)$$

on A , hence of dimension $r - 1$, and so we can apply Proposition 2.15 and the induction hypothesis. \square

2.19. If Conjecture A holds for CM abelian varieties over \mathbb{Q}^{al} , then Conjecture C holds for the same class. Once the arguments in §5 have been completed, this will imply that Conjecture C holds for all abelian varieties over \mathbb{C} with good reduction at w .

Thus, an affirmative answer to Question 2.17 would allow us to extend Deligne's theory of absolute Hodge classes on abelian varieties to characteristic p . As Tate once wrote in a similar context,¹⁴ "we have a completely down-to-earth question which could be explained to a bright freshman and which should be settled one way or the other."

A VARIANT INDUCTIVE ARGUMENT

2.20. Of course, there are variations of the above argument, for example, by dropping CM in the definition of liftable.

NOTES

2.21. Let (A, λ) be a polarized abelian variety over a field k . Then

$$\text{NS}^0(A) \simeq \{\alpha \in \text{End}^0(A) \mid \alpha^\dagger = \alpha\}$$

(Mumford 1970, p. 208). Thus, Questions 2.17 can be restated in terms of symmetric endomorphisms, or even in terms of the subalgebras they generate. Not all subfields of endomorphism algebras of abelian varieties can be lifted to characteristic zero. For example, a subfield E of $\text{End}^0(A)$ such that $[E : \mathbb{Q}] = 2 \dim A$ must be CM in characteristic zero, but need not be so in characteristic p . In particular, a real quadratic subfield of the endomorphism algebra of an elliptic curve does not lift to characteristic zero. However, with some obvious restrictions, every endomorphism lifts (up to isogeny). See, for example, Zink 1983, 2.7.

2.22. It may be possible to use the natural Jordan algebra structure on $\text{NS}^0(A)$ (Mumford 1970, p. 208).

2.23. Let A be an abelian variety over \mathbb{F} . If A_1 is a model of A over a finite subfield k of \mathbb{F} such that $\text{End}_k^0(A_1) = \text{End}_{\mathbb{F}}^0(A)$, then we let $\mathbb{Q}\{\pi\}$ denote the \mathbb{Q} -subalgebra of $\text{End}_{\mathbb{F}}^0(A)$ generated by the Frobenius endomorphism of A_1 (relative to k). It is independent of the choice of the model.

¹⁴Tate 1965, p. 107. Tate's question was answered (negatively) by Mumford.

THEOREM 2.24. *Let A_0 be a simple abelian variety over \mathbb{F} , and let L be a CM subfield of $\text{End}^0(A)$ such that*

- (a) L contains $\mathbb{Q}\{\pi\}$,
- (b) L splits $\text{End}^0(A)$, and
- (c) $[L : \mathbb{Q}] = 2 \dim A$.

Then, up to isogeny, A_0 lifts to an abelian variety A in characteristic zero such that $L \subset \text{End}^0(A)$.

PROOF. See [Tate 1968](#), Thm 2. □

THEOREM 2.25 (?). *Let (A, ν, λ) be a Weil triple over \mathbb{Q}^{al} with respect to E . Assume that the degree of λ is prime to p and that p is unramified in E . Let $R \subset \text{End}_E^0(A)$ be a product of CM fields respecting the polarization and of degree $2 \dim A$ over \mathbb{Q} . There exists a Weil triple (A', ν', λ') over \mathbb{Q}^{al} in the same family as (A, ν, λ) equipped with an action of R such that $(A', \nu', \lambda', R)_0$ is isogenous to $(A, \nu, \lambda, R)_0$.*

PROOF. Compare [Zink 1983](#), especially Theorem 2.7. □

I expect that it will be possible to use [Theorem 2.18](#) to prove Conjecture A for abelian varieties with CM by a field E unramified over p . The general case is less certain.

A variational approach to proving Conjecture A for Weil classes

As explained in the introduction, the key to proving the conjectures is to prove that split Weil classes are w -rational. Not having a proof, here we simply make some remarks.

Let (A_1, ν_1, λ_1) be a split Weil triple over \mathbb{Q}^{al} (relative to a CM algebra E) with good reduction at w .

2.26. If (A_1, ν_1, λ_1) is of the form $(B \otimes \mathcal{O}_E, \dots)$, then the Weil classes on A_1 are w -rational because they are algebraic ([Deligne 1982](#), 4.5).

2.27. Consider the Weil family $f : A \rightarrow S$ over \mathbb{Q}^{al} containing (A_1, ν_1, λ_1) (see [1.43](#)). If $(A_1, \nu_1, \lambda_1)_0$ lifts (up to isogeny) to a triple in the family whose Weil classes are w -rational, then the Weil classes on A_1 are w -rational (obviously).

2.28. Let δ_1 be a Lefschetz class on $(A_1)_0$. If B and the family $\bar{f} : \bar{S} \rightarrow C$ in [1.48](#) can be chosen so that δ_1 extends to a section δ of $R^{2d'} \bar{f}_* \mathbb{A}(d')$ satisfying the hypothesis of [3.29](#) (or [3.30](#)), then, for all global sections γ of $W_E(A/S)$,

$$\langle (\gamma_1)_0 \cdot \delta_1 \rangle = \langle (\gamma_2)_0 \cdot \delta_2 \rangle,$$

which lies in \mathbb{Q} because γ_2 is algebraic ([2.26](#)) and δ_2 is Lefschetz ([3.29](#)).

3 Conjecture B

Every CM abelian variety A over \mathbb{Q}^{al} has good reduction at w to an abelian variety A_0 over \mathbb{F} . The Hodge classes on A define a \mathbb{Q} -structure on the part of $H_{\mathbb{A}}^{2*}(A)(*)$ fixed by the Mumford–Tate group of A , and the Lefschetz classes on A_0 define a \mathbb{Q} -structure on the part of $H_{\mathbb{A}}^{2*}(A_0)(*)$ fixed by the Lefschetz group of A_0 . The weak rationality conjecture says that the two structures are compatible.

In this section, we state the weak rationality conjecture (Conjecture B), which is a variant of Conjecture A, and we suggest an variational proof of it for CM abelian varieties.

Statement of the conjecture

Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w . Let γ be a Hodge class in $H_{\mathbb{A}}^{2*}(A)(*)$ and γ_0 its image in $H_{\mathbb{A}}^{2*}(A_0)(*)$.

DEFINITION 3.1. We say that γ is *w-Lefschetz* if γ_0 is Lefschetz and *weakly w-Lefschetz* if γ_0 is weakly Lefschetz.

Thus γ is *w-Lefschetz* (resp. *weakly w-Lefschetz*) if γ_0 is in the \mathbb{Q} -algebra (resp. \mathbb{A} -algebra) generated by the divisor classes.

CONJECTURE (B). *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w , and let γ be a Hodge class on A . If γ is weakly w-Lefschetz, then it is w-Lefschetz.*

PROPOSITION 3.2. *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w , and let γ be a Hodge class on A . If γ is w-rational (e.g., algebraic) and weakly w-Lefschetz, then it is w-Lefschetz.*

PROOF. Let γ be a w-rational Hodge class of codimension r on A . Choose a \mathbb{Q} -basis e_1, \dots, e_t for the space of Lefschetz classes of codimension r on A_0 , and let f_1, \dots, f_t be the dual basis for the space of Lefschetz classes of complementary dimension (here we use 1.29). If γ is weakly w-Lefschetz, then $\gamma_0 = \sum c_i e_i$ with $c_i \in \mathbb{A}$. Now

$$\langle \gamma_0 \cup f_j \rangle = c_j,$$

and c_j lies in \mathbb{Q} because γ is w-rational. □

COROLLARY 3.3. *If Conjecture A holds for A , then so does Conjecture B.*

PROOF. Conjecture A says that all the Hodge classes on A are w-rational □

If Conjecture B holds for all CM abelian varieties over \mathbb{Q}^{al} , then so does Conjecture A (see §4).

Homomorphisms in families

In this subsection, we study how homomorphisms of abelian varieties behave in families.

We shall need one trivial lemma and two theorems.

LEMMA 3.4. *Let Q be a field and R a Q -algebra. Consider a commutative diagram of linear maps*

$$\begin{array}{ccc} W & \longrightarrow & W' \\ \downarrow a & & \downarrow b \\ V & \longrightarrow & V' \end{array} \quad \begin{array}{l} W, V \text{ } \mathbb{Q}\text{-vector spaces} \\ W', V' \text{ } R\text{-modules,} \end{array}$$

If either a or b is injective and the horizontal arrows are such that

$$W \otimes_Q R \xrightarrow{\cong} W', \quad V \otimes_Q R \hookrightarrow V', \quad (**)$$

then both a and b are injective, and

$$W = V \cap W' \quad (\text{intersection in } V').$$

PROOF. If b is injective, then a is injective because $W \rightarrow W' \xrightarrow{b} V'$ is injective. Consider the diagram

$$\begin{array}{ccccc} W & \longrightarrow & W \otimes_Q R & \xrightarrow{\simeq} & W' \\ \downarrow a & & \downarrow a \otimes 1 & & \downarrow b \\ V & \longrightarrow & V \otimes_Q R & \hookrightarrow & V' \end{array}$$

If a is injective, then b is injective because $W \otimes R \xrightarrow{a \otimes 1} V \otimes R \rightarrow V'$ is injective and $W \otimes R \rightarrow W'$ is surjective. For the remaining statement, we may replace W' and V' with $W \otimes R$ and $V \otimes R$. Let $V = W \oplus U$, and let $v = w + u \in V$. Then

$$v \otimes 1 = w \otimes 1 + u \otimes 1 \in (W \otimes R) \oplus (U \otimes R).$$

If $v \otimes 1 \in W \otimes R$, then $u = 0$ and so $v = w \in W$. □

THEOREM 3.5. *Let A and B be abelian schemes over a connected noetherian normal scheme S . Every homomorphism $A_\eta \rightarrow B_\eta$ of the generic fibres extends uniquely to a homomorphism $A \rightarrow B$ over S .*

PROOF. When $\dim(S) = 1$, B is the Néron model of B_η , so this follows from the universal property of such models. For a proof that the general case follows from this case, see [Chai and Faltings 1990](#), I, Proposition 2.7. □

THEOREM 3.6 (TATE, DE JONG). *Let G and H be p -divisible groups over a connected noetherian normal scheme S . Every homomorphism $G_\eta \rightarrow H_\eta$ of the generic fibres extends uniquely to a homomorphism $G \rightarrow H$ over S .*

PROOF. Let $\eta = \text{Spec } K$. When K has characteristic zero, this is Theorem 4 of [Tate 1967](#), and when K has characteristic $p \neq 0$, it is Theorem 2 of [de Jong 1998](#). □

For an abelian scheme A over a scheme S and integer $n > 0$, we let

$$A_n = \text{Ker}(n : A \rightarrow A).$$

This is a finite flat group scheme over S , and we let TA denote the projective system $(A_n)_n$. Then $A \rightsquigarrow TA$ is a faithful functor, compatible with base change.

THEOREM 3.7. *Let A and B be abelian schemes over a connected normal scheme S of finite type over a field k , and let $u : TA \rightarrow TB$ be a homomorphism. If there exists a closed point $s \in S$ such that $u_s : TA_s \rightarrow TB_s$ equals Tw for some $w : A_s \rightarrow B_s$, then there exists an integer $n > 0$ and a homomorphism $v : A \rightarrow B$ such that $Tv = nu$.*

PROOF. Grothendieck (1966, p. 60) states this as a conjecture, but remarks that it is a consequence of the Tate conjecture. We explain how. In proving the theorem, we may

suppose that the field k is a finitely generated field (ibid. 2.2).¹⁵ Consider the diagram

$$\begin{array}{ccc}
 \mathrm{Hom}(A_\eta, B_\eta) & \longrightarrow & \mathrm{Hom}(TA_\eta, TB_\eta) \\
 \uparrow \simeq & & \uparrow \simeq \\
 \mathrm{Hom}(A, B) & \longrightarrow & \mathrm{Hom}(TA, TB) \\
 \downarrow a & & \downarrow b \\
 \mathrm{Hom}(A_s, B_s) & \longrightarrow & \mathrm{Hom}(TA_s, TB_s).
 \end{array}$$

The restriction maps

$$\begin{array}{ccc}
 \mathrm{Hom}(A, B) & \rightarrow & \mathrm{Hom}(A_\eta, B_\eta) \\
 \mathrm{Hom}(TA, TB) & \rightarrow & \mathrm{Hom}(TA_\eta, TB_\eta)
 \end{array}$$

are bijective by Theorems 3.5 and 3.6. The top and bottom horizontal maps induce isomorphisms

$$\mathrm{Hom}^0(A_\eta, B_\eta) \otimes_{\mathbb{Q}} \mathbb{A} \rightarrow \mathrm{Hom}(TA_\eta, TB_\eta)_{\mathbb{Q}} \quad (3)$$

$$\mathrm{Hom}^0(A_s, B_s) \otimes_{\mathbb{Q}} \mathbb{A} \rightarrow \mathrm{Hom}(TA_s, TB_s)_{\mathbb{Q}}. \quad (4)$$

by the Tate conjecture (proved in this case by Tate, Zarhin, and Faltings). The map a is injective because $\mathrm{Hom}(T_l A, T_l B) \rightarrow \mathrm{Hom}(T_l A_s, T_l B_s)$ is obviously injective. On applying Lemma 3.4 to the bottom square, we find that

$$\mathrm{Hom}^0(A, B) = \mathrm{Hom}^0(A_s, B_s) \cap \mathrm{Hom}(TA, TB)_{\mathbb{Q}}$$

(intersection inside $\mathrm{Hom}(TA_s, TB_s)_{\mathbb{Q}}$) as required. \square

COROLLARY 3.8. *With the notation of the theorem,*

$$\mathrm{Hom}^0(A, B) = \mathrm{Hom}^0(A_s, B_s) \cap \mathrm{Hom}(TA_\eta, TB_\eta) \otimes \mathbb{Q}$$

(intersection inside $\mathrm{Hom}(TA_s, TB_s) \otimes \mathbb{Q}$).

REMARK 3.9. A dévissage (Grothendieck 1966, 1.2) shows that the Theorem 3.7 is true over any reduced connected scheme S , locally of finite type over $\mathrm{Spec}(\mathbb{Z})$ or a field.

Divisor classes in families

Let A be an abelian variety. We say that an element of $H_{\mathbb{A}}^{2n}(A)(n)$ is *algebraic* if it is in the \mathbb{Q} -span of the classes of algebraic cycles, i.e., if it is in the image of the cycle class map $\mathrm{CH}^n(A) \otimes \mathbb{Q} \rightarrow H_{\mathbb{A}}^{2n}(A)(n)$.

THEOREM 3.10. *Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme S of finite type over a field k , and let γ be a global section of $R^2 f_* \mathbb{A}(1)$. If $\gamma_s \in H^2(A_s, \mathbb{A}(1))$ is algebraic for one closed $s \in S$, then it is algebraic for all closed $s \in S$.*

¹⁵Alternatively, replace $\mathrm{Hom}(TA, TB)$ etc. with $\varinjlim \mathrm{Hom}(TA', TB')$, where the limit runs over the models of $A \rightarrow S, B \rightarrow S$ over finitely generated subfields of k .

As Grothendieck (1966, p. 66) notes, because of the correspondence between endomorphisms of abelian varieties and divisor classes, this is essentially equivalent to Theorem 3.7.

We explain how to prove Theorem 3.10. Recall that $M \xrightarrow{\otimes} N$ means that M is a \mathbb{Q} -structure on N , i.e., that $M \otimes_{\mathbb{Q}} \mathbb{A} \simeq N$.

For an abelian scheme A over S , we let

$$\mathrm{Pic}(A/S) \stackrel{\mathrm{def}}{=} \mathrm{Pic}_{A/S}(S) = \frac{\mathrm{Pic}(A)}{\mathrm{Pic}(S)}.$$

Recall that, for abelian schemes A and B over a scheme S ,

$$\mathrm{DC}_S(A, B) \stackrel{\mathrm{def}}{=} \frac{\mathrm{Pic}(A \times_S B/S)}{\mathrm{pr}_1^* \mathrm{Pic}(A/S) + \mathrm{pr}_2^* \mathrm{Pic}(B/S)}$$

$$\mathrm{DC}_S(A, B) \simeq \mathrm{Hom}_S(A, B^\vee),$$

and that the map $\mu^* - \mathrm{pr}_1^* - \mathrm{pr}_2^* : \mathrm{Pic}(A/S) \rightarrow \mathrm{Pic}(A \times_S A/S)$ factors through an injection

$$\mathrm{NS}(A/S) \hookrightarrow \mathrm{Pic}(A \times_S A/S).$$

Consider the diagram

$$\begin{array}{ccc} \mathrm{NS}(A_\eta) & \xleftarrow[\Delta^*]{\mu^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*} & \mathrm{DC}(A_\eta, A_\eta) \simeq \mathrm{Hom}(A_\eta, A_\eta^\vee) \\ \uparrow & & \uparrow \simeq \\ \mathrm{NS}(A/S) & \xleftarrow[\Delta^*]{\mu^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*} & \mathrm{DC}_S(A, A) \simeq \mathrm{Hom}_S(A, A^\vee) \\ \downarrow & & \downarrow \\ \mathrm{NS}(A_S) & \xleftarrow[\Delta^*]{\mu^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*} & \mathrm{DC}(A_S, A_S) \simeq \mathrm{Hom}(A_S, A_S^\vee). \end{array}$$

The composite of each map $\mu^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*$ with Δ^* is multiplication by 2. Therefore, after tensoring with \mathbb{Q} , we get the left hand side of the following diagram,

$$\begin{array}{ccccc} \mathrm{NS}^0(A_\eta) & \xrightarrow{\otimes} & H_{\mathbb{A}}^2(A_{\bar{\eta}})(1)^{\pi_1(\eta)} & & \\ \simeq \uparrow & & \uparrow e & & \\ \mathrm{NS}^0(A/S) & \xrightarrow{d} & H^0(S, R^2 f_* \mathbb{A}(1)) & \xrightarrow{\simeq} & H_{\mathbb{A}}^2(A_{\bar{\eta}})(1)^{\pi_1(S)} \\ \downarrow & & \downarrow & & \\ \mathrm{NS}^0(A_S) & \xrightarrow{\otimes} & H_{\mathbb{A}}^2(A_{\bar{S}})(1)^{\pi_1(S)}. & & \end{array}$$

As in the previous case, in proving the theorem, we may suppose that the field k is finitely generated, which allows us to apply the Tate conjecture (known in this case) to the top and bottom rows of the diagram. From the diagram, we see that d is injective. The map e is injective, and it follows from the diagram that it is an isomorphism. Hence the map d induces an isomorphism

$$\mathrm{NS}^0(A/S) \otimes \mathbb{A} \longrightarrow H^0(S, R^2 f_* \mathbb{A}(1)).$$

On applying Lemma 3.4 to the bottom square, we obtain the theorem.

COROLLARY 3.11. *With the notation of the theorem, for any closed point s of S ,*

$$\mathrm{NS}^0(A/S) = \mathrm{NS}^0(A_s) \cap H^0(S, R^2 f_* \mathbb{A}(1))$$

(intersection inside $H_{\mathbb{A}}^2(A_s)(1)$).

REMARK 3.12. When $f : A \rightarrow S$ is an abelian scheme over a connected normal scheme S , we define $\mathcal{D}^1(A/S)$ to be the image of $\mathrm{NS}^0(A/S)$ in $H^0(S, R^2 f_* \mathbb{A}(1))$. In the above proof, when S is of finite type over a field, we obtained a diagram

$$\begin{array}{ccc} \mathcal{D}^1(A/S) & \xrightarrow{\otimes} & H^0(S, R^2 f_* \mathbb{A}(1)) \\ \downarrow & & \downarrow \\ \mathcal{D}^1(A_s) & \xrightarrow{\otimes} & H_{\mathbb{A}}^2(A_s(1))^{\pi_1(s)}. \end{array}$$

ASIDE 3.13. See also Conjecture 1.4 of Grothendieck 1966 and Theorem 0.2 (= Theorem 1.4) of Morrow 2019.

Algebraic classes in families

For an abelian variety A , we let $\mathcal{A}^*(A)$ denote the \mathbb{Q} -algebra of algebraic classes in $H_{\mathbb{A}}^{2*}(A)(n)$.

Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme S over a finite field k . Let s be a closed point of S and η the generic point. Consider the diagram

$$\begin{array}{ccccc} \mathcal{A}^n(A_\eta) & \xrightarrow{\otimes} & H_{\mathbb{A}}^{2n}(A_\eta)(n)^{\pi_1(\eta)} & & \\ c \uparrow & & e \uparrow & & \\ \mathcal{A}^n(A/S) & \xrightarrow{d} & H^0(S, R^{2n} f_* \mathbb{A}(n)) & \xrightarrow{\simeq} & H_{\mathbb{A}}^{2n}(A_\eta)(n)^{\pi_1(S)} \\ \downarrow & & \downarrow & & \\ \mathcal{A}^n(A_s) & \xrightarrow{\otimes} & H_{\mathbb{A}}^{2n}(A_s)^{\pi_1(s)} & & \end{array}$$

The maps d and e are injective, so c is injective.

THEOREM 3.14. *Assume that $\mathcal{A}^n(A/S) \rightarrow \mathcal{A}^n(A_\eta)$ is surjective and that the Tate conjectures holds for algebraic cycles of codimension n on A_η and A_s . Then*

$$\mathcal{A}^n(A/S) = \mathcal{A}^n(A_s) \cap H^0(S, R^{2n} f_* \mathbb{A}(n))$$

(intersection inside $H_{\mathbb{A}}^{2n}(A_s)^{\pi_1(s)}$).

PROOF. Under the assumptions, c and e are isomorphisms, so d becomes an isomorphism when $\mathcal{A}^n(A/S)$ is tensored with \mathbb{A} . Now apply Lemma 3.4 to the lower square. \square

COROLLARY 3.15. *With the assumptions of the theorem, if $\gamma \in H^0(S, R^{2n} f_* \mathbb{A}(n))$ is algebraic for one closed s , then it is algebraic for all closed s .*

PROOF. If γ is algebraic for one s , then the theorem shows that it lies in $\mathcal{A}^n(A/S)$, and hence its image in $H_{\mathbb{A}}^{2n}(A_s)$ lies in $\mathcal{A}^n(A_s)$ for all s . \square

NOTES

3.16. This section is only of heuristic significance. We certainly do not want to assume the Tate conjecture.

Weakly Lefschetz classes in families

3.17. Let G be a group (abstract, profinite, algebraic, ...) acting on a finite-dimensional vector space V over a field k of characteristic 0. The k -algebra $(\otimes^* V)^G$ is generated by G -invariant tensors of degree 2 in each of the following cases:

- (a) $G = \text{Sp}(\phi)$ with ϕ a nondegenerate skew-symmetric form on V ;
- (b) $G = O(\phi)$ with ϕ a nondegenerate symmetric form on V ;
- (c) $G = \text{GL}(W)$ and $V = W \oplus W^\vee$;
- (d) T is a torus and the weights ξ_1, \dots, ξ_{2m} of T on V can be numbered in such a way that the \mathbb{Z} -module of relations among the ξ_i is generated by the relations $\xi_i + \xi_{i+1} = 0$, $i = 1, \dots, m$.

See [Milne 1999b](#), 3.6, 3.8.

3.18. Let G be a group acting on a finite-dimensional vector space V over a field k . If the k -algebra $(\otimes^* V)^G$ is generated by G -invariant tensors of degree 2, then the same is true of $(\wedge^* V)^G$.

QUESTION 3.19. Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme of finite type over an algebraically closed field k , and let γ be a global section of $R^{2n}f_*\mathbb{Q}_\ell(n)$. If γ_s is weakly Lefschetz for one closed $s \in S$, then is it weakly Lefschetz for all closed $s \in S$?

After replacing k with a finitely generated subfield, we have a diagram

$$\begin{array}{ccccc} \mathcal{D}^1(A/S) & \longrightarrow & \mathcal{D}^1(A/S) \otimes \mathbb{Q}_\ell & \xrightarrow{\simeq} & H^0(S, R^2f_*\mathbb{Q}_\ell(1)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}^1(A_s) & \longrightarrow & \mathcal{D}^1(A_s) \otimes \mathbb{Q}_\ell & \xrightarrow{\simeq} & H^2(A_s, \mathbb{Q}_\ell(1))^{\pi_1(s)} \end{array}$$

(see the proof of [3.10](#)), which shows that the answer is positive for $n = 1$. Let $V = H^1(A_{\bar{\eta}}, \mathbb{Q}_\ell)$. The question comes down to the following. Let $\gamma \in H^0(S, R^{2n}f_*\mathbb{Q}_\ell(n)) = (\wedge^{2n} V)(n)^{\pi_1(S)}$. Suppose that, when regarded as an element of $(\wedge^{2n} V)(n)^{\pi_1(S)}$, γ lies in the \mathbb{Q}_ℓ -algebra generated by $\wedge^2 V(1)^{\pi_1(S)}$. Does this imply that γ lies in the \mathbb{Q}_ℓ -algebra generated by $\wedge^2 V(1)^{\pi_1(S)}$? The answer is surely negative in general, but there may be useful conditions on S that ensure that the answer is positive.

Lefschetz groups in families

3.20. Let A be an abelian variety over a separably closed field k , and let ℓ be a prime number $\neq \text{char}(k)$. Let $C_\ell(A)$ denote the centralizer of $\text{End}^0(A)$ in $\text{End}_{\mathbb{Q}_\ell}(V_\ell A)$. This

is a semisimple algebra over \mathbb{Q}_ℓ with an involution \dagger (defined by any Rosati involution). The Lefschetz group of A (relative to H_ℓ) is the algebraic group $L(A)$ over \mathbb{Q}_ℓ with

$$L(A)(\mathbb{Q}_\ell) = \{a \in C_\ell(A) \mid a^\dagger a \in \mathbb{Q}_\ell^\times\}$$

(see Milne 1999c, 4.4). When $k = \mathbb{F}$,

$$C_\ell(A) = C_0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell,$$

where $C_0(A)$ is the centre of $\text{End}^0(A)$, so $C_0(A) = \mathbb{Q}\{\pi\}$.

DEFINITION 3.21. Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme of finite type over an algebraically closed field k , and let $s \in S(k)$. When we identify $V_\ell(A_s)$ with $V_\ell(A_\eta)$, we have $C_\ell(A_s) \subset C_\ell(A_\eta)$. We say that f is *general* if the \mathbb{Q}_ℓ -algebras $C_\ell(A_s)$, $s \in S(k)$, generate $C_\ell(A_{\bar{\eta}})$.

EXAMPLE 3.22. Let $k = \mathbb{F}$. For $s \in S(k)$, we have

$$\text{End}^0(A_s) \longleftarrow \text{End}^0(A/S) \xrightarrow{\cong} \text{End}^0(A_\eta),$$

so $C_0(A_s) \subset C_\ell(A_\eta)$. So f is general if the \mathbb{Q} -algebras $C_0(A_s)$ generate the centralizer of $\text{End}^0(A_{\bar{\eta}})$ in $\text{End}_{\mathbb{Q}_\ell}(V_\ell(A_\eta))$.

EXAMPLE 3.23. Let f be the universal elliptic curve over the affine line ($k = \mathbb{F}$). Then $C_\ell(A_{\bar{\eta}}) = \text{End}(V_\ell A_{\bar{\eta}}) \approx M_2(\mathbb{Q}_\ell)$. On the other hand, $C_0(A_s)$ is either \mathbb{Q} or F , where F is an imaginary quadratic number field, and all quadratic imaginary number fields occur. Therefore, f is general.

EXAMPLE 3.24. Let $f : A \rightarrow S$ be a constant abelian variety ($k = \mathbb{F}$). Then $\text{End}(A_s) = \text{End}(A_\eta)$ for all closed $s \in S$. Therefore $C(A_s) = C(A_\eta)$ for all closed s , and f is again general.

PROPOSITION 3.25. *If $f : A \rightarrow S$ is general, then the algebraic group $L(A_{\bar{\eta}})$ is generated by its subgroups $L(A_s)$, $s \in S(k)$.*

PROOF. This follows from the above description on $L(A)$. □

PROPOSITION 3.26. *Let S be a smooth projective curve over \mathbb{F} and $f : A \rightarrow S$ an abelian scheme such that the $k(\eta)/k$ -trace of A_η is zero. Then f is general.*

PROOF. This follows from applying a Chebotarev density theorem to a model of f over a finite subfield of \mathbb{F} . □

QUESTION 3.27. Are all abelian schemes general?

REMARK 3.28. There is an extensive literature on Mumford–Tate groups and their variation in families (see Milne 2013, §6, for a summary), much of which carries over to Lefschetz groups.

Lefschetz classes in families

THEOREM 3.29. *Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme of finite type over an algebraically closed field k , and let δ be a global section of $R^{2n}f_*\mathbb{A}(n)$ such that δ is fixed by $L(A_{\bar{\eta}})$. If δ_s is Lefschetz for one closed $s \in S$, then it is Lefschetz for all closed $s \in S$.*

PROOF. Let $\mathcal{D}^*(A/S)$ denote the \mathbb{Q} -subalgebra of

$$H^0(S, R^{2*}f_*\mathbb{A}(*)) \simeq H_{\mathbb{A}}^{2n}(A_{\bar{\eta}})(n)^{\pi_1(S)}$$

generated by the image of $\text{NS}(A/S)$. Consider the diagram

$$\begin{array}{ccc} \mathcal{D}^n(A_{\bar{\eta}}) & \xrightarrow{\otimes} & H^{2n}(A_{\bar{\eta}}, \mathbb{A}(n))^{L(A_{\bar{\eta}})} \\ a \uparrow & & c \uparrow \\ \mathcal{D}^n(A/S) & \xrightarrow{b} & H^0(S, R^{2n}f_*\mathbb{A}(n))^{L(A_{\bar{\eta}})} \\ \downarrow & & \downarrow \\ \mathcal{D}^n(A_s) & \xrightarrow{\otimes} & H^{2n}(A_s, \mathbb{A}(n))^{L(A_s)}. \end{array}$$

For the top and bottom arrows, see 1.32. The map a is surjective when $n = 1$ (see the proof of Theorem 3.10), and so it is surjective for all n . The maps b and c are injective, from which it follows that a and c are isomorphisms and that b induces an isomorphism

$$\mathcal{D}^n(A/S) \otimes \mathbb{A} \rightarrow H^0(S, R^{2n}f_*\mathbb{A}(n))^{L(A_{\bar{\eta}})}.$$

On applying Lemma 3.4 to the bottom square, we find that

$$\mathcal{D}^n(A/S) = \mathcal{D}^n(A_s) \cap H^0(S, R^{2n}f_*\mathbb{A}(n))^{L(A_{\bar{\eta}})}.$$

If the element δ is such that $\delta_s \in \mathcal{D}^n(A_s)$ for one s , then it lies in $\mathcal{D}^n(A/S)$, and so $\delta_s \in \mathcal{D}^n(A_s)$ for all s . □

COROLLARY 3.30. *Let $f : A \rightarrow S$ be as in the statement of the theorem. Assume that f is general, and let δ be a global section of $R^{2n}f_*\mathbb{A}(n)$. If δ is weakly Lefschetz for all closed $s \in S$ and Lefschetz for one closed $s \in S$, then it is Lefschetz for all closed s .*

PROOF. The hypotheses imply that δ is fixed by $L(A_{\bar{\eta}})$. □

A variational approach to proving Conjecture B for Weil classes.

The key to proving the conjectures is to prove that Conjecture B holds for split Weil classes. Not having a proof, we simply make some remarks.

Let (A_1, ν_1, λ_1) be a split Weil triple over \mathbb{Q}^{al} (relative to a CM-algebra E) with good reduction at w .

3.31. If (A_1, ν_1, λ_1) is of the form $(B \otimes \mathcal{O}_E, \dots)$, then the Weil classes on A_1 are w -Lefschetz if they are weakly w -Lefschetz (because they are algebraic).

3.32. Consider the universal Weil family $f : A \rightarrow S$ over \mathbb{Q}^{al} containing (A_1, ν_1, λ_1) . If the Weil classes on A_1 are weakly w -Lefschetz, and $(A_1, \nu_1, \lambda_1)_0$ lifts (up to isogeny) to a triple in the family whose Weil classes are w -Lefschetz, then the Weil classes on A_1 are w -Lefschetz (obviously).

3.33. Assume that the Weil classes on A_1 are weakly w -Lefschetz. If B and the family $\bar{f} : \bar{A} \rightarrow C$ in 1.48 can be chosen so that the Weil classes γ on \bar{A} (i.e, the global sections of $W_E(\bar{A}/C)$) are fixed by $L(\bar{A}_{\bar{\eta}})$, then γ_s is Lefschetz for all closed $s \in C$. In particular, the Weil classes on A_1 are w -Lefschetz.

CM abelian varieties

Let \mathbb{Q}^{al} be the algebraic closure of \mathbb{Q} in \mathbb{C} , and let w be a prime of \mathbb{Q}^{al} lying over p .

THEOREM 3.34. *Let A be an abelian variety over \mathbb{Q}^{al} of CM-type. There exist abelian varieties A_{Δ} of split Weil type and homomorphisms $f_{\Delta} : A \rightarrow A_{\Delta}$ such that every Hodge class γ on A can be written as a sum $\gamma = \sum f_{\Delta}^*(\gamma_{\Delta})$ with γ_{Δ} a Weil class on A_{Δ} . If γ is weakly w -Lefschetz on A , then the γ_{Δ} can be chosen to be weakly w -Lefschetz on A_{Δ} .*

PROOF. Let E_0 be the centre of $\text{End}(A_0)$ and L_0 its Lefschetz group. Then the \mathbb{Q} -vector space of weakly w -Lefschetz class is $B^p(A)^{L_0}$. This is equal to the sum $\sum_{\Delta} f_{\Delta}^*(W_K(A_{\Delta}))$, where Δ runs over the classes Δ satisfying (2) and such that the elements of

$$\bigoplus_{t \in T} H^{2p}(A_{\Delta})_{\Delta \times \{t\}}$$

are fixed by L_0 . See 1.39. □

COROLLARY 3.35. *If Conjecture B holds for split Weil classes then it holds for all Hodge classes on CM abelian varieties.*

PROOF. The pullback of a w -Lefschetz class by a morphism of abelian varieties is w -Lefschetz (because the pullback of a Lefschetz class is Lefschetz). □

REMARK 3.36. It is not true that all Weil classes on abelian varieties of Weil type over \mathbb{Q}^{al} are weakly w -Lefschetz because that would imply that all Hodge classes on CM abelian varieties specialize to Lefschetz classes, which is false.

4 Conjecture A implies C and D

For CM abelian varieties, there is a proof (Milne 2009, §4), using tannakian categories, that Conjecture B implies Conjecture D, hence also Conjecture C. In the second subsection below, we discuss an elementary approach to proving that Conjecture B implies Conjecture C, and in the third subsection we explain the tannakian proof. Recall that Conjecture A implies Conjecture B.

Preliminaries

4.1. Recall that a representation of an affine group scheme over R on an R -module V corresponds to co-action $\rho : V \rightarrow V \otimes \mathcal{O}(G)$ of the Hopf algebra $\mathcal{O}(G)$ of G on V . We define

$$V^G = \{v \in V \mid \rho(v) = v \otimes 1\}.$$

Its definition commutes with base change. See [Milne 2017](#), 4.i.

4.2. Let Q be a field and R a Q -algebra. Let M be an R -module. Recall that a Q -structure on M is a Q -submodule V of M such that the map $v \otimes r \mapsto rm : V \otimes_Q R \rightarrow M$ is an isomorphism. We write $V \xrightarrow{\otimes} M$ to signify that V is a Q -structure on M .

4.3. Let A and B be abelian varieties such that $\text{Hom}(A, B) = 0$. Then

$$\mathcal{D}^*(A \times B) \simeq \mathcal{D}^*(A) \otimes \mathcal{D}^*(B).$$

The corresponding statement for Hodge classes, Tate classes, rational Tate classes... is false. However, for example, $\mathcal{B}^*(A) = f^*(\mathcal{B}^*(A \times B))$, where $f : A \rightarrow A \times B$ is the map $a \mapsto (a, 0)$.

Conjecture B implies Conjecture C

4.4. Let A be a CM abelian variety over \mathbb{Q}^{al} and A_0 its reduction over \mathbb{F} . The inclusion $\text{End}^0(A) \hookrightarrow \text{End}^0(A_0)$ maps the centre $C(A)$ of $\text{End}^0(A)$ onto a \mathbb{Q} -subalgebra of $\text{End}^0(A_0)$ containing its centre $C(A_0)$, and hence it defines an inclusion $L(A_0) \hookrightarrow L(A)$. We have a commutative diagram

$$\begin{array}{ccc} \text{MT}(A) & \hookrightarrow & L(A) \\ \updownarrow & & \updownarrow \\ P(A_0) & \hookrightarrow & L(A_0), \end{array}$$

where $P(A_0)$ is the smallest algebraic subgroup of $L(A_0)$ containing a Frobenius element for A_0 (cf. [Milne 2009](#), 3.2).

Let $\mathcal{H}^r(A) = H_{\mathbb{A}}^{2r}(A)(r)$. Recall ([Deligne 1982](#)) that

$$\mathcal{B}^r(A) \xrightarrow{\otimes} \mathcal{H}^r(A)^{\text{MT}(A)},$$

and (1.32)

$$\mathcal{D}^r(A_0) \xrightarrow{\otimes} \mathcal{H}^r(A_0)^{L(A_0)}.$$

Using the projection $\mathcal{H}^r(A) \rightarrow \mathcal{H}^r(A_0)$, we identify $\mathcal{B}^r(A)$ with its image in $\mathcal{H}^r(A_0)$, and then

$$\mathcal{B}^r(A) \xrightarrow{\otimes} \mathcal{H}^r(A_0)^{\text{MT}(A)} \tag{5}$$

PROPOSITION 4.5. *Conjecture B for A implies that*

$$\mathcal{B}^r(A) \cap \mathcal{D}^r(A_0) \xrightarrow{\otimes} \mathcal{H}^r(A_0)^{\text{MT}(A) \cdot L(A_0)}$$

(intersection in $\mathcal{H}^r(A_0)$).

PROOF. On taking fixed points with respect to $L(A_0)$ in (5), we find that

$$\mathcal{B}^r(A)^{L(A_0)} \xrightarrow{\otimes} \mathcal{H}^r(A_0)^{\text{MT}(A) \cdot L(A_0)}.$$

If $a \in \mathcal{B}^r(A)$ is fixed by $L(A_0)$, then it is weakly w -Lefschetz, hence w -Lefschetz by Conjecture B. Therefore $\mathcal{B}^r(A)^{L(A_0)} = \mathcal{B}^r(A) \cap \mathcal{D}^r(A_0)$. \square

Assume that Conjecture B holds for A . From the split-exact sequence

$$0 \longrightarrow \mathcal{H}^r(A_0) \xrightarrow{x \mapsto (x,x)} \mathcal{H}^r(A_0) \oplus \mathcal{H}^r(A_0) \xrightarrow{(x,y) \mapsto x-y} \mathcal{H}^r(A_0) \longrightarrow 0,$$

we get the bottom row of the following exact commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{B}^r(A) \cap \mathcal{D}^r(A_0) & \longrightarrow & \mathcal{B}^r(A) \oplus \mathcal{D}^r(A_0) & \longrightarrow & \mathcal{B}^r(A) + \mathcal{D}^r(A_0) \longrightarrow 0 \\ & & \downarrow \otimes & & \downarrow \otimes & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}^r(A_0)^{\text{MT}(A) \cdot L(A_0)} & \longrightarrow & \mathcal{H}^r(A_0)^{\text{MT}(A)} \oplus \mathcal{H}^r(A_0)^{L(A_0)} & \longrightarrow & \mathcal{H}^r(A_0)^{\text{MT}(A) \cap L(A_0)} \end{array}$$

From the diagram, we see that the dashed arrow exists and that

$$(\mathcal{B}^r(A) + \mathcal{D}^r(A_0)) \otimes \mathbb{A} \hookrightarrow \mathcal{H}^r(A_0)^{\text{MT}(A) \cap L(A_0)}.$$

As $P(A_0) \subset \text{MT}(A) \cap L(A_0)$, we obtain an injection

$$(\mathcal{B}^r(A) + \mathcal{D}^r(A_0)) \otimes \mathbb{A}_f \hookrightarrow \mathcal{J}^r(A).$$

This will not usually be an isomorphism.

QUESTION 4.6. If $P(A_0) = \text{MT}(A) \cap L(A_0)$, is $\mathcal{R}^*(A_0)$ the smallest \mathbb{Q} -subalgebra of $\mathcal{J}^*(A_0)$ containing $\mathcal{B}^*(A)$ and $\mathcal{D}^*(A_0)$?

The question only makes sense because $(\mathcal{B}^*(A) + \mathcal{D}^*(A_0)) \otimes \mathbb{A}_f \rightarrow \mathcal{J}^*(A_0)$ is injective (assuming Conjecture B).

Let K be a CM subfield of \mathbb{C} that is finite and Galois over \mathbb{Q} . We say that K splits a CM abelian variety A if $\text{End}^0(A) \otimes_{\mathbb{Q}} K$ is a product of matrix algebras over K . Let A^K be a CM abelian variety over \mathbb{Q}^{al} split by K and so large that every simple abelian variety over \mathbb{Q}^{al} split by K is isogenous to an abelian subvariety of A . Then A_0^K is an abelian variety over \mathbb{F} such that every simple abelian variety over \mathbb{F} split by K is isogenous to an abelian subvariety of A_0^K . For A^K , we have

$$P(A_0^K) = \text{MT}(A^K) \cap L(A_0^K),$$

(Milne 1999c, Theorem 6.1). Is the \mathbb{Q} -subalgebra of $\mathcal{J}^*(A_0^K)$ generated by $\mathcal{B}^*(A^K)$ and $\mathcal{D}^*(A_0^K)$ a \mathbb{Q} -structure on $\mathcal{J}^*(A_0^K)$? If so, we could define $\mathcal{R}^*(A_0)$ to be this \mathbb{Q} -algebra. For any other abelian variety B over \mathbb{F} , there exists a homomorphism $f : B \rightarrow A_0^K$ with finite kernel for some K , and we could define

$$\mathcal{R}^*(B) = f^* \mathcal{R}^*(A_0)$$

This would give an elementary construction of the family of rational Tate classes.

SUMMARY 4.7. Assume that conjecture B holds for CM abelian varieties over \mathbb{Q}^{al} . Then there exists a unique family $\mathcal{R}^*(A)$ of \mathbb{Q} -structures on the \mathbb{A}_f -algebras $\mathcal{J}^*(A)$, indexed by the abelian varieties over \mathbb{F} , satisfying (R1) and (R2) and such that Hodge classes on CM abelian varieties over \mathbb{Q}^{al} specialize to rational Tate classes.

Without Conjecture B, the map $(\mathcal{B}^r(A) + \mathcal{D}^r(A_0)) \otimes \mathbb{A}_f \rightarrow \mathcal{J}^r(A_0)$ need not be injective.

Conjecture B implies Conjecture D

In this section, we assume that Conjecture B holds for all CM abelian varieties over \mathbb{Q}^{al} , and we construct the category of motives $\text{Mot}(\mathbb{F})$ over \mathbb{F} . This section is largely a review of earlier work of the author.

STATEMENTS

4.8. Assuming Conjecture B for CM abelian varieties, we construct commutative diagrams

$$\begin{array}{ccc}
 S & S_{\mathbb{Q}_l} & \text{CM}(\mathbb{Q}^{\text{al}}) \\
 \uparrow & \uparrow & \downarrow R \\
 P & P_{\mathbb{Q}_l} & \text{Mot}(\mathbb{F}) \\
 & \swarrow & \searrow \xi_l \\
 & P_l & R_l(\mathbb{F})
 \end{array} \quad l = 2, \dots, p, \dots, \quad (6)$$

where

- ◊ $\text{CM}(\mathbb{Q}^{\text{al}})$ is the subcategory of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ of motives of CM-type;
- ◊ $\text{Mot}(\mathbb{F})$ is a tannakian category over \mathbb{Q} with fundamental group P ;
- ◊ $P \rightarrow S$ is the Shimura–Taniyama homomorphism (1.14)
- ◊ $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ is a quotient functor bound by $P \rightarrow S$;
- ◊ $\xi_l : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow R_l(\mathbb{F})$ is the realization functor (1.57, 1.58).

A CONSTRUCTION

Let $\text{LCM}(\mathbb{Q}^{\text{al}})$ denote the tannakian subcategory of $\text{LMot}(\mathbb{Q}^{\text{al}})$ generated by the abelian varieties of CM-type. There are canonical exact tensor functors $J : \text{LCM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{CM}(\mathbb{Q}^{\text{al}})$ and $R : \text{LCM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{LMot}(\mathbb{F})$ giving rise to homomorphisms $S \hookrightarrow T$ and $L \hookrightarrow T$ of (commutative) fundamental groups. We shall construct quotient functors $q : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}'(\mathbb{F})$ and $q' : \text{LMot}(\mathbb{F}) \rightarrow \text{Mot}'(\mathbb{F})$ with the following properties:

- (a) the diagram at left commutes and corresponds to the diagram of fundamental groups at right

$$\begin{array}{ccc}
 \text{CM}(\mathbb{Q}^{\text{al}}) & \xleftarrow{J} & \text{LCM}(\mathbb{Q}^{\text{al}}) & & S & \hookrightarrow & T \\
 \downarrow q & & \downarrow R & & \uparrow & & \uparrow \\
 \text{Mot}'(\mathbb{F}) & \xleftarrow{q'} & \text{LMot}(\mathbb{F}) & & P & \hookrightarrow & L
 \end{array}$$

- (b) the functors $\xi_l : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow R_l(\mathbb{F})$ factor through q .

The functors R and J are both quotient functors, and so correspond to \mathbb{Q} -valued functor ω^R and ω^J on $\text{LCM}(\mathbb{Q}^{\text{al}})^L$ and $\text{LCM}(\mathbb{Q}^{\text{al}})^S$ respectively (see 1.2). Conjecture B for CM abelian varieties says exactly that these two functors restrict to the same fibre functor ω_1 on $\text{LCM}(\mathbb{Q}^{\text{al}})^{L \cdot S}$ and that ω_1 is a \mathbb{Q} -structure on the adélic fibre functor on $\text{LCM}(\mathbb{Q}^{\text{al}})^{L \cdot S}$ defined by the standard Weil cohomology theories. As $P = S \cap L$ (Milne 1999c, 6.1), the sequence

$$0 \rightarrow S/P \rightarrow T/L \rightarrow T/(L \cdot S) \rightarrow 0$$

is exact. Therefore $J| : \text{LCM}(\mathbb{Q}^{\text{al}})^L \rightarrow \text{CM}(\mathbb{Q}^{\text{al}})^P$ is a quotient functor and $(\text{LCM}(\mathbb{Q}^{\text{al}})^L)^{S/P} = \text{LCM}(\mathbb{Q}^{\text{al}})^{L \cdot S}$:

$$\begin{array}{ccccc}
 \text{CM}(\mathbb{Q}^{\text{al}})^P & \xleftarrow{J|} & \text{LCM}(\mathbb{Q}^{\text{al}})^L & \xleftarrow{\quad} & \text{LCM}(\mathbb{Q}^{\text{al}})^{L \cdot S} & & S/P & \longrightarrow & T/L & \longrightarrow & T/L \cdot S \\
 \downarrow & & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{CM}(\mathbb{Q}^{\text{al}}) & \xleftarrow{J} & \text{LCM}(\mathbb{Q}^{\text{al}}) & \xleftarrow{\quad} & \text{LCM}(\mathbb{Q}^{\text{al}})^S & & S & \longrightarrow & T & \longrightarrow & T/S \\
 \downarrow q & & \downarrow R & & & & \uparrow & & \uparrow & & \\
 \text{Mot}'(\mathbb{F}) & \xleftarrow{q'} & \text{LMot}(\mathbb{F}) & & & & P & \longrightarrow & L & &
 \end{array}$$

From ω_1 and the equality $\omega_1 = \omega^R|$, we get a fibre functor ω_0 on $\text{CM}(\mathbb{Q}^{\text{al}})^P$ (see 1.3) such that

- (a) $\omega_0|_{\text{LCM}(\mathbb{Q}^{\text{al}})^L} = \omega^R$,
- (b) ω_0 is a \mathbb{Q} -structure on x_l for $l = 2, \dots, p, \dots$.

We define $\text{Mot}'(\mathbb{F})$ to be the quotient $\text{CM}(\mathbb{Q}^{\text{al}})/\omega_0$. Because of (a), the functor $q \circ J$ factors through R , say, $q \circ J = q' \circ R$. The triple $(\text{Mot}'(\mathbb{F}), q, q')$ has the properties (a) and (b). See Milne 2009, §4, for more details.

5 Extending the reduction functor

In this section, we assume that Conjecture B holds for all CM abelian varieties over \mathbb{Q}^{al} , and we investigate whether the reduction functor $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ extends to $\text{Mot}^w(\mathbb{Q}^{\text{al}})$.

Statements

Let $\text{Mot}'(\mathbb{Q}^{\text{al}})$ be a tannakian subcategory of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ containing $\text{CM}(\mathbb{Q}^{\text{al}})$, and consider the following statements.

THEOREM 5.1. *The reduction functor $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ extends uniquely to a functor $R : \text{Mot}'(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ with the following properties:*

- (a) if $hA \in \text{ob Mot}'(\mathbb{Q}^{\text{al}})$, then $R(hA) = hA_0$, and
- (b) the diagrams

$$\begin{array}{ccc}
 \text{Mot}'(\mathbb{Q}^{\text{al}}) & & \\
 \downarrow R & \searrow \xi_l & \\
 \text{Mot}(\mathbb{F}) & \xrightarrow{\eta_l} & R_l(\mathbb{F})
 \end{array}$$

commute for all prime numbers l .

If $\text{Mot}'(\mathbb{Q}^{\text{al}})$ is generated by the abelian varieties it contains, then the uniqueness is obvious.

COROLLARY 5.2. *Let A be an abelian variety over \mathbb{Q}^{al} . If hA lies in $\text{Mot}'(\mathbb{Q}^{\text{al}})$, then Hodge classes on A specialize to rational Tate classes on A_0 .*

PROOF. Obvious from the definitions. \square

COROLLARY 5.3. *Conjecture A holds for all abelian varieties over \mathbb{Q}^{al} such that $hA \in \text{Mot}'(\mathbb{Q}^{\text{al}})$.*

PROOF. Obvious from Corollary 5.2. \square

For an abelian motive M over $\mathbb{Q}^{\text{al}} \subset \mathbb{C}$, we let G_M denote the Mumford–Tate group of the rational Hodge structure $\omega_B(M)$.

COROLLARY 5.4. *Let M be a motive in $\text{Mot}'(\mathbb{Q}^{\text{al}})$. The Galois representation attached to any model of M over a sufficiently large algebraic number field in \mathbb{Q}^{al} takes values in G_M and is strictly compatible.*

PROOF. The first part of the statement is obvious, and the second follows from the properties of the motive $R(M)$. See 6.9. \square

When $\text{Mot}'(\mathbb{Q}^{\text{al}}) = \text{CM}(\mathbb{Q}^{\text{al}})$, Theorem 5.1 says nothing new. When $\text{Mot}'(\mathbb{Q}^{\text{al}})$ is the category generated by the abelian varieties with very good reduction, we prove this in 5.15 below. When $\text{Mot}'(\mathbb{Q}^{\text{al}})$ consists of the abelian motives with visibly good reduction, we suggest two approaches to proving it. When $\text{Mot}'(\mathbb{Q}^{\text{al}}) = \text{Mot}^w(\mathbb{Q}^{\text{al}})$, it is left as an exercise for the reader.

NOTES

5.5. For a collection \mathfrak{s} of abelian varieties over \mathbb{Q}^{al} with good reduction at w , we define $\text{Mot}^{\mathfrak{s}}(\mathbb{Q}^{\text{al}})$ to be the tannakian subcategory of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ generated by \mathfrak{s} . Clearly Theorem 5.1 holds for $\text{Mot}^{\mathfrak{s}}(\mathbb{Q}^{\text{al}})$ if and only if the Hodge classes on the abelian varieties in \mathfrak{s} specialize to rational Tate classes.

5.6. We would like to prove Theorem 5.1 in the general case using as little of the theory of Shimura varieties as possible. One of the goals of this article is to *recover* the theory of Shimura varieties from the theory of motives, not merely enhance it.

5.7. Let A be an abelian variety over a number field K . Kisin and Zhou 2025b show that, after replacing K with a finite extension, the Weil–Deligne representation attached to A takes values in the Mumford–Tate group of A and is strictly compatible. In other words, a version of Corollary 5.4 holds for *all* abelian varieties over \mathbb{Q}^{al} , not just those with good reduction at w . This suggests, as noted elsewhere, that many statements concerning abelian varieties with good reduction at w should extend mutatis mutandis to all abelian varieties over \mathbb{Q}^{al} .

CM lifts

Up to isogeny, every abelian variety over \mathbb{F} lifts to a CM abelian variety in characteristic zero (2.24). There is the following more precise conjecture.

CONJECTURE 5.8. *Let A be an abelian variety over \mathbb{Q}^{al} with good reduction at w and let γ be a Hodge class on A . There exist a CM abelian variety A' over \mathbb{Q}^{al} and a Hodge class γ' on A' such that $(A, \gamma)_0 \sim (A', \gamma')_0$, i.e., such that there exists an isogeny $A_0 \rightarrow A'_0$ sending γ_0 to γ'_0 .*

5.9. If Conjecture 5.8 holds for all γ on the abelian variety A , then A satisfies Conjecture A. Indeed, the condition implies that γ_0 is a rational Tate class on A_0 , and intersections of rational Tate classes of complementary dimension are rational numbers.

5.10. An abelian motive M over \mathbb{Q}^{al} is said to have *visibly good reduction* if it can be expressed in the form $h(A, e, m)$ with A an abelian variety with good reduction at w . We write $\text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})$ for the category of abelian motives over \mathbb{Q}^{al} (tannakian subcategory of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$).

5.11. Conjecture 5.8 implies that Conjecture B holds for all abelian varieties over \mathbb{Q}^{al} with good reduction at w . The same argument as in §5 then allows us to define $\text{Mot}(\mathbb{F})$ to be $\text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})/\omega$ for a suitable fibre functor ω on $\text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})^P$, and the functor $\text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})$ induces an equivalence $\text{CM}(\mathbb{Q}^{\text{al}})/\omega \rightarrow \text{Mot}^{\text{vis}}(\mathbb{Q}^{\text{al}})/\omega$. Therefore Conjecture 5.8 implies Theorem 5.1 for abelian motives with visibly good reduction.

5.12. There is a converse to the last statement. Let $\text{Sh}_p(G, X)$ be a Shimura variety abelian type with rational weight satisfying the condition to have good reduction at p . From Theorem 5.1 for $\text{Mot}^w(\mathbb{F})$, we obtain an integral canonical model of the Shimura variety and a description of it as a moduli variety for abelian motives (see the next section). Proceeding as in [Langlands and Rapoport 1987](#), we then attach to each point of $\text{Sh}_p(\mathbb{F})$ an admissible homomorphism $\mathfrak{B} \rightarrow \mathfrak{G}_G$. Now a cohomological argument (assuming G^{der} is simply connected) shows that φ is the homomorphism attached to a special point (op. cit. 5.3). In this way, we see that every point of $\text{Sh}_p(\mathbb{F})$ lifts to a special point. Cf. §4 of [Milne 1992](#), especially Theorem 4.6.

5.13. There are many results in the literature concerning Conjecture 5.8. For example, [Kisin and Zhou 2025a](#) prove that every point in the μ -ordinary locus of the special fibre of a Shimura variety lifts to a special point.

Nifty abelian varieties

Recall that an abelian variety A over \mathbb{Q}^{al} with good reduction at w is nifty if $\text{MT}(A) \cdot L(A_0) = L(A)$.

PROPOSITION 5.14. *Hodge classes on nifty abelian varieties specialize to rational Tate classes.*

PROOF. Omitted for the moment. □

Abelian varieties with very good reduction

We say that an abelian variety A has *very good reduction* at w if it has good reduction at w and the adjoint group of $\text{MT}(A)$ is unramified at p . Note that products of abelian varieties with very good reduction have very good reduction, and that all CM abelian varieties over \mathbb{Q}^{al} have very good reduction.

Let $\text{Mot}^{\text{vg}}(\mathbb{Q}^{\text{al}})$ denote the category of abelian motives over \mathbb{Q}^{al} generated by the abelian varieties with very good reduction. We explain in this subsection how to extend R to $\text{Mot}^{\text{vg}}(\mathbb{Q}^{\text{al}})$.

Let A have very good reduction at w , and let (G, h) be the Mumford-Tate group of A . Let X be the conjugacy class of h . Then (G, X) is a Shimura datum, and so, for every compact open $K \subset G(\mathbb{A}_f)$, we have a variety $\text{Sh}_K(G, X)$ over \mathbb{C} . We assume that $K = K_p \times K^p$ with K_p a hyperspecial subgroup of $G(\mathbb{Q}_p)$ and K^p a sufficiently small subgroup of $G(\mathbb{A}_f^p)$. The action of G on the \mathbb{Q} -vector space $V \stackrel{\text{def}}{=} H_1(A, \mathbb{Q})$ allows us to realize $\text{Sh}_K(G, X)$ as the solution to a moduli problem over \mathbb{C} . The moduli problem is defined over \mathbb{Q}^{al} , and descent theory shows that $\text{Sh}_K(G, X)$ has a canonical model over \mathbb{Q}^{al} , which we denote $S_{\mathbb{Q}^{\text{al}}}$, and which is the solution to a moduli problem over \mathbb{Q}^{al} . Specifically, $S_{\mathbb{Q}^{\text{al}}}$ parametrizes triples $(B, \mathfrak{f}, \lambda)$ where B is an abelian variety over \mathbb{Q}^{al} , \mathfrak{f} is a family of Hodge tensors on B and its powers, and λ is a level structure on B . We choose \mathfrak{f} to be the family of all Hodge classes on A and its powers. We assume that K has been chosen small enough to force B to have good reduction at w . Results of Kisin and Vasiu, show that $S_{\mathbb{Q}^{\text{al}}}$ extends to a smooth canonical model S over \mathcal{O}_w , and that the point of $S(\mathbb{F})$ defined by $(A, \mathfrak{f}, \lambda)$ is isogenous to the reduction of a special point. Specifically, this means that there exists an abelian variety B over \mathbb{Q}^{al} and an isogeny $A_{\mathbb{F}} \rightarrow B_{\mathbb{F}}$ such that

- (a) B is of CM-type;
- (b) let γ be a Hodge class on $A_{\mathbb{Q}^{\text{al}}}$ and γ' the corresponding Hodge class on B ; then, under the isogeny, γ_l maps to γ'_l for all l .

Therefore, there exists an exact \mathbb{Q} -linear tensor functor $\langle A \rangle^{\otimes} \rightarrow \text{Mot}(\mathbb{F})$ such that the diagram

$$\begin{array}{ccc} \langle A \rangle^{\otimes} & & \\ \downarrow R & \searrow \xi_l & \\ \text{Mot}(\mathbb{F}) & \xrightarrow{\eta_l} & R_l(\mathbb{F}) \end{array}$$

commutes. Let A' be a second abelian variety over \mathbb{Q}^{al} with very good reduction at w . On repeating the argument for $A \times A'$, we can extend the above diagram to a diagram

$$\begin{array}{ccc} \langle A \times A' \rangle^{\otimes} & & \\ \downarrow R & \searrow \xi_l & \\ \text{Mot}(\mathbb{F}) & \xrightarrow{\eta_l} & \text{Mot}(\mathbb{F}). \end{array}$$

As the set of isogeny classes of abelian varieties over \mathbb{Q}^{al} with very good reduction is countable, this will eventually lead to a functor $R : \text{Mot}^{\text{vg}}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ such that the diagrams (7) commute (by the axiom of dependent choice). We have proved the following statement.

THEOREM 5.15. *The reduction functor $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ extends uniquely to $\text{Mot}^{\text{vg}}(\mathbb{Q}^{\text{al}})$, and makes the diagrams (7) commute.*

COROLLARY 5.16. *Let A be an abelian variety over \mathbb{Q}^{al} . If A has very good reduction at w , then Hodge classes on A specialize to rational Tate classes on A_0 .*

COROLLARY 5.17. *Conjecture A holds for abelian varieties with very good reduction.*

PROOF. Obvious from Corollary 5.16. □

COROLLARY 5.18. *Let M be a motive in $\text{Mot}^{\text{vg}}(\mathbb{Q}^{\text{al}})$. The Galois representation attached to any model of M over a sufficiently large algebraic number field in \mathbb{Q}^{al} takes values in G_M and is strictly compatible.*

REMARK 5.19. The hypothesis that we have made in this section that $\text{MT}(A)^{\text{ad}}$ be unramified at p , i.e., quasi-split over \mathbb{Q}_p and splits over an unramified extension, is unnecessarily strong. For example, if Conjecture 1 of [Kisin et al. 2022](#) holds, then we can replace it with the requirement that $\text{MT}(A)$ be quasi-split over \mathbb{Q}_p . See also Reimann 1997, B3.12.

Abelian motives with visibly good reduction

In this subsection, we investigate the following statement, which implies that Theorem 5.1 holds for abelian motives over \mathbb{Q}^{al} with visibly good reduction at w (cf. 5.10, 5.11).

5.20. For every abelian variety A over \mathbb{Q}^{al} with good reduction at w , Hodge classes on A specialize to rational Tate classes.

FIRST APPROACH

5.21. As in §6 of [Deligne 1982](#), embed (A, γ) in a family of abelian varieties with additional structure over \mathbb{C} . In particular, γ extends to a global section of the family. Now the family is defined over a number field, and specializes to a family over \mathbb{F} . Complete the proof by showing that A_0 lifts to a CM abelian variety in the family (this seems to be more general than known, or even conjectured, results).

SECOND APPROACH

We saw earlier (Theorem 6.19) that 5.20 holds under some hypotheses.

Abelian motives with good reduction

As mentioned earlier, all statements in this article for abelian varieties over \mathbb{Q}^{al} with good reduction at w should hold *mutatis mutandis* also for those with bad reduction. In particular, the second approach in the last subsection should yield a proof of Theorem 5.1 for $\text{Mot}^w(\mathbb{Q}^{\text{al}})$.

6 Consequences of Conjecture A for motives

In this section, we assume the conjectures A, B, C, D with \mathfrak{s} the collection of CM abelian varieties, and we investigate some of the applications of the conjectures.

The category of motives over \mathbb{F}

We define the category of motives¹⁶ over \mathbb{F} and show that it has most of the properties that Grothendieck's category of numerical motives would have if the Tate and standard conjectures were known over \mathbb{F} .

6.1. Let $\text{Mot}(\mathbb{F})$ denote the category of motives based on the abelian varieties over \mathbb{F} using the rational Tate classes as correspondences. Specifically, its objects are triples (A, e, m) , where

- ◊ A is a variety over \mathbb{F} each of whose components admits the structure of an abelian variety,
- ◊ $e \in \mathcal{R}^{\dim A}(A \times A)$ is such that $e^2 = e$, and
- ◊ $m \in \mathbb{Z}$

and the Homs are given by

$$\text{Hom}((A, e, m), (B, f, n)) = f \cdot \mathcal{R}^{\dim A + n - m}(A \times B) \cdot e.$$

We use that the Künneth components of the diagonal are rational Tate classes to modify the commutativity constraint.

6.2. The category $\text{Mot}(\mathbb{F})$ is a tannakian category over \mathbb{Q} with canonical functors

$$\eta_l : \text{Mot}(\mathbb{F}) \rightarrow R_l(\mathbb{F}), \quad l \text{ a prime number.}$$

Its band is the Weil-number protorus P . The category $\text{Mot}(\mathbb{F})$ has a canonical structure of a Tate triple.

6.3. There is a canonical functor R making the diagrams in 4.8 commute. There is a unique polarization on $\text{Mot}(\mathbb{F})$ compatible with the canonical polarization on $\text{CM}(\mathbb{Q}^{\text{al}})$. This can be proved as in [Milne 2002](#).

6.4. Grothendieck's standard conjectures hold for abelian varieties over \mathbb{F} and rational Tate classes. For the Lefschetz standard conjecture, this follows from the fact that the inverse Lefschetz operator is even Lefschetz. For the Hodge standard conjecture, it is a restatement of 6.3

6.5. The functor $\text{Mot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_l)$ defines an equivalence

$$\text{Mot}(\mathbb{F})_{(\mathbb{Q}_l)} \rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_l).$$

Thus, we can regard the tannakian category $\text{Mot}(\mathbb{F})$ as being (simultaneously) a \mathbb{Q} -structure on the tannakian categories being a $\text{Mot}(\mathbb{F})_{(\mathbb{Q}_l)} \rightarrow V_l(\mathbb{F})$ are equivalences of categories.

¹⁶The reader may ask why we call this the category of motives over \mathbb{F} rather than the category of abelian motives. Conjecturally, the two are the same. Specifically, when we assume the Tate and standard conjectures over \mathbb{F} , the simple objects of $\text{Mot}(\mathbb{F})$ are classified by the conjugacy classes of Weil numbers, and the abelian motives exhaust the possible Weil numbers ([Milne 1994a](#), 2.7).

The category of motives over a finite field.

6.6. In Grothendieck's motivic paradise, $\text{Mot}(\mathbb{F}_q)$ is a tannakian category over \mathbb{Q} with fundamental group $P(q)$ (see 1.12), and $\text{Mot}(\mathbb{F})$ is a tannakian category over \mathbb{Q} with fundamental group the Weil-number protorus P (see Milne 1994a). The functor $\text{Mot}(\mathbb{F}_q) \rightarrow \text{Mot}(\mathbb{F})$ identifies $\text{Mot}(\mathbb{F}_q)$ with the category whose objects are pairs consisting of an object of $\text{Mot}(\mathbb{F})$ and an action of $P(q)$ on the object consistent with the action of P (see the author's book on Tannakian Categories). To give such an action, it suffices to endow the object with a suitable Frobenius endomorphism.

6.7. In particular, the preceding remark suggests the following *definition*. Every object $\text{Mot}(\mathbb{F})$ is equipped with an action of P . In particular, it has a germ of Frobenius endomorphisms. We define $\text{Mot}(\mathbb{F}_q)$ to be the category whose objects are the pairs (M, π_M) , where M is an object of $\text{Mot}(\mathbb{F})$ and π_M is a Frobenius endomorphism representing the germ and such that $\rho(\pi_M) \cdot \overline{\rho(\pi_M)} = q^m$. The resulting category $\text{Mot}(\mathbb{F}_q)$ has essentially all the properties that the category of numerical motives has in Grothendieck's motivic paradise. (See Milne 1994a.)

6.8. The category $\text{Mot}(\mathbb{F}_q)$ is semisimple. The Frobenius element π_M of a simple motive M is a Weil q -number of weight m . The isomorphism classes of simple motives are classified by the orbits of $\Gamma \stackrel{\text{def}}{=} \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ acting on $W(q)$,

$$\Sigma(\text{Mot}(\mathbb{F}_q)) \simeq \Gamma \backslash W(q).$$

Let M be a simple motive over \mathbb{F}_q . Then $E \stackrel{\text{def}}{=} \text{End}(M)$ is a simple \mathbb{Q} -algebra with centre π_M , and its invariant at a prime v of $\mathbb{Q}[\pi_M]$ is given by

$$\text{inv}_v(E) = \begin{cases} 1/2 \text{ if } v \text{ is real and } M \text{ has odd weight,} \\ \frac{\text{ord}_v(\pi_X)}{\text{ord}_v(q)} \cdot [\mathbb{Q}[\pi_X]_v : \mathbb{Q}_p] \text{ if } v|p, \\ 0 \text{ otherwise.} \end{cases}$$

Abelian motives in characteristic zero

Assume now that the reduction functor R extends to $\text{Mot}^w(\mathbb{Q}^{\text{al}})$. This allows us to prove results, otherwise unknown, about abelian motives in characteristic zero (in the sense of Deligne 1982).

Let P be the Weil-number protorus and let $G = \mathcal{A}ut^{\otimes}(\omega_B | \text{Mot}^w(\mathbb{Q}^{\text{al}}))$ — they are both affine group schemes over \mathbb{Q} . The reduction functor

$$R : \text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$$

defines a morphism of the associated bands

$$bP \rightarrow bG.$$

We make this explicit. The choice of a \mathbb{Q}^{al} -valued fibre functor on $\text{Mot}(\mathbb{F})$ and an isomorphism $\omega_B \otimes \mathbb{Q}^{\text{al}} \rightarrow \omega \circ R$ defines a homomorphism

$$P_{\mathbb{Q}^{\text{al}}} \rightarrow G_{\mathbb{Q}^{\text{al}}}$$

that is independent of the choices up to an inner automorphism of $G_{\mathbb{Q}^{\text{al}}}$. In this way we get a conjugacy class of homomorphisms $P_{\mathbb{Q}^{\text{al}}} \rightarrow G_{\mathbb{Q}^{\text{al}}}$ stable under $\text{Gal}(\mathbb{Q}^{\text{al}}/K)$. Let clG denote the scheme of conjugacy classes in G (quotient of G by its action on itself by inner automorphisms). The composite $P_{\mathbb{Q}^{\text{al}}} \rightarrow G_{\mathbb{Q}^{\text{al}}} \rightarrow (clG)_{\mathbb{Q}^{\text{al}}}$ is independent of all choices and is defined over \mathbb{Q} ,

$$\begin{array}{ccccc} P_{\mathbb{Q}^{\text{al}}} & \longrightarrow & G_{\mathbb{Q}^{\text{al}}} & \longrightarrow & (clG)_{\mathbb{Q}^{\text{al}}} \\ \uparrow & & & & \uparrow \\ P & \longrightarrow & & \longrightarrow & clG. \end{array}$$

We need a variant of this. Let $L \subset \mathbb{Q}^{\text{al}}$ be a number field, and let $\text{Mot}^w(L)$ be the category of abelian motives over L with good reduction at $w|L$, i.e., which satisfy the Néron condition. Let $\mathbb{F}_q \subset \mathbb{F}$ be the residue field at $w|L$. The functor $R : \text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ defines an exact tensor functor

$$R : \text{Mot}^w(L) \rightarrow \text{Mot}(\mathbb{F}_q),$$

and hence a morphism of bands

$$P(q) \rightarrow G, \quad G \stackrel{\text{def}}{=} \mathcal{A}ut(\omega_B | \text{Mot}^w(L)).$$

As before, this defines a morphism of schemes over \mathbb{Q} ,

$$P(q) \rightarrow clG.$$

Let $M \in \text{ob } \text{Mot}^w(L)$, and let $G_M = \mathcal{A}ut^{\otimes}(\omega_B | \langle M_{\mathbb{Q}^{\text{al}}} \rangle^{\otimes})$. Let M_0 denote the specialization of M in $\text{Mot}(\mathbb{F}_q)$. Let $\ell \neq p$ be a prime number. After possibly replacing L with a finite extension, we obtain a representation

$$\rho_{\ell} : \text{Gal}(\mathbb{Q}^{\text{al}}/L) \rightarrow G_M(\mathbb{Q}_{\ell}).$$

Let γ_{ℓ} denote the image of Frob_w in $(clG_M)(\mathbb{Q}_{\ell})$.

THEOREM 6.9. *There exists a $\gamma \in (clG_M)(\mathbb{Q})$ that maps to $\gamma_{\ell} \in (clG_M)(\mathbb{Q}_{\ell})$, all $\ell \neq p$.*

PROOF. We have a morphism (over \mathbb{Q}),

$$P(q) \rightarrow clG \rightarrow clG_M.$$

We can take γ to be the image of the universal element $\pi_{\text{univ}} \in P(q)(\mathbb{Q})$. □

NOTES. Theorem 6.9 is proved for abelian varieties in [Kisin and Zhou 2025a](#) after earlier work of Laskar and Noot. See also [Commelin 2019](#).

Integral motives

Let $k = \mathbb{F}_q$ or \mathbb{F} . For the definition of the categories $R^+(k; \hat{\mathbb{Z}})$ and $R^+(k; \mathbb{A}_f)$, we refer the reader to [Milne and Ramachandran 2004](#). We let $\text{Mot}^+(k)$ denote the subcategory of $\text{Mot}(k)$ of effective motives (triples (A, e, m) with $m \geq 0$).

DEFINITION 6.10. The category of effective integral motives $\text{Mot}^+(k, \mathbb{Z})$ over k is the full subcategory of the fibre product category

$$\mathbb{R}^+(k; \hat{\mathbb{Z}}) \times_{\mathbb{R}^+(k; \mathbb{A}_f)} \text{Mot}^+(k)$$

whose objects (X_f, X_0, x_f) are those for which the prime-to- p torsion subgroup of X_f is finite.

Thus, an effective integral motive is a triple (X_f, X_0, x_f) consisting of

- (a) an object $X_f = (X_l)_l$ of $\mathbb{R}^+(k; \mathbb{Z})$ such that X_l is torsion-free for almost all l ,
- (b) as effective motive X_0 , and
- (c) an isomorphism $x_f : (X_f)_{\mathbb{Q}} \rightarrow \omega_f(X_0)$ in $\mathbb{R}^+(k; \mathbb{A}_f)$.

For M in $\mathbb{R}_p^+(\mathbb{F}_q)$, let $r(M)$ denote the rank of M and $s(M)$ the sum of the slopes of M . Thus, if

$$P_M(T) = T^h + \dots + c,$$

then $r(M) = h$ and $s(M) = \text{ord}_p(c) / \text{ord}_p(q)$.

THEOREM 6.11. *Let X and Y be effective motives over \mathbb{F}_q (i.e., objects of $\text{Mot}^+(\mathbb{F}_q)$). The group $\text{Ext}^1(X, Y)$ is finite, and*

$$\lim_{s \rightarrow 0} \frac{\zeta(X^\vee \otimes Y)}{(1 - q^{-s})^{\rho(X, Y)}} = q^{-\chi(X, Y)} \frac{[\text{Ext}^1(X, Y)] \cdot D(X, Y)}{[\text{Hom}(X, Y)_{\text{tors}}] \cdot [\text{Hom}(Y, X)_{\text{tors}}]},$$

where

- ◊ $\chi(X, Y) = s(X_p)r(Y_p)$,
- ◊ $D(X, Y)$ is the discriminant of the pairing

$$\text{Hom}(Y, X) \times \text{Hom}(X, Y) \rightarrow \text{End}(Y) \xrightarrow{\text{trace}} \mathbb{Z}.$$

PROOF. See [Milne and Ramachandran 2004](#), 10.1. □

ASIDE 6.12. Compare 6.11 with the following result ([Milne 1968](#)). If A and B are abelian varieties over \mathbb{F}_q , then

$$q^{\dim(A)\dim(B)} \prod_{a_i \neq b_j} \left(1 - \frac{a_i}{b_j}\right) = [\text{Ext}^1(A, B)] \cdot D(A, B)$$

where

- ◊ $(a_i)_{1 \leq i \leq 2 \dim A}$ and $(b_i)_{1 \leq i \leq 2 \dim B}$ are the roots of the characteristic polynomials of the Frobenius endomorphisms of A and B ,
- ◊ $D(A, B)$ is the discriminant of the pairing

$$\text{Hom}(B, A) \times \text{Hom}(A, B) \rightarrow \text{End}(B) \xrightarrow{\text{trace}} \mathbb{Z}.$$

Almost rational Tate classes

6.13. Assuming Conjecture B for CM abelian varieties over \mathbb{Q}^{al} , we have shown how to construct tannakian categories of abelian motives $\text{Mot}(\mathbb{F}_{p^n})$ for all p and n . We now explain how to obtain tannakian categories of abelian motives $\text{Mot}(k)$ for all fields k . For simplicity, we take k to be algebraically closed.

6.14. Let A be an abelian variety over k . An *almost-RT class* of codimension n on A is an element $\gamma \in H_{\mathbb{A}}^{2n}(A)(n)$ such that there exists a cartesian square

$$\begin{array}{ccc} X & \longleftarrow & A \\ \downarrow f & & \downarrow \\ S & \longleftarrow & \text{Spec}(k) \end{array}$$

and a global section $\tilde{\gamma}$ of $R^{2n}f_*\mathbb{A}(n)$ satisfying the following conditions

- ◊ S is a connected normal scheme of finite type over $\text{Spec } \mathbb{Z}$;
- ◊ $f : X \rightarrow S$ is an abelian scheme over S ;
- ◊ the fibre of $\tilde{\gamma}$ over $\text{Spec}(k)$ is γ , and the specialization of $\tilde{\gamma}$ at s is rational Tate for all closed points s in a dense open subset U of S .

Note that the residue field $\kappa(s)$ at a closed point of S is finite, so it makes sense to require $\tilde{\gamma}_s$ to be rational Tate.

6.15. Let $\text{Mot}(k)$ denote the category of motives based on the abelian varieties over k using the almost-RT classes as correspondences. Then $\text{Mot}(k)$ is a tannakian category over \mathbb{Q} with many of the properties anticipated for Grothendieck's category of abelian motives.

QUESTION 6.16. Does the Tate conjecture for almost-RT classes on abelian varieties hold over finitely generated fields?

QUESTION 6.17. Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme S of finite type over \mathbb{Z} . Is the set of closed $s \in S$ such that γ_s is rational Tate closed?

PROPOSITION 6.18. *Assume that 6.16 and 6.17 have positive answers. Let $f : A \rightarrow S$ be an abelian scheme over a connected normal scheme of finite type over \mathbb{F} , and let γ be a global section of $R^{2n}f_*\mathbb{A}(n)$. If $\gamma_s \in H_{\mathbb{A}}^{2n}(A_s)(n)$ is a rational Tate class for one $s \in S(\mathbb{F})$, then it is a rational Tate class for all $s \in S(\mathbb{F})$.*

PROOF. As in the proof of 3.7, we may replace \mathbb{F} with a finite subfield. Consider the diagram

$$\begin{array}{ccc} \mathcal{R}^n(A_\eta) & \xrightarrow{\otimes} & H_{\mathbb{A}}^{2n}(A_\eta)(n)^{\pi_1(\eta)} \\ \simeq \uparrow & & \uparrow c \\ \mathcal{R}^n(A/S) & \xrightarrow{b} & H^0(S, R^{2n}f_*\mathbb{A}(n)) \\ \downarrow & & \downarrow \\ \mathcal{R}^n(A_s) & \xrightarrow{\otimes} & H_{\mathbb{A}}^{2n}(A_s)(n)^{\pi_1(s)}. \end{array}$$

Because the Tate conjecture holds for rational Tate classes,

$$\mathcal{R}^n(A_s) \otimes_{\mathbb{Q}} \mathbb{A} \xrightarrow{\cong} H_{\mathbb{A}}^{2n}(A_s)(n)^{\pi_1(s)}.$$

Let $\mathcal{R}^n(A_\eta)$ denote the space of almost-RT classes on A_η . Because we are assuming the Tate conjecture for almost-RT classes (6.16),

$$\mathcal{R}^n(A_\eta) \otimes_{\mathbb{Q}} \mathbb{A} \xrightarrow{\cong} H_{\mathbb{A}}^{2n}(A_\eta)(n)^{\pi_1(\eta)}.$$

Define $R^n(A/S)$ to be the space of global sections γ of $R^{2n}f_*\mathbb{A}(n)$ such that $\gamma_s \in H_{\mathbb{A}}^{2n}(A_s)(n)$ is a rational Tate class for all $s \in S(\mathbb{F}_q)$. The map

$$\mathcal{R}^n(A/S) \rightarrow \mathcal{R}^n(A_\eta)$$

is injective, and, because of (6.17), it is surjective. Now we can apply Lemma 3.4 to obtain the equality,

$$\mathcal{R}^n(A/S) = \mathcal{R}^n(A_s) \cap H^0(S, R^{2n}f_*\mathbb{A}(n))$$

(intersection inside $H_{\mathbb{A}}^{2n}(A_s)(n)$). Thus, if γ_s is rational Tate for one s , it lies in $\mathcal{R}^n(A/S)$, which means that γ_s is rational Tate for all s . \square

THEOREM 6.19. *Assume that 6.16 and 6.17 have positive answers. All Hodge classes on abelian varieties over \mathbb{Q}^{al} with good reduction at w specialize to rational Tate classes.*

PROOF. First prove this for split Weil classes (see the proof of Theorem 3.33). Then deduce it for Hodge classes on CM abelian varieties (apply 1.39). Finally, deduce the general case by the argument in §6 of Deligne 1982. \square

COROLLARY 6.20. *Under the assumptions of the theorem, all Hodge classes on abelian varieties over fields of characteristic zero are almost-RT.*

PROOF. For \mathbb{Q}^{al} , this follows from Theorem 6.19. For the general case, specialize first to \mathbb{Q}^{al} . \square

Comparison with the constructions in Langlands and Rapoport 1987

Recall that we have canonically-defined tannakian categories and quotient functors,

$$\text{CM}(\mathbb{Q}^{\text{al}}) \xleftarrow{J} \text{LCM}(\mathbb{Q}^{\text{al}}) \xrightarrow{R} \text{LMot}(\mathbb{F}).$$

Let ω^R be the functor on $\text{LCM}(\mathbb{Q}^{\text{al}})^L$ defined by R , so

$$\omega^R(X) = \text{Hom}(\mathbf{1}, R(X)).$$

We have the following statement.

THEOREM 6.21. *There exists a unique \mathbb{Q} -valued fibre functor ω_0 on $\text{CM}(\mathbb{Q}^{\text{al}})^P$ such that*

$$\omega_0(J(X)) = \omega^R(X)$$

for all X in $\text{LCM}(\mathbb{Q}^{\text{al}})$. Moreover, ω_0 provides a \mathbb{Q} -structure for $\omega_{\mathbb{A}}$.

Because $\text{LCM}(\mathbb{Q}^{\text{al}})^L \rightarrow \text{CM}(\mathbb{Q}^{\text{al}})^P$ is a quotient functor, the uniqueness is obvious. That ω_0 is a \mathbb{Q} -structure on $\omega_{\mathbb{A}}$ follows from the fact that ω^R is a \mathbb{Q} -structure on $\omega_{\mathbb{A}}$. The proof of the existence requires Conjecture B (in fact, is equivalent to it).

Using cohomology, it is possible to prove only the following weaker result.

THEOREM 6.22. *There exists a \mathbb{Q} -valued fibre functor ω on $\text{CM}(\mathbb{Q}^{\text{al}})^P$ such that $\omega \otimes_{\mathbb{Q}} \mathbb{Q}_l \approx \omega_l$ for all l . Any two become isomorphic on any algebraic subcategory of $\text{CM}(\mathbb{Q}^{\text{al}})$. The set of isomorphism classes of such ω is a principal homogeneous space for $\varprojlim_{\mathcal{F}} C(K)$, where \mathcal{F} is the set of CM-subfields of \mathbb{Q}^{al} finite over \mathbb{Q} and $C(K)$ is the ideal class group of K .*

PROOF. As $\text{CM}(\mathbb{Q}^{\text{al}})$ has a canonical fibre functor ω_B , the isomorphism classes of \mathbb{Q} -valued fibre functors on $\text{CM}(\mathbb{Q}^{\text{al}})^P$ are classified by the cohomology group $H^1(\mathbb{Q}, S/P)$. The proof of the existence of ω occupies a large part of the article [Langlands and Rapoport 1987](#). For the rest, see Theorem 4.1 of [Milne 2003](#). \square

We can now choose a \mathbb{Q} -valued subfunctor ω_0 of $\omega_{\mathbb{A}}$ such that $\omega_0 \otimes_{\mathbb{Q}} \mathbb{A} = \omega_{\mathbb{A}}$. We define $\text{Mot}(\mathbb{F})$ to be the quotient $q : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ of $\text{CM}(\mathbb{Q}^{\text{al}})$ corresponding to the functor ω_0 on $\text{CM}(\mathbb{Q}^{\text{al}})^P$ (see [Milne 2007](#)). As $\text{CM}(\mathbb{Q}^{\text{al}})$ is semisimple, this has an explicit description (ibid. 2.12 et seq.). Apart from involving a choice, this definition does not give an object with the wished for properties.

Comparison with Grothendieck's categories of motives

Let $\text{CM}_{\text{num}}(\mathbb{Q}^{\text{al}})$ and $\text{Mot}_{\text{num}}(\mathbb{F})$ be the categories of CM and abelian motives defined using algebraic cycles modulo numerical equivalence.

PROPOSITION 6.23. *The following statements are equivalent:*

- (a) *the functor $\text{CM}_{\text{num}}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ factors through the functor $\text{CM}_{\text{num}}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}_{\text{num}}(\mathbb{F})$;*
- (b) *an object M of $\text{Mot}(\mathbb{F})$ is trivial if and only if the Frobenius element $\pi_M = 1$;*
- (c) *the Tate conjecture holds for all abelian varieties over \mathbb{F} .*

PROOF. For the equivalence of (a) and (b), see [Milne 2007d](#). For the equivalence of (b) and (c), see [Geisser 1998](#). \square

Comparison with André's categories

6.24. Fix a prime number $\ell \neq p$, and let $\text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell})$ denote the \mathbb{Q}_{ℓ} -linear category based on abelian varieties and using Tate classes as correspondences. For some countable subfield Q of \mathbb{Q}_{ℓ} , André defines a Q -linear category $\text{Mot}(\mathbb{F}; Q)$ based on abelian varieties and using motivated classes as correspondences. There are canonical exact tensor functors

$$\text{Mot}(\mathbb{F}) \rightarrow \text{Mot}(\mathbb{F}; Q) \rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell})$$

such that

$$\begin{aligned} \text{Mot}(\mathbb{F})_{(Q)} &\rightarrow \text{Mot}(\mathbb{F}; Q) \\ \text{Mot}(\mathbb{F}; Q)_{(\mathbb{Q}_{\ell})} &\rightarrow \text{Mot}(\mathbb{F}; \mathbb{Q}_{\ell}) \end{aligned}$$

are equivalences of tensor categories.

To prove this, note that André (2006a, 2006b) shows that motivated classes on abelian varieties satisfy the conditions (R1), (R2), (R3), and (R4) of Conjecture C. Since the spaces $\mathcal{R}^*(A) \otimes_{\mathbb{Q}} \mathbb{Q}$ also satisfy these conditions, the argument in the proof of Theorem 0.2 shows that the two families coincide.

SUMMARY 6.25. Every rational Tate class on an abelian variety over \mathbb{F} becomes motivated over \mathbb{Q} , and the space of rational Tate classes is a \mathbb{Q} -structure on the \mathbb{Q} -space of motivated classes.

7 Shimura varieties of abelian type

In this section, we assume that there exists a commutative diagram

$$\begin{array}{ccccc}
 \text{LMot}^w(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}^w(\mathbb{Q}^{\text{al}}) & & \\
 \downarrow R & & \downarrow R & \searrow \xi_f & \\
 \text{LMot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}; \mathbb{A}_f) \\
 & \searrow & \text{---} & \nearrow & \\
 & & & \xi_f &
 \end{array} \tag{7}$$

such that the functor

$$\text{Mot}(\mathbb{F})_{(\mathbb{A}_f)} \longrightarrow \text{Mot}(\mathbb{F}; \mathbb{A}_f)$$

is a tensor equivalence, and we investigate its applications to Shimura varieties.

Introduction

7.1. Since the 1970s, Deligne has been promoting the idea that Shimura varieties with rational weight should be thought of as moduli schemes for motives with additional structure. Indeed this is a powerful tool for discovery, which has been used most prominently by Langlands in his work on understanding the zeta functions of Shimura varieties. In his Corvallis article (1979), Langlands applied it to find a conjectural description of the conjugate of a Shimura variety — this is needed to compute the factors at infinity of the zeta function. In his article with Rapoport (1987), Langlands applied it to find a conjectural description of the points of the Shimura variety modulo p — this is needed to compute the factors at the finite places of the zeta function.

7.2. Recall that the Shimura varieties of abelian type are, by definition, exactly those for which Deligne proved the existence of a canonical model in his Corvallis article (1979). Let (G, X) be a Shimura datum of abelian type with rational weight, and chose an algebra (V, t) over \mathbb{Q} such that $G = \text{Aut}(V, t)$.¹⁷ Such a choice realizes $\text{Sh}(G, X)$ as a moduli scheme (in the category of complex analytic spaces) for polarizable rational Hodge structures with algebra and level structure. It follows from Theorem 1.7 that the

¹⁷See Milne 2020a. By an algebra over \mathbb{Q} we mean a finite-dimensional \mathbb{Q} -vector space V together with a linear map $V \otimes V \rightarrow V$ (no conditions). Readers may prefer to take any V and family of tensors determining G .

pair (G, X) is of abelian type if and only if the Hodge structures in the moduli family are in the essential image of the (fully faithful) Betti functor

$$\text{Mot}(\mathbb{C}) \rightarrow \text{Hdg}_{\mathbb{Q}}.$$

From the theorem of Borel (1972), it follows that, when (G, X) is of abelian type, this realization becomes a realization of $\text{Sh}(G, X)$ as a moduli scheme (in the category of algebraic schemes over \mathbb{C}) for abelian motives with algebra and level structure. From this modular realization, it is possible to read off a proof Langlands's conjecture on conjugates. Elementary descent theory gives a proof of the existence of canonical models that is both simpler and more natural than the original — the Shimura variety is defined over the number field because the moduli problem is defined over the number field. Moreover, this approach provides a *description* of the canonical model as a moduli scheme, whereas Deligne's original approach (1979) provides only a *characterization* of it in terms of reciprocity laws at the special points.

7.3. When the existence of the diagram (7, p. 61) is assumed, the theory outlined above extends to characteristic p . Specifically, suppose that G is unramified at p , and suppose that the \mathbb{Q} -algebra (V, t) is chosen to satisfy the condition 3.2.3 of [Kisin 2020](#). The moduli problem over the reflex field can be extended over its ring of integers, and the corresponding moduli scheme is smooth. This gives us a smooth integral model of $\text{Sh}(G, X)$ and a modular interpretation of its functor of points. The modular description of the points with coordinates in \mathbb{F} can be regarded as a categorification of the conjectural description in Langlands and Rapoport (1987). To obtain their original description is an exercise in tannakian theory. Besides the integral model of the Shimura variety, one obtains in this way an integral model of the standard principal bundle, and hence integral models of the automorphic vector bundles on the Shimura variety.

Characteristic zero

7.4. Recall that, for a field k of characteristic zero, $\text{Mot}(k)$ denotes the category of abelian motives over k (defined using absolute Hodge classes). Let S be a connected smooth algebraic scheme over a field k of characteristic zero, and let η be its generic point. We define an abelian motive M over S to be an abelian motive M_{η} over $k(\eta)$ such that the action of $\pi_1(\eta, \bar{\eta})$ on $\omega_f(M)$ factors through $\pi_1(S, \bar{\eta})$. See [Milne 1994b](#), 2.37.¹⁸

7.5. Let (G, X) be a Shimura datum. In order for the Shimura variety $\text{Sh}(G, X)$ to be a moduli variety for motives, it is necessary that every special point be CM. This is true when (G, X) satisfies the conditions:

- (a) the central character w_X is defined over \mathbb{Q} and
- (b) the connected centre of G is split by a CM field.

See [Milne 1988](#), A.3. From now on, we always assume the condition (b).¹⁹

¹⁸For hints on how to extend the definition to nonsmooth schemes, see *ibid.* 2.45.

¹⁹It is the author's view that pairs (G, X) failing (b) are pathological and should be excluded, but Deligne disagrees.

7.6. Let (G, X) be a Shimura datum such that w_X is rational. Let K be a (small) compact open subgroup of $G(\mathbb{A}_f)$, and let (V, t) be an algebra such that $G = \text{Aut}(V, t)$.²⁰ Then $\text{Sh}_K(G, X)$ is the solution of a moduli problem \mathcal{H}_K on the category of smooth algebraic schemes over \mathbb{C} . More precisely, for such a scheme S , $\text{Sh}_K(G, X)(S)$ classifies certain triples (\mathbb{V}, t, η) , where \mathbb{V} is a variation of Hodge structures on S , $t : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V}$ is an algebra structure on \mathbb{V} , and η is a K -level structure. If the largest \mathbb{R} -split torus in $Z(G)$ is already split over \mathbb{Q} , then $\text{Sh}_K(G, X)$ is a fine moduli scheme. See [Milne 1994b](#), 3.10, 3.11.

7.7. The elements of $\mathcal{H}_K(\mathbb{C})$ (in 7.6) are the Betti realizations of abelian motives if and only if (G, X) is of abelian type.²¹ When this is the case, $\text{Sh}_K(G, X)$ is a moduli scheme over \mathbb{C} for abelian motives with additional structure, and a fine moduli scheme if $Z(G)$ satisfies the condition in 7.6. See [Milne 1994b](#), 3.13.

7.8. Let (G, X) be a Shimura datum of abelian type with rational weight. From (7.7), it is possible to read off a proof Langlands's conjugation conjecture, except with the Taniyama group in place of Langlands's group. See [Milne 1990](#), 4.2. To complete the proof, one needs to use that the two groups are equal (1.17).

7.9. Let (G, X) be a Shimura datum of abelian type with rational weight, and let F be a number field such that the moduli problem in 7.7 is defined over F . Then an elementary descent argument ([Milne 1999a](#)) shows that $\text{Sh}_K(G, X)$ has model over F that is a solution to the moduli problem. When $F \subset \mathbb{C}$ is the reflex field, we get the canonical model of $\text{Sh}_K(G, X)$ in the original sense of [Deligne 1979](#); otherwise, we get the canonical model in the sense of [Sempliner and Taylor 2025](#).

7.10. Let (G, X) be a Shimura datum, and let G^c denote the quotient of G by the largest subtorus of $Z(G)$ that is split over \mathbb{R} but has no subtorus that is split over \mathbb{Q} . Let

$$P(G, X) = G(\mathbb{Q}) \backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f) / G(\mathbb{Q})^-.$$

It is principal bundle G^c -bundle with a flat connection, called the standard principal bundle. There is a canonical equivariant morphism $\gamma : P(G, X) \rightarrow X^\vee$. The automorphic vector bundles are obtained as follows: start with a G^c -vector bundle on X^\vee , pull it back to $P(G, X)$, and descend it to $\text{Sh}(G, X)$. See [Milne 1990](#), III.

Now assume that (G, X) is of abelian type and is a fine moduli scheme for abelian motives with additional structure (7.7). The system

$$\text{Sh}_K(G, X) \longleftarrow P_K(G, X) \longrightarrow X^\vee \tag{8}$$

can be re-constructed from the universal abelian motive over $\text{Sh}_K(G, X)$ (cf. [Milne 1990](#), 3.3). From this, we can read off

- (a) a description of the conjugate of the entire system (8) by an automorphism of \mathbb{C} , extending Langlands's description of the conjugate of $\text{Sh}_K(G, X)$ (but necessarily expressed in terms of the period torsor);

²⁰See [Milne 2020a](#). Here and elsewhere, the reader may prefer to take any vector space V and family of tensors determining G .

²¹This can be proved by comparing Deligne's characterization of the Shimura varieties he considers with the characterization of the algebraic quotients of G in [Theorem 1.7](#).

- (b) the existence of a canonical model of the system (8), extending the existence of the canonical model of $\mathrm{Sh}(G, X)$.

7.11. For an arbitrary Shimura variety $\mathrm{Sh}(G, X)$ of abelian type, not necessarily with rational weight, there are morphisms

$$\mathrm{Sh}(G, X) \times \mathrm{Sh}(Z_*, \epsilon) \rightarrow \mathrm{Sh}(G_*, X_*),$$

where Z_* is a torus and $\mathrm{Sh}(G_*, X_*)$ is of abelian type with rational weight. These can be used to deduce statements about $\mathrm{Sh}(G, X)$ from statements about $\mathrm{Sh}(G_*, X_*)$. See [Milne 1994b](#), 3.33–3.37.

NOTES. For more details, see [Milne 1990](#), II,3; [Milne 1994b](#), §3; [Milne 2013](#).

Mixed characteristic

7.12 (SERRE-TATE). Let $S = \mathrm{Spec} R$ be an artinian local scheme with (closed) point s such that $k \stackrel{\mathrm{def}}{=} \kappa(s)$ has characteristic $p > 0$. The functor $A \rightsquigarrow (A_s, T_p A, \mathrm{id})$ is an equivalence from the category of abelian schemes over S to the category of triples (A_0, X, φ) , where A_0 is an abelian variety over k , X is a p -divisible group over S , and φ is an isomorphism $X_s \rightarrow T_p(A_0)$.

7.13 (FONTAINE). Let S be an artinian local scheme, as in 7.12. The functor sending a p -divisible group X over S to (L, M) , where M is the covariant Dieudonné module of X_s and L is an R -submodule of $M \otimes_R k = VM/pM$, is an equivalence of categories.

7.14. Let S be an artinian local scheme, as in 7.12, but with k equal \mathbb{F}_q or \mathbb{F} . On combining the last two statements, we see that, to give an abelian scheme over S is equivalent to giving an abelian variety A_0 over k and a lifting of the filtration on the covariant Dieudonné module of A_0 . This suggests *defining* an abelian motive over S to be an abelian motive M over k (object of $\mathrm{Mot}(k)$) and a lifting of the filtration on the crystalline homology groups of M .

7.15. There is a similar statement (and definition) when S is the spectrum of a complete noetherian local ring with residue field \mathbb{F}_q or \mathbb{F} .

7.16. More generally, we define an abelian motive over a perfectoid space S over \mathbb{F} to be a triple (M_0, X, φ) , where M_0 is an abelian motive over \mathbb{F} , X is a p -adic shtuka over S , and φ is an isomorphism from X_s to the p -adic shtuka of M_0 .

7.17. Let (G, X) be a Shimura datum of abelian type satisfying the conditions (a) and (b) of 7.5 (to be a moduli scheme). Assume that G is unramified at p , and let $\mathrm{Sh}_p(G, X) = \mathrm{Sh}_{K_p K_p}(G, X)$ with K_p hyperspecial. As explained above, when we write $G = \mathrm{Aut}(V, t)$, then we obtain a model of $\mathrm{Sh}_p(G, X)$ over the reflex field E and a description of it as a moduli scheme for abelian motives with additional structure. Now assume that (V, t) can be chosen to satisfy the condition 3.2.3 of [Kisin 2020](#). Then the moduli problem extends to schemes over \mathcal{O}_w , where \mathcal{O} is the ring of integers in E , and standard methods show that it has a solution that is a smooth scheme over \mathcal{O}_w . In this way, we get

- (a) an integral canonical model of $\mathrm{Sh}_p(G, X)$ over \mathcal{O}_w ;
- (b) a description of the model as a moduli variety for abelian motives with algebra and level structure (in particular, a categorification of the conjecture of Langlands and Rapoport);
- (c) an integral model of the system (8), extending that of $\mathrm{Sh}_p(G, X)$;
- (d) an integral theory of automorphic vector bundles on $\mathrm{Sh}_p(G, X)$.

SUMMARY 7.18. In conclusion, our theory of motives allows us to identify the functor of points of a Shimura variety of abelian type with rational weight (which would delight Grothendieck), but for those without rational weight we are forced to apply the trick of Shimura (which would delight Serre).

8 Shimura varieties not of abelian type

What about Shimura varieties not of abelian type? It remains an open, and very interesting question, whether the polarizable Hodge structures arising from all Shimura varieties are motivic. Absent a proof of that, we are stuck with the old methods for proving Langlands's conjugacy conjecture and the existence of canonical models. Concerning these, here is my response (June 14, 2025) to a query from Richard Taylor (slightly edited for clarity).

Rather than revisiting the ad hoc methods I use in §6 of my 1983 paper to prove compatibility for different special points, I think one should instead use the following beautiful result of Borovoi.

THEOREM 1. *Let G be a simply connected semisimple algebraic group over a totally real algebraic number field F . Assume that G has an anisotropic maximal torus T that splits over some totally imaginary quadratic extension K of the field F . Let Π be a base of the root system $R = R(G_K, T_K)$. Then $G(F^{\mathrm{rc}})$ is generated by the subgroups $G_\alpha(F^{\mathrm{rc}})$, $\alpha \in \Pi$ (here F^{rc} is a totally real closure of F).*

THEOREM 2. *Under the conditions of Theorem 1, assume that G is a geometrically simple group of totally hermitian type that is not totally compact. Then $G(F^{\mathrm{rc}})$ is generated by the subgroups $G_\alpha(F^{\mathrm{rc}})$, $\alpha \in R^{\mathrm{rtc}}$.*

In his 1983/84 paper, Borovoi made use of a stronger statement, which still hasn't been proved, but later he did prove Theorems 1 and 2 with the help of his Russian colleagues. See,

The group of points of a semisimple group over a totally real closed field
Borovoi, M. V., *Selecta Math. Soviet.* 9 (1990), no. 4, 331–338.

In the last two sections of my 1988 *Inventiones* paper, I gave two proofs of the compatibility, one with and one without Borovoi's statement.

But, as I mentioned briefly at the [Tate 100] conference, I think the whole business of canonical models (in the general case) needs to be rethought.

At present we

- (a) use Kazhdan's theorem that conjugates of arithmetic varieties are arithmetic varieties (arithmetic variety = $\text{bsd}/\text{arithmetic group}$);
- (b) deduce the conjugation theorem for Shimura varieties (Borovoi–Milne);
- (c) prove the conjugation theorem for the standard principal bundle (Milne, 1988, Inventiones).

What Nori and Raghunathan (1993) show is that Kazhdan asked the wrong question. Let D be a bounded symmetric domain and G a real algebraic group such that $G(\mathbb{R})$ acts transitively on D with compact isotropy groups. From an arithmetic Γ we get a system

$$(Y, P, \nabla, D^\vee, \gamma), \quad Y \leftarrow (P, \nabla) \xrightarrow{\gamma} D^\vee,$$

where $Y = \Gamma \backslash D$, P is the principal bundle $\Gamma \backslash D \times G(\mathbb{C})$, ∇ is a flat connection, D^\vee is the compact dual, and $\gamma : P(\Gamma) \rightarrow D^\vee$ is defined by the Borel map. This system is algebraic, and the correct question to ask is that the conjugate of such a system be again such a system. Nori and Raghunathan characterize such systems and show that the characterizing properties are preserved under conjugation. This is much much simpler than Kazhdan.

From a Shimura datum (G, X) , we get a similar system

$$(S, P, \nabla, X^\vee, \gamma), \quad S \leftarrow (P, \nabla) \xrightarrow{\gamma} X^\vee$$

and I think, similarly, that one should work directly with such systems instead of just the Shimura variety. This should make everything simpler — the conjugation conjecture, canonical models, and even integral canonical models. I talked about this at the Borel conference at Hangzhou in 2004, but haven't worked out the details.

A little history. Langlands made little progress in understanding the zeta functions of Shimura varieties until Deligne explained to him his axioms (especially $h!$) and that he should think of them as moduli varieties of motives. Langlands stated his Corvallis conjecture in order to understand the factors of the zeta function at infinity, and his conjecture with Rapoport to understand the factors at finite places. When I asked Langlands how he came up with the "cocycle" for the conjugation conjecture, he just said that it was the only thing he could think of. When he explained it to Deligne, Deligne realized that it conjecturally gave an explicit description of the Taniyama group, something that he and others had been searching for.

9 Mixed Shimura varieties

To be continued.

10 Appendix: The cohomological approach.

Tannakian categories and the functors between them are classified by cohomology sets. In particular, our conjectures predicts the existence of certain cohomology classes with particular properties. Here, we briefly look at three examples. The results are positive, but inconclusive, and do not seem to be helpful in proving the existence of the categories of motives.

Tannakian theory and cohomology (0)

Recall (Theorem 1.18) that we have a commutative diagram of bands

$$\begin{array}{ccc} G & \longrightarrow & G_{\mathbb{Q}^{\text{al}}} \\ \uparrow & & \uparrow \\ P & \longrightarrow & P_{\mathbb{Q}^{\text{al}}}. \end{array}$$

A necessary condition for this to extend to a commutative diagram of tannakian categories

$$\begin{array}{ccc} \text{Mot}^w(\mathbb{Q}^{\text{al}}) & \longrightarrow & \text{Mot}(\mathbb{Q}^{\text{al}}; \mathbb{Q}_l) \\ \downarrow & & \downarrow \\ \text{Mot}(\mathbb{F}) & \longrightarrow & \text{Mot}(\mathbb{F}; \mathbb{Q}_l) \end{array}$$

is that the class of $\text{Mot}(\mathbb{F})$ in $H^2(\mathbb{Q}, P)$ map to the trivial class in $H^2(\mathbb{Q}, G)$. Certainly, it maps to the trivial class in $H^2(\mathbb{Q}_l, G)$ for all l (including $l = p, \infty$), so the necessary condition holds if the Hasse principle holds for $H^2(\mathbb{Q}, G)$. There is an exact sequence

$$1 \rightarrow G^{\text{der}} \rightarrow G \rightarrow S \rightarrow 1.$$

For any CM field K , $H^2(G, S^K)$ satisfies the Hasse principle and we can apply the following theorem of Douai to the algebraic quotients of G^{der} .

THEOREM 10.1. *Let G be a semisimple algebraic group over a field k . The gerbes over Aff_k locally bound by G are trivial in each of the following cases:*

- (a) k is a local field with finite residue field;
- (b) k is a number field with no real prime;
- (c) k is a global field and G is simply connected;
- (d) k is a number field and G is a semisimple group of type ${}^2A_n, B_n, C_n, {}^1D_{2n}, {}^2D_{2n+1}, {}^2E_6, E_7, E_8, F_4$, or G_2 .

PROOF. See [Douai 1975](#). □

Tannakian theory and cohomology (1)

We apply the theory of tannakian categories to the problem of extending the reduction functor from $\text{CM}(\mathbb{Q}^{\text{al}})$ to $\text{Mot}^w(\mathbb{Q}^{\text{al}})$. We begin with a preliminary.

PROPOSITION 10.2. *Let G be a semisimple algebraic group over a field k and T a torus in G . If G is simply connected, then so also is the derived group of $C_G(T)$.*

PROOF. See, for example, [Conrad 2014](#), 6.5.2(iv). □

Let \mathcal{T} be a tannakian category over a field k . Then

$$\text{FIB}(\mathcal{T}) \stackrel{\text{def}}{=} \text{HOM}(\mathcal{T}, \text{Vec}_k)$$

is a gerbe over Aff_k . We call its band G the band of \mathcal{T} . For any fibre functor ω of \mathcal{T} , G is represented by $\text{Aut}^{\otimes}(\omega)$ equipped with its natural descent datum up to inner

automorphisms. The gerbe $\text{FIB}(T)$ defines a class in $H^2(k, G)$, which is trivial if and only if T is neutral.

More generally, let C and D be tannakian categories over k , and let $u : H \rightarrow G$ be a morphism from the band of D to that of C . The morphisms of tannakian categories (exact tensor functors) $C \rightarrow D$ banded by u form a gerbe

$$\text{HOM}_u(C, D)$$

over Aff_k . Its band is the centralizer C_u of u (apply [Giraud 1971](#), IV, 2.3.2, to $\text{FIB}(C)$ and $\text{FIB}(D)$). The gerbe $\text{HOM}_u(C, D)$ defines a class in $H^2(k, C_u)$, which is trivial if and only if there exists a morphism of tannakian categories $C \rightarrow D$ banded by u .

Now let $\text{Mot}'(\mathbb{Q}^{\text{al}})$ denote the tannakian subcategory of $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ generated by the abelian varieties in $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ whose Mumford–Tate group has simply connected derived group (cf. [Theorem 1.8](#)). The band G of $\text{Mot}'(\mathbb{Q}^{\text{al}})$ has simply connected derived group. Let $u : P \rightarrow G$ be the morphism of bands underlying the canonical morphisms $P_{\mathbb{Q}_l} \rightarrow G_{\mathbb{Q}_l}$ (see [1.18](#)). Consider the commutative diagram of gerbes over Aff_k ,

$$\begin{array}{ccc} \text{HOM}_u(\text{Mot}'(\mathbb{Q}^{\text{al}}), \text{Mot}(\mathbb{F})) & \longrightarrow & \text{HOM}_v(\text{CM}(\mathbb{Q}^{\text{al}}), \text{Mot}(\mathbb{F})) \\ \downarrow & & \downarrow \\ \text{HOM}_u(\text{Mot}'(\mathbb{Q}^{\text{al}}), \mathbb{V}_l(\mathbb{F})) & \longrightarrow & \text{HOM}_v(\text{CM}(\mathbb{Q}^{\text{al}}), \mathbb{V}_l(\mathbb{F})). \end{array}$$

Here $v : P \rightarrow S$ is the Shimura–Taniyama homomorphism, the horizontal morphisms are defined by the inclusion $\text{CM}(\mathbb{Q}^{\text{al}}) \hookrightarrow \text{Mot}'(\mathbb{Q}^{\text{al}})$, and the vertical morphisms are defined by the exact tensor functors $\eta_l : \text{Mot}(\mathbb{F}) \rightarrow \mathbb{V}_l(\mathbb{F})$. We have an object $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ in the gerbe at top-right and an object $\xi_l : \text{Mot}'(\mathbb{Q}^{\text{al}}) \rightarrow \mathbb{V}_l(\mathbb{F})$ in the gerbe at bottom-left, which map to the same object in the gerbe at bottom-right, and we seek an object in the gerbe at top-left that maps to both R and ξ_l (for all l). Let $C = C_u$ (centralizer of u). Then C has a filtration whose quotients are

$$C^{\text{der}}, \quad \text{a torus}, \quad S.$$

According to [Proposition 10.2](#), C^{der} is simply connected.

To simplify things, we now pretend that the torus is trivial. Also, we ignore some of the complexities of nonabelian cohomology. What follows is only heuristic.

PROPOSITION 10.3 (?). *There exists a morphism of tannakian categories $R' : \text{Mot}'(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ such that*

- (a) $R' | \text{CM}(\mathbb{Q}^{\text{al}}) = R$,
- (b) $R' \otimes \mathbb{Q}_l \approx \xi_l$ for all l .

PROOF. Consider the maps

$$H^2(\mathbb{Q}, P) \rightarrow H^2(\mathbb{Q}, C) \rightarrow H^2(\mathbb{Q}, S).$$

Because $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ exists, the class of $\text{Mot}(\mathbb{F})$ in $H^2(\mathbb{Q}, P)$ maps to “zero” in $H^2(\mathbb{Q}, S)$, but, by Douai’s theorem, the map $H^2(\mathbb{Q}, C) \rightarrow H^2(\mathbb{Q}, S)$ is injective, and so it maps to zero in $H^2(\mathbb{Q}, C)$. Thus, R extends to a morphism of tannakian categories

$\text{Mot}'(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$. Choose one extension R' . On comparing $R' \otimes \mathbb{Q}_l$ to ξ_l , we get a class $a_l \in H^1(\mathbb{Q}_l, C)$. The (fake) exact sequence

$$0 \rightarrow C^{\text{der}} \rightarrow C \rightarrow S \rightarrow 0,$$

gives rise to an exact commutative diagram

$$\begin{array}{ccccc} H^1(\mathbb{Q}, C^{\text{der}}) & \longrightarrow & H^1(\mathbb{Q}, C) & \longrightarrow & H^1(\mathbb{Q}, S) \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ H^1(\mathbb{R}, C^{\text{der}}) & \longrightarrow & \prod_l H^1(\mathbb{Q}_l, C) & \longrightarrow & \prod_l H^1(\mathbb{Q}_l, S), \end{array}$$

in which the product signs mean restricted product. As C^{der} is simply connected, $H^1(\mathbb{Q}_l, C^{\text{der}}) = 0$ for all $l \neq \infty$ and the first vertical arrow is an isomorphism (e.g., [Milne 2017](#), 25.61, 25.63). Moreover, the map $H^1(\mathbb{Q}, C) \rightarrow H^1(\mathbb{Q}, S)$ is surjective by the theorem of Douai, and the map $H^1(\mathbb{Q}, S) \rightarrow \prod_l H^1(\mathbb{Q}_l, S)$ is an isomorphism, at least on the finite level (see [Milne 2003](#), §3). Now a diagram chase starting from $(a_l)_l \in \prod_l H^1(\mathbb{Q}_l, C)$ gives us an element of $H^1(\mathbb{Q}, C)$ that can be used to modify R' to obtain a morphism with the required properties. \square

Note that [Proposition 10.3](#) is not what we need. Specifically, we need the statement with $\text{Mot}^w(\mathbb{Q}^{\text{al}})$ for $\text{Mot}'(\mathbb{Q}^{\text{al}})$ and, more crucially, with $R' \otimes \mathbb{Q}_l = \xi_l$ (not isomorphic).

REMARK 10.4. It would be better to fix a “large” CM subfield K of \mathbb{C} , finite and Galois over \mathbb{Q} , and consider only abelian varieties over \mathbb{Q}^{al} such that $\text{End}^0(A) \otimes_{\mathbb{Q}} K$ is a product of matrix algebras over K . Also, things simplify when we consider only abelian varieties with good ordinary reduction.

EXERCISE 10.5. Strengthen the above argument

- (a) to take account of the torus;
- (b) to obtain a morphism $\text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ whose image in $\text{HOM}_v(\text{CM}(\mathbb{Q}^{\text{al}}), \text{Mot}(\mathbb{F}))$ is equal to R and whose image in $\text{HOM}_u(\text{Mor}'(\mathbb{Q}^{\text{al}}), \mathbb{V}_l(\mathbb{F}))$ is equal to ξ_l .

EXERCISE 10.6. We are given a morphism of tannakian categories over \mathbb{A} ,

$$\text{Mot}^w(\mathbb{Q}^{\text{al}})_{(\mathbb{A})} \rightarrow \text{Mot}(\mathbb{F})_{(\mathbb{A})}.$$

Use a super-version of [Lemma 1.19](#) to deduce that this morphism arises from a morphism of tannakian categories over \mathbb{Q} ,

$$\text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F}).$$

NOTES. Results obtained using cohomology, while comforting — if [Proposition 10.3](#) were false, then at least one of the Hodge, Tate, or standard conjectures would be false — are not strong enough to allow to construct a motivic paradise because, as in the above proposition, they only allow us to prove that two objects are isomorphic whereas we need that they are equal. Nevertheless, they can be useful, for example, they allowed Langlands and Rapoport to state their conjecture, and perhaps can be combined with geometric results to allow us to obtain the true results.

Tannakian theory and cohomology (2)

Recall that we have a morphism of bands $P \rightarrow G$, and hence an action of P on $\text{Mot}^w(\mathbb{Q}^{\text{al}})$. A quotient morphism $R : \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ corresponds to a \mathbb{Q} -valued fibre functor ω_0 on $\text{Mot}(\mathbb{F})^P$. To extend R to a quotient morphism $\text{Mot}^w(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$, we need to extend ω_0 to $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$. We briefly discuss this problem.

We begin with some preliminaries. For simplicity, we assume that k has characteristic zero.

10.7. Let G be a semisimple algebraic group over a field k , and let G_1, \dots, G_n be the set of almost-simple normal algebraic subgroups of G . The multiplication map

$$G_1 \times \cdots \times G_n \rightarrow G$$

is a central isogeny (Milne 2017, 24.3). If G is simply connected, then the map is an isomorphism and each G_i is simply connected; moreover, every connected normal subgroup N of G is a product of some of the G_i , and so both N and G/N are simply connected semisimple algebraic groups.

Now let G be a reductive group. Let T be a torus in G , and let N be the smallest normal subgroup of G containing T (intersection of the normal algebraic subgroups of G containing T).

10.8. The algebraic group N is connected. Indeed, N° contains T and is normal in G (because it is a characteristic subgroup of N , *ibid.* 1.52).

10.9. The algebraic group N is reductive. We may suppose that k is algebraically closed. Let $R_u(N)$ be the unipotent radical of N (largest connected normal unipotent algebraic subgroup of N). From its definition, $R_u(N)$ is clearly a characteristic subgroup of N , and so it is normal in G . Thus $R_u(N) \subset R_u(G) = e$.

Now assume that G^{der} is simply connected.

10.10. We have $N^{\text{der}} \subset G^{\text{der}}$ (obviously). Therefore N^{der} is simply connected, and $G^{\text{der}}/N^{\text{der}}$ is simply connected. If $N^{\text{der}} = G^{\text{der}} \cap N$, then we have an exact sequence

$$1 \rightarrow G^{\text{der}}/N^{\text{der}} \rightarrow G/N \rightarrow S/\pi_0(N) \rightarrow 1$$

with $G/N^{\text{der}} = (G/N)^{\text{der}}$ and simply connected.

10.11. Is $G^{\text{der}} \cap N$ connected? If so, then, as it is normal in G^{der} , the quotient $G^{\text{der}}/G^{\text{der}} \cap N$ is simply connected. Now we can use the exact sequence

$$1 \rightarrow G^{\text{der}}/G^{\text{der}} \cap N \rightarrow G/N \rightarrow S/\pi_0(N) \rightarrow 1$$

in the next section. Alas, this doesn't seem to be always true. Consider, for example, $G = \text{GL}_2$, and $T = \mathbb{G}_m$ the centre of G . Then $N = T$ and $G^{\text{der}} \cap N = \mu_2$. Hence $G^{\text{der}}/G^{\text{der}} \cap N = \text{SL}_2/\mu_2 = \text{PGL}_2$, which is not simply connected.

Let \mathbb{Q}^{al} be the algebraic closure of \mathbb{Q} in \mathbb{C} , and let $G = \text{Aut}^\otimes(\omega_B)$, where ω_B is the Betti fibre functor.

THEOREM 10.12 (?). Let ω_0 be the \mathbb{Q} -valued fibre functor on $\text{CM}(\mathbb{Q}^{\text{al}})^P$ defined by the quotient functor $R: \text{CM}(\mathbb{Q}^{\text{al}}) \rightarrow \text{Mot}(\mathbb{F})$ (see 1.2). Up to isomorphism, ω_0 extends to a \mathbb{Q} -valued fibre functor ω on $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$ such that

$$\omega \otimes \mathbb{Q}_l \approx \xi_l$$

for all l .

PROOF. Let N be the normal closure of P in G . Then $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P = \text{Mot}^w(\mathbb{Q}^{\text{al}})^N$. Using the Betti fibre functors, we can identify fibre functors with torsors, which are classified by their cohomology classes. When we do this, the theorem becomes the following statement: let γ_0 be class of ω_0 in $H^1(\mathbb{Q}, S/P)$; then γ_0 lifts to a class in γ in $H^1(\mathbb{Q}, G/N)$ such that $\gamma_l = \gamma_l$ for all l (including p and ∞). Let G' be the derived group of G and let $N' = G' \cap N'$. From the snake lemma applied to

$$\begin{array}{ccccccc} N' & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & S, \end{array}$$

we get an exact sequence

$$1 \rightarrow G'/N' \rightarrow G/N \rightarrow S/P \rightarrow 1.$$

We assume that G'/N' is simply connected. Consider the diagram

$$\begin{array}{ccccc} H^1(\mathbb{Q}, G'/N') & \longrightarrow & H^1(\mathbb{Q}, G/N) & \longrightarrow & H^1(\mathbb{Q}, S/P) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathbb{R}, G'/N') & \longrightarrow & \prod_l H^1(\mathbb{Q}_l, G/N) & \longrightarrow & \prod_l H^1(\mathbb{Q}_l, S/P). \end{array}$$

As G'/N' is simply connected, $H^1(\mathbb{Q}_l, G'/N') = 0$ for all $l \neq \infty$ and the first vertical arrow is an isomorphism (Milne 2017, 25.61, 25.63). Moreover, the map $H^1(\mathbb{Q}, G/N) \rightarrow H^1(\mathbb{Q}, S/P)$ is surjective by the theorem of Douai. Start with the given element a of $\prod_l H^1(\mathbb{Q}_l, G/N)$. Its image b in $\prod_l H^1(\mathbb{Q}_l, S/P)$ is the image of an element c of $H^1(\mathbb{Q}, S/P)$. According to Douai's theorem, this lifts to an element d of $H^1(\mathbb{Q}, G/N)$. Let d' be the image of d in $\prod_l H^1(\mathbb{Q}_l, G/N)$. Now $a - d'$ is the image of an element e in $H^1(\mathbb{R}, G'/N')$, which in turn is the image of an element f in $H^1(\mathbb{Q}, G'/N')$. Let f' be the image of f in $H^1(\mathbb{Q}, G'/N')$. Now $d + f'$ is the element sought. \square

Of course, we want to extend ω_0 (on the nose) to a fibre functor ω on $\text{Mot}^w(\mathbb{Q}^{\text{al}})^P$ such that $\omega \otimes \mathbb{Q}_l$ and ξ_l are equal (not simply isomorphic).

We now drop the assumption that G'/N' is simply connected. Assume G' is simply connected. We can still sometimes apply Douai's theorem.

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