1986a Values of Zeta Functions of Varieties over Finite Fields

(Am. J. Math. 108, 292–360).

Section 2, p314. In his thesis[1], Berthelot defined the crystalline cohomology class of a closed subvariety only when the subvariety is smooth. In the early 1980s, I pointed out to Berthelot, Gabber, Illusie, and others, that the definition could be extended to all singular subvarieties by using the ν_n-cohomology (thereby proving that crystalline cohomology is a Weil cohomology). This section represents my exposition of this idea. A more comprehensive exposition of it can be found in the thesis of Illusie's student Gros (Mem. Soc. Math. France, 21, 1985).[2] There is a different proof that crystalline cohomology is a Weil cohomology in Gillet and Messing 1987.[3]

p320, 2.12. This conjecture is proved, even for the Zariski topology, in Gros and Suwa, Duke Math. J. 57 (1988). See also the remark pp72–73 in my paper, Compos. math. 68 (1988), 59-102.

Erratum

In (0.4a) q^{1-s} should be q^{-s}.

For some corrections (most notably, a term was dropped from the definition of $\alpha_r(X)$ on p. 299), see pages 94 and 100–101 of the sequel, Motivic cohomology and values of zeta functions, Compositio Math. 68 (1988), 59–102.

Addendum

As the article was being completed,[4] Lichtenbaum conjectured that there should exist complexes $\mathbb{Z}(r)$ of sheaves for the étale topology on a smooth variety X extending the sequence $\mathbb{Z}, \mathbb{G}_m[-1], \ldots$ (see Afterthought 10.7). It is now (2009) generally accepted that, for smooth varieties, $\mathbb{Z}(r)$ should be taken to be the complex of étale sheaves given by Bloch's higher Chow groups[5] (see for example the survey article Geisser 2005, 1.2.2[6]). The main theorem of my paper (Theorem 0.1) has a beautiful restatement in terms of the Weil-étale cohomology of the complex $\mathbb{Z}(r)$, which I now explain.

Review of abelian groups

In this subsubsection, we review some elementary results on abelian groups.

[1] Berthelot, Pierre, Cohomologie cristalline des schémas de caractéristique $p > 0$. Lecture Notes in Mathematics, Vol. 407. Springer-Verlag, Berlin-New York, 1974. 604 pp.

[2] The closest Gros comes to acknowledging the origin of the idea is in the statement p7: "Je remercie également O. Gabber qui m'a communiqué certains résultats qui ont servi de catalyseur à ceux présentés ici."

[3] Gillet, Henri; Messing, William. Cycle classes and Riemann-Roch for crystalline cohomology. Duke Math. J. 55 (1987), no. 3, 501–538.

[4] It was submitted in September 1983.

[5] There is an alternative complex defined by Suslin and Voevodsky, but Voevodsky (Int. Math. Res. Not. 2002, no. 7, 351–355) proves (but doesn't state) that the two complexes are canonically isomorphic on smooth varieties.

[6] Geisser, Thomas. Motivic cohomology, K-theory and topological cyclic homology. Handbook of K-theory. Vol. 1, 2, 193–234, Springer, Berlin, 2005.

LEMMA 1. *Let M be a subgroup of an abelian group N. If M is bounded (i.e., $nM = 0$ for some $n \geq 1$) and pure (i.e., $M \cap nN = nM$ for all $n \geq 1$), then M is a direct summand of N.*

PROOF. Kaplansky 1954, Theorem 7, p18, or Fuchs 1970, 27.5. □

LEMMA 2. *Let M be a subgroup of N, and let l^n be a prime power. If $M \cap l^n N = 0$ and M is maximal among subgroups with this property, then M is pure (hence a direct summand of N).*

PROOF. As $l^n M \subset M \cap l^n N = 0$, M is a bounded l-group. To prove that it is pure, one shows by induction on $k \in \mathbb{N}$ that $M \cap l^k N \subset l^k M$. See Fuchs 1970, 27.7. □

Every abelian group M contains a largest divisible subgroup M_{div}, which is (obviously) contained in the first Ulm subgroup $U(M) \stackrel{\text{def}}{=} \bigcap_{n \geq 1} nM$ of M.

PROPOSITION 3. *If M/nM is finite for all $n \geq 1$, then $U(M) = M_{\text{div}}$.*

PROOF. (Cf. Milne 1988, 3.3) If $U(M)$ is not divisible, then there exists an $x \in U(M)$ and a prime l such that x is not divisible by l in $U(M)$. For each $n \geq 1$, there exists an element x_n of M such that $l^n x_n = x$ (because $x \in U(M)$), and $l^{n-1} x_n \notin U(M)$ (because $x \notin lU(M)$). Therefore, x_n has order exactly l^n in $M/U(M)$, and so $M/U(M)$ contains finite subgroups S with $\dim_{\mathbb{F}_l} S^{(l)}$ arbitrarily large.

We claim that $U(M/U(M)) = 0$. Let x be an element of M that becomes divisible by n in $M/U(M)$. Then there exists a $y \in M$ such that $ny - x \in U(M)$, and so $ny - x = ny'$ for some $y' \in M$. Now $x = n(y - y')$, and so x is divisible by n in M. This proves the claim.

Let S be a finite subgroup of $M/U(M)$ of l-power order. As $U(M/U(M)) = 0$ and S is finite, there exists an n such that $S \cap l^n(M/U(M)) = 0$. By Zorn's lemma, there exists a subgroup N of $M/U(M)$ that is maximal among the subgroups satisfying (a) $N \supset S$ and (b) $N \cap l^n(M/U(M)) = 0$. Moreover, N is maximal with respect to (b) alone. Therefore N is a direct summand of $M/U(M)$ (Lemma 2), and so

$$N^{(l)} \hookrightarrow (M/U(M))^{(l)} \simeq M^{(l)}.$$

Hence

$$\dim_{\mathbb{F}_l} M^{(l)} \geq \dim_{\mathbb{F}_l} N^{(l)} \geq \dim_{\mathbb{F}_l} S^{(l)},$$

which contradicts the hypothesis. □

COROLLARY 4. *If $TM = 0$ and all quotients M/nM are finite, then $U(M)$ is uniquely divisible (= divisible and torsion-free=a $\mathbb{Q}$-vector space).*

PROOF. The first condition implies that M_{div} is torsion-free, and the second that $U(M) = M_{\text{div}}$. □

For an abelian group M, we let M_l denote the completion of M with respect to the l-adic topology. Any continuous homomorphism from M into a complete separated group factors uniquely through M_l. In particular, the quotient maps $M \to M/l^n M$ extend to homomorphisms $M_l \to M/l^n M$, and these induce an isomorphism $M_l \to \varprojlim_n M/l^n M$. The kernel of $M \to M_l$ is $\bigcap_n l^n M$. See Fuchs 1970, §13.

LEMMA 5. *Let N be a torsion-free abelian group such that N/lN is finite. The l-adic completion $N_l = \varprojlim_n N/l^n N$ of N is a free finitely generated $\mathbb{Z}_l$-module.*

PROOF. Let $y_1, \ldots, y_r$ be elements of N that form a basis for N/lN. Then

$$N = \sum \mathbb{Z} y_i + lN = \sum \mathbb{Z} y_i + l(\sum \mathbb{Z} y_i + lN) = \cdots = \sum \mathbb{Z} y_i + l^n N,$$

and so $y_1, \ldots, y_r$ generate $N/l^n N$. As $N/l^n N$ has order l^{nr}, it is in fact a free $\mathbb{Z}/l^n\mathbb{Z}$-module with basis $\{y_1, \ldots, y_r\}$. Let $a \in N_l$, say $a = (a_n)_{n\in\mathbb{N}}$ with $a_n \in N/l^{n+1}N$. Then

$$a_n = c_{n,1} y_1 + \cdots + c_{n,r} y_r$$

for some $c_{n,i} \in \mathbb{Z}/l^{n+1}\mathbb{Z}$. As a_n maps to a_{n-1} in $N/l^n N$ and the $c_{n,i}$ are unique, $c_{n,i}$ maps to $c_{n-1,i}$ in $\mathbb{Z}/l^n\mathbb{Z}$. Hence $(c_{n,i})_{n\in\mathbb{N}} \in \mathbb{Z}_l$, and it follows that $\{y_1, \ldots, y_r\}$ is a basis for N_l as a $\mathbb{Z}_l$-module. □

PROPOSITION 6. *Let M and N be abelian groups and let*

$$\phi: M \times N \to \mathbb{Z}$$

be a bi-additive map. Assume that, for some prime l, $\bigcap_n l^n M = 0$ and N/lN is finite. If the pairing

$$\phi_l: M_l \times N_l \to \mathbb{Z}_l$$

defined by ϕ on the l-adic completions of M and N has trivial left kernel, then M is finitely generated.

PROOF. As $\bigcap_n l^n M = 0$, the map $M \to M_l$ is injective. Choose elements $y_1, \ldots, y_r$ of N that form a basis for N/lN. According to the proof of (5), their images form a basis for N_l as a $\mathbb{Z}_l$-module. Consider the map

$$x \mapsto (\phi(x, y_1), \ldots, \phi(x, y_r)): M \to \mathbb{Z}^r.$$

If x is in the kernel of this map, then $\phi_l(x, y) = 0$ for all $y \in N_l$, and so $x = 0$. Therefore M injects into $\mathbb{Z}^r$, which shows that it is free and finitely generated. □

VALUES OF ZETA FUNCTIONS

For reference, we state the following conjectures (X is a smooth projective variety over a finite field k).

$T^r(X)$ (Tate conjecture): The order of the pole of the zeta function $Z(X,t)$ at $t = q^{-r}$ is equal to the rank ρ_r of the group of numerical equivalence classes of algebraic cycles of codimension r on X.

$T^r(X,l)$ (l-Tate conjecture): The map $\mathrm{CH}^r(X) \otimes \mathbb{Q}_l \to H^{2r}(\bar{X}_{\text{ét}}, \mathbb{Q}_l(r))^{\Gamma}$ is surjective.

$S^r(X,l)$ (semisimplicity at 1): The map $H^{2r}(\bar{X}_{\text{ét}}, \mathbb{Q}_l(r))^{\Gamma} \to H^{2r}(\bar{X}_{\text{ét}}, \mathbb{Q}_l(r))_{\Gamma}$ induced by the identity map is bijective.

The statement $T^r(X)$ is equivalent to the conjunction of the statements $T^r(X,l)$, $T^{d-r}(X,l)$, and $S^r(X,l)$ for one (or all) l (see Tate 1994, 2.9).

Let X be a variety over a finite field k. To give a sheaf on $X_{\text{ét}}$ is the same as giving a sheaf on $\bar{X}_{\text{ét}}$ together with a continuous action of Γ. Let Γ_0 be the subgroup of Γ generated by the Frobenius element (so $\Gamma_0 \simeq \mathbb{Z}$). The Weil-étale topology is defined so that to give a sheaf on

$X_{\text{Wét}}$ is the same as giving a sheaf on $\bar{X}_{\text{ét}}$ together with an action of Γ_0 (Lichtenbaum 2005). For example, for $X = \mathbb{A}^0$, the sheaves on $X_{\text{ét}}$ are the discrete Γ-modules, and the sheaves on $X_{\text{Wét}}$ are the Γ_0-modules. For the Weil-étale topology, the Hochschild-Serre spectral sequence becomes

$$H^i(\Gamma_0, H^j(\bar{X}_{\text{ét}}, F)) \implies H^{i+j}(X_{\text{Wét}}, F). \tag{1}$$

Since[7]

$$H^i(\Gamma_0, M) = M^{\Gamma_0}, M_{\Gamma_0}, 0, 0, \ldots \text{ for } i = 0, 1, 2, 3, \ldots, \tag{2}$$

this gives exact sequences

$$0 \to H^{i-1}(\bar{X}_{\text{ét}}, F)_{\Gamma_0} \to H^i(X_{\text{Wét}}, F) \to H^i(\bar{X}_{\text{ét}}, F)^{\Gamma_0} \to 0, \quad \text{all } i \geq 0.$$

If F is a sheaf on $X_{\text{ét}}$ such that the groups $H^j(\bar{X}_{\text{ét}}, F)$ are torsion, then the Hochschild-Serre spectral sequence for the étale topology gives exact sequences

$$0 \to H^{i-1}(\bar{X}_{\text{ét}}, F)_{\Gamma} \to H^i(X_{\text{ét}}, F) \to H^i(\bar{X}_{\text{ét}}, F)^{\Gamma} \to 0, \quad \text{all } i \geq 0.$$

Thus, for such a sheaf F, the canonical maps $H^i(X_{\text{ét}}, F) \to H^i(X_{\text{Wét}}, F)$ are isomorphisms.

There is an obvious element in $H^1(\text{Spec}(k)_{\text{Wét}}, \mathbb{Z}) \simeq H^1(\Gamma_0, \mathbb{Z}) \simeq \text{Hom}(\Gamma_0, \mathbb{Z})$, which defines, by cup-product, canonical maps

$$e^i \colon H^i(X_{\text{Wét}}, \mathbb{Z}(r)) \to H^{i+1}(X_{\text{Wét}}, \mathbb{Z}(r)),$$

and we let

$$\chi(X_{\text{Wét}}, \mathbb{Z}(r)) = \prod\nolimits_{i \neq 2r, 2r+1} [H^i(X_{\text{Wét}}, \mathbb{Z}(r))]^{(-1)^i} z(e^{2r})$$

when all terms are defined and finite. Let

$$\chi(X, \mathcal{O}_X, r) = \sum\nolimits_{\substack{0 \leq i \leq r \\ 0 \leq j \leq d}} (-1)^{i+j} \dim H^j(X, \Omega^i)$$

where $d = \dim X$.

For an abelian group M, let $M' = M/U(M)$, and let $\chi'(X_{\text{Wét}}, \mathbb{Z}(r))$ equal $\chi(X_{\text{Wét}}, \mathbb{Z}(r))$ but with each group $H^i(X_{\text{Wét}}, \mathbb{Z}(r))$ replaced by $H^i(X_{\text{Wét}}, \mathbb{Z}(r))'$.

THEOREM 7. *Let X be a smooth projective variety over a finite field. If $T^r(X)$ holds for some integer $r \geq 0$, then $\chi'(X_{\text{Wét}}, \mathbb{Z}(r))$ is defined, and*

$$\zeta(X, s) \sim \pm \chi'(X_{\text{Wét}}, \mathbb{Z}(r)) \cdot q^{\chi(X, \mathcal{O}_X, r)} \cdot (1 - q^{r-s})^{-\rho_r} \text{ as } s \to r. \tag{3}$$

In particular, the groups $H^i(X_{\text{Wét}}, \mathbb{Z}(r))'$ are finite for $i \neq 2r, 2r+1$. For $i = 2r, 2r+1$, they are finitely generated. For all i, $U(H^i(X_{\text{Wét}}, \mathbb{Z}(r)))$ is uniquely divisible.

PROOF. We begin with a brief review of Milne 1986. Let $\nu_s(r)$ be the sheaf of logarithmic de Rham-Witt differentials on $X_{\text{ét}}$ killed by p^s (ibid. p307). For an integer $n = n_0 p^s$ with $\gcd(p, n_0) = 1$,

$$H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r)) \overset{\text{def}}{=} H^i(X_{\text{ét}}, \mu_{n_0}^{\otimes r}) \times H^{i-r}(X_{\text{ét}}, \nu_s(r)), \text{ and}$$
$$H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \overset{\text{def}}{=} \varprojlim_n H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r))$$

[7] The sequence $0 \to I \to \mathbb{Z}[\Gamma_0] \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ regarded as Γ_0-module with trivial action.

(ibid. p309). There is an obvious element in $H^1(\mathrm{Spec}(k)_{\text{ét}}, \hat{\mathbb{Z}}) \simeq H^1(\Gamma, \hat{\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \hat{\mathbb{Z}})$, which defines, by cup-product, canonical maps $\epsilon^i: H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \to H^{i+1}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$, and

$$\chi(X, \hat{\mathbb{Z}}(r)) \overset{\text{def}}{=} \prod\nolimits_{i \neq 2r, 2r+1} [H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r))]^{(-1)^i} z(\epsilon^{2r})$$

when all terms are defined and finite (ibid. p298). Theorem 0.1 (ibid. p298) states that $\chi(X, \hat{\mathbb{Z}}(r))$ is defined if and only if $S^r(X,l)$ holds for all l, in which case

$$\zeta(X,s) \sim \pm\chi(X, \hat{\mathbb{Z}}(r)) \cdot q^{\chi(X, \mathcal{O}_X, r)} \cdot (1 - q^{r-s})^{-\rho_r} \text{ as } s \to r. \tag{4}$$

In particular, if $S^r(X,l)$ holds for all l, then the groups $H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$ are finite for all $i \neq 2r, 2r+1$.

The first property of $\mathbb{Z}(r)$ that we shall need is the following.[8]

(A)$_{n_0}$. For any integer n_0 prime to the characteristic of k, the cycle class map

$$\left(\mathbb{Z}(r) \xrightarrow{n_0} \mathbb{Z}(r)\right) \to \mu_{n_0}^{\otimes r}[0]$$

is a quasi-isomorphism (Geisser and Levine 2001, 1.5).

(A)$_p$. For any integer $s \geq 1$, the cycle class map

$$\left(\mathbb{Z}(r) \xrightarrow{p^s} \mathbb{Z}(r)\right) \to \nu_s(r)[-r-1]$$

is a quasi-isomorphism (Geisser and Levine 2000, Theorem 15).

For each $n \geq 1$ and $i \geq 0$, (A) gives an exact sequence

$$0 \to H^i(X_{\text{Wét}}, \mathbb{Z}(r))^{(n)} \to H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r)) \to H^{i+1}(X_{\text{Wét}}, \mathbb{Z}(r))_n \to 0.$$

in which the middle term is finite. On passing to the inverse limit, we obtain an exact sequence

$$0 \to H^i(X_{\text{Wét}}, \mathbb{Z}(r))^{\wedge} \to H^i(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \to TH^{i+1}(X_{\text{Wét}}, \mathbb{Z}(r)) \to 0 \tag{5}$$

in which the middle term is finite for $i \neq 2r, 2r+1$. As $TH^{i+1}(X_{\text{Wét}}, \mathbb{Z}(r))$ is torsion-free, it must be zero for $i \neq 2r, 2r+1$. In other words, $TH^i(X_{\text{Wét}}, \mathbb{Z}(r)) = 0$ for $i \neq 2r+1, 2r+2$.

So far we have only used conjecture $S^r(X,l)$ (all l) and property (A) of $\mathbb{Z}(r)$. To continue, we need to use $T^r(X,l)$ (all l) and the following property of $\mathbb{Z}(r)$.

(B) There exists a cycle class map $\mathrm{CH}^r(X) \to H^{2r}(X_{\text{ét}}, \mathbb{Z}(r))$ compatible (via (A)) with the cycle class map into $H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$.

The l-Tate conjecture $T^r(X,l)$ for all l implies that the cokernel of the map $\mathrm{CH}^r(X) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \to H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$ is torsion. As this map factors through $H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))^{\wedge}$, it follows that $TH^{2r+1}(X_{\text{Wét}}, \mathbb{Z}(r)) = 0$ and $H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))^{\wedge} \simeq H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r))$. Consider the diagram

$$\begin{array}{ccc} H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))^{\wedge} & \xrightarrow{\simeq} & H^{2r}(X_{\text{ét}}, \hat{\mathbb{Z}}(r)) \\ \downarrow \widehat{e^{2r}} & & \downarrow \epsilon^{2r} \\ H^{2r+1}(X_{\text{Wét}}, \mathbb{Z}(r))^{\wedge} & \longrightarrow & H^{2r+1}(X_{\text{ét}}, \hat{\mathbb{Z}}(r)). \end{array}$$

[8] For an abelian group M, it is customary to define $M(r)$ to be $\mathbb{Z}(r) \otimes M$ (Geisser 2005, p196). When $M = \mathbb{Z}/n\mathbb{Z}$, this gives another definition of the groups $H^i(X_{\text{ét}}, (\mathbb{Z}/n\mathbb{Z})(r))$ which happily, because of property (A), coincides with the previous definition.

As ϵ^{2r} has finite cokernel, so does the bottom arrow, and so $TH^{2r+2}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))=0$. We have now shown that

$$TH^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))=0 \text{ for all } i$$

and so

$$\begin{cases} H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))\hat{}\simeq H^i(X_{\mathrm{\acute{e}t}},\hat{\mathbb{Z}}(r)) \\ U(H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r)) \text{ is uniquely divisible} \end{cases} \text{for all } i.$$

In particular, we have proved the first statement of the theorem except that each group $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ has been replaced by its completion. It remains to prove that $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ is finite for $i\neq 2r,2r+1$ and is finitely generated for $i=2r,2r+1$ (for then $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))\hat{}\simeq H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'\otimes\hat{\mathbb{Z}}$).

The maps $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'\to H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))\hat{}$ are injective, and so $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ is finite for $i\neq 2r,2r+1$.

We next show that the groups $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ and $H^{2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ are finitely generated. For this we shall need one last property of $\mathbb{Z}(r)$.

(C). There exist pairings

$$\mathbb{Z}(r)\otimes^L\mathbb{Z}(s)\to\mathbb{Z}(r+s)$$

compatible (via $(\mathrm{A})_n$) with the natural pairings

$$\mu_n^{\otimes r}\times\mu_n^{\otimes s}\to\mu_n^{\otimes r+s},\quad \gcd(n,p)=1;$$

moreover, there exists a trace map $H^{2d+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(d))\to\mathbb{Z}$ compatible (via $(\mathrm{A})_n$) with the usual trace map in étale cohomology.

For a fixed prime $l\neq p$, these pairings give rise to a commutative diagram

$$\begin{array}{ccccc} H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'/\mathrm{tors} & \times & H^{2d-2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(d-r))'/\mathrm{tors} & \to & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ H^{2r}(X_{\mathrm{\acute{e}t}},\mathbb{Z}_l(r))/\mathrm{tors} & \times & H^{2d-2r+1}(X_{\mathrm{\acute{e}t}},\mathbb{Z}_l(d-r))/\mathrm{tors} & \to & \mathbb{Z}_l \end{array}$$

to which we wish to apply Proposition 6. The bottom pairing is nondegenerate, $U(H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))')=0$ by Corollary 4, and the group $H^{2d-2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(d-r))^{(l)}$ is finite, and so the Proposition shows that $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'/\mathrm{tors}$ is finitely generated. Because $U(H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))')=0$, the torsion subgroup of $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ injects into the torsion subgroup of $H^{2r}(X_{\mathrm{\acute{e}t}},\hat{\mathbb{Z}}(r))$, which is finite (Gabber 1983). Hence $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ is finitely generated. The group $H^{2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ can be treated similarly. □

REMARK 8. In the proof, we didn't use the full force of $T^r(X)$, but only that $T^r(X,l)$ and $S^r(X,l)$ hold for all primes l.

We shall need the following standard result.

LEMMA 9. *Let A be a (noncommutative) ring and let $\bar{A}$ be the quotient of A by a nil ideal I (i.e., a two-sided ideal in which every element is nilpotent). Then:*

(a) *an element of A is invertible if it maps to an invertible element of $\bar{A}$;*
(b) *every idempotent in $\bar{A}$ lifts to an idempotent in A, and any two liftings are conjugate by an element of A lying over $1_{\bar{A}}$;*

(c) *let $a \in A$; every decomposition of $\overline{a}$ into a sum of orthogonal idempotents in $\overline{A}$ lifts to a similar decomposition of a in A.*

PROOF. We denote $a+I$ by $\overline{a}$.

(a) It suffices to consider an element a such that $\overline{a} = 1_{\overline{A}}$. Then $(1-a)^N = 0$ for some $N > 0$, and so

$$\overbrace{(1-(1-a))}^{a}\left(1+(1-a)+(1-a)^2+\cdots+(1-a)^{N-1}\right) = 1.$$

(b) Let a be an element of A such that $\overline{a}$ is idempotent. Then $(a-a^2)^N = 0$ for some $N > 0$, and we let $a' = (1-(1-a)^N)^N$. A direct calculation shows that $a'a' = a'$ and that $\overline{a}' = \overline{a}$.

Let e and e' be idempotents in A such that $\overline{e} = \overline{e}'$. Then $a \stackrel{\text{def}}{=} e'e + (1-e')(1-e)$ lies above $1_{\overline{A}}$ and satisfies $e'a = e'e = ae$.

(c) Follows easily from (b). □

THEOREM 10. *Let X be a smooth projective variety over a finite field such that the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil for some prime l (e.g., X is an abelian variety), and let $r \in \mathbb{N}$. Then $H^i(X_{\mathrm{et}}, \mathbb{Z}(r))$ is torsion for all $i \neq 2r$. If $T^r(X)$ holds, then $H^{2r}(X_{\mathrm{et}}, \mathbb{Z}(r)) \approx \mathbb{Z}^{\rho_r}$ modulo torsion.*

PROOF. This is essentially proved in Jannsen 2007, pp131–132, and so we only sketch the argument. Set $d = \dim X$ and let $q = [k]$.

According to Lemma 9c, there exist orthogonal idempotents $\pi_0, \ldots, \pi_{2d}$ in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ lifting the Künneth components of the diagonal in the l-adic topology. Let $h^i X = (hX, \pi_i)$. Let $P_i(T)$ be the characteristic polynomial $\det(T - \varpi_X | H^i(\overline{X}_{\mathrm{et}}, \mathbb{Q}_l)$ of the Frobenius endomorphism ϖ_X of X acting on $H^i(\overline{X}_{\mathrm{et}}, \mathbb{Q}_l)$. Then $P_i(\varpi_X)$ acts as zero $h^i_{\mathrm{hom}} X$, and so $P_i(\varpi_X)^N$ acts as zero on $h^i X$ for some $N \geq 1$. But ϖ_X acts as multiplication by q^r on $K_{2r-i}(X)^{(r)} \simeq H^i(X_{\mathrm{et}}, \mathbb{Q}(r))$ where $K_{2r-i}(X)^{(r)}$ is the subspace of $K_{2r-i}(X)_{\mathbb{Q}}$ on which the nth Adams operator acts as n^r for all r. Therefore $H^i(X_{\mathrm{et}}, \mathbb{Q}(r))$ is killed by $P_i(q^r)^N$, which is nonzero for $i \neq 2r$ (by the Weil conjectures).

Tate's conjecture implies that $P_{2r}(T) = Q(T) \cdot (T - q^r)^{\rho_r}$ where $Q(q^r) \neq 0$. As before, $P_{2r}(\omega_X)^N$ acts as zero on $h^{2r} X$ for some $N \geq 1$. Now

$$1 = q(T) Q(T)^N + p(T)(T - q^r)^{N\rho_r}, \quad \text{some } q(T),\ p(T) \in \mathbb{Q}[T].$$

Therefore $q(\varpi_X) Q(\varpi_X)^N$ and $p(\varpi_X)(\varpi_X - q^r)^{N\rho_r}$ are orthogonal idempotents in $\mathrm{End}(h^{2r} X)$ with sum 1, and correspondingly $h^{2r} X = M_1 \oplus M_2$. Now $H^{2r}(M_1, \mathbb{Q}(r)) = 0$ because $Q(\varpi_X)^N$ is zero on M_1 and $Q(q^r) \neq 0$. On the other hand, M_2 is isogenous to $(\mathbb{L}^{\otimes r})^{\rho_r}$ (Jannsen 2007, p132), and so $H^{2r}(M_2, \mathbb{Z}(r))$ differs from $H^{2r}(\mathbb{L}^{\otimes r}, \mathbb{Z}(r))^{\rho_r} \simeq H^{2r}(\mathbb{P}^d, \mathbb{Z}(r))^{\rho_r} \simeq \mathbb{Z}^{\rho_r}$ by a torsion group. □

THEOREM 11. *Let X be a smooth projective variety over a finite field such that the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)$ is nil for some l (e.g., X is an abelian variety). If the Tate conjecture $T^r(X)$ holds, then $\chi(X_{W\acute{e}t}, \mathbb{Z}(r))$ is defined, and*

$$\zeta(X, s) \sim \pm \chi(X_{W\acute{e}t}, \mathbb{Z}(r)) \cdot q^{\chi(X, \mathcal{O}_X, r)} \cdot (1 - q^{r-s})^{-\rho_r} \text{ as } s \to r. \tag{6}$$

In particular, the groups $H^i(X_{W\acute{e}t}, \mathbb{Z}(r))$ are finite for $i \neq 2r, 2r+1$. For $i = 2r, 2r+1$, they are finitely generated.

PROOF. We have to show that the groups $U^{i} \stackrel{\text{def}}{=} U(H^{i}(X_{\text{Wét}}, \mathbb{Z}(r)))$ are zero. If a Γ_0-module M is finitely generated modulo torsion, then so also are its cohomology groups $H^{i}(\Gamma_0, M)$ (see (2)). Since the groups $H^{i}(\bar{X}_{\text{et}}, \mathbb{Z}(r))$ are finitely generated modulo torsion (10), the spectral sequence (1) shows that the groups $H^{i}(X_{\text{Wét}}, \mathbb{Z}(r))$ are also finitely generated modulo torsion. In particular, $H^{i}(X_{\text{Wét}}, \mathbb{Z}(r))$ does not contain a nonzero $\mathbb{Q}$-vector space, and so $U^{i} = 0$.

Alternative proof for $i = 2r$. The hypotheses imply that the composite of the maps

$$\mathrm{CH}^{r}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \xrightarrow{a} H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r)) \otimes_{\mathbb{Z}} \mathbb{Q}_l \xrightarrow{b} H^{2r}(X_{\text{et}}, \mathbb{Q}_l(r))$$

is an isomorphism (Kimura 2005, 7.6; Jannsen 2007, 6.1.4). The kernel of b is $U^{2r} \otimes_{\mathbb{Q}} \mathbb{Q}_l$, and the map a is an isomorphism (see below; not quite). It follows that $U^{2r} = 0$, and so $H^{2r}(X_{\text{Wét}}, \mathbb{Z}(r))$ is finitely generated. □