On the integral Tate conjecture for abelian varieties

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Abstract

Recently Engel et al. (2025) have shown that the integral Hodge conjecture fails for very general abelian varieties. Using Deligne's theory of absolute Hodge cycles, we deduce a similar statement for the integral Tate conjecture.

For a smooth projective variety X over a subfield k of $\mathbb{C}$, we let $C^r(X)$ denote the $\mathbb{Z}$-module of algebraic cycles of codimension r on X modulo (Betti) homological equivalence. When k is finitely generated over $\mathbb{Q}$, the integral Tate conjecture[1] says that the cycle class map

$$C^r(X) \otimes \mathbb{Z}_\ell \xrightarrow{c'} H^{2r}_{\text{et}}(\bar{X}, \mathbb{Z}_\ell(r))^{\text{Gal}(\bar{k}/k)},$$

is surjective. Here $\bar{k}$ is the algebraic closure of k in $\mathbb{C}$ and $\bar{X} = X \times \operatorname{Spec} \bar{k}$.

THEOREM *Let A_0 be an abelian variety over a subfield k of $\mathbb{C}$ finitely generated over $\mathbb{Q}$. If the integral Hodge conjecture fails for $A \overset{\text{def}}{=} (A_0)_{\mathbb{C}}$, then the integral Tate conjecture fails for A_0 over some finite extension of k.*

PROOF. We use that the cohomology of abelian varieties has no torsion to identify the integral cohomology groups with subgroups of the rational groups.

Let $\gamma \in H^{2r}(A, \mathbb{Z}(r)) \cap H^{0,0}$ be an integral Hodge class on A that is not algebraic, i.e., not in the image of the cycle class map

$$C^r(X) \xrightarrow{c} H^{2r}(A, \mathbb{Z}(r)).$$

For some ℓ, $\gamma \otimes 1$ is not in the image of

$$C^r(X) \otimes \mathbb{Z}_\ell \xrightarrow{c \otimes 1} H^{2r}(A, \mathbb{Z}(r)) \otimes \mathbb{Z}_\ell. \tag{*}$$

According to the main theorem of Deligne 1982, the image of γ in $H^{2r}_{\mathbb{A}}(A)(r)$ is an absolute Hodge class on A. According Proposition 2.9, ibid., γ arises from an absolute Hodge class γ_0 on $\bar{A}_0$ and $\text{Gal}(\bar{k}/k)$ acts on γ_0 through a finite quotient. After passing to a finite extension of k, we may suppose that $\text{Gal}(\bar{k}/k)$ fixes γ_0. Then the ℓ-component $\gamma_0(\ell)$ of γ_0 is a Tate class in $H^{2r}(\bar{A}_0, \mathbb{Q}_\ell(r))$. Consider the commutative diagram

$$
\begin{array}{ccc}
C^r(A) \otimes \mathbb{Z}_\ell & \xrightarrow{\ c \otimes 1\ } & H^{2r}(A, \mathbb{Z}(r)) \otimes \mathbb{Q}_\ell \\
\big\uparrow & & \big\uparrow{\scriptstyle \simeq} \\
C^r(A_0) \otimes \mathbb{Z}_\ell & \xrightarrow{\ c'\ } & H^{2r}_{\text{et}}(\bar{A}_0, \mathbb{Q}_\ell(r)),
\end{array}
$$

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[1]Not stated by Tate. At the time Tate stated his conjectures, Atiyah and Hirzebruch had already found their counterexample to the integral Hodge conjecture.

where the isomorphism at right is the composite of the canonical isomorphisms

$$H_{\text{et}}^{2r}(\bar{A}_0, \mathbb{Q}_\ell(r)) \simeq H_{\text{et}}^{2r}(A, \mathbb{Q}_\ell(r)) \simeq H^{2r}(A, \mathbb{Z}(r)) \otimes \mathbb{Q}_\ell.$$

Under this isomorphism $\gamma_0(\ell)$ corresponds to $\gamma \otimes 1$, so $\gamma_0(\ell)$ lies in $H^{2r}(\bar{A}_0, \mathbb{Z}_\ell(r))$ and is an integral Tate class on A_0. If $\gamma_0(\ell)$ is in the image of c', then $\gamma \otimes 1$ is in the image of $c \otimes 1$, contradicting (*). We have shown that the integral Tate conjecture fails for A_0. □

Engel et al. 2025 show that all very general principally polarized abelian varieties over $\mathbb{C}$ fail the integral Hodge conjecture. Thus, a principally polarized abelian variety over a finitely generated subfield k of $\mathbb{C}$ fails the integral Tate conjecture over some finite extension of k if, over $\mathbb{C}$, it is "very general" in their sense.

PROBLEM Exhibit an abelian variety over a number field that fails the integral Tate conjecture.

References

Deligne, P. 1982. Hodge cycles on abelian varieties (notes by J.S. Milne), pp. 9–100. In Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin.

Engel, P., de Gaay Fortman, O., Schreieder, S., 2025. Matroids and the integral Hodge conjecture for abelian varieties, arXiv:2507.15704v2.