The Tate and Standard Conjectures for Certain Abelian Varieties

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Abstract

In two earlier articles, we proved that, if the Hodge conjecture is true for *all* CM abelian varieties over $\mathbb{C}$, then both the Tate conjecture and the standard conjectures are true for abelian varieties over finite fields. Here we rework the proofs so that they apply to a single abelian variety. As a consequence, we prove (unconditionally) that the Tate and standard conjectures are true for many abelian varieties over finite fields, including abelian varieties for which the algebra of Tate classes is *not* generated by divisor classes. The article is partly expository.

Contents

In earlier articles (1999b, 2002), we proved that, if the Hodge conjecture holds for *all* CM abelian varieties, then the Tate and standard conjectures hold for abelian varieties over finite fields.[1] In this article, we rework the proofs so that they apply to a single abelian variety. We prove (Theorem 3.1) that if an abelian variety A satisfies a certain condition (*), then the Hodge conjecture for the powers of A implies both the Tate and standard conjectures for the powers of A mod p. Using this, we obtain a number of new results, among which is the theorem below.

We say that an abelian variety A over an algebraically closed field k of characteristic zero (resp. over $\mathbb{F}$) is neat if no power of it supports an exotic Hodge class (resp. an exotic Tate class). The theorems of Lefschetz and Tate show that the Hodge and Tate conjectures hold for neat abelian varieties. Let $\mathbb{Q}^{\mathrm{al}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and $\mathbb{F}$ its residue field at a prime dividing p.

[1]Thus, if the Hodge standard conjecture fails for a single abelian variety, then everything — the Hodge, Tate, and Grothendieck conjectures — fails. From a more optimistic perspective, with the proof of the algebraicity of Weil's classes in the first interesting case, namely, for fourfolds with determinant 1, we may hope that this will be proved for a widening collection of abelian varieties. Then the methods of this paper will show that the Tate and standard conjectures are also true for a widening collection of abelian varieties. At some point we may dare to believe that the conjectures are true for all abelian varieties.

THEOREM 0.1. *Let A be a simple abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of dimension n with good reduction to a simple abelian variety A_0 over $\mathbb{F}$, and let B be a CM elliptic curve over $\mathbb{Q}^{\mathrm{al}}$ with good reduction B_0. Suppose that both A and A_0 are neat, but that neither $A \times B$ nor $A_0 \times B_0$ is neat.*[2]

(a) *For some m, there is an imaginary quadratic field $Q \subset \mathrm{End}^0(A \times B^m)$ such that $(A \times B^m, Q)$ is of Weil type.*

(b) *If the Weil classes on $(A \times B^m, Q)$ are algebraic, then, for all $r, s \in \mathbb{N}$,*

  i) *the Hodge conjecture holds for the abelian varieties $A^r_{\mathbb{C}} \times B^s_{\mathbb{C}}$;*
  ii) *the Tate conjecture holds for the abelian varieties $A^r \times B^s$;*
  iii) *the Tate conjecture holds for the abelian varieties $A^r_0 \times B^s_0$;*
  iv) *the standard conjectures hold for the abelian varieties $A^r_0 \times B^s_0$.*

When $n = 3$, a recent theorem of Markman shows that the Weil classes on $(A \times B, Q)$ are algebraic. Thus we obtain new cases of the Hodge, Tate, and standard conjectures. Apart from Ancona 2021, these are the first unconditional results on the Hodge standard conjecture since it was stated over fifty years ago.

We give examples of pairs A, B satisfying the hypotheses of the theorem.

We assume that the reader is familiar with Milne 2001, especially Appendix A, whose notation we adopt. In particular, ι or $a \longmapsto \bar{a}$ denotes complex conjugation on $\mathbb{C}$ and its subfields, and $\mathbb{N} = \{0, 1, ...\}$. For a CM algebra E, we let U_E denote the torus such that $U_E(\mathbb{Q}) = \{a \in E^\times \mid a\bar{a} = 1\}$. We regard abelian varieties as objects of the category whose morphisms are $\mathrm{Hom}^0(A, B) \overset{\text{def}}{=} \mathrm{Hom}(A, B) \otimes \mathbb{Q}$. Throughout, algebraic varieties are connected.

We assume that the reader is familiar with Deligne and Milne 1982, especially §5, and Deligne 1989, §5. The fundamental group $\pi(\mathsf{C})$ of a tannakian category C is a group *in* C such that $\omega(\pi(\mathsf{C})) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ for all fibre functors ω, and γ^C is the functor $\mathrm{Hom}(\mathbb{1}, -)$ from $\mathsf{C}^{\pi(\mathsf{C})}$ to vector spaces (an equivalence of tensor categories).

1 Characteristic zero

Let A be an abelian variety over an algebraically closed field of characteristic zero. By a Hodge class on A, we mean an absolute Hodge class in the sense of Deligne 1982. Such a class is said to be ***exotic*** if it is not in the $\mathbb{Q}$-algebra generated by the Hodge classes of degree 1. According to a theorem of Lefschetz, the nonexotic Hodge classes are algebraic — we call them ***Lefschetz classes***. We let $B^*(A)$ denote the $\mathbb{Q}$-algebra of Hodge classes on A and $D^*(A)$ the $\mathbb{Q}$-subalgebra generated by the divisor classes (the algebra of Lefschetz classes).

a. *The Lefschetz group*

Let A be an abelian variety over an algebraically closed subfield k of $\mathbb{C}$. The centralizer $C(A)$ of $\mathrm{End}^0(A)$ in $\mathrm{End}(H_1(A_{\mathbb{C}}, \mathbb{Q}))$ is a $\mathbb{Q}$-algebra stable under the involution † defined by an ample divisor D of A, and the restriction of † to $C(A)$ is independent of the choice

[2] To make the statement interesting.

D. The ***Lefschetz group*** $L(A)$ of A is the algebraic group over $\mathbb{Q}$ such that

$$L(A)(\mathbb{Q}) = \{\alpha \in C(A)^\times \mid \alpha\alpha^\dagger \in \mathbb{Q}^\times\}.$$

If A is CM, i.e., if the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$ contains an étale $\mathbb{Q}$-subalgebra of degree $2\dim A$, then $C(A)$ is the centre of $\mathrm{End}^0(A)$, and this definition makes sense over any algebraically closed field (not necessarily of characteristic zero).

b. *The Mumford–Tate group*

Let A be an abelian variety over an algebraically closed subfield k of $\mathbb{C}$, and let $V = H_1(A_{\mathbb{C}}, \mathbb{Q})$. When we let $L(A)$ act on the cohomology groups $H^{2r}(A^s_{\mathbb{C}}, \mathbb{Q})(r)$, $r, s \in \mathbb{N}$, throught the homomorphism

$$\alpha \longmapsto (\alpha, \alpha\alpha^\dagger)\colon L(A) \to \mathrm{GL}_V \times \mathbb{G}_m,$$

it becomes the algebraic subgroup of $\mathrm{GL}_V \times \mathbb{G}_m$ fixing the Lefschetz classes on all powers of A. We define

$$\mathrm{MT}(A) \subset M(A) \subset L(A)$$

to be the algebraic subgroups of $L(A)$ fixing, respectively, the Hodge classes and the algebraic classes on the powers of A. Then the Lefschetz classes are *exactly* the cohomology classes fixed by $L(A)$, and similarly for the other groups. For the ***Mumford–Tate group*** $\mathrm{MT}(A)$, this is easy to prove; for $M(A)$, it follows from the fact that the abelian motives modulo numerical equivalence form a tannakian category (Lieberman 1968, Jannsen 1992);[3] for the Lefschetz group $L(A)$, it is proved by case-by-case checking (Milne 1999a).

The kernel of the natural homomorphism $\mathrm{MT}(A) \to \mathbb{G}_m$ (resp. $L(A) \to \mathbb{G}_m$) is called the ***Hodge group*** (resp. the ***special Lefschetz group***) and denoted by $\mathrm{Hg}(A)$ (resp. $S(A)$). The abelian variety A is isogenous to a product $A_1^{s_1} \times \cdots \times A_m^{s_m}$ with each A_i simple and no two isogenous, and

$$\begin{aligned} \mathrm{Hg}(A) &= \mathrm{Hg}(A_1 \times \cdots \times A_m) = \text{a subproduct of } \mathrm{Hg}(A_1) \times \cdots \times \mathrm{Hg}(A_m), \\ S(A) &= S(A_1 \times \cdots \times A_m) = S(A_1) \times \cdots \times S(A_m). \end{aligned}$$

The kernel of $M(A) \to \mathbb{G}_m$ is denoted by $M'(A)$.

PROPOSITION 1.1. *The following conditions on A are equivalent:*

(a) *no power of A supports an exotic Hodge class;*
(b) $\mathrm{MT}(A) = L(A)$*;*
(c) $\mathrm{Hg}(A) = S(A)$*.*

PROOF. The equivalence of (a) and (b) follows from the above discussion, and the equivalence of (b) and (c) follows from the five-lemma. See Milne 1999a, 4.8. □

An abelian variety satisfying the equivalent conditions of the proposition is said to be ***neat***.[4] For example, if A is an abelian variety A such that $E \stackrel{\text{def}}{=} \mathrm{End}^0(A)$ is a field, then

[3] See the Appendix.

[4] In other words, the neat abelian varieties are those for which the Hodge conjecture is true for trivial reasons.

A is neat in each of the following cases: E is totally real and $\dim(A)/[E:\mathbb{Q}]$ is odd; E is CM and $\dim(A)$ is prime; E is imaginary quadratic and the representation of $E \otimes \mathbb{C}$ on $\mathrm{Tgt}_0(A)$ is of the form $m\rho \oplus n\bar{\rho}$ with $\gcd(m,n) = 1$ (Tankeev; Ribet 1983). It follows that all simple abelian varieties of odd prime dimension are neat. Abelian surfaces and products of elliptic curves are also neat.

c. *Abelian varieties of Weil type*

In the early 1960s, Mumford and Tate tried to prove the Hodge conjecture for abelian varieties by showing that they are all neat, but then Mumford found a nonneat simple abelian fourfold (Pohlmann 1968 §2). Mumford's variety was CM, but as Weil (1977) explained, the important fact was that the abelian variety was acted on by an imaginary quadratic field.

Let A be an abelian variety over $\mathbb{C}$, and let Q be an imaginary quadratic subfield[5] of the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$. Let $\beta \in Q$ be such that $\bar{\beta} = -\beta$, so $Q = \mathbb{Q}[\beta]$ and $\beta\bar{\beta} = b \in \mathbb{Q}$.

LEMMA 1.2. *There exists a polarization of (A,Q), i.e., a polarization of A whose Rosati involution stabilizes Q and acts on it as complex conjugation.*

PROOF. Let ψ be a Riemann form for A, i.e., an element ψ of $\mathrm{Hom}(\bigwedge^2_{\mathbb{Q}} V, \mathbb{Q}) \simeq H^2(A,\mathbb{Q})$ such that $\psi_{\mathbb{R}}(Jx,Jy) = \psi_{\mathbb{R}}(x,y)$ and $\psi_{\mathbb{R}}(x,Jx) > 0$ for $x,y \in V_{\mathbb{R}}$, $x > 0$. Then β acts on $B^1(A)(-1) \subset H^2(A,\mathbb{Q})$ with eigenvalues in $\{d,-d\}$. Let $\psi = \psi_+ + \psi_-$ be the decomposition of ψ into eigenvectors. Then ψ_+ is again a Riemann form, and the condition $\beta^*\psi_+ = d\psi_+$ implies that $\psi_+(\beta x,y) = \psi_+(x,\bar{\beta}y)$, as required. □

Let λ be a polarization of (A,Q), and

$$\psi\colon H_1(A,\mathbb{Q}) \times H_1(A,\mathbb{Q}) \to \mathbb{Q}$$

its Riemann form. Define $\phi\colon H_1(A,\mathbb{Q}) \times H_1(A,\mathbb{Q}) \to Q$ by

$$\phi(x,y) = \psi(x,\beta y) + \beta\psi(x,y).$$

Then ϕ is the unique hermitian form on the Q-vector space $H_1(A,\mathbb{Q})$ such that

$$\phi(x,y) - \overline{\phi(x,y)} = 2\beta\psi(x,y).$$

The discriminant of ϕ is an element of $\mathbb{Q}^\times/\mathrm{Nm}(Q^\times)$, called the ***determinant*** of (A,Q,λ).

PROPOSITION 1.3. *Let A be an abelian variety of dimension $2m$ and Q an imaginary quadratic subfield of $\mathrm{End}^\circ(A)$. The following conditions on (A,Q) are equivalent:*

(a) $\mathrm{Tgt}_0(A)$ *is a free $Q \otimes_{\mathbb{Q}} \mathbb{C}$-module;*

(b) *the one-dimensional Q-vector space*[6]

$$W(A,Q) \overset{\text{def}}{=} \Big(\bigwedge\nolimits^{2m}_Q H^1(A,\mathbb{Q})\Big)(m) \subset H^{2m}(A,\mathbb{Q})(m)$$

consists of Hodge classes.

[5] By this we mean that it is a $\mathbb{Q}$-subalgebra; in particular, the identity elements coincide.

[6] As before, let $V = H_1(A,\mathbb{Q})$. We can identify $H^{2m}(A,\mathbb{Q})$ with the space of $\mathbb{Q}$-multilinear alternating forms $V \times \cdots \times V \to \mathbb{Q}$ and $W(A,Q)(-m)$ with the subspace of those that are Q-balanced.

PROOF. Straightforward; see Deligne 1982, 4.4. □

A pair (A,Q) satisfying the equivalent conditions of the proposition is said to be of ***Weil type***. The elements of $W(A,Q)$ are the ***Weil classes*** on A. The hermitian form ϕ attached to a polarization λ of (A,Q) has signature (m,m), and so

$$(-1)^m \cdot \det(A,Q,\lambda) > 0.$$

PROPOSITION 1.4. *For any imaginary quadratic field Q, integer $m \geq 1$, and element $a \in \mathbb{Q}^\times/\operatorname{Nm}(Q^\times)$ with sign $(-1)^m$, the $2m$-dimensional polarized abelian varieties (A,Q,λ) of Weil type with determinant a form an m^2-dimensional family.*

PROOF. The underlying variety of the family is the connected Shimura variety attached to $\mathrm{SU}(\phi)$. See Weil 1977, Deligne 1982, 4.8, or van Geemen 1994. □

Weil showed that, in general, the Weil classes are exotic, and suggested that they formed a good test case for the Hodge conjecture. The first interesting case is $\dim(A) = 4$. Concerning this, there is the following result.

THEOREM 1.5 (MARKMAN). *Let (A,Q,λ) be a polarized abelian fourfold of Weil type with determinant 1 in $\mathbb{Q}^\times/\operatorname{Nm}(Q^\times)$. Then the Weil classes on A are algebraic.*

PROOF. See Markman 2021, 1.5. The proof uses methods from the deformation theory of hyperkähler manifolds. □

PROPOSITION 1.6. *Let $(A \times B, Q)$ be of Weil type, where A is an abelian n-fold and B an elliptic curve. Then there exists a polarization λ of $(A \times B, Q)$ with determinant $(-1)^{(n+1)/2}$ modulo $\operatorname{Nm}(Q^\times)$.*

PROOF. Let λ_A (resp. λ_B) be a polarization of (A,Q) (resp. (B,Q)). Let ϕ_A and ϕ_B be the corresponding Q-valued hermitian forms. Then $\lambda_A \times \lambda_B$ is a polarization of $(A \times B, Q)$ with hermitian form $\phi_A \oplus \phi_B$, which has determinant some $a \in \mathbb{Q}^\times/\operatorname{Nm}(Q^\times)$ with $(-1)^{(n+1)/2}a > 0$. There exists a $c \in \mathbb{Z}$, $c > 0$, such that $ac = (-1)^{(n+1)/2}$ in $\mathbb{Q}^\times/\operatorname{Nm}(Q^\times)$. Now $\lambda_A \times c\lambda_B$ is a polarization of $(A \times B, Q)$ with determinant $(-1)^{(n+1)/2}$ modulo $\operatorname{Nm}(Q^\times)$. □

ASIDE 1.7. In the above discussion, Q can be replaced by any CM field. If all Weil classes in this more general sense are algebraic, then the Hodge conjecture holds for all CM abelian varieties (Deligne 1982, §5).

d. *Almost-neat abelian varieties*

Let (A,Q) be of Weil type. Then $S(A)$ acts on $W(A,Q)$ through a "determinant" homomorphism $\rho\colon S(A) \to U_Q$, and $\operatorname{Hg}(A)$ is contained in the kernel of ρ. We say that the pair (A,Q) is ***almost-neat*** if the sequence

$$1 \to \operatorname{Hg}(A) \longrightarrow S(A) \xrightarrow{\rho} U_Q \to 1 \tag{1}$$

is exact. As $W(A,Q)$ has weight 0 and $L(A) = w(\mathbb{G}_m) \cdot S(A)$, it also acts on $W(A,Q)$ through a homomorphism $\rho\colon L(A) \to U_Q$. A pair (A,Q) of Weil type is almost-neat if and only if the sequence

$$1 \to \operatorname{MT}(A) \to L(A) \xrightarrow{\rho} U_Q \to 1 \tag{2}$$

is exact. If (A,Q) is almost-neat, then the Weil classes on A are exotic.

THEOREM 1.8. *If (A,Q) is almost-neat and the Weil classes on A are algebraic, then the Hodge conjecture holds for A and its powers.*

PROOF. As the Weil classes on A are algebraic, $M(A)$ is contained in the kernel of ρ, and so equals $\mathrm{MT}(A)$. Thus the Hodge classes on the powers of A coincide with the algebraic classes. □

Recall that that $W(A,Q)$ has dimension 1 as a Q-vector space. If $W(A,Q)$ contains a single nonzero algebraic class, then it consists of algebraic classes because the action of the endomorphisms of A on its cohomology groups preserves algebraic classes.

LEMMA 1.9. *Let E be a CM field and Q an imaginary quadratic subfield of E. Then*

$$U_{E/Q} \stackrel{\text{def}}{=} \mathrm{Ker}(\mathrm{Nm}_{E/Q} \colon U_E \to U_Q)$$

is a subtorus of U_E of codimension 1, and every subtorus of U_E of codimension 1 is of this form for a unique imaginary quadratic subfield of E.

PROOF. That $U_{E/Q}$ is a subtorus of codimension 1 can be checked on the character groups. Conversely, if U is a one-dimensional quotient of U_E, then its splitting field Q is an imaginary quadratic extension of $\mathbb{Q}$, and $U = U_Q$. That the kernel of $U_E \to U$ is isomorphic to $U_{E/Q}$ can be checked on the character groups.[7] □

PROPOSITION 1.10. *Let A be a neat simple abelian variety and B a CM elliptic curve such that $A\times B$ is not neat.*

(a) There is an exact sequence

$$1 \to \mathrm{Hg}(A\times B) \to S(A\times B) \to U \to 1$$

in which the projection $S(A) \to U$ is surjective and $S(B) \to U$ an isomorphism.

(b) There exists an embedding of $Q \stackrel{\text{def}}{=} \mathrm{End}^0(B)$ into $\mathrm{End}^0(A)$ and an $m \in \mathbb{N}$ such that $(A\times B^m, Q)$ is of Weil type (hence almost-neat).

PROOF. Because A is neat but $A\times B$ is not,

$$S(A)\times S(B) = S(A\times B) \supsetneqq \mathrm{Hg}(A\times B) \twoheadrightarrow \mathrm{Hg}(A) = S(A).$$

It follows that $S(A)\times S(B) = \mathrm{Hg}(A\times B)\cdot S(B)$, and so there is an exact sequence

$$1 \to \mathrm{Hg}(A\times B) \to S(A)\times S(B) \to U \to 1$$

in which U one-dimensional and the projection $S(B) \to U$ is surjective. The kernel of $S(B) \to U$ is trivial because $\mathrm{Hg}(A\times B)$ is connected.

The projection $S(A) \to U$ is surjective, because otherwise $\mathrm{Hg}(A\times B) = S(A)^\circ$, which is not possible because $\mathrm{Hg}(A\times B) \to \mathrm{Hg}(B) = S(B)$ is surjective.

It follows that A is of type IV, because otherwise the algebraic group $S(A)^\circ$ would be semisimple (Milne 1999a, §2), and so the centre of $\mathrm{End}^0(A)$ is a CM-field E. The centre of $C(A)$ equals that of $\mathrm{End}^0(A)$,[8] and the homomorphism $S(A) \to U$ induces a

[7] This lemma is well known.

[8] Let A and B be subalgebras of an algebra, each the centralizer of the other. Then $A\cap B$ is the common centre of A and B.

surjection $U_E \to U$. According to the Lemma 1.9, there is an imaginary quadratic field Q and an embedding of Q into E such that $U_E \to U$ is given by the norm map $E \to Q$. Clearly $Q \approx \mathrm{End}^0(B)$.

Let $\rho \colon Q \to \mathbb{C}$ be an embedding, and let the representation of $Q \otimes \mathbb{C}$ on $\mathrm{Tgt}_0(A)$ be $n_1\rho \oplus n_2\bar{\rho}$. We may suppose that $n_1 > n_2$ (otherwise replace ρ with $\bar{\rho}$), and let $m = n_1 - n_2$. Choose the isomorphism $Q \to \mathrm{End}^0(B)$ so that $Q \otimes \mathbb{C}$ acts on $\mathrm{Tgt}_0(B)$ as $\bar{\rho}$. When we let Q act diagonally on $A \times B^m$, the pair $(A \times B^m, Q)$ is of Weil type, and (a) shows that it is almost-neat. □

Let A be an abelian variety over $\mathbb{C}$ and Q an imaginary quadratic subfield of $\mathrm{End}^0(A)$. Let $n_1\rho \oplus n_2\bar{\rho}$ be the representation of $Q \otimes \mathbb{C}$ on $\mathrm{Tgt}_0(A)$. If $Q = \mathrm{End}^0(A)$, then $n_1, n_2 > 0$ (Shimura; see Ribet 1983, p. 525). If A is CM, then $n_1, n_2 > 0$ unless A is isogenous to a power of an elliptic curve.

COROLLARY 1.11. *Let A be a simple abelian threefold and B a CM elliptic curve. The Hodge conjecture holds for all varieties $A^r \times B^s$, $r, s \in \mathbb{N}$.*

PROOF. Recall that A and B are neat. If $A \times B$ is neat, then certainly the Hodge conjecture holds for the varieties $A^r \times B^s$. Otherwise, there exists an imaginary quadratic subfield Q of $\mathrm{End}^0(A \times B)$ such that $(A \times B, Q)$ is almost-neat (see 1.10). The pair $(A \times B, Q)$ admits a polarization λ such that $\det(A \times B, Q, \lambda) = 1$ in $\mathbb{Q}^\times / \mathrm{Nm}(Q^\times)$ (see 1.6), and so the Weil classes on $A \times B$ are algebraic (1.5). Therefore the Hodge conjecture holds for the varieties $A^r \times B^s$ (see 1.8). □

Let A and B be as in the corollary, but with $\mathrm{End}^0(B) = \mathbb{Q}$. If A is not of type IV or is CM, then $A \times B$ is neat (Hazama 1989, 0.1, 3.1). In the remaining case, $\mathrm{End}^0(A)$ is an imaginary quadratic field, and it follows from Goursat's lemma that $\mathrm{Hg}(A \times B) = \mathrm{Hg}(A) \times \mathrm{Hg}(B)$, and so $A \times B$ is again neat.

EXAMPLE 1.12. Let E be a CM field of degree $2m$, $m \geq 3$, over $\mathbb{Q}$ containing an imaginary quadratic field Q. Choose an embedding $\rho_0 \colon Q \to \mathbb{Q}^{\mathrm{al}}$, and let $\{\varphi_0, \dots, \varphi_{m-1}\}$ be the set of extensions of ρ_0 to E. Then $\Phi_0 \overset{\text{def}}{=} \{\varphi_0, \iota\circ\varphi_1, \dots, \iota\circ\varphi_{m-1}\}$ is a CM-type on E, and we let (A, E) denote an abelian variety over $\mathbb{C}$ of CM-type (E, Φ_0).

Let (B, Q) be an elliptic curve over $\mathbb{C}$ of CM-type (Q, ρ_0), and let Q act diagonally on $A \times B^{m-2}$. Then

$$\mathrm{Tgt}_0(A \times B^{m-2}) \simeq \mathrm{Tgt}_0(A) \oplus (m-2)\,\mathrm{Tgt}_0(B)$$

is a free $Q \otimes \mathbb{C}$-module, and so $(A \times B^{m-2}, Q)$ is of Weil type.

The abelian variety A is neat, and the pair $(A \times B^{m-2}, Q)$ is almost-neat (Milne 2001). In particular, if $m = 3$, then the Hodge conjecture holds for the varieties $A^r \times B^s$, $r, s \in \mathbb{N}$.

REMARK 1.13. Let A be a neat abelian variety (not necessarily simple) and B a CM elliptic curve such that $A \times B$ is not neat. Then there exists a simple isogeny factor A' of A such $A' \times B$ is not neat, and so Proposition 1.10 shows that there exists a homomorphism from $Q \overset{\text{def}}{=} \mathrm{End}^0(B)$ into a direct factor of the centre of $\mathrm{End}^0(A)$.

e. *Hodge classes on general abelian varieties of Weil type*

Let (A,Q,λ) be a polarized abelian variety of Weil type, and let $W(A,Q)$ be the Q-vector space of Weil classes on A. The action of $S(A)$ on $W(A,Q)$ defines a homomorphism $\rho\colon S(A)\to U_Q$. Let $\phi\colon V\times V\to Q$ be the hermitian form attached to (A,Q,λ). There is an exact sequence of algebraic groups

$$1\to \mathrm{SU}(\phi)\to \mathrm{U}(\phi)\xrightarrow{\det} U_Q\to 1, \qquad (3)$$

where $\mathrm{U}(\phi)$ is the subgroup of $\mathrm{GL}_Q(V)$ of elements fixing ϕ.

THEOREM 1.14 (WEIL). *If (A,Q,λ) is general,*[9] *then the sequence*

$$1\to \mathrm{Hg}(A)\longrightarrow S(A)\xrightarrow{\rho} U_Q\to 1,$$

coincides with the exact sequence (3). In particular, (A,Q) is almost neat.

PROOF. See Weil 1977; also van Geemen 1994. □

In a family of abelian varieties, the Mumford-Tate group stays constant on the complement of a countable union of closed subvarieties where it can only shrink (see, for example, Milne 2013, §6). The theorem says that, in the Weil family, the general Mumford–Tate and Lefschetz groups are as large as possible given the obvious constraints.

REMARK 1.15. If (A,Q,λ) is general, then it follows from invariant theory that

$$B^*(A)=D^*(A)\oplus W(A,Q).$$

See, for example, van Geemen 1994, 6.12.

THEOREM 1.16. *If (A,Q,λ) is general, then the Weil classes are exotic, and the Hodge conjecture holds for the powers of A if they are algebraic.*

PROOF. This is obvious from the exact sequence

$$1\to \mathrm{MT}(A)\longrightarrow L(A)\xrightarrow{\rho} U_Q\to 1.$$

□

COROLLARY 1.17. *Let (A,Q,λ) be a general polarized abelian fourfold of Weil type. If $\det(A,Q,\lambda)=1$ in $\mathbb{Q}^\times/\mathrm{Nm}(Q^\times)$, then the Hodge conjecture holds for the powers of A.*

PROOF. According to Theorem 1.5, the Weil classes on A are algebraic. □

REMARK 1.18. Let (A_i,Q_i,λ_i), $1\le i\le n$, be general polarized abelian varieties of Weil type such that no two of the algebraic groups $\mathrm{SU}(\phi_i)$ are isomorphic. Then

$$\mathrm{Hg}(A)=\mathrm{Hg}(A_1)\times\cdots\times\mathrm{Hg}(A_n)$$

(Goursat's lemma). Hence, if the (A_i,Q_i,λ_i) are fourfolds with determinant 1, then the Hodge conjecture holds for all varieties of the form $A_1^{m_1}\times\cdots\times A_n^{m_n}$, $m_i\in\mathbb{N}$.

SUMMARY 1.19. Let A be a simple abelian fourfold. Either A is neat, so the Hodge conjecture holds for the powers of A, or there exists an imaginary quadratic field $Q\subset \mathrm{End}^0(A)$ such that (A,Q) is of Weil type. In the second case, the Hodge conjecture holds for the powers of A if there exists a polarization λ such that $\det(A,Q,\lambda)=1$ and (A,Q,λ) is general.

[9]That is, in the complement of a countable union of proper closed subvarieties of the moduli space.

f. *The Tate conjecture*

Let X be a smooth projective variety over an algebraically closed field k of of characteristic zero. Recall that $H^{2i}(X,\mathbb{Q}_\ell(i)) \simeq H^{2i}(\rho X,\mathbb{Q}_\ell(i))$ for all embeddings $\rho\colon k\hookrightarrow\mathbb{C}$.

CONJECTURE 1.20 (DELIGNE). *Let $\gamma\in H^{2i}(X,\mathbb{Q}_\ell(i))$ some ℓ. If γ becomes a Hodge class*[10] *in $H^{2i}(\rho X,\mathbb{Q}_\ell(i))$ for one ρ, then it does so for all ρ.*

THEOREM 1.21 (DELIGNE). *Conjecture 1.20 is true for abelian varieties.*

PROOF. This is the main theorem of Deligne 1982 □

From now on, the field k will be the algebraic closure of a subfield finitely generated over $\mathbb{Q}$. Let X_1 be a model of X over a finitely generated subfield k_1 of k with algebraic closure k. An element of the étale cohomology group $H^{2i}(X,\mathbb{Q}_\ell)(i)$ is a ***Tate class*** if it is fixed by some open subgroup of $\mathrm{Gal}(k/k_1)$. This definition is independent of the choice of the model X_1/k_1. A Tate class is ***exotic*** if it is not in the $\mathbb{Q}_\ell$-algebra generated by the Tate classes of degree 1. According to a theorem of Faltings (1983), the nonexotic Tate classes on an abelian variety are algebraic, i.e., in the $\mathbb{Q}_\ell$-span of the cohomology classes of algebraic cycles.

TATE CONJECTURE. *All ℓ-adic Tate classes on X are algebraic.*

THEOREM 1.23 (PIATETSKI-SHAPIRO, DELIGNE). *Let A be an abelian variety over $k\subset\mathbb{C}$. If the Tate conjecture holds for A, then the Hodge conjecture holds for $A_{\mathbb{C}}$.*

PROOF. Let $V\subset H^{2i}(A,\mathbb{Q}_\ell(i))$ be the $\mathbb{Q}$-subspace spanned by the classes on A that become Hodge on $A_{\mathbb{C}}$. Let A_1 be a model of A over a finitely generated subfield k_1 of k with algebraic closure k. Theorem 1.21 implies that the action of $\mathrm{Gal}(k/k_0)$ on $H^{2i}(A,\mathbb{Q}_\ell(i))$ stabilizes V. As V is countable, it follows that the action factors through a finite quotient. Therefore V consists of Tate classes, which are algebraic if the Tate conjecture holds for A. □

REMARK 1.24. Let X be a smooth projective variety over k. We say that $\gamma\in H^{2i}(X,\mathbb{Q}_\ell(i))$ is ***absolutely Hodge*** if it becomes Hodge under every embedding $\rho\colon k\hookrightarrow\mathbb{C}$. The argument in the proof of 1.23 shows that, if the Tate conjecture holds for X, then all absolutely Hodge classes on X are algebraic. Therefore,

$$\text{Deligne conjecture} + \text{Tate conjecture} \implies \text{Hodge conjecture}.$$

Let A be an abelian variety over k, and let A_1 be a model of A over a finitely generated subfield k_1 with algebraic closure k. Some open subgroup U of $\mathrm{Gal}(k/k_1)$ acts trivially on the Hodge classes in $\bigoplus_{r,s} H^{2r}(A^s,\mathbb{Q}_l)(r)$ (see the proof of Theorem 1.23), and it follows that $\mathrm{MT}(A)(\mathbb{Q}_\ell)$ contains U.

MUMFORD–TATE CONJECTURE. *The algebraic group $\mathrm{MT}(A)$ is generated by the subgroup U, i.e., if G is an algebraic subgroup of $\mathrm{MT}(A)$ such that $G(\mathbb{Q}_\ell)\supset U$, then $G=\mathrm{MT}(A)$.*

If the conjecture is true for one U contained in $\mathrm{MT}(A)(\mathbb{Q}_\ell)$, then it is true for all U.

[10]That is, an element of $H^{2i}(\rho X,\mathbb{Q}(i))\cap H^{0,0}\subset H^{2i}(\rho X,\mathbb{Q}_\ell(i))$.

THEOREM 1.26. *If the Mumford–Tate conjecture is true for A, then the Tate conjecture holds for A if and only if the Hodge conjecture holds for $A_{\mathbb{C}}$.*

PROOF. Since the Tate classes in $H^{2*}(A, \mathbb{Q}_\ell(*))$ are those fixed by any sufficiently small U, and Hodge classes are those fixed by $\mathrm{MT}(A)$, equivalently $\mathrm{MT}(A)(\mathbb{Q}_\ell)$, this is obvious. □

The Mumford–Tate conjecture is known for many abelian varieties, for example, for elliptic curves, abelian varieties of prime dimension (many authors; Chi 1991), most abelian fourfolds (Lesin 1994), and all CM abelian varieties (Shimura, Taniyama; Pohlmann 1968). It is true for a product of abelian varieties if it is true for each factor (Vasiu 2008, Commelin 2019).

2 Characteristic p

a. *Statement of the folklore conjecture*

Let X be a smooth projective variety over an algebraically closed field k, and let ℓ be a prime number distinct from the characteristic of k.

FOLKLORE CONJECTURE. *Numerical equivalence coincides with ℓ-adic homological equivalence in the cohomology of X.*[11]

In characteristic zero, this has been proved for abelian varieties (Lieberman 1968). In characteristic p, we have only the following result.

THEOREM 2.2 (CLOZEL). *Let A be an abelian variety over $\mathbb{F}$ (algebraic closure of the field of p elements). There exists a set $s(A)$ of primes ℓ of density > 0 such that the folklore conjecture holds for A and the ℓ in $s(A)$.*

PROOF. This is the main theorem of Clozel 1999. □

The set $s(A)$ can be chosen to depend only on the set of simple isogeny factors of A (Milne 2001, B.2). In particular, $s(A) = s(A^n)$.

b. *Statement of the Tate conjecture over $\mathbb{F}$*

Let ℓ be a prime number $\neq p$. Let X be a smooth projective variety over $\mathbb{F}$, and let X_1 be a model of X over a finite subfield $\mathbb{F}_q$ of $\mathbb{F}$. An element of the étale cohomology group $H^{2i}(X, \mathbb{Q}_\ell)(i)$ is a ***Tate class*** if it is fixed by an open subgroup of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. This definition is independent of the choice of the model $X_1/\mathbb{F}_q$. A Tate class is ***exotic*** if it is not in the $\mathbb{Q}_\ell$-algebra generated by the Tate classes of degree 1. According to a theorem of Tate (1966), the nonexotic Tate classes on an abelian variety are algebraic, i.e., in the $\mathbb{Q}_\ell$-span of the cohomology classes of algebraic cycles.

TATE CONJECTURE. *All ℓ-adic Tate classes on X are algebraic.*

[11] This was stated by Tate in the talk at Woods Hole, 1964, in which he announced his conjectures, and so can be considered to be part of the Tate conjectures. It is also a consequence of the standard conjecture, and is sometimes referred to as the homological standard conjecture.

THEOREM 2.4. *Let X be a smooth projective variety over $\mathbb{F}$. If the Tate and folklore conjectures are true for one $\ell \neq p$, then they are true for all.*

PROOF. Folklore; see Tate 1994, 2.9. □

c. *Tate classes on abelian varieties*

Let A be an abelian variety over $\mathbb{F}$. A model A_1 of A over a finite subfield $\mathbb{F}_q$ of $\mathbb{F}$ defines a Frobenius element $\pi_1 \in \operatorname{End}^0(A)$. The group $P(A)$ is defined to be the smallest algebraic subgroup of $L(A)$ containing some power of π_1 — it is independent of the choice of the model $A_1/\mathbb{F}_q$. There is a canonical homomorphism $P(A) \to \mathbb{G}_m$, and we let $P'(A)$ denote its kernel.

THEOREM 2.5. *Let A be an abelian variety over $\mathbb{F}$. The following conditions on A are equivalent:*

(a) *no power of A supports an exotic Tate class;*

(b) *$P(A) = L(A)$.*

PROOF. Let ℓ be a prime $\neq p$. Almost by definition, the Tate classes are the ℓ-adic cohomology classes fixed by $P(A)$. On the other hand, the cohomology classes fixed by $L(A)$ are exactly those in the $\mathbb{Q}_\ell$-algebra generated by the Tate classes of degree 1 (Milne 1999a, 3.2). Therefore (b) implies (a), and the converse is true because both groups are determined by their fixed tensors. □

An abelian variety over $\mathbb{F}$ satisfying the equivalent conditions of the proposition is said to be ***neat***. Many abelian varieties are known to be neat, for example, products of elliptic curves, and abelian varieties satisfying certain conditions on their Newton polygons (Lenstra, Spiess, Zarhin; see Milne 2001, A.7).

d. *Statement of the standard conjectures*

Let X be a smooth projective variety of dimension n over an algebraically closed field k (possibly of characteristic zero). For $\ell \neq \operatorname{char}(k)$, let $L\colon H^i(X,\mathbb{Q}_\ell) \to H^{i+2}(X,\mathbb{Q}_\ell)(1)$ denote the Lefschetz operator on ℓ-adic étale cohomology defined by an ample divisor. According to the strong Lefschetz theorem (Deligne 1980), the map

$$L^{n-2i}\colon H^{2i}(X,\mathbb{Q}_\ell)(i) \to H^{2n-2i}(X,\mathbb{Q}_\ell)(n-i)$$

is an isomorphism for all $i \leq n/2$. Let $A^i_\ell(X)$ denote the $\mathbb{Q}$-subspace of $H^{2i}(X,\mathbb{Q}_\ell)(i)$ spanned by the classes of algebraic cycles.

LEFSCHETZ STANDARD CONJECTURE. *The map*

$$L^{n-2i}\colon A^i_\ell(X) \to A^{n-i}_\ell(X)$$

is an isomorphism for all $i \leq n/2$.

The map L^{n-2i} is always injective, and it is surjective, for example, if the folklore conjecture holds for X and ℓ (because then $A^i_\ell(X)$ and $A^{n-i}_\ell(X)$ are finite dimensional and dual).

Assuming the Lefschetz standard conjecture, we get a decomposition

$$A^i_\ell(X) = P^i_\ell(X) \oplus LP^{i-1}_\ell(X) \oplus \cdots,$$

where $P^j_\ell(X) = \operatorname{Ker}\left(L^{n-2j+1} : A^j_\ell(X) \to A^{n-j+1}_\ell(X)\right)$.

HODGE STANDARD CONJECTURE. *The pairing*

$$a, b \longmapsto (-1)^i \langle L^{n-2i} a \cdot b \rangle : P^i_\ell(X) \times P^i_\ell(X) \to \mathbb{Q} \tag{4}$$

is positive definite for all $i \leq n/2$.

In characteristic zero, the Hodge standard conjecture follows from Hodge theory.

THEOREM 2.8. *Let X be a smooth projective variety over $\mathbb{F}$. If the Tate and standard conjectures hold for X and one $\ell \neq p$, then they hold for X and all $\ell \neq p$.*

PROOF. Suppose that the Tate and standard conjectures hold for X and ℓ_0. The standard conjecture of Hodge type for X and ℓ_0 implies the folklore conjecture for X and ℓ_0. Because the Tate and folklore conjectures hold for X and ℓ_0, they hold for X and all $\ell \neq p$ (see 2.4). Because the folklore conjecture is true for all $\ell \neq p$, the standard conjectures are independent of ℓ. □

The Lefschetz standard conjecture is known for abelian varieties (Kleiman, Lieberman) — we even know that the correspondence is given by a Lefschetz class (Milne 1999a, 5.9). For the Hodge standard conjecture, the first interesting case is abelian fourfolds. Concerning this, there is the following result.

THEOREM 2.9 (ANCONA). *Let A be an abelian fourfold over an algebraically closed field k. The pairing (4) is positive definite on the algebraic cycles modulo numerical equivalence.*

PROOF. This is the main theorem of Ancona 2021. □

COROLLARY 2.10. *Let A be an abelian fourfold over $\mathbb{F}$. The Hodge standard conjecture holds for A and all $\ell \in s(A)$ (see 2.2). If the Tate conjecture holds for A and one $\ell \in s(A)$, then the Hodge standard conjecture holds for A and all $\ell \neq p$.*

PROOF. The first assertion is obvious, and the second follows from Theorem 2.8. □

e. *A criterion for the Hodge standard conjecture*

Let k be an algebraically closed field, and let $\mathsf{Mot}(k; \mathcal{S})$ be the category of motives modulo numerical equivalence generated by a collection $\mathcal{S}$ of smooth projective varieties over k. Suppose that, for some prime ℓ_0, the Lefschetz standard conjecture and the folklore conjecture hold for the varieties in $\mathcal{S}$. Then $\mathsf{Mot}(k; \mathcal{S})$ is a tannakian category with a fibre functor ω_{ℓ_0}, and it has a natural structure of a Tate triple.

For every variety X in $\mathcal{S}$ and $i \in \mathbb{N}$, there exists a subobject $p^i(X)$ of $h^{2i}(X)(i)$ and a pairing $\phi^i : p^i(X) \otimes p^i(X) \to \mathbb{1}$, both fixed by the fundamental group of $\mathsf{Mot}(k)$, such that $\omega_{\ell_0}(\phi^i)$ is the pairing in the statement of the Hodge standard conjecture.

PROPOSITION 2.11. *The Hodge standard conjecture holds for the varieties in $\mathcal{S}$ and ℓ_0 if and only if there exists a polarization on* $\mathsf{Mot}(k;\mathcal{S})$ *for which the forms ϕ^i are positive.*

PROOF. $\Rightarrow$: If the Hodge standard conjecture holds for all $X \in \mathcal{S}$, then there is a canonical polarization Π on $\mathsf{Mot}(k;\mathcal{S})$ for which the bilinear forms

$$\varphi^i : h^i(X) \otimes h^i(X) \overset{\mathrm{id}\otimes *}{\to} h^i(X) \otimes h^{2n-i}(X)(n-i) \to h^{2n}(X)(n-i) \simeq \mathbb{1}(-i)$$

are positive (Saavedra 1972, VI 4.4) — here $X \in \mathcal{S}$ has dimension n and $*$ is defined by an ample divisor of X. The restriction of $\varphi^{2i} \otimes \mathrm{id}_{\mathbb{1}(2i)}$ to the subobject $p^i(X)$ of $h^{2i}(X)(i)$ is the form ϕ^i, which is therefore positive for Π (Deligne and Milne 1982, 4.11b).

$\Leftarrow$: We let V denote the category of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces V equipped with a semilinear automorphism a such that $a^2 v = (-1)^n v$ ($v \in V^n$); it has a natural structure of a Tate triple over $\mathbb{R}$ (Deligne and Milne 1982, 5.3). Let Π be a polarization on $\mathsf{Mot}(k;\mathcal{S})$ for which the forms ϕ^i are positive. There exists a morphism of Tate triples $\xi : \mathsf{Mot}(k;\mathcal{S}) \to \mathsf{V}$ such that ξ maps Π to the canonical polarization Π^V on V; in particular, for X of weight 0 and $\phi \in \Pi(X)$, $(\gamma^V \circ \xi)(\phi)$ is a positive definite symmetric form on $(\gamma^V \circ \xi)(X)$ (ibid. 5.20). The restriction of $\gamma^V \circ \xi$ to $\mathsf{Mot}(k;\mathcal{S})^\pi$ is (uniquely) isomorphic to γ^{Mot}, and so $\gamma^{\mathrm{Mot}}(\phi^i)$ is positive definite. □

Let A be an abelian variety over $\mathbb{F}$, and let $\langle A\rangle^\otimes$ be the category of motives modulo numerical equivalence generated by A and $\mathbb{P}^1$; it has a natural structure of a Tate triple. Note that $\langle A\rangle^\otimes$ contains the motives of the powers of A. Recall that the Lefschetz standard conjecture holds for abelian varieties.

COROLLARY 2.12. *Let $\ell \in s(A)$; the Hodge standard conjecture holds for ℓ and the powers of A if and only if there exists a polarization on the Tate triple $\langle A\rangle^\otimes$ such that the forms $\phi^i(A^r) : p^i(A^r) \otimes p^i(A^r) \to \mathbb{1}$ are positive, all $i, r \in \mathbb{N}$.*

PROOF. Immediate consequence of the proposition. □

ASIDE 2.13. Besides abelian varieties, the most natural varieties to "test" these conjectures on are $K3$ surfaces. The Hodge conjecture is known for squares of $K3$ surfaces with complex multiplication (Mukai; Buskin 2019, 1.3), and the Tate conjecture is known for $K3$ surfaces over $\mathbb{F}$ and their squares (many authors; Ito et al. 2021).

3 Mixed characteristic

Let $\mathbb{Q}^{\mathrm{al}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let w_0 be a prime of $\mathbb{Q}^{\mathrm{al}}$ dividing p.[12] The residue field at w_0 is an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction to an abelian variety A_0 over $\mathbb{F}$. If A is of CM-type, then there is a unique homomorphism $L(A_0) \to L(A)$ compatible with the actions of the groups on cohomology and with the specialization isomorphisms $H^i(A^s, \mathbb{Q}_\ell) \simeq H^i(A_0^s, \mathbb{Q}_\ell)$. As

[12]In the following, it is possible to replace $\mathbb{Q}^{\mathrm{al}}$ with an algebraic closure C of its completion at w_0, and then choose an embedding of C into $\mathbb{C}$, but this requires the axiom of choice instead of the more benign axiom of dependent choice.

absolute Hodge classes are Tate classes, some Frobenius endomorphism for A_0 will lie in $\mathrm{MT}(A)(\mathbb{Q})$, and so $P(A_0) \subset \mathrm{MT}(A)$. Thus we have a commutative diagram

$$\begin{tikzcd} \mathrm{MT}(A) \arrow[r, hook] & L(A) \\ P(A_0) \arrow[u, hook] \arrow[r, hook] & L(A_0). \arrow[u, hook] \end{tikzcd} \tag{5}$$

If A is not CM, the diagram still exists, but only as a diagram of groups in the tannakian category of Lefschetz motives generated by A_0.

THEOREM 3.1. *Let A be a CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w_0 to an abelian variety A_0 over $\mathbb{F}$. Assume that*

$$P(A_0) = L(A_0) \cap \mathrm{MT}(A) \qquad \textit{(intersection inside } L(A)\textit{).} \tag{*}$$

If the Hodge conjecture holds for A and its powers, then

(a) *the Tate and folklore conjectures hold for A_0 and its powers;*

(b) *the Hodge standard conjecture holds for A_0 and its powers.*

For any A, there exists a CM abelian variety B such that $A \times B$ satisfies (*) (see 3.5 below).

PROOF. (a) Let $\ell_0 \in s(A_0)$ (so the folklore conjecture holds for ℓ_0 and the powers of A_0). Let $M(A_0)$ be the algebraic subgroup of $L(A_0)$ fixing the ℓ_0-adic algebraic classes on the powers of A_0. Because algebraic classes are Tate, $P(A_0) \subset M(A_0)$, and the assumption of the Hodge conjecture implies that $M(A_0) \subset \mathrm{MT}(A)$. Now (*) implies that $M(A_0) = P(A_0)$, and so the ℓ_0-adic Tate classes on the powers of A_0 are algebraic.[13] According to Theorem 2.4, this implies that the Tate and folklore conjectures hold for the powers of A_0 and all $\ell \neq p$.

(b) We defer the proof to later in this section. □

COROLLARY 3.2. *Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w_0 to an abelian variety A_0 over $\mathbb{F}$.*

(a) *If A is neat, then the Hodge conjecture holds for the powers of A.*

(b) *If A_0 is neat, then (*) holds for A and the Tate conjecture holds for the powers of A_0.*

(c) *If A_0 is neat and* $\mathrm{End}^0(A) = \mathrm{End}^0(A_0)$, *then A is neat.*

(d) *If both A and A_0 are neat, then the Hodge standard conjecture for the powers of A_0.*

PROOF. Statements (a) and (b) are true almost by definition, (c) follows from the diagram (5), and (d) follows from (b) of Theorem 3.1 and the next remark. □

REMARK 3.3. In Theorem 3.1, it is not necessary to assume that the Hodge classes on A are algebraic, only almost-algebraic. Part (a) of the theorem also holds for non-CM abelian varieties A with essentially the same proof.

REMARK 3.4. If $L(A_0) = L(A)$, then (*) holds if an only if π_{A_0} generates $\mathrm{MT}(A)$. Here π_{A_0} denotes a sufficiently high power of the Frobenius element defined by a some model of A_0 over a finite field.

[13] This uses that the cohomology classes fixed by $M(A)$ are algebraic — see the Appendix

We list some abelian varieties for which (*) holds.

EXAMPLE 3.5. Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$, and let A be a CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with reflex field contained in K and such that every simple CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with reflex field contained in K is an isogeny factor of A. Then (*) holds[14]

In this case, (*) becomes the formula $P^K = L^K \cap S^K$ of Milne 1999b, Theorem 6.1. There it is proved under the hypothesis that K contains an imaginary quadratic field in which p splits, but this assumption is unnecessary (see the Appendix).

EXAMPLE 3.6. Let $(A \times B^{m-2}, Q)$ be as in Example 1.12. Let K be the subfield of $\mathbb{Q}^{\mathrm{al}}$ generated by the conjugates of E in $\mathbb{Q}^{\mathrm{al}}$. Let $H = \mathrm{Gal}(K/\varphi_0 E)$ and let $D(w_0) \subset \mathrm{Gal}(K/\mathbb{Q})$ be the decomposition group of $w_0|K$. Assume that p splits in Q and that

$$H \cdot D(w_0) = D(w_0) \cdot H.$$

Then (*) holds for $A \times B$ (see Milne 2001). Thus Theorem 3.1 applies to $A \times B$.

EXAMPLE 3.7. We give an example where (*) fails. Let π be a Weil q-integer of degree 6 with the following properties:

(a) for all $v|p$, $\frac{v(\pi)}{v(q)}[\mathbb{Q}[\pi]_v : \mathbb{Q}_p] \equiv 0 \bmod 1$;

(b) there exists an imaginary quadratic field $Q \subset \mathbb{Q}[\pi]$ such that $\mathrm{Nm}_{\mathbb{Q}[\pi]/Q}(\pi^2/q) = 1$.

Such a π exists. The simple abelian variety A_0 over $\mathbb{F}_q$ with Weil integer π has dimension 3 and endomorphism algebra $E \overset{\mathrm{def}}{=} \mathbb{Q}[\pi]$. Up to isogeny, there exists a lift A of A_0 to $\mathbb{Q}^{\mathrm{al}}$ with complex multiplication by E (Tate 1968, Thm 2). As A has dimension 3, it is neat. On the other hand,

$$P(A_0) \subset \mathrm{Ker}(L(A_0) \to (\mathbb{G}_m)_{Q/\mathbb{Q}})$$

and so $P(A_0) \neq L(A_0) \cap \mathrm{MT}(A)$. The group $P(A_0)$ acts trivially on

$$\Big(\bigwedge\nolimits^{6}_{Q\otimes\mathbb{Q}_\ell} H^1(A^2, \mathbb{Q}_\ell)\Big)(3),$$

which therefore consists of exotic Tate classes.

a. *Proof of (b) of Theorem 3.1*

Let $R\colon \mathsf{C}_1 \to \mathsf{C}_2$ be a functor of Tate triples. We say that R maps a polarization Π_1 on C_1 to a polarization Π_2 on C_2 if $\psi \in \Pi_1(X)$ implies $R(\psi) \in \Pi_2(R(X))$, i.e., if all Weil forms positive for Π_1 map to Weil forms positive for Π_2.

Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w_0 to an abelian variety A_0 over $\mathbb{F}$. Assume that (*) holds and that the Hodge conjecture holds for A and its powers. According to Theorem 2.8, it suffices to prove the Hodge standard conjecture for a single ℓ. Implicitly, we choose $\ell \in s(A)$.

Let $\langle A\rangle^{\otimes}$ be the category of motives generated by A and $\mathbb{P}^1$ using Hodge classes as correspondences, and let $\langle A_0\rangle^{\otimes}$ be the category of numerical motives generated by A_0 and $\mathbb{P}^1$. We regard both as Tate triples with their natural additional structures. Because

[14]For example, for each CM-type Φ on K, let A_Φ be an abelian variety of $\mathbb{Q}^{\mathrm{al}}$ of CM-type (K,Φ). Then $A \overset{\mathrm{def}}{=} \prod A_\Phi$ has the required property.

we are assuming the Hodge conjecture for the powers of A, there is a reduction functor $q\colon \langle A\rangle^{\otimes} \to \langle A_0\rangle^{\otimes}$. This realizes $\langle A_0\rangle^{\otimes}$ as a quotient of $\langle A\rangle^{\otimes}$ in the sense of Milne 2007, 2.2. Hodge theory provides the Tate triple $\langle A\rangle^{\otimes}$ with a canonical polarization Π.

LEMMA 3.8. *To prove (b) of Theorem 3.1, it suffices to show that there exists a polarization Π_0 on $\langle A_0\rangle^{\otimes}$ such that q maps Π to Π_0.*

PROOF. Let D be an ample divisor on A, and use it to define the bilinear forms

$$\phi^i(A^r)\colon\ p^i(A^r)\otimes p^i(A^r)\to \mathbb{1},\quad i,r\in\mathbb{N}.$$

Then $\phi^i(A^r)$ is positive (by definition) for Π, and $q(\phi^i(A^r))$ is the bilinear form $\phi^i(A_0^r)$ defined by the ample divisor D_0 on A_0. As q maps Π to Π_0, the forms $\phi_0^i(A^r)$ are positive for Π_0, which implies that the Hodge standard conjecture holds for the varieties in $\langle A_0\rangle^{\otimes}$ (see 2.12). □

LEMMA 3.9. *There exists a polarization Π_0 on $\langle A_0\rangle^{\otimes}$ such that q maps Π to Π_0 if there exists an $X\in\left(\langle A\rangle^{\otimes}\right)^{P(A_0)}$ and a $\psi\in\Pi(X)$ such that $\mathrm{MT}(A)/P(A_0)$ acts faithfully on X and $q(\psi)$ is a positive definite form on the vector space $q(X)$.*

PROOF. Apply Milne 2002, 1.5. □

We now prove (b) of Theorem 3.1 by showing that there exists a pair (X,ψ) satisfying the conditions of 3.9. Let $X=\underline{\mathrm{End}}(h_1A)^{P(A_0)}$. In Lemma 3.10 below, we show that $L(A)/L(A_0)$ acts faithfully on $\underline{\mathrm{End}}(h_1A)^{L(A_0)}$. As (*) implies that $\mathrm{MT}(A)/P(A_0)$ injects into $L(A)/L(A_0)$, this shows that $\mathrm{MT}(A)/P(A_0)$ acts faithfully on X.

Let $\phi\colon\ h_1A\otimes h_1A\to \mathbb{1}(-1)$ be the form defined by an ample divisor D on A, and let ψ be the symmetric bilinear form on $\mathrm{End}(h_1A)$ defined by ϕ (see Milne 2002, 1.1). Write $\psi|$ for the restriction of ψ to X. Then $\psi|\in\Pi(X)$ and it remains to show that $q(\psi|)$ is positive definite. But $q(X)=\mathrm{End}^0(A_0)$ and $q(\psi)$ is the trace pairing $u,v\longmapsto \mathrm{Tr}(u\cdot v^{\dagger})$ of the Rosati involution defined by the divisor D_0 on A_0, which is positive definite by Weil 1948, Théorème 38.

LEMMA 3.10. *Let A be a CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w_0 to an abelian variety A_0 over $\mathbb{F}$. The action of $L(A)/L(A_0)$ on $\underline{\mathrm{End}}(h_1A)^{L(A_0)}$ is faithful.*

PROOF. First an elementary remark. Let T be a torus acting on a finite dimensional vector space V, and let L be a subtorus of T. Let $\chi_1,\dots,\chi_n$ be the characters of T occurring in V. Then T acts faithfully on V if and only if $\chi_1,\dots,\chi_n$ span $X^*(T)$ as a $\mathbb{Z}$-module — assume this. The characters of T occurring in $\mathrm{End}(V)$ are $\{\chi_i-\chi_j\}$, and the set of those occurring in $\mathrm{End}(V)^L$ is

$$\{\chi_i-\chi_j \mid \chi_i|L=\chi_j|L\}. \tag{*}$$

On the other hand,

$$X^*(T/L)=\left\{\sum a_i\chi_i \,\middle|\, \sum a_i\chi_i|L=0\right\}. \tag{**}$$

Thus, T/L will act faithfully on $\mathrm{End}(V)^L$ if the set (*) spans the $\mathbb{Z}$-module (**).

It suffices to prove the lemma for a simple A. Let K be a (sufficiently large) CM field, Galois over $\mathbb{Q}$, with Galois group Γ. Then A corresponds to a Γ-orbit Ψ of CM-types on K (Milne 1999b, 2.3). Each $\psi\in\Psi$ defines a character of $L(A)$, and $L(A)$ acts on $\omega_\ell(h_1A)$

through the ψ in Ψ. For $\psi \in \Psi$, let $\pi(\psi)$ denote the Weil germ attached to ψ by the Taniyama formula (Tate 1966, Thm 5).

The abelian variety A_0 is a isotypic, and hence corresponds to a Γ-orbit Π of Weil germs (Milne 1999b, 4.1). In fact, $\Pi = \{\pi(\psi) \mid \psi \in \Psi\}$ (ibid. 5.4). Each $\pi \in \Pi$ defines a character of $X^*(L(A_0))$, and the homomorphism $X^*(L(A)) \to X^*(L(A_0))$ sends ψ to $\pi(\psi)$. The elements of Ψ can be numbered $\psi_1, \ldots, \psi_n, \bar{\psi}_{n+1}, \ldots, \bar{\psi}_{2n}$, in such a way that $\pi(\psi_1) = \cdots = \pi(\psi_d) = \pi_1$, $\pi(\psi_{d+1}) = \cdots = \pi(\psi_{2d}) = \pi_2$, etc.. Now $\sum a_i \cdot \psi_i | L(A_0) = \sum a_i \cdot \pi(\psi_i)$, which is zero if and only if $\sum_{i=1}^{d} a_i = 0$, $\sum_{i=d+1}^{2d} a_i = 0$, etc. But then

$$\sum_{i=1}^{n} a_i \psi_i = \sum_{i=1}^{d} a_i(\psi_i - \psi_1) + \cdots,$$

which (by the remark) shows that

$$S(A)/S(A_0) \simeq L(A)/L(A_0)$$

acts faithfully on $\underline{\mathrm{End}}(h_1 A)^{L(A_0)}$. □

ASIDE 3.11. Strictly, the proof only shows that the Hodge standard conjecture holds for a Lefschetz operator on A_0 coming from A. However, recall the following theorem. Let (A, λ) be a polarized abelian variety over $\mathbb{F}$. For some discrete valuation ring R containing the ring of Witt vectors $W(\mathbb{F})$ and finite over $W(\mathbb{F})$, there exists a polarized abelian scheme (B, μ) over R whose generic fibre has complex multiplication and whose special fibre is isogenous to (A, λ). Indeed, Mumford 1970, Corollary 1, p. 234, allows us to assume that the polarization λ is principal, in which case we can apply Zink 1983, 2.7 (with $L = \mathbb{Q}$).

b. *Proof of Theorem 0.1*

Let A and B be as in the statement of Theorem 0.1. As A is neat but $A \times B$ is not neat, there exists an embedding of $Q \overset{\text{def}}{=} \mathrm{End}^0(B)$ into $\mathrm{End}^0(A)$ and an $m \in \mathbb{N}$ such $(A \times B^m, Q)$ is of Weil type and

$$1 \to \mathrm{Hg}(A \times B) \longrightarrow S(A \times B) \xrightarrow{\rho} U_Q \to 1$$

is exact (1.10). Thus, the Hodge conjecture holds for the powers of $A_{\mathbb{C}} \times B_{\mathbb{C}}$ if the Weil classes are algebraic (Theorem 1.8).

As A and B are neat, they satisfy the Mumford–Tate conjecture, and therefore $A \times B$ does too. Thus, the Tate conjecture for the powers of $A \times B$ follows from the Hodge conjecture for the powers of $(A \times B)_{\mathbb{C}}$ (Theorem 1.26).

As A_0 is neat but $A_0 \times B_0$ is not neat, B_0 must be ordinary. Now the same argument as in the proof of Proposition 1.10 gives us an exact sequence

$$1 \to P'(A \times B) \to S(A \times B) \to U \to 1$$

in which U is one-dimensional and the projection $S(B) \to U_0$ is surjective. Thus, we have a commutative diagram with exact rows,

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Hg}(A \times B) \arrow[r] & S(A) \times S(B) \arrow[r, "\rho"] & U_Q \arrow[r] & 1 \\
1 \arrow[r] & P'(A_0 \times B_0) \arrow[r] \arrow[u] & S(A_0) \times S(B_0) \arrow[r] \arrow[u] & U_0 \arrow[r] \arrow[u] & 1.
\end{tikzcd}$$

The maps $S(B_0) \to S(B) \to U$ are isomorphisms. A diagram chase now shows that $U_0 \to U$ is injective, and so

$$P'(A_0 \times B_0) = S(A_0 \times B_0) \cap \mathrm{Hg}(A \times B).$$

Thus (*) holds. As the Hodge conjecture holds for the powers of $A \times B$, the Tate, folklore, and Hodge standard conjecture hold for the powers of $A_0 \times B_0$ (Theorem 3.1).

4 Some examples

We list a few examples.

a. *Products of elliptic curves*

THEOREM 4.1. *Let A be a product of elliptic curves over an algebraically closed field k.*

(a) If k has characteristic zero, then A is neat; in particular, it satisfies the Hodge conjecture.

(b) If k is the algebraic closure of a field finitely generated over $\mathbb{Q}$, then A is neat; in particular, it satisfies the Tate conjecture.

(c) If $k = \mathbb{F}$, then A is neat; in particular, it satisfies the Tate conjecture.

(d) For any field k, A satisfies the Hodge standard conjecture.

PROOF. (a) Let $A = A_1 \times \cdots \times A_m$. Suppose first that the A_i are CM. Choose a prime number p such that the A_i have good ordinary reduction at p. Then the statement follows from (c) and 3.2(c).

As elliptic curves are neat, in the general case it suffices to prove that, if $A_1, \ldots, A_m$ are elliptic curves no two of which are isogenous, then

$$\mathrm{Hg}(A_1 \times \cdots \times A_m) = \mathrm{Hg}(A_1) \times \cdots \times \mathrm{Hg}(A_m). \tag{6}$$

This can be proved using Goursat's lemma (Imai 1976, p. 367).

(b) As the Mumford–Tate conjecture is known in this case, the statement follows from (a).

(c) This is the main theorem of Spieß 1999.

(d) By a specialization argument, it suffices to prove this over $\mathbb{F}$. Given a product of elliptic curves over $\mathbb{F}$, we may lift it to a product of CM elliptic curves over $\mathbb{Q}^{\mathrm{al}}$, and apply 3.2(d). □

b. *Abelian surfaces*

Abelian surfaces are neat, both over $\mathbb{F}$ and over fields of characteristic zero. Thus, in characteristic zero, powers of abelian surfaces satisfy the Hodge and Tate conjectures, and, in characteristic p, they satisfy the Tate and standard conjectures (see Corollary 3.2).

c. *Abelian threefolds*

In characteristic zero, abelian threefolds are neat. Over $\mathbb{F}$, abelian threefolds satisfy the Tate conjecture, and they are neat except in the following case: $E \overset{\text{def}}{=} \mathrm{End}^0(A_0)$ is a field containing an imaginary quadratic field Q and $\mathrm{Nm}_{E/Q}(\pi_A^2/q) = 1$. See Zarhin 1994.

Let A_0 be an exceptional abelian threefold over $\mathbb{F}$, and let B_0 be an elliptic curve over $\mathbb{F}$ with $Q = \mathrm{End}^0(B_0)$. There exist lifts A and B of A_0 and B_0 (up to isogeny) as in 1.12. Therefore, the Tate and standard conjectures hold for the varieties $A^r \times B^s$ (see 3.6).

d. *Products of threefolds and elliptic curves over* $\mathbb{F}$

THEOREM 4.2. *Let A_0 (resp. B_0) be a simple abelian threefold (resp. elliptic curve) over $\mathbb{F}$. If A_0 is neat, then the Tate and standard conjectures hold for the abelian varieties $A_0^r \times B_0^s$ ($r, s \in \mathbb{N}$).*

PROOF. Lift A_0 and B_0 (up to isogeny) to abelian varieties A and B over $\mathbb{Q}^{\mathrm{al}}$. If neither $A \times B$ nor $A_0 \times B_0$ is neat, then we can apply Theorem 0.1 to prove the statement. If $A \times B$ and $A_0 \times B_0$ are both neat, we can apply Corollary 3.2. We leave the remaining cases as an exercise. □

e. *Simple abelian fourfolds*

THEOREM 4.3. *Let A be a CM abelian fourfold over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w_0 to an abelian variety A_0 over $\mathbb{F}$. Suppose that* $\mathrm{End}^0(A_0) = \mathbb{Q}[\pi_A]$ *and that π_A generates* $\mathrm{MT}(A)$. *If either*

(a) *A is not of Weil type, or*

(b) *A is of Weil type relative to $Q \subset \mathbb{Q}[\pi_A]$ and* $\det(A, Q, \lambda) = 1$ *for some polarization λ,*

then the Hodge conjecture holds for the powers of $A_{\mathbb{C}}$ and the Tate and standard conjectures hold for the powers of A_0.

PROOF. The hypotheses imply that $L(A_0) = L(A)$ and that (*) holds. Under either (a) or (b), the Hodge conjecture holds for the powers of A, and so we can apply Theorem 3.1. □

f. *General abelian varieties of Weil type*

Let (A, Q, λ) be a general abelian variety of Weil type over $\mathbb{Q}^{\mathrm{al}}$. We saw in Theorem 1.16 that the Hodge conjecture holds for the powers of A if the Weil classes on (A, Q) are algebraic. I expect that if A has good reduction to an abelian variety A_0 over $\mathbb{F}$, then (*) holds and the Tate and standard conjectures hold for the powers of A_0 under the same hypothesis on the Weil classes. Similar statements should hold for products of general abelian varieties of Weil type (see 1.18).

A Appendix

We add some commentary and details.

a. *Proof of the tannakian property of $M(A)$*

Let A be an abelian variety over $\mathbb{C}$, and let $M(A)$ denote the algebraic subgroup of $L(A)$ fixing the algebraic classes in $H \overset{\text{def}}{=} \bigoplus_{r,s} H^{2r}(A^s,\mathbb{Q})(r)$. Then every element of H fixed by $M(A)$ is algebraic.

In the 1980s, this statement was considered to be beyond reach; in 1990s, it was considered obvious. What changed was that Jannsen proved that the ring of correspondences modulo numerical equivalence is semisimple (see below).

Before proving the statement, I illustrate its importance. Let A be an abelian variety, and let G by the subgroup of $\mathrm{GL}_{V(A)}$ fixing some algebraic classes on A. Then *every* cohomology class fixed by G is algebraic. This follows from the fact that G (obviously) contains $M(A)$. Note that this application does not require us to know $M(A)$.

Let $\mathsf{Mot}(\mathbb{C})$ denote the category of abelian motives over $\mathbb{C}$ modulo numerical equivalence. This is a pseudo-abelian tensor category, and Jannsen's result shows that it is abelian. As the folklore conjecture is known for abelian varieties over $\mathbb{C}$, there is a Betti fibre functor ω_B on $\mathsf{Mot}(\mathbb{C})$. Therefore $\mathsf{Mot}(\mathbb{C})$ is a neutral tannakian category. Let $\langle A\rangle^{\otimes}$ denote the tannakian subcategory of $\langle A\rangle^{\otimes}$ generated by A and $\mathbb{P}^1$. Then $M(A) = \underline{\mathrm{Aut}}^{\otimes}(\omega_B|\langle A\rangle^{\otimes})$, and the above statement is simply an expression of tannakian duality.

The same statement holds over $\mathbb{F}$ when we replace Betti cohomology with ℓ-adic cohomology with $\ell \in s(A)$ (see 2.2).

b. *Proof that the ring of correspondences is semisimple*

We prove that the existence of a Weil cohomology theory implies that the ring of correspondences modulo numerical equivalence of an algebraic variety is semisimple.

Theorem A.4 below is extracted from Jannsen 1992. Throughout, H is a Weil cohomology theory with coefficient field Q. We let $A^r_H(X)$ denote the $\mathbb{Q}$-subspace of $H^{2r}(X)(r)$ spanned by the algebraic classes, and $A^r_{\mathrm{num}}(X)$ the quotient of $A^r_H(X)$ be the left kernel of the intersection pairing

$$A^r_H(X) \times A^{d-r}_H(X) \to A^d_H(X) \simeq \mathbb{Q}.$$

A.1. The $\mathbb{Q}$-vector space $A^r_{\mathrm{num}}(X)$ is finite-dimensional over $\mathbb{Q}$: if $f_1,\dots,f_s \in A^{d-r}_H(X)$ span the subspace $Q\cdot A^{d-r}_H(X)$ of $H^{2d-2r}(X)(d-r)$, then the map

$$x \longmapsto (x\cdot f_1,\dots,x\cdot f_s)\colon A^r_H(X) \to \mathbb{Q}^s$$

has image $A^{d-r}_{\mathrm{num}}(X)$.

A.2. Let $A^r_H(X,Q) = Q\cdot A_H$. Define $A^r_{\mathrm{num}}(X,Q)$ to be the quotient of $A^r_H(X,Q)$ by the left kernel of the pairing

$$A^r_H(X,Q)\times A^{d-r}_H(X,Q) \to A^d_H(X,Q) \simeq Q$$

induced by cup product. Then $A^r_H(X) \to A^r_{\mathrm{num}}(X,Q)$ factors through $A^r_{\mathrm{num}}(X)$,

$$\begin{tikzcd}
A^r_H(X) \arrow[r] \arrow[d] & A^r_{\mathrm{num}}(X) \arrow[d, dashed] \\
A^r_H(X,Q) \arrow[r] & A^r_{\mathrm{num}}(X,Q),
\end{tikzcd}$$

and I claim that the map

$$Q \otimes A^r_{\mathrm{num}}(X) \to A^r_{\mathrm{num}}(X, Q)$$

is an isomorphism. As $A^r_{\mathrm{num}}(X, Q)$ is spanned by the image of $A^r_H(X)$, the map is obviously surjective. Let $e_1, \ldots, e_m$ be a $\mathbb{Q}$-basis for $A^r_{\mathrm{num}}(X)$, and let $f_1, \ldots, f_m$ be the dual basis in $A^{d-r}_{\mathrm{num}}(X)$. If $\sum_{i=1}^m a_i e_i$, $a_i \in Q$, is zero in $A^r_{\mathrm{num}}(X, Q)$, then $a_j = (\sum a_i e_i) \cdot f_j = 0$ for all j.

A.3. Recall that the ***radical*** $R(A)$ of a ring A is the intersection of the maximal left ideals in A. Equivalently it is the intersection of the annihilators of simple A-modules. It is a two-sided ideal in A. Every left (or right) nil ideal[15] is contained in $R(A)$. For any ideal $\mathfrak{a}$ of A contained in $R(A)$, $R(A/\mathfrak{a}) = R(A)/\mathfrak{a}$. The radical of an artinian ring A is nilpotent, and it is the largest nilpotent two-sided ideal in A. (Bourbaki, A, VIII, §6.)

THEOREM A.4 (JANNSEN). *Let X be a smooth projective variety over a field k. Then the $\mathbb{Q}$-algebra $A^*_{\mathrm{num}}(X \times X)$ is semisimple.*

PROOF. Let $B = A^*_{\mathrm{num}}(X \times X)$. Recall that B has finite dimension over $\mathbb{Q}$, and that multiplication in B is composition $\circ$ of correspondences. By definition of numerical equivalence, the pairing

$$f, g \longmapsto \langle f \cdot g \rangle : B \times B \to \mathbb{Q}$$

is nondegenerate. Let f be an element of the radical $R(B)$ of B. We have to show that $\langle f \cdot g \rangle = 0$ for all $g \in B$.

Let H be a Weil cohomology with coefficient field Q. Let $A = A^*_H(X \times X, Q)$; then A is a finite-dimensional Q-algebra, and there is a surjective homomorphism

$$A \stackrel{\mathrm{def}}{=} A^d_H(X \times X, Q) \to A^d_{\mathrm{num}}(X \times X, Q) \simeq Q \otimes B$$

(see A.2). This maps the radical of A onto that of $Q \otimes B$ (see A.3). Therefore, there exists an $f' \in R(A)$ mapping to $1 \otimes f$. For all $g \in A$,

$$\langle f' \cdot g^t \rangle = \sum_i (-1)^i \operatorname{Tr}(f' \circ g \mid H^i(X)) \tag{7}$$

(Kleiman 1968, 1.3.6). As $f' \circ g$ lies in $R(A)$, it is nilpotent (see A.3), and so (7) shows that $\langle f' \cdot g^t \rangle = 0$. □

c. *Goursat's Lemma*

Let $B_1, \ldots, B_n$ be algebraic groups.[16] A subdirect product of $B_1 \times \cdots \times B_n$ is an algebraic subgroup A such that the projections $A \to B_i$ are all faithfully flat.

GOURSAT'S LEMMA A.5. *Let A be a subdirect product of $B_1 \times B_2$, and let N_1 and N_2 be the kernels of the projections $A \to B_2$ and $A \to B_1$ regarded as (normal) algebraic subgroups of B_1 and B_2. Let $\bar{A}$ be the image of A in $B_1/N_1 \times B_2/N_2$. Then the projections $\bar{A} \to B_1/N_1$ and $\bar{A} \to B_2/N_2$ are isomorphisms.*

[15] An ideal is nil if all of its elements are nilpotent. A finitely generated nil ideal is nilpotent.

[16] That is, a group schemes of finite type over a field.

PROOF. By symmetry, it suffices to show that the projection $\bar{A} \to B_1/N_1$ is an isomorphism. It is a faithfully flat by hypothesis, and we prove that it is injective by showing that the two homomorphisms $A \to \bar{A} \to B_1/N_1$ and $A \to \bar{A}$ have the same kernel. For a k-algebra R, the R-points of the kernels are

$$\{(a_1, a_2) \in A(R) \mid a_1 \in N_1(R)\} \text{ and } \{(a_1, a_2) \in A(R) \mid a_1 \in N_1(R) \text{ and } a_2 \in N_2(R)\}.$$

Let (a_1, a_2) lie in the first kernel. To say that $a_1 \in N_1(R)$ means that $(a_1, 1) \in \operatorname{Ker}(A(R) \to B_2(R))$. In particular, $(a_1, 1) \in A(R)$. Thus $(1, a_2) \in A(R)$, and so it lies in $N_2(R)$. Hence $(a_1, a_2) \in N_1(R) \times N_2(R)$, and so it lies in the second kernel. □

The lemma says that the image $\bar{A}$ of A in $B_1/N_1 \times B_2/N_2$ is the graph of an isomorphism $B_1/N_1 \simeq B_2/N_2$. If B_1 and B_2 are almost-simple, then either $A = B_1 \times B_2$ or it is the graph of an isogeny $B_1 \to B_2$.

LEMMA A.6. *Let A be a normal algebraic subgroup of $B_1 \times B_2$ such that the projections $A \to B_1$ and $A \to B_2$ are isomorphisms. Then B_1 and B_2 are commutative.*

PROOF. By assumption, A is the graph of an isomorphism $\varphi : B_1 \to B_2$. Let R be a k-algebra, and let $(b_1, \varphi(b_1)) \in A(R)$. For all $b_2 \in B_2(R)$, $(1, b_2)$ normalizes $(b_1, \varphi(b_1))$, and so b_2 centralizes $\varphi(b_1)$. This shows that B_2 is commutative. □

LEMMA A.7. *Let A be a subdirect product of $B_1 \times \cdots \times B_n$. If A is normal in $B_1 \times \cdots \times B_n$ and the B_i are perfect, then $A = B_1 \times \cdots \times B_n$.*

PROOF. We first take $n = 2$. With the notation of Goursat's lemma, the algebraic subgroup $\bar{A}$ of $B_1/N_1 \times B_2/N_2$ satisfies the hypotheses of the last lemma, and so B_1/N_1 and B_2/N_2 are commutative, hence trivial. As A contains N_1 and N_2, it equals $B_1 \times B_2$.

We prove the general case by induction on n. On applying the induction hypothesis to the image of A in $B_1 \times \cdots \times B_{n-1}$, we find that the projection map $A \to B_1 \times \cdots \times B_{n-1}$ is faithfully flat. Now A is a subdirect product of $(B_1 \times \cdots \times B_{n-1}) \times B_n$, and the case $n = 2$ shows that $A = B_1 \times \cdots \times B_n$. □

LEMMA A.8. *Let A be an algebraic subgroup of $B_1 \times \cdots \times B_n$. If each B_i is perfect and each projection $A \to B_i \times B_j$, $1 \le i < j \le n$, is faithfully flat, then $A = B_1 \times \cdots \times B_n$.*

PROOF. The lemma is certainly true for $n = 2$. Assume that $n > 2$, and that the statement is true for $n - 1$. Then the projection $A \to B_1 \times \cdots \times B_{n-1}$ is faithfully flat, and so the kernel N of the projection $A \to B_n$ is a normal algebraic subgroup of $B_1 \times \cdots \times B_{n-1}$. By the last lemma, it equals $B_1 \times \cdots \times B_{n-1}$, and so there is an exact commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & B_1 \times \cdots \times B_{n-1} \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, hook] & B_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & B_1 \times \cdots \times B_{n-1} \arrow[r] & B_1 \times \cdots \times B_n \arrow[r] & B_n \arrow[r] & 1,
\end{tikzcd}$$

from which it follows that $A = B_1 \times \cdots \times B_n$. □

The results in this section are well-known.

d. *Proof of the equality $P^K = L^K \cap S^K$.*

Let K be a CM field, Galois over $\mathbb{Q}$, with a given p-adic prime. Let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$, and let $D \subset \Gamma$ be the decomposition group of the given prime. We write P, S, L, T for P^K, S^K, L^K, T^K. Recall that we have diagrams

$$\begin{tikzcd} T & S \arrow[l] \\ L \arrow[u] & P \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} X^*(T) \arrow[r, "j"] \arrow[d, "r"] & X^*(S) \arrow[d, "s"] \\ X^*(L) \arrow[r, "i"] & X^*(P) \end{tikzcd}$$

and have to show that the induced map $S/P \to T/L$ is injective or, equivalently, that the map

$$j\colon\ \mathrm{Ker}(r) \to \mathrm{Ker}(s)$$

is surjective.

Recall that ι denotes complex conjugation (on K say), and so $\langle\iota\rangle$ is the subgroup $\{1, \iota\}$ of Γ. For a finite set Y with an action of $\langle\iota\rangle$ and a positive integer d, define

$$\begin{aligned} \mathbb{Z}[Y] &= \text{free abelian group on } Y = \{\text{maps } f\colon\ Y \to \mathbb{Z} \text{ or sums } \textstyle\sum f(y)y\}, \\ \mathbb{Z}[Y]^d &= \{f \in \mathbb{Z}[Y] \mid \exists c \in \mathbb{Z} \text{ such that } f + \iota f = d \cdot c \text{ (constant function)}\}, \\ \mathbb{Z}[Y]_0 &= \mathbb{Z}[Y]/\{f \mid f = \iota f \text{ and } \textstyle\sum f(y) = 0\}. \end{aligned}$$

Then (see §§1–5 of the Milne 1999b; also Milne 2001, especially the diagram in A.8) the second of the above diagrams can be identified with

$$\begin{tikzcd} \mathbb{Z}[\mathcal{S}]_0 \arrow[r, "j"] \arrow[d, "r"] & \mathbb{Z}[\Gamma]^1 \arrow[d, "s"] \\ \mathbb{Z}[\mathcal{P}]_0 \arrow[r, "i"] & \mathbb{Z}[\Gamma/D]^d. \end{tikzcd}$$

Here $\mathcal{S}$ is the set of CM-types on K, i.e., functions $\varphi\colon\ \Gamma \to \{0,1\}$ such that $\varphi + \iota\varphi = 1$, and $\mathcal{P}$ is the set[17] of functions $\pi\colon\ \Gamma/D \to \{0, 1, \dots, d\}$, $d = (D\colon\ 1)$, such that $\pi + \iota\pi = d$. The horizontal maps send a formal sum to a sum of functions, e.g., j sends the formal sum $\sum f(\varphi)\varphi$ to the function $\tau \longmapsto \sum f(\varphi)\varphi(\tau)\colon\ \Gamma \to \mathbb{Z}$. The map r sends φ to $r(\varphi)$ where $r(\varphi)(\tau D) = \sum_{\sigma\in D} \varphi(\tau\sigma)$, and s sends f to $s(f)$ where $s(f)(\tau D) = \sum_{\sigma \in D} f(\tau\sigma)$.

Case I: $\iota \in D$. Choose a set B of coset representatives for $\langle\iota\rangle$ in D and a set A of coset representatives for D in Γ. Then, as a map of $\langle\iota\rangle$-sets, $\Gamma \to \Gamma/D$ can be identified with the projection map

$$A \times B \times \{0,1\} \to A \times \{0,1\}.$$

[17] Let v be the given p-adic prime on K, so that $d = [K_v\colon\ \mathbb{Q}_p]$. The map $\tau \longmapsto \tau v$ defines a bijection of Γ/D onto the set X of p-adic primes of K. Let $\pi \in W(p^\infty)$, and let $s_\pi\colon\ X \to \mathbb{Q}$ be the corresponding slope function (Milne 2001, A.6). For $\pi \in W^K(p^\infty)$, $d \cdot s_\pi$ takes values in $\mathbb{Z}$, and the correspondence

$$d \cdot s_\pi \leftrightarrow \pi$$

identifies $\mathcal{P}$ with the set of integral elements of weight 1 (or maybe -1, depending on conventions) in $W^K(p^\infty)$, i.e., with the Weil numbers in the sense of Tate 1968/69 modulo roots of 1 corresponding to abelian varieties over $\mathbb{F}$ whose endomorphism algebra is split by K.

Here ι acts only on $\{0,1\}$. Thus, the diagram becomes

$$\begin{array}{ccc}
\mathbb{Z}[\mathcal{S}]_0 & \longrightarrow & \mathbb{Z}[A\times B\times\{0,1\}]^1 \\
\downarrow r & & \downarrow s \\
\mathbb{Z}[\mathcal{P}]_0 & \longrightarrow & \mathbb{Z}[A\times\{0,1\}]^d.
\end{array}$$

With this notation, a CM-type on K is a function $\varphi: A\times B\times\{0,1\}\to\{0,1\}$ such that

$$\varphi(a,b,0)+\varphi(a,b,1)=1,\quad \text{all } a,b,$$

and an element of $\mathcal{P}$ is a function $\pi: A\times\{0,1\}\to\{0,\dots,d\}$ such that

$$\pi(a,0)+\pi(a,1)=d,\quad \text{all } a.$$

For each a,b, let $\varphi_{a,b}$ be the CM-type such that

$$\varphi_{a,b}(x,y,0)=1 \iff (x,y)=(a,b),$$

and let φ' be the CM-type such that

$$\varphi(x,y,z)=z.$$

Then φ' and the $\varphi_{a,b}$ generate $\mathbb{Z}[A\times B\times\{0,1\}]^1$: if f is a function on $A\times B\times\{0,1\}$ such that $f+\iota f=c$ ($c\in\mathbb{Z}$), then

$$f=\sum_{a,b} f(a,b,0)\varphi_{a,b}+\Big(c-\sum_{a,b} f(a,b,0)\Big)\varphi'.$$

Let $\pi_a=r(\varphi)_{a,b}$ and $\pi'=r(\varphi')$: thus π_a is the element of $\mathcal{P}$ such that $\pi_a(x,0)=1$ for $x=a$ and is zero otherwise, and $\pi'(x,z)=d\cdot z$. Clearly, the π_a and π' are linearly independent.

We now prove this case of the theorem. Let $f\in\mathbb{Z}[A\times B\times\{0,1\}]^1$, and write it

$$f=\sum_{a,b} n(a,b)\cdot\varphi_{a,b}+n\cdot\varphi',$$

so

$$s(f)=\sum_a\Big(\sum_b n(a,b)\Big)\pi_a+n\pi'.$$

If $s(f)=0$, then, because of the linear independence of the π_a and π',

$$\sum_{b\in B} n(a,b)=0,\quad \text{all } a\in A, \text{ and}$$
$$n=0.$$

The first equation implies that $\sum_b n(a,b)\varphi_{a,b}$ is in the kernel of r, which completes the proof.

Case II: $\iota\notin D$. This case is so similar to the preceding that it should be left as an exercise to the reader.

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