Shimura Varieties and Motives

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Abstract

Deligne has expressed the hope that a Shimura variety whose weight is defined over $\mathbb{Q}$ is the moduli variety for a family of motives. Here we prove that this is the case for "most" Shimura varieties. As a consequence, for these Shimura varieties, we obtain an explicit interpretation of the canonical model and a modular description of its points in any field containing the reflex field. Moreover, when we assume the existence of a sufficiently good theory of motives in mixed characteristic, we are able to obtain a description of the points on the Shimura variety modulo a prime of good reduction.

Contents

Introduction

A Shimura variety $\mathrm{Sh}(G,X)$ is a projective system of algebraic varieties over $\mathbb{C}$. The data needed to define it are a reductive group G over $\mathbb{Q}$ together with a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h{:}\,\mathbb{C}^{\times} \to G(\mathbb{R})$ satisfying conditions sufficient to ensure that X is, in a natural way, a finite union of bounded symmetric domains.

In a small number of cases, the Shimura variety can be interpreted as a moduli variety for abelian varieties with the additional structure of an endomorphism ring, a polarization, and a level structure. Such a Shimura variety is said to be of PEL-type. For example, the Shimura variety defined by a group of symplectic similitudes (a Siegel modular variety) or by the group GL_2 over a totally real field (a Hilbert-Blumenthal variety) is of PEL-type. When such a modular interpretation exists, it is a great help in studying the variety, for example, in

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except that the TeX has been updated, some corrections and a few minor editorial changes made, and some footnotes added. Significant changes to the original article have been noted in footnotes.

constructing a model of the variety over a number field, or, better, the ring of integers in the number field, and in studying the compactification of the model. See Faltings and Chai 1990 for Siegel modular varieties, and Rapoport 1978 for Hilbert-Blumenthal varieties.[1] In fact, the realization of elliptic modular curves as the moduli varieties of elliptic curves with level structure has been extraordinarily fruitful both for the study of the modular curves and for the study of elliptic curves.

If the group G can be embedded into a group of symplectic similitudes $G(\psi)$ in such a way that the elements h of X define on V a Hodge structure of type $\{(-1,0),(0,-1)\}$ for which $\pm 2\pi i\psi$ is a polarization, then $\mathrm{Sh}(G,X)$ has an interpretation as a moduli variety for abelian varieties with absolute Hodge cycle and level structure. Such a Shimura variety is said to be of Hodge type. Unfortunately, this interpretation is valid only in characteristic zero because absolute Hodge cycles only make sense there (so long as the Hodge conjecture for abelian varieties remains open), but the interpretation can again be used to study models and compactifications of the Shimura variety over number fields—see Brylinski 1983.

A Shimura variety whose weight is defined over $\mathbb{Q}$ can always be interpreted (over $\mathbb{C}$) as a parameter space for Hodge structures, and Deligne notes: "Pour interpréter des structures de Hodge de type plus compliqué, on aimerait remplacer les variétés abéliennes par des "motifs" convenables, mais il ne s'agit encore que d'un rêve." (Deligne 1979, p. 248). The main purpose of this article is to provide such an interpretation when G has no factors of type E_6, E_7, or certain types D, and hence to realize the Shimura variety as a moduli variety for motives. As for Shimura varieties of Hodge type, the interpretation is valid only in characteristic zero and depends in a crucial way on Deligne's theorem that all Hodge cycles on abelian varieties are absolutely Hodge (Deligne 1982).

I now describe the contents of the article in more detail.

Betti cohomology provides a functor from the category of motives over $\mathbb{C}$ (defined using algebraic cycles) to the category $\mathsf{Hdg}_{\mathbb{Q}}$ of polarizable rational Hodge structures. The Hodge conjecture predicts that the functor is fully faithful, but there is no description, not even conjectural, for its essential image.

Define the category of abelian motives $\mathsf{Mot}^{\mathrm{ab}}(\mathbb{C})$ over $\mathbb{C}$ to be the tensor subcategory of the category of motives over $\mathbb{C}$ (defined using absolute Hodge cycles) generated by the motives of abelian varieties. The main theorem of Deligne 1982 implies that the Betti fibre functor

$$\mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \to \mathsf{Hdg}_{\mathbb{Q}}$$

is fully faithful. In §1 we describe the essential image of this functor, i.e., we describe the Hodge structures that are the Betti realization of an abelian motive, and we classify the reductive groups that arise as the Mumford-Tate group of an abelian motive. The key ingredients in the proof of the classification are Satake's results on symplectic embeddings of semisimple groups and the well-known fact that every polarizable Hodge structure with commutative Mumford-Tate group is the Betti realization of an abelian motive.

In §2 we investigate, along the lines of Griffiths 1970 and Deligne 1979, the problem of realizing a motive over $\mathbb{C}$, endowed with the structure provided by a family of tensors, as a member of a universal family. In general it is not known how to do this, but we show that it is possible when the motive is abelian.

[1] Added 2017. For general Shimura varieties of PEL-type, see Lan, Kai-Wen, *Arithmetic compactifications of PEL-type Shimura varieties.* London Mathematical Society Monographs Series, 36. Princeton University Press, Princeton, NJ, 2013.

The study of the moduli of motives leads very naturally to the notion of a Shimura variety, and in §3 we classify the Shimura varieties that are moduli varieties for abelian motives; the class includes all but those whose defining group has factors of type E_6, E_7, or certain types D. For these Shimura varieties we are able to obtain a more direct proof of the existence of canonical models than that in Deligne 1979, and, for those whose weight is rational, we deduce a modular interpretation of the canonical model.

As noted above, the theory is restricted to characteristic zero. However, when we assume the existence of a sufficiently good theory of motives in mixed characteristic, we can extend the description of the Shimura variety as a moduli variety for abelian motives to characteristic p, and use this to obtain an explicit description of the points of a reduction of the Shimura variety with coordinates in the algebraic closure of a finite field. The statement we arrive at is (essentially) the main conjecture of Langlands and Rapoport 1987.

Roughly speaking, when one combines the results of §4 of this paper with the results of §§5–7 of Milne 1992 and Theorem 7.1 of Kottwitz 1990, then one arrives at the following statement: for Shimura varieties of abelian type and rational weight, Langlands's conjecture on the contribution of the variety itself (i.e., ignoring its boundary) to the local component of the zeta function at a good prime is a consequence of standard conjectures in algebraic geometry and representation theory, the most significant of which are the existence of a good theory of motives in mixed characteristic and the fundamental lemma.[2]

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Notation and conventions

An affine group scheme over a field k is said to be algebraic when it is of finite type over k. Every affine group scheme is the projective limit of its algebraic quotients. A simple algebraic group is a nontrivial semisimple algebraic group with only finite normal proper subgroups. For an algebraic group G over a field k, G° denotes the connected component of G containing 1 for the Zariski topology; when $k = \mathbb{R}$, $G(\mathbb{R})^+$ denotes the connected component containing 1 for the *real topology*.

For a reductive group G, G^{der} denotes the derived group of G, G^{sc} the simply connected covering group of G^{der}, $Z(G)$ the centre of G, G^{ad} the adjoint group $G/Z(G)$ of G, and G^{ab} the maximal abelian quotient G/G^{der} of G. If $G = G^{\mathrm{ad}}$, then G is called an adjoint group. This notation extends in an obvious way to affine group schemes: write $G = \varprojlim G'$ (limit over the algebraic quotients of G), and set $G^* = \varprojlim G'^*$. The map $\mathrm{ad}\colon G \to \mathit{Aut}(G)$, sending an element of G to the inner automorphism it defines, factors through G^{ad},

$$G \xrightarrow{\mathrm{ad}} G^{\mathrm{ad}} \to \mathit{Aut}(G).$$

The map sending an element g of G to the differential of $\mathrm{ad}(g)$,

$$G \to \mathrm{GL}(\mathfrak{g}), \quad \mathfrak{g} = \mathrm{Lie}(G),$$

is denoted by Ad.

Let $F : \mathsf{A} \to \mathsf{B}$ be a functor, and let A be an object of A. A map $\beta : B \to F(A)$ is said to ***generate*** A if the following holds: for every subobject $A' \hookrightarrow A$ of A such that β factors through $F(A') \to F(A)$, the map $A' \to A$ is an isomorphism.

[2] Added 2017. The fundamental lemma has now been proved.

When S is a set of objects in a Tannakian category T over a field k, we define the ***tensor category generated by*** S to be the smallest full subcategory T' of T containing S and closed under the formation of subobjects, quotient objects, direct sums, duals, and finite tensor products (in particular, it contains with any object X of T, all objects isomorphic to X). It is again a Tannakian category over k.

We often write $V(R')$ for $V \otimes_R R'$. Other notation agrees with that in Milne 1994 except that here we denote the field of fractions of the Witt vectors by B.

1 Abelian Motives and Their Mumford-Tate Groups

Throughout this section, k is an algebraically closed field of characteristic zero.

Hodge structures: definitions

We write $\mathbb{S}$ for the torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ over $\mathbb{R}$; thus

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times, \quad \mathbb{S}_\mathbb{C} = \mathbb{G}_m \times \mathbb{G}_m.$$

The last identification is made in such a way that the map

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C})$$

induced by $\mathbb{R} \hookrightarrow \mathbb{C}$ is $z \mapsto (z, \bar{z})$. Let $U^1 = \mathrm{Ker}(\mathbb{S} \xrightarrow{\mathrm{Nm}} \mathbb{G}_m)$, so that

$$U^1(\mathbb{R}) = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\}.$$

With any homomorphism $h\colon \mathbb{S} \to G$ of real algebraic groups there are associated homomorphisms

$$\mu_h\colon \mathbb{G}_m \to G_\mathbb{C}, \quad \mu_h(z) = h_\mathbb{C}(z,1), \quad z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times,$$

and

$$w_h\colon \mathbb{G}_m \to G, \quad w_h(r) = h(r)^{-1}, \quad r \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times \subset \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$$

(the ***weight homomorphism***). The following formulas are useful:

$$h_\mathbb{C}(z_1, z_2) = \mu_h(z_1) \cdot \bar{\mu}_h(z_2); \quad h(z) = \mu(z) \cdot \overline{\mu(z)} \tag{1.1.1}$$

$$h(i) \equiv \mu_h(-1) \mod w_h(\mathbb{G}_m) \tag{1.1.2}$$

A ***real Hodge structure*** on an $\mathbb{R}$-vector space V can be variously defined as:

(1.2.1) a representation h of $\mathbb{S}$ on V;

(1.2.2) a (Hodge) decomposition $V \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$ all p,q;

(1.2.3) a (weight) gradation $V = \bigoplus_{m \in \mathbb{Z}} V_m$ and a descending (Hodge) filtration

$$\cdots \supset F^p \supset F^{p+1} \supset \cdots$$

such that $V_m = (V_m \cap F^p) \oplus (V_m \cap \overline{F^q})$ for all m, p, q with $p + q = m + 1$.

To pass from one definition to another, use the following rules:

$$v \in V^{p,q} \iff h(z)v = z^{-p} \cdot \bar{z}^{-q} v, \text{ all } z \in \mathbb{C}^{\times};$$
$$V_m \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q}, \quad F^p = \bigoplus_{p' \geq p} V^{p',q'};$$
$$V^{p,q} = V_{p+q} \cap F^p \cap \overline{F^q}.$$

Note that the weight gradation is defined by w_h. From the definition (1.2.1) it is clear that the real Hodge structures form a Tannakian category $\mathsf{Hdg}_{\mathbb{R}}$ over $\mathbb{R}$ with the forgetful functor ω as a fibre functor, and that $Aut^{\otimes}(\omega) = \mathbb{S}$.

A ***rational Hodge structure*** is a vector space V over $\mathbb{Q}$ together with a representation h of $\mathbb{S}$ on $V \otimes \mathbb{R}$ such that w_h is defined over $\mathbb{Q}$. Thus to give a rational Hodge structure on V is the same as giving a gradation $V = \bigoplus V_m$ of V together with a real Hodge structure of weight m on $V_m \otimes \mathbb{R}$ for each m. The rational Hodge structure $\mathbb{Q}(m)$ has $(2\pi i)^m \mathbb{Q}$ as its underlying vector space with $h(z)$ acting as multiplication by $(z\bar{z})^m$ (so $\mathbb{Q}(m)$ is of type $\{(-m,-m)\}$). There are similar definitions with $\mathbb{Q}$ replaced by a subring $R \subset \mathbb{R}$.

A ***polarization*** of a real Hodge structure (V,h) is a family of morphisms of Hodge structures

$$\psi_m : V_m \times V_m \longrightarrow \mathbb{R}(-m), \quad m \in \mathbb{Z},$$

such that

$$(x,y) \mapsto (2\pi i)^m \psi_m(x, h(i)y) : V_m \times V_m \longrightarrow \mathbb{R}$$

is symmetric and positive-definite for each m; equivalently, such that $(2\pi i)^m \psi_m$ is symmetric or skew-symmetric according as m is even or odd and $(2\pi i)^m \psi_m(x, h(i)x) > 0$ all $x \neq 0$. A ***polarization*** of a rational Hodge structure is a family of morphisms of rational Hodge structures $\psi_m : V_m \times V_m \to \mathbb{Q}(-m)$ such that the family $(\psi_m \otimes \mathbb{R})_m$ is a polarization of real Hodge structures. The polarizable rational Hodge structures form a Tannakian category $\mathsf{Hdg}_{\mathbb{Q}}$ over $\mathbb{Q}$ with the forgetful functor ω as fibre functor; we let $G_{\mathrm{Hdg}} = Aut^{\otimes}(\omega)$. The tensor functor

$$\mathsf{Hdg}_{\mathbb{Q}} \to \mathsf{Hdg}_{\mathbb{R}}, \quad V \rightsquigarrow V \otimes \mathbb{R},$$

defines a homomorphism $h_{\mathrm{Hdg}} : \mathbb{S} \to G_{\mathrm{Hdg}}$.

The conditions (SV)

We list some conditions on a homomorphism $h : \mathbb{S} \to G$ of real algebraic groups:

(SV1) the Hodge structure on the Lie algebra $\mathfrak{g}$ of G defined by $\mathrm{Ad} \circ h : \mathbb{S} \to \mathrm{GL}(\mathfrak{g})$ is of type $\{(1,-1),(0,0),(-1,1)\}$;

(SV2) $\mathrm{ad}\, h(i)$ is a Cartan involution of G^{ad}.

When G is connected, (SV1) implies that $w_h(\mathbb{G}_m) \subset Z(G)$. In the presence of this condition, we sometimes need to consider a stronger form of (SV2):

(SV2*) $\mathrm{ad}\, h(i)$ is a Cartan involution of $G/w_h(\mathbb{G}_m)$.

Note that (SV2*) implies that G is reductive.

Let G be an algebraic group over $\mathbb{Q}$, and let h be a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$. We say that (G,h) satisfies the condition (SVx) when $(G_{\mathbb{R}},h)$ satisfies (SVx). For such a pair, we shall also need to consider the condition:

(SV4) [3]The weight homomorphism $w_h\colon \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$ and maps into the centre of G.

Finally, when G is an affine group scheme over $\mathbb{Q}$, we say that (G,h) satisfies (SVx) if $(H, q_{\mathbb{R}} \circ h)$ satisfies (SVx) for every algebraic quotient $q\colon G \to H$ of G.

Abelian motives: definition

Let $\mathsf{Mot}(k)$ denote the category of motives over k defined using absolute Hodge cycles (see Deligne and Milne 1982, §6). We shall be concerned with the tensor subcategory $\mathsf{Mot}^{\mathrm{ab}}(k)$ of $\mathsf{Mot}(k)$ generated by the motives $h_1(A)$ for A an abelian variety over k. An object of $\mathsf{Mot}^{\mathrm{ab}}(k)$ will be called an ***abelian motive*** over k.

EXAMPLE 1.3. (a) The Tate motive, being isomorphic to $\bigwedge^2 h_1(E)$ for any elliptic curve E, is an abelian motive.

(b) Let X be a smooth projective variety over k. Then $h(X)$ is an abelian motive if X is a curve, a unirational variety of dimension ≤ 3, a Fermat hypersurface, or a $K3$-surface (Deligne and Milne 1982, 6.26).

(c) Recall that the ***level*** of a pure Hodge structure (V,h) is the maximum value of $|q-p|$ for which $V^{p,q} \neq 0$. Let (V,h) be a polarizable Hodge structure of level ≤ 1. If (V,h) has even weight $2m$, then it is isomorphic to a sum of copies of $\mathbb{Q}(-m)$; if it has odd weight $2m-1$, then $V \otimes \mathbb{Q}(-m)$ is of type $\{(-1,0),(0,-1)\}$ which Riemann's theorem shows to equal $h_1(A)$ for some abelian variety. In either case, (V,h) is the Betti realization of an abelian motive.

(d) Write $V_n(a_1,\ldots,a_d)$ for the complete intersection of d smooth hypersurfaces of degrees $a_1,\ldots,a_d$ in general position in $\mathbb{P}^{n+d}$ over $\mathbb{C}$. The varieties $V_n(2)$, $V_n(2,2)$, $V_2(3)$, $V_n(2,2,2)$ (n odd), $V_3(3)$, $V_3(2,3)$, $V_5(3)$, $V_3(4)$ have rational cohomology groups with Hodge structures of level ≤ 1 (see Rapoport 1972), and so, if all Hodge cycles are absolutely Hodge, their motives are abelian.

By definition, the abelian motives over k form a Tannakian category over $\mathbb{Q}$, and Betti cohomology provides a fibre functor ω_B over $\mathbb{Q}$ once we choose an embedding $k \to \mathbb{C}$. Let $G_{\mathrm{Mab}} = \mathit{Aut}^{\otimes}(\omega_B)$. The functor

$$M \rightsquigarrow \omega_B(M) \otimes \mathbb{R}\colon \mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \longrightarrow \mathsf{Hdg}_{\mathbb{R}}$$

defines a homomorphism $h_{\mathrm{Mab}}\colon \mathbb{S} \to G_{\mathrm{Mab}}$. In 1.34 below, we shall exhibit a universal property for the pair $(G_{\mathrm{Mab}}, h_{\mathrm{Mab}})$.

Polarizable rational Hodge structures

Let (V,h) be a polarizable rational Hodge structure, and let $G = \mathit{Aut}^{\otimes}(\omega)$, where ω is the forgetful functor on the tensor category generated by (V,h). Then G can be identified with a subgroup of $\mathrm{GL}(V)$, and h can be regarded as a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$. We call the pair (G,h) the ***Mumford-Tate group*** of (V,h).[4]

[3]Added 2017. I have changed the numbering to agree with that in my article *Introduction to Shimura varieties* (2005).

[4]Sometimes the Mumford-Tate group of (V,h) is defined to be the group attached to the tensor category generated by (V,h) and $\mathbb{Q}(1)$. For the relation between the two notions, see the penultimate subsection of this section.

LEMMA 1.4. *The following conditions on a $t \in V^{\otimes r} \otimes V^{\vee \otimes s}$ are equivalent:*

(a) t *is of type* $(0,0)$*;*

(b) t *is fixed by* $h(\mathbb{C}^{\times})$*;*

(c) t *is fixed under the action of* G *on* $V^{\otimes r} \otimes V^{\vee \otimes s}$.

PROOF. The implications (a) $\iff$ (b) and (c) $\implies$ (b) are obvious. For (a) $\implies$ (c), note that if t is of type (0,0), then the map $\mathbb{Q}(0) \to V^{\otimes r} \otimes V^{\vee \otimes s}$ sending 1 to t is a morphism of Hodge structures, and that, by definition, the action of G commutes with morphisms of Hodge structures. ■

A tensor t satisfying the equivalent conditions in the lemma is called a ***Hodge tensor*** of V.

Let G be a real algebraic group, and let C be an element of $G(\mathbb{R})$ whose square is central. A ***C-polarization*** of a real representation (V, ξ) of G is a G-invariant bilinear form $\psi: V \times V \to \mathbb{R}$ such that $(x, y) \mapsto \psi(x, Cy)$ is symmetric and positive-definite.

LEMMA 1.5. *Let* G *and* C *be as above. The following conditions are equivalent:*

(a) $\mathrm{ad}(C)$ *is a Cartan involution of* G*;*

(b) *every real representation of* G *is* C*-polarizable;*

(c) G *admits a faithful representation that is* C*-polarizable.*

PROOF. See Deligne 1972, 2.8.[5] ■

PROPOSITION 1.6. [6] *Let* G *be a connected algebraic group over* $\mathbb{Q}$*, and let* h *be a homomorphism* $\mathbb{S} \to G_{\mathbb{R}}$*. The pair* (G, h) *is the Mumford-Tate group of a polarizable rational Hodge structure if and only if it satisfies the conditions (SV2*,4) and* G *is generated by* h *(i.e., there is no proper* $\mathbb{Q}$*-rational subgroup* H *of* G *such that* $\mathrm{Im}(h) \subset H_{\mathbb{R}}$*).*

PROOF. Let (G, h) be the Mumford-Tate group of a polarizable rational Hodge structure (V, h). That w_h is defined over $\mathbb{Q}$ is part of the definition of a rational Hodge structure. For all $a \in \mathbb{Q}^{\times}$, $w_h(a): V \to V$ is a morphism of Hodge structures and hence commutes with the action of G. Therefore (G, h) satisfies (SV4).

Let H be a subgroup of G such that $H_{\mathbb{R}}$ contains $h(\mathbb{S})$. Then

$$t \text{ fixed by } H \implies t \text{ is a Hodge tensor} \implies t \text{ fixed by } G,$$

and it follows that $H = G$ because they have the same fixed tensors in spaces $V^{\otimes r} \otimes V^{\vee \otimes s}$, G is reductive, and every character of H extends[7] to G (Deligne 1982, 3.1, 3.5). Thus h generates G.

Let $C = h(i)$. Then $C^2 = h(-1) = w_h(-1)$, which lies in the centre of $G(\mathbb{R})$. Let (W, ξ) be a representation of $G/w_h(\mathbb{G}_m)$. The rational Hodge structure $(W, \xi \circ h)$ is in the tensor category generated by (V, h), and so it is polarizable. Let $\psi: W \otimes W \to \mathbb{Q}(0)$ be a

[5] Added 2017. Or 1.20 of my article *Introduction to Shimura varieties*, which incorporates many results from Deligne 1971b, 1972, and 1979.

[6] Added. The proposition describes the pairs (G, h) that arise as Mumford-Tate groups of polarizable rational Hodge structures. It therefore answers the question raised later by Green, Griffiths, and Kerr (*Mumford-Tate Groups and Domains*, PUP, 2012) and Moonen 1999. The description of G can be made more explicit as in the proof of Theorem 1.27 below.

[7] Added. For this, we need to take H to be the smallest such subgroup.

polarization of $(W, \xi \circ h)$ as a rational Hodge structure. Then ψ is fixed under the action of $h(\mathbb{S})$, and because $G/w_h(\mathbb{G}_m)$ is generated by h, it is also fixed under the action of $G/w_h(\mathbb{G}_m)$, and so it is a C-polarization. Therefore Lemma 1.5 shows that $\operatorname{ad} C$ is a Cartan involution for $(G/w_h(\mathbb{G}_m))_{\mathbb{R}}$, i.e., that (G,h) satisfies (SV2*).

Conversely, let (G,h) be a pair satisfying (SV2*,4) and such that G is generated by h. We first show that, for every representation $\xi: G \to \mathrm{GL}(V)$ of G on a $\mathbb{Q}$-vector space, the rational Hodge structure $(V, \xi \circ h)$ is polarizable. Again let $C = h(i)$. If $(V, \xi \circ h)$ has weight 0, then every C-polarization of (V, ξ) is also a polarization of $(V, \xi \circ h)$. If the weight is nonzero, there will be a smallest $m > 0$ such that $\mathbb{Q}(m)$ lies in the tensor category generated by (V,h), and we let G_1 be the subgroup of G that acts trivially on $\mathbb{Q}(m)$. The element C acts as 1 on $\mathbb{Q}(m)$, and so lies in $G_1(\mathbb{R})$. The map $G_1 \to G/w(\mathbb{G}_m)$ is an isogeny, and so the condition (SV2*) for (G,h) implies that $\operatorname{ad} C$ is a Cartan involution of $G_1(\mathbb{R})$. Therefore there is a G_1-invariant C-polarization ψ of V. After replacing V with a homogeneous component, we may suppose that $(V, \xi \circ h)$ has weight n, and then the map

$$(2\pi i)^{-n}\psi: V \otimes V \longrightarrow \mathbb{Q}(-n)$$

is a polarization of $(V, \xi \circ h)$.

Now choose ξ to be a faithful representation of G, and let G' be the Mumford-Tate group of the polarizable rational Hodge structure $(V, \xi \circ h)$. Both G and G' are algebraic subgroups of $\mathrm{GL}(V)$ generated by h, and so they must be equal. ■

COROLLARY 1.7. *The pair $(G_{\mathrm{Hdg}}, h_{\mathrm{Hdg}})$ satisfies the conditions (SV2*,4); moreover, for any algebraic group G and map h satisfying these conditions, there is a unique homomorphism $\rho(h): G_{\mathrm{Hdg}} \to G$ such that $h = \rho(h)_{\mathbb{R}} \circ h_{\mathrm{Hdg}}$.*

PROOF. The algebraic quotients of $(G_{\mathrm{Hdg}}, h_{\mathrm{Hdg}})$ are precisely the Mumford-Tate groups of polarizable rational Hodge structures, and so the proposition shows that $(G_{\mathrm{Hdg}}, h_{\mathrm{Hdg}})$ satisfies (SV2*,4) and is generated by h_{Hdg}. Let (G,h) be a pair satisfying (SV2*,4), and let G' be the subgroup of G generated by h. Then (G',h) is the Mumford-Tate group of a polarizable Hodge structure, and so there is a homomorphism $\rho(h): G_{\mathrm{Hdg}} \to G' \subset G$ such that $h = \rho(h)_{\mathbb{R}} \circ h_{\mathrm{Hdg}}$. It is unique because h generates G_{Hdg}. ■

Note that the conditions in the corollary determine the pair $(G_{\mathrm{Hdg}}, h_{\mathrm{Hdg}})$ uniquely (up to a unique isomorphism).

The Mumford-Tate group of a polarizable Hodge structure is reductive (because it satisfies (SV2*)) and connected (because it is generated by h and $\mathbb{S}$ is connected). Consequently G_{Hdg} is pro-reductive and connected.

The functor

$$\omega_B: \mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \longrightarrow \mathsf{Hdg}_{\mathbb{Q}}$$

induces a homomorphism

$$\rho: G_{\mathrm{Hdg}} \longrightarrow G_{\mathrm{Mab}}$$

such that $\rho_{\mathbb{R}} \circ h_{\mathrm{Hdg}} = h_{\mathrm{Mab}}$, and ρ is the unique homomorphism satisfying this condition. Because ω_B is fully faithful, ρ is surjective (i.e., faithfully flat).

Hodge structures of CM-type

A polarizable rational Hodge structure is said to be of ***CM-type*** if its Mumford-Tate group is commutative, and hence a torus. The Hodge structures of CM-type form a Tannakian subcategory $\mathsf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}}$ of $\mathsf{Hdg}_{\mathbb{Q}}$ over $\mathbb{Q}$.

PROPOSITION 1.8. *Every Hodge structure of CM-type is the Betti realization of an abelian motive.*

PROOF. This is well known. For a proof, see Milne 1994, 4.6. ∎

COROLLARY 1.9. *The kernel of $\rho\colon G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}}$ is contained in $(G_{\mathrm{Hdg}})^{\mathrm{der}}$.*

PROOF. Let $S = Aut^{\otimes}(\omega)$, where ω is the forgetful functor on $\mathsf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}}$. The proposition shows that the inclusion $\mathsf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}} \hookrightarrow \mathsf{Hdg}_{\mathbb{Q}}$ factors through $\mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \hookrightarrow \mathsf{Hdg}_{\mathbb{Q}}$, and so the homomorphism $G_{\mathrm{Hdg}} \to S$ factors through $G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}}$. Thus,

$$\begin{array}{ccccc} \mathsf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}} & \hookrightarrow & \mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) & \hookrightarrow & \mathsf{Hdg}_{\mathbb{Q}} \\ S & \twoheadleftarrow & G_{\mathrm{Mab}} & \twoheadleftarrow & G_{\mathrm{Hdg}}. \end{array}$$

Hence $\mathrm{Ker}(\rho) \subset \mathrm{Ker}(G_{\mathrm{Hdg}} \twoheadrightarrow S) = (G_{\mathrm{Hdg}})^{\mathrm{der}}$. ∎

The pro-torus S in the proof is called the ***Serre group.*** For a description of it in terms of its character group, see, for example, Milne 1994, §4.

Dynkin diagrams

1.10. For future reference, we provide a table of Dynkin diagrams on the next page. As will be explained later, solid nodes are special, and nodes marked by stars correspond to symplectic representations. The number in parenthesis indicates the position of the special node.[8]

Special Hodge structures

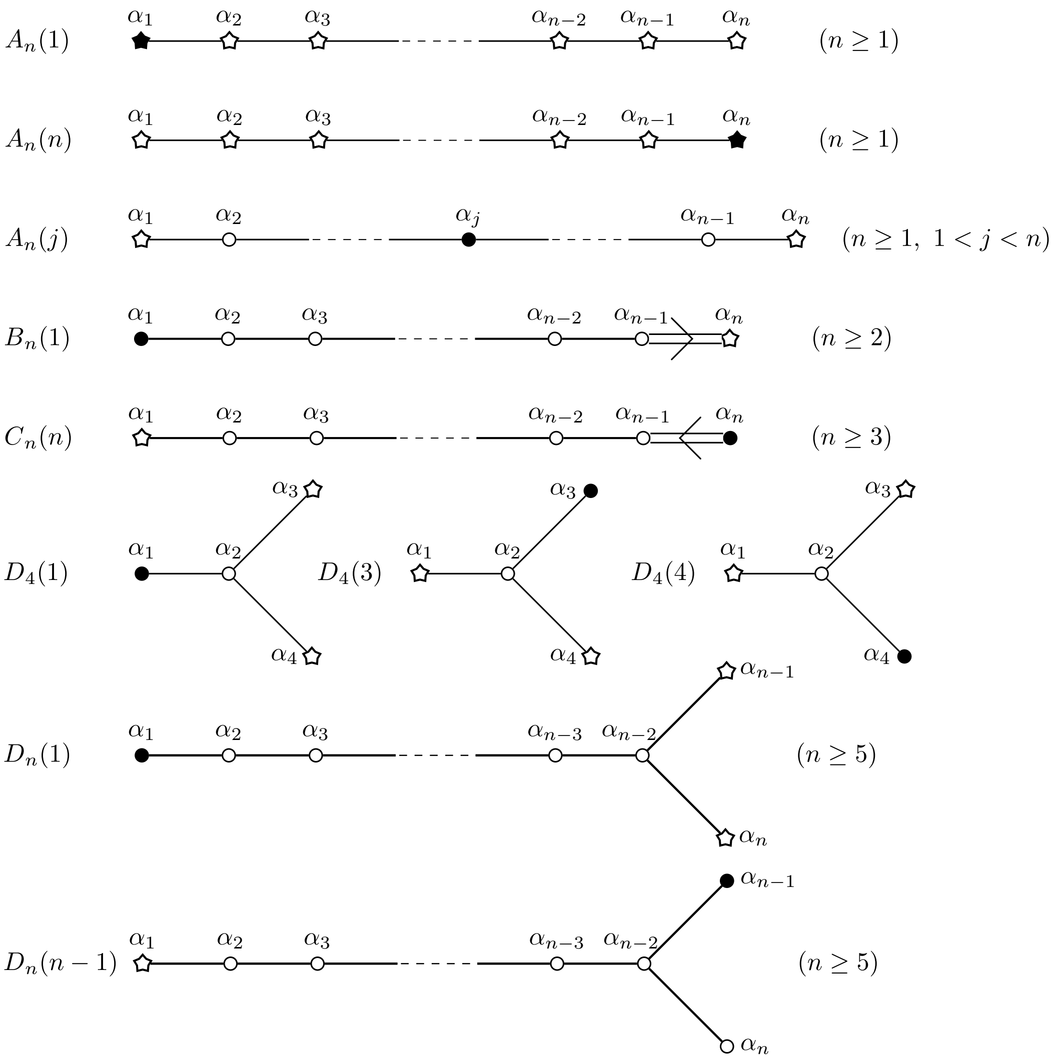
A ***special*** Hodge structure is a polarizable rational Hodge structure whose Mumford-Tate group (G,h) satisfies (SV1). The next two statements are from Deligne 1972, 7.3.

PROPOSITION 1.11. *The special Hodge structures form a Tannakian subcategory of* $\mathsf{Hdg}_{\mathbb{Q}}$.

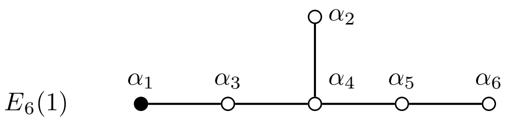
PROOF. A direct sum of special Hodge structures is obviously special. Let (V,h) be a special Hodge structure, and let (G,h) be its Mumford-Tate group. The Mumford-Tate group of any Hodge structure in the tensor category generated by (V,h) is a quotient of (G,h), and hence satisfies (SV1). ∎

PROPOSITION 1.12. *The Betti realization of an abelian motive is special.*

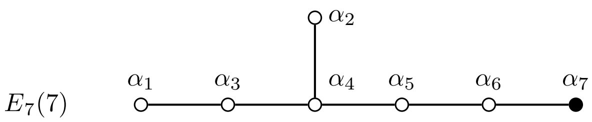
[8]Added. The table corrects errors in the original article and in Deligne 1979, Table 1.3.9. For a detailed explanation of the table, see Milne 2013.



$D_n(n)$ Same as $D_n(n-1)$ but with α_{n-1} and α_n interchanged (reflection)



$E_6(6)$ Same as $E_6(1)$ but with (α_1, α_3) interchanged with (α_6, α_5) (reflection)



PROOF. After Proposition 1.11, it suffices to prove this for an abelian variety. The Betti realization (V,h) of an abelian variety[9] is of type $\{(-1,0),(0,-1)\}$; let (G,h) be its Mumford-Tate group. Then $\mathfrak{g} \subset \mathrm{End}(V) \simeq V \otimes V^{\vee}$, which is of type $\{(-1,1),(0,0),(1,-1)\}$. ■

It is reasonable to hope that the following statement may be true.

HYPOTHESIS 1.13. *Every special Hodge structure is the Betti realization of a motive.*

More explicitly, this means the following: for any special Hodge structure (V,h), there exists a projective algebraic variety X and an integer m such that (V,h) is a direct factor of $H_B(X)(m)$ and the projection $H_B(X)(m) \to V \subset H_B(X)(m)$ is an absolute Hodge cycle on X (e.g., the class of an algebraic cycle).

A motive whose Betti realization is special will be called a ***special motive.*** The next example indicates that they are indeed exceptional among motives. Apparently, no special motive is known that is not already abelian, although the hypothesis predicts their existence. In §3 we shall see that Deligne's hope that all Shimura varieties with rational weight are moduli varieties for motives implies the hypothesis.

EXAMPLE 1.14. Let $X \to \mathbb{P}^1$ be a Lefschetz pencil of hypersurfaces of degree d and odd dimension $2r-1$ over $\mathbb{C}$, and let X_s be the fibre over $s \in \mathbb{P}^1(\mathbb{C})$. It is known (see Deligne 1972, 7.6) that for s outside a countable subset of $\mathbb{P}^1(\mathbb{C})$, the Mumford-Tate group of the rational Hodge structure $H^{2r-1}(X_s,\mathbb{Q})$ is the full group of symplectic similitudes. It follows that the Hodge structure is not special unless it has level ≤ 1.

PROPOSITION 1.15. *A pair (G,h) is the Mumford-Tate group of a special Hodge structure if and only if it satisfies (SV1,2*,4) and h generates G.*

PROOF. Immediate consequence of Proposition 1.6 and the definition of a special Hodge structure. ■

CLASSIFICATION

Following Deligne 1979, we classify the pairs (G,h) as in Proposition 1.15 with G a simple adjoint group. Note that for an adjoint group, (SV4) simply says that h is a homomorphism $\mathbb{S}/\mathbb{G}_m \to G_{\mathbb{R}}$, and that (SV2) implies (SV2*).

LEMMA 1.16. *A simple adjoint group G over $\mathbb{Q}$ for which there exists a homomorphism $h\colon \mathbb{S}/\mathbb{G}_m \to G_{\mathbb{R}}$ satisfying (SV2) is of the form $\mathrm{Res}_{F/\mathbb{Q}} G_0$ for some absolutely simple group G_0 over a totally real number field F.*

PROOF. Every simple adjoint group over $\mathbb{Q}$ is of the form $\mathrm{Res}_{F/\mathbb{Q}} G_0$ for some absolutely simple group G_0 over a number field F, and so the only problem is to show that F is totally real. A compact simple group over $\mathbb{R}$ is absolutely simple, and an inner form of an absolutely simple group is also absolutely simple. The condition (SV2) implies that $G_{\mathbb{R}}$ is an inner form of its compact form, and hence its simple factors are absolutely simple. Since

$$G_{\mathbb{R}} = \prod_{v|\infty} \mathrm{Res}_{F_v/\mathbb{R}} G_{0,F_v},$$

this shows that F must be totally real. ■

[9] Added. More accurately, of h^1 of an abelian variety.

Thus to give a pair (G,h) as in Proposition 1.15 with G simple and adjoint is the same as giving an absolutely simple adjoint group G_0 over a totally real field F together with homomorphisms

$$h_v\colon \mathbb{S}/\mathbb{G}_m \to G_v, \quad G_v \overset{\text{def}}{=} G_0 \otimes_F F_v, \quad v \text{ a real prime of } F,$$

satisfying (SV1,2) and such that at least one h_v is nontrivial.

We fix a simple adjoint group G_∞ over $\mathbb{C}$, and consider the triples (G,γ,h) consisting of a real inner form[10][11] (G,γ) of the compact form G_c of G_∞ and a nontrivial homomorphism $h\colon \mathbb{S}/\mathbb{G}_m \to G$ satisfying (SV1,2).

Choose a maximal torus T in G_∞. Let $R \subset X^*(T)$ be the corresponding system of roots, and choose a system of simple roots B. The nodes of the Dynkin diagram D are parametrized by the elements of B. Recall (Bourbaki 1981, VI.1.8) that there is a unique (highest) root $\widetilde{\alpha} = \sum_{\alpha\in B} n(\alpha)\alpha$ with the property that, for every other root $\sum_{\alpha\in B} m(\alpha)\alpha$, the coefficient $n(\alpha) \geq m(\alpha)$ for all $\alpha \in B$. We call a node s_α of D ***special*** if $n(\alpha) = 1$.

From (G,γ,h) we obtain a $G_\infty(\mathbb{C})$-conjugacy class of cocharacters $\gamma \circ \mu_h$ of G_∞. This class contains a unique element $\mu \in X_*(T)$ such that

$$\langle \alpha,\mu\rangle \ \geq \ 0 \text{ for all } \alpha \in B.$$

The condition (SV1) implies that

$$\langle \alpha,\mu\rangle \ \in \{1,0,-1\} \text{ for } \alpha \in R.$$

Since μ is nontrivial, not all the values $\langle \alpha,\mu\rangle$ can be zero, and so these conditions imply that $\langle \alpha,\mu\rangle = 1$ for exactly one $\alpha \in B$, which must in fact be special (otherwise $\langle \widetilde{\alpha},\mu\rangle > 1$); moreover, this condition is also sufficient for (SV1) to hold.

PROPOSITION 1.17. *The map $(G,\gamma,h) \mapsto s_\alpha$ defines a one-to-one correspondence between the set of isomorphism classes of triples (as above) and the set of special nodes of D. The isomorphism class of the pair (G,γ) itself determines s_α unless the opposition involution τ moves s_α, in which case*

$$(G,\gamma,h) \leftrightarrow s_\alpha, \quad (G,\gamma,h^{-1}) \leftrightarrow \tau s_\alpha.$$

PROOF. We explain only how to construct the triple (G,γ,h) corresponding to a special node s_α—see Deligne 1979, 1.2 for more details. There is a unique $\mu \in X_*(T)$ such that

$$\langle \alpha,\mu\rangle = 1, \quad \langle \alpha',\mu\rangle = 0 \text{ for all } \alpha' \in B, \quad \alpha' \neq \alpha. \tag{1.17.1}$$

[10]Let G_0 be an algebraic group over a field k_0 of characteristic zero and let k be an algebraic closure of k_0. An ***inner form*** of G_0 is an algebraic group G over k_0 together with a $G_0(k)$-conjugacy class γ of isomorphisms $c\colon G_{0,k} \to G_k$ such that $c^{-1} \circ \tau c$ is an inner automorphism of $G_{0,k}$ for all $\tau \in \mathrm{Gal}(k/k_0)$. Two inner forms (G,γ) and (G',γ') are ***isomorphic*** if there is an isomorphism of algebraic groups $\varphi\colon G \to G'$ (over k_0) such that

$$c \in \gamma \implies \varphi \circ c \in \gamma'.$$

Such a φ is uniquely determined up to an inner automorphism of G over k_0. If (G,γ) is an inner form of G_0 and $c \in \gamma$, then $c_\tau = c^{-1} \circ \tau c$ is a 1-cocycle for G_0^{ad} whose cohomology class does not depend on the choice of c. In this way, the set of isomorphism classes of inner forms of G_0 becomes identified with $H^1(k_0, G_0^{\mathrm{ad}})$.

[11]Added. It is equivalent, but less clumsy, to say that an ***inner form*** of G_0 is a pair (G,c) consisting of an algebraic group G over k_0 and an isomorphism $c\colon G_{0,k} \to G_k$ such that $c^{-1} \circ \tau c$ is inner for all τ. An ***isomorphism*** $(G,c) \to (G',c')$ of inner forms is an isomorphism $\varphi\colon G \to G'$ such that c' differs from $\varphi_k \circ c$ by an inner automorphism of G_{0k}.

Let (G,γ) be the real form of the compact form G_c corresponding to the Cartan involution $\mathrm{ad}\,\mu(-1)$, i.e., such that

$$c(\mu(-1)\cdot\iota g\cdot\mu(-1)) = \iota c(g), \quad g\in G_c(\mathbb{C}), \quad \text{some } c\in\gamma,$$

and set $h(z_1,z_2)=\mu(z_1)\cdot(\iota\mu)(z_2)$ (cf. 1.1.1, 1.1.2). Then h, when regarded as a map into $G_{\mathbb{C}}$, is defined over $\mathbb{R}$, and because s_α is special, (G,h) satisfies (SV1). The final statement follows from Deligne 1979, 1.2.7. ∎

An examination of the tables in Bourbaki 1981, pp. 250–275 reveals that every node of the Dynkin diagram of type A_n is special, that the Dynkin diagrams of type B_n, C_n, and E_7 each have one special node, that the Dynkin diagrams of type D_n each have three special nodes, and that the Dynkin diagram of type E_6 has two special nodes. This is illustrated in Table 1.10, where the special nodes are filled. The Dynkin diagrams of type E_8, F_4, and G_2 have no special nodes, and so groups of these types cannot occur as factors of the adjoint group of the Mumford-Tate group of a special Hodge structure.

Following Deligne, we write $D_n^{\mathbb{R}}$ for the diagrams $D_n(1)$, $D_4(3)$, and $D_4(4)$, and we write $D_n^{\mathbb{H}}$ for the remaining diagrams of type D_n. A simple adjoint group G over $\mathbb{R}$ will be said to be of type A_n, B_n, C_n, $D_n^{\mathbb{R}}$, $D_n^{\mathbb{H}}$, E_6, or E_7 if it corresponds to a diagram of that type in Table 1.10.

Let G be a simple group over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ is noncompact. We say that G is of type A_n, B_n, C_n, $D_n^{\mathbb{R}}$, $D_n^{\mathbb{H}}$, E_6, or E_7 if all the noncompact factors of $G_{\mathbb{R}}^{\mathrm{ad}}$ are of this type. When noncompact factors of type $D_n^{\mathbb{R}}$ and $D_n^{\mathbb{H}}$ both occur, we say G is of mixed type D.

Symplectic representations

In this subsection, we review the symplectic representations of groups. These were studied by Satake in a series of papers (see especially Satake 1965, 1967, 1980). Our exposition follows that of Deligne 1979. [12]

A ***symplectic space*** (V,ψ) over a field k is a finite-dimensional vector space V over k together with a nondegenerate alternating form ψ on V. The corresponding ***symplectic group*** is the subgroup of $\mathrm{GL}(V)$ such that

$$\mathrm{Sp}(\psi)(k)=\{g\in\mathrm{GL}(V)\mid \psi(gx,gy)=\psi(x,y),\ \text{all } x,y\in V\},$$

and the group of ***symplectic similitudes***, $G(\psi)$, is $\mathrm{Sp}(\psi)\cdot\mathbb{G}_m$ (here $\mathbb{G}_m$ is identified with the group of nonzero diagonal matrices). Write $\mathrm{PSp}(\psi)$ for the adjoint group of $\mathrm{Sp}(\psi)$. The ***Siegel upper half-space*** $X(\psi)^+$ corresponding to a real symplectic space (V,ψ) is the set of Hodge structures h on V of type $\{(-1,0),(0,-1)\}$ for which $2\pi i\psi$ is a polarization. Each $h\in X(\psi)^+$ factors through $G(\psi)$, and the map $h\mapsto\bar{h}=\mathrm{ad}\circ h$ identifies $X(\psi)^+$ with an $\mathrm{Sp}(\psi)(\mathbb{R})$-conjugacy class of maps $\mathbb{S}/\mathbb{G}_m\to\mathrm{PSp}(\psi)$.

THE REAL CASE

Let H be a semisimple group over $\mathbb{R}$, and let $\bar{h}$ be a homomorphism $\mathbb{S}/\mathbb{G}_m\to H^{\mathrm{ad}}$, none of whose components are trivial, satisfying (SV1,2).

[12] Added 2017. See §10 of Milne 2013 for a detailed exposition of this material.

We shall say that a representation $\xi\colon H \to \mathrm{GL}(V)$ with finite kernel is ***symplectic*** if there exists a real reductive group G, a homomorphism $h\colon \mathbb{S} \to G$, a nondegenerate alternating form ψ on V, and a factorization

$$H \to G \to \mathrm{GL}_V$$

of ξ such that

(1.18.1) H has image G^{der} in G (so $H^{\mathrm{ad}} \simeq G^{\mathrm{ad}}$), and $\bar{h} = \mathrm{ad} \circ h$;

(1.18.2) (G,h) maps into $(G(\psi), X(\psi)^{+})$.

Assume H is simply connected, and set $\mu = \mu_{\bar{h}}$. Choose a maximal torus T in $H_{\mathbb{C}}$, and let $R \subset X^*(T)$ be the corresponding system of roots. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots such that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in B$. Recall that the lattice of weights is

$$P(R) = \{\varpi \in X^*(T) \otimes \mathbb{Q} \mid \langle \varpi, \alpha^\vee \rangle \in \mathbb{Z} \text{ all } \alpha^\vee \in R^\vee\},$$

that the fundamental weights are the elements of the dual basis $\{\varpi_1, \ldots, \varpi_n\}$ to $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$, and that the dominant weights are the elements $\sum n_i \varpi_i$, $n_i \in \mathbb{N}$. The quotient $P(R)/Q(R)$ of $P(R)$ by the lattice $Q(R)$ generated by R is the character group of $Z(H)$. Write τ for the opposition involution acting on the Dynkin diagram (or on R or on the set of fundamental weights): it preserves each connected component of the diagram, and acts as the unique nontrivial involution on a component of type A_n, D_n (n odd), or E_6, and trivially on the other components.

PROPOSITION 1.19. *Let ξ be an irreducible representation of H on a real vector space, and let ϖ be the highest weight of an irreducible component of $\xi_{\mathbb{C}}$. The representation ξ is symplectic if and only if*

$$\langle \varpi + \tau\varpi, \bar{\mu} \rangle = 1. \tag{1.19.1}$$

PROOF. See Deligne 1979, 1.3.6, or Milne 2013, 10.6. ∎

COROLLARY 1.20. *If ξ is symplectic, then ϖ is a fundamental weight. Therefore the representation factors through a simple factor of H.*

PROOF. For every dominant weight ϖ, $\langle \varpi + \tau\varpi, \mu \rangle \in \mathbb{N}$ because $\varpi + \tau\varpi \in Q(R)$. If $\varpi \neq 0$, $\langle \varpi + \tau\varpi, \mu \rangle > 0$ unless μ kills all the weights of the representation corresponding to ϖ. Hence a dominant weight satisfying (1.19.1) cannot be a sum of two dominant weights. ∎

PROPOSITION 1.21. *Let H be a simply connected simple group over $\mathbb{R}$, and let $\bar{h}\colon \mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}$ be a nontrivial homomorphism satisfying (SV1,2). There exists a nontrivial symplectic representation of $(H, \bar{h})$ if and only if H is of type A, B, C, or D. If H is of type A, B, C, or $D^{\mathbb{R}}$, then the symplectic representations form a faithful family of representations of H; if H is of type $D^{\mathbb{H}}$ they form a faithful family of representations of the double covering of the adjoint group corresponding to the subgroup of $P(R)/Q(R)$ generated by ϖ_1.*

PROOF. The proof proceeds by an examination of the tables. We treat only the cases B, D, and E_6, since the remaining cases are similar. In each case, $(H, \bar{h})$ corresponds to a special simple root α of $H_{\mathbb{C}}$, and $\langle \varpi_i, \mu \rangle$ is the coefficient of α in the expression of ϖ_i as a $\mathbb{Q}$-linear combination of the simple roots (see 1.17.1).

(B_n). In this case $\mu \leftrightarrow \alpha_1$, and the opposition involution acts trivially on the Dynkin diagram. Thus we seek a fundamental weight ϖ_i such that $\varpi_i = \frac{1}{2}\alpha_1 + \cdots$. According to the tables in Bourbaki,

$$\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_n) \quad (1 \le i < n)$$
$$\varpi_n = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n),$$

and so only ϖ_n has this property. Because ϖ_n generates P/Q, the representation with highest weight ϖ_n is a faithful representation of H.

$(D_n^{\mathbb{R}_n})$. Suppose first that $n = 4$ and $\mu \leftrightarrow \alpha_4$. The opposition involution acts trivially, and so ϖ_1 and ϖ_3 give rise to symplectic representations; they generate P/Q. The case $\mu \leftrightarrow \alpha_3$ is similar. Otherwise $\mu \leftrightarrow \alpha_1$. The opposition involution acts trivially if n is even, and switches α_{n-1} and α_n if n is odd. In the first case, ϖ_n and ϖ_{n-1} give rise to symplectic representations and together they generate P/Q. In the second case, ϖ_n and ϖ_{n-1} give rise to symplectic representations, and each generates P/Q.

$(D_n^{\mathbb{H}_n})$. In this case, $\mu \leftrightarrow \alpha_n$ (or α_{n-1}). Only ϖ_1 gives rise to a symplectic representation, and it generates a subgroup of order 2 (and index 2) in $P(R)/Q(R)$. The corresponding representation factors through H/C, where C is the kernel of ϖ_1 regarded as a character of $Z(H)$.

(E_6). In this case, $\mu \leftrightarrow \alpha_1$ or α_6. No fundamental weight qualifies. ■

REMARK 1.22. Let H be the identity component of the group of automorphisms of a nondegenerate skew-hermitian form on a vector space of dimension n over a quaternion algebra $\mathbb{H}$ over $\mathbb{R}$. Then H is an inner form of SO$(2n)$, and it is of type $D_n^{\mathbb{H}}$. It is the double covering of G^{ad} corresponding to the subgroup of $P(R)/Q(R)$ generated by ϖ_1.

THE RATIONAL CASE

Now let H be a semisimple group over $\mathbb{Q}$, and let $\bar{h}\colon \mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2) and generating H^{ad}. We choose the maximal torus T in $H_{\mathbb{C}}$ to be rational over $\mathbb{Q}^{\mathrm{al}}$.

We shall say that a representation $\xi\colon H \to \mathrm{GL}(V)$ (over $\mathbb{Q}$) with finite kernel is ***symplectic*** if there exists a reductive group G, a homomorphism $h\colon \mathbb{S} \to G_{\mathbb{R}}$, a nondegenerate alternating form ψ on V, and a factorization

$$H \to G \to \mathrm{GL}_V$$

of ξ such that

(1.23.1) H has image G^{der} in G (so $G^{\mathrm{ad}} \simeq H^{\mathrm{ad}}$), and $\bar{h} = \mathrm{ad} \circ h$;

(1.23.2) (G, h) maps into $(G(\psi), X(\psi)^+)$.

LEMMA 1.24. *Let H be a simply connected and simple group over $\mathbb{Q}$, and let $\bar{h}\colon \mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ be a nontrivial homomorphism satisfying (SV1,2). If $(H, \bar{h})$ has a symplectic representation over $\mathbb{Q}$, then H cannot be of exceptional type or of mixed type D.*

PROOF. Because $(H, \bar{h})$ satisfies (SV2), $H = \mathrm{Res}_{F/\mathbb{Q}} H_0$ for some absolutely simple group H_0 over a totally real field F (see 1.16). Thus

$$H_{\mathbb{R}} = \prod_{v \in I} H_v, \quad H_v = H_0 \otimes_{F,v} \mathbb{R}, \quad I = \mathrm{Hom}(F, \mathbb{R}).$$

Let I_{nc} be the subset of I of v for which H_v is not compact, and let $H_{\mathrm{nc}} = \prod_{v \in I_{\mathrm{nc}}} H_v$. Because $\bar{h}$ generates H^{ad}, I_{nc} is nonempty. The Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts on the Dynkin diagram of $H_{\mathbb{C}}$ in a manner consistent with its projection to I.

Let (V,ξ) be a symplectic representation of $(H,\bar{h})$. The restriction of $\xi_{\mathbb{R}}$ to H_{nc} is a real symplectic representation of H_{nc}, and so (by 1.20, 1.21) every nontrivial irreducible component of $\xi_{\mathbb{C}}|H_{\mathrm{nc}}$ factors through H_v for some $v \in I_{\mathrm{nc}}$ and corresponds to a node of the Dynkin diagram D_v of H_v marked in (1.10) with a star.

An irreducible component W of $\xi_{\mathbb{C}}$ is of the form $\bigotimes_{v \in T} W_v$, where T is a subset of I and W_v is the representation of $G_{v\mathbb{C}}$ corresponding to a node $s_v \in D_v$. Let $\mathcal{S}(\xi)$ be the set of all nodes that arise in this fashion from an irreducible component of $\xi_{\mathbb{C}}$. Then $\mathcal{S}(\xi)$ is nonempty, stable under the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and, if $s \in \mathcal{S}(\xi) \cap D_v$, $v \in I_{nc}$, then s is marked by a star in (1.10). Since the diagrams for E_6 and E_7 have no starred nodes, no such set exists in this case. If G is of mixed type D, then there is no such set $\mathcal{S}(\xi)$ because there is no automorphism of a Dynkin diagram of type D_n, $n \geq 5$, carrying the node s_1 into either s_{n-1} or s_n. ■

We shall need to consider the following condition on a semisimple group H over $\mathbb{Q}$:

(1.25) there exists an isogeny $H' \to H$ with H' a product of simple groups H'_i such that either

(a) H'_i is simply connected of type A, B, C, or $D^{\mathbb{R}}$, or,

(b) H'_i is of type $D_n^{\mathbb{H}}$ $(n \geq 5)$ and equals $\mathrm{Res}_{F/\mathbb{Q}} H_0$ for H_0 the double covering of an adjoint group that is a form of $\mathrm{SO}(2n)$ (cf. 1.22).

THEOREM 1.26. *Let H be a semisimple group over $\mathbb{Q}$, and let $\bar{h}$ be a homomorphism $\mathbb{S}/\mathbb{G}_m \to H_{\mathbb{R}}^{\mathrm{ad}}$ satisfying (SV1,2) and generating H^{ad}. There exists an isogeny $H' \to H$ such that $(H',\bar{h})$ admits a faithful family of symplectic representations if and only if H satisfies (1.25).*

PROOF. Suppose that H satisfies (1.25), and let $H' \to H$ be an isogeny as in the statement of (1.25). It suffices to show that H' admits a faithful family of symplectic representations, and for this it suffices show that each simple factor of H' admits such a family. This is proved in (Deligne 1979, 2.3.10).

Conversely, suppose that H has a covering H' such that $(H',\bar{h})$ admits a faithful family of symplectic representations. According to Lemma 1.24, H' (hence H) cannot be of exceptional type or mixed type D. Let H^{sc} be the universal covering group of H' (hence of H), and let H'' be the quotient of H^{sc} by the intersection of the kernels of the rational symplectic representations of H^{sc}. Then H'' is still a covering of H', and it follows from (1.20, 1.21) that it satisfies (1.25). ■

Abelian motives: Mumford-Tate groups

As we noted above, Deligne's theorem (Deligne 1982) shows that $\omega_B\colon \mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \to \mathsf{Hdg}_{\mathbb{Q}}$ is fully faithful, and so the homomorphism $G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}}$ it defines is surjective. When we identify rational Hodge structures with representations of G_{Hdg} on $\mathbb{Q}$-vector spaces, abelian motives become identified with those representations that factor through G_{Mab} (cf. Deligne 1990, 8.17).

THEOREM 1.27. *Let G be an algebraic group over $\mathbb{Q}$, and let $h:\mathbb{S}\to G_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2*,4) and generating G. The pair (G,h) is the Mumford-Tate group of an abelian motive if and only if G^{der} satisfies (1.25).*

The proof will occupy the rest of this subsection.

PROPOSITION 1.28. *For every semisimple group H over $\mathbb{Q}$ and homomorphism $\bar{h}:\mathbb{S}/\mathbb{G}_m\to H^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SV1,2), there exists a reductive group G with $G^{\mathrm{der}}=H$ and a homomorphism $h:\mathbb{S}\to G_{\mathbb{R}}$ lifting $\bar{h}$ and satisfying (SV1,2*,4).*

PROOF. (Milne 1988, A.2) For every finite extension L of $\mathbb{Q}$ splitting H, there exists a central extension defined over $\mathbb{Q}$

$$1\longrightarrow N\longrightarrow G\longrightarrow H^{\mathrm{ad}}\longrightarrow 1$$

such that $G^{\mathrm{der}}=H$ and N is a product of copies of $(\mathbb{G}_m)_{L/\mathbb{Q}}$ (the $\mathbb{Q}$-torus obtained from $\mathbb{G}_{m,L}$ by restriction of scalars). For a proof, see Milne and Shih 1982, 3.1.

Assume first that $\bar{h}$ "special", i.e., that it factors through $T_{\mathbb{R}}$ for some maximal torus T in H^{ad}. Then (SV2) implies that $T_{\mathbb{R}}$ is anisotropic, and so T splits over a CM-field L, which we may choose to be Galois over $\mathbb{Q}$. Construct G as above using this L. According to Borel 1991, 12.4, 13.17, there is a maximal torus $T'\subset G$ mapping onto T. Since T' is its own centralizer, it contains N, which is therefore the kernel of $T'\to T$. Hence $X_*(T')\to X_*(T)$ is surjective, and we can choose $\mu\in X_*(T')$ mapping to $\mu_{\bar{h}}\in X_*(T)$. The weight $w =_{df} -\mu-\iota\mu$ of μ lies in $X_*(N)$. Because $X_*(N)$ is an induced Galois module, its cohomology groups are zero; in particular, the zeroth Tate (modified) group

$$H^0_{\mathrm{Tate}}(\mathrm{Gal}(\mathbb{C}/\mathbb{R}),X_*(N))\overset{\mathrm{def}}{=}\frac{X_*(N)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}}{(\iota+1)X_*(N)}=0.$$

Clearly $\iota w=w$, and so there exists a $\mu_0\in X_*(N)$ such that $(\iota+1)\mu_0=w$. When we replace μ with $\mu+\mu_0$, then we find that the weight becomes 0; in particular, it is defined over $\mathbb{Q}$. Choose h so that $h(z)=\mu(z)\cdot\overline{\mu(z)}$.

For a general $\bar{h}$, there will exist a $\bar{g}\in H^{\mathrm{ad}}(\mathbb{R})$ such that $\mathrm{ad}\,\bar{g}\circ\bar{h}$ is special (Deligne 1982, p. 75). Construct G and h as in the last paragraph corresponding to $\mathrm{ad}\,\bar{g}\circ\bar{h}$. Because $H^1(\mathbb{R},N)=H^1(L\otimes_{\mathbb{Q}}\mathbb{R},\mathbb{G}_m)=0$, $\bar{g}$ will lift to an element $g\in G(\mathbb{R})$, and we take the pair $(G,\mathrm{ad}\,g^{-1}\circ h)$.

The pair (G,h) we have constructed satisfies (SV1,2,4), and its centre is split by a CM-field. Let T be the subtorus of G^{ab} generated by h. Then $T_{\mathbb{R}}$ is anisotropic, and when we replace G with the inverse image of T, we obtain a pair (G,h) satisfying (SV1,2*,4). ∎

COROLLARY 1.29. *Let H be a semisimple group over $\mathbb{Q}$, and let $\bar{h}:\mathbb{S}/\mathbb{G}_m\to H^{\mathrm{ad}}_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2). There exists a unique homomorphism $\rho(H,\bar{h}):(G_{\mathrm{Hdg}})^{\mathrm{der}}\to H$ such that the following diagram commutes:*

$$\begin{tikzcd}
(G_{\mathrm{Hdg}})^{\mathrm{der}} \arrow[r, "\rho(H,\bar{h})"] \arrow[d, hook] & H \arrow[d, two heads] \\
G_{\mathrm{Hdg}} \arrow[r, "\rho(\bar{h})"] & H^{\mathrm{ad}}
\end{tikzcd}$$

Here $\rho(\bar{h})$ is the unique homomorphism such that $\bar{h}=\rho(\bar{h})_{\mathbb{R}}\circ h_{\mathrm{Hdg}}$ (see 1.7).

PROOF. Two such homomorphisms $\rho(H,\bar{h})$ would differ by a map into $Z(H)$. Because $(G_{\mathrm{Hdg}})^{\mathrm{der}}$ is connected, every such map is constant, and so the homomorphisms will be equal.

For the existence, choose a pair (G,h) as in 1.28, and take $\rho(H,\bar{h}) = \rho(h)|(G_{\mathrm{Hdg}})^{\mathrm{der}}$. ■

REMARK 1.30. Let G be a reductive group over $\mathbb{Q}$, and let h be a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$ satisfying (SV1,2*,4). Let $H = G^{\mathrm{der}}$ and let $\bar{h} = \mathrm{ad} \circ h$. The restriction of $\rho(h)$ to $(G_{\mathrm{Hdg}})^{\mathrm{der}}$ satisfies the condition of 1.29 relative to $(H,\bar{h})$, and hence equals $\rho(H,\bar{h})$.

LEMMA 1.31. *The assignment $(H,\bar{h}) \mapsto \rho(H,\bar{h})$ is functorial: if $\alpha: H \to H'$ is a homomorphism mapping $Z(H)$ into $Z(H')$ and carrying $\bar{h}$ to $\bar{h}'$, then $\rho(H',\bar{h}') = \alpha \circ \rho(H,\bar{h})$.*

PROOF. After replacing H and H' with the subgroups generated by h and h', we may assume that α is surjective. Choose (G,h) for $(H,\bar{h})$ as in 1.28, and let $G' = G/\mathrm{Ker}(\alpha)$. Write α again for the projection $G \to G'$ and let $h' = \alpha_{\mathbb{R}} \circ h$. On restricting the maps in the equality

$$\rho(h') = \alpha \circ \rho(h)$$

to $(G_{\mathrm{Hdg}})^{\mathrm{der}}$ we obtain the equality

$$\rho(H',\bar{h}') = \alpha \circ \rho(H,\bar{h}).$$

■

LEMMA 1.32. *Let H be a semisimple group over $\mathbb{Q}$, and let $\bar{h}$ be a homomorphism $\mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SV1,2) and generating H. If $(H,\bar{h})$ has a faithful family of symplectic representations, then $\rho(H,\bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$.*

PROOF. It is clear from the definition of a symplectic representation 1.23 that $\rho(H,\bar{h})$ maps $\mathrm{Ker}(G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}})$ into the kernel of every symplectic representation of H, but, by assumption, the intersection of these kernels is trivial. ■

LEMMA 1.33. *Let H be a semisimple group over $\mathbb{Q}$, and let $\bar{h}$ be a homomorphism $\mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SV1,2). The homomorphism $\rho(H,\bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$ if and only if H satisfies (1.25).*

PROOF. Suppose H satisfies (1.25). According to (1.26), there is a finite covering $\alpha: H' \to H$ such that $(H',\bar{h})$ has a faithful family of symplectic representations. By (1.32), $\rho(H',\bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$, and therefore so also does $\rho(H,\bar{h}) = \alpha \circ \rho(H',\bar{h})$.

Conversely, suppose $\rho(H,\bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$. There will be an algebraic quotient (G,h) of $(G_{\mathrm{Mab}},h_{\mathrm{Mab}})$ such that $(H,\bar{h})$ is a quotient of $(G^{\mathrm{der}},\mathrm{ad} \circ h)$. Consider the category of abelian motives M for which the action of G_{Mab} on $\omega_B(M)$ factors through G. By definition, this category is contained in the tensor category generated by $h_1(A)$ for some abelian variety A. We can replace G with the Mumford-Tate group of A. Then $(G^{\mathrm{der}},\mathrm{ad} \circ h)$ has a symplectic embedding, and according to (1.26), this implies that G^{der} satisfies (1.25). Since H is a quotient of G^{der}, it also satisfies (1.25). ■

We can now complete the proof of the Theorem 1.27. From (1.9), we know that $\rho(h)$ factors through G_{Mab} if and only if $\rho(G^{\mathrm{der}},\mathrm{ad} \circ h)$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$, and from (1.33) we know that this is true if and only if G^{der} satisfies (1.25). ■

COROLLARY 1.34. *For every pair (G,h) satisfying (SV1,2*,4) and such that G^{der} satisfies (1.25), there is a unique homomorphism $\rho(h)\colon G_{\mathrm{Mab}} \to G$ such that $\rho(h)_{\mathbb{R}} \circ h_{\mathrm{Mab}} = h$.*

PROOF. Let G' be the subgroup of G generated by h. The theorem implies that (G',h) is the Mumford-Tate group of an abelian motive, and so it is a quotient of $(G_{\mathrm{Mab}}, h_{\mathrm{Mab}})$. ∎

COROLLARY 1.35. *Let (V,h) be a special Hodge structure, and let (G,h) be its Mumford-Tate group. Then (V,h) is the Betti realization of an abelian motive if and only if G^{der} satisfies (1.25).*

PROOF. From the theorem we know that the action of G_{Hdg} on V factors through G_{Mab} if and only if G^{der} satisfies (1.25). ∎

REMARK 1.36. In order to prove (1.13), it suffices to prove the following: let H be a simple simply connected group over $\mathbb{Q}$, and let $\bar{h}\colon \mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2); then $\rho(H,\bar{h})$ factors through the group G_{Mot} attached to $\mathsf{Mot}(\mathbb{C})$ and the Betti fibre functor. If we knew that all Hodge cycles were absolutely Hodge, this would be equivalent to showing that such a pair is of the form $(G^{\mathrm{der}}, \mathrm{ad}\circ h)$ with (G,h) the Mumford-Tate group of the Betti realization of a motive. Of course, this has to be shown only for groups H not satisfying (1.25), i.e., for (simply connected) groups of type $D^{\mathbb{H}}$, mixed type D, and the exceptional types E_6 and E_7.

The extended Mumford-Tate group

Let (V,h) be a polarizable rational Hodge structure, and let (G',h') be the Mumford-Tate group of $V \oplus \mathbb{Q}(1)$. The action of G' on $\mathbb{Q}(1)$ determines a homomorphism $t\colon G' \to \mathbb{G}_m$ (defined over $\mathbb{Q}$) such that $t \circ w_{h'} = -2$, and we define the ***extended Mumford-Tate group*** of (V,h) to be the triple (G',h',t). We want to relate (G',h',t) to the Mumford-Tate group (G,h) of (V,h).

Suppose first that (V,h) has weight zero. Then $G' = G \times \mathbb{G}_m$, h' is the map

$$z \mapsto (h(z), |z|^{-2}), \quad z \in \mathbb{C}^{\times},$$

and t is the projection map.

When the weight is not zero, there is a smallest $m > 0$ for which $\mathbb{Q}(m)$ is in the tensor category generated by (V,h), and there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Ker}(t) \arrow[r] \arrow[d, equal] & G' \arrow[r, "t"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "m"] & 1 \\
1 \arrow[r] & \mathrm{Ker}(t_m) \arrow[r] & G \arrow[r, "t_m"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

in which t_m is the map defined by the action of G on $\mathbb{Q}(m)$ and the right hand square is cartesian, i.e.,

$$G' = G \underset{t_m, \mathbb{G}_m, m}{\times} \mathbb{G}_m.$$

Moreover, $G = G'/\mathrm{Ker}(m \circ t)$. Using these statements, it is possible to express the earlier results in terms of extended Mumford-Tate groups.

The variation of Mumford-Tate groups in families

[13] Let S be a complex manifold. A ***holomorphic family of rational Hodge structures on*** S is a triple $(\mathbb{V}, F, W)$ consisting of a local system $\mathbb{V}$ of $\mathbb{Q}$-vector spaces on S together with a decreasing (Hodge) filtration of the vector bundle $\mathcal{V} \stackrel{\text{def}}{=} \mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{V}$ by holomorphic subbundles

$$\cdots \supset F^p\mathcal{V} \supset F^{p+1}\mathcal{V} \supset \cdots$$

and a (weight) gradation $\mathbb{V} = \bigoplus W_m V$ of $\mathbb{V}$ such that, for every $s \in S$ and every $m \in \mathbb{Z}$, the $\mathbb{Q}$-vector space $W_m\mathbb{V}_s$ together with the filtration induced by F is a rational Hodge structure of weight m.

PROPOSITION 1.37. *Let $\mathbb{V} = (\mathbb{V}, F, W)$ be a holomorphic family of rational Hodge structures on a complex manifold S, and let G_s be the Mumford-Tate group of $\mathbb{V}_{\sim}$, $s \in S$. Then there exists a subset U of S with thin complement such that $s \mapsto G_s$ is locally constant on U.*

This is a consequence of the following more precise result.

PROPOSITION 1.38. *Let $(\mathbb{V}, F, W)$ be a holomorphic family of rational Hodge structures on a connected complex manifold S. Assume that $\mathbb{V}$ is the constant sheaf with stalk V, and regard G_s as a subgroup of $\mathrm{GL}(V)$. Then there exists a subset U of S with thin complement such that G_u is constant for $u \in U$, say $G_u = G$, and $G \supset G_s$ for $s \notin U$.*

PROOF. A tensor $t \in V^{\otimes r} \otimes V^{\vee \otimes r'}$ is a Hodge tensor for a Hodge structure h on V if and only if it has weight 0 and lies in $F_h^0(V^{\otimes r} \otimes V^{\vee\otimes r'})$. Thus the $s \in S$ where t is a Hodge tensor form a closed analytic set. Let G be the subgroup of $\mathrm{GL}(V)$ fixing all tensors $t \in V^{\otimes r} \otimes V^{\vee \otimes r}$ that are Hodge tensors for all $s \in S$. We can take U to be the set of s such that $G = G_s$. ∎

2 Moduli of Motives

We discuss the problem of realizing a motive as a member of a universal family.

The concept of a moduli variety

Let Ω be an algebraically closed field. A ***moduli problem over*** Ω is a contravariant functor $\mathcal{M}$ from the category of algebraic varieties over Ω to the category of sets together with equivalence relations $\sim$ on each of the sets $\mathcal{M}(T)$ such that

$$m \sim m' \text{ in } \mathcal{M}(T) \implies \phi^*(m) \sim \phi^*(m') \text{ in } \mathcal{M}(T') \text{ for all morphisms } \phi\colon T' \to T.$$

A point t of a variety T with coordinates in Ω, i.e., a morphism $\mathrm{Spec}(\Omega) \to T$, defines a map

$$m \mapsto m_t \stackrel{\text{def}}{=} t^*m\colon \mathcal{M}(T) \to \mathcal{M}(\Omega).$$

A ***solution*** (S,α) ***to the moduli problem*** is an algebraic variety S over Ω and a bijection

$$\alpha\colon \mathcal{M}(\Omega)/{\sim} \to S(\Omega)$$

with the following properties:

[13] Added. For a detailed treatment of this topic, see §6 of Milne 2013. The subset U in Proposition 1.37 can be chosen so that its complement is a countable union of proper analytic subspaces.

(2.1.1) for every variety T over Ω and $m \in \mathcal{M}(T)$, the map $t \mapsto \alpha(m_t): T(\Omega) \to S(\Omega)$ is a morphism of algebraic varieties over Ω;

(2.1.2) for some open covering $\{S_i\}$ of S, there exist elements $m_i \in \mathcal{M}(S_i)$ such that $\alpha((m_i)_s) = s$ for all $s \in S$.

These conditions determine (S,α) uniquely up to a unique isomorphism, for if (S',α') is a second pair satisfying the conditions, then (2.1.2) for S and (2.1.1) for S' show that the map $\alpha' \circ \alpha^{-1}: S(\Omega) \to S'(\Omega)$ becomes a morphism when restricted to the members of some open covering of S, and is therefore a morphism $S \to S'$. Similarly, there is a morphism $S' \to S$, which is inverse to $S \to S'$ because it is on the on points with coordinates in Ω.

A solution (S,α) to the moduli problem is said to be ***fine*** (and S is called a ***fine moduli variety***) if

(2.2.1) for every variety T over Ω, the equivalence class of $m \in \mathcal{M}(T)$ is determined by the equivalence classes of the elements m_t, $t \in T(\Omega)$;

(2.2.2) there exists an element $m_0 \in \mathcal{M}(S)$ such that $\alpha(m_{0s}) = s$ for all $s \in S(\Omega)$.

Let m_0 be as in (2.2.2). Then (2.2.1) implies that, for every $m \in \mathcal{M}(T)$, there is a unique morphism $\varphi: T \to S$ such that $\varphi^*(m_0) \sim m$. Therefore the pair $(S,[m_0])$ represents the functor $T \rightsquigarrow \mathcal{M}(T)/\sim$. Here $[m_0]$ is the equivalence class of m_0. Conversely, if there exists an $m_0 \in \mathcal{M}(S)$ such that $(S,[m_0])$ represents $\mathcal{M}/\sim$, then $(\mathcal{M},\sim)$ is a fine moduli problem and (S,α), with α the inverse of $s \mapsto [m_{0s}]$, is a solution to the moduli problem.

There are variants of these definitions that are also useful. For example, we could replace the category of varieties over Ω with that of smooth varieties over Ω or we could allow the covering in (2.1.2) to be with respect to the étale or flat topologies. Alternatively, we could replace the category of algebraic varieties with that of complex manifolds.

In §3, we shall also need the notion of a moduli variety over a nonalgebraically closed field k. To avoid problems with inseparability, we assume k to be of characteristic zero. A moduli problem over k is a pair $(\mathcal{M},\sim)$ as before, but with $\mathcal{M}$ a functor from the category of algebraic varieties over k to that of sets. Fix an algebraically closed field Ω containing k, and define

$$\mathcal{M}(\Omega) = \varinjlim \mathcal{M}(R),$$

where the limit is over the subalgebras R of Ω that are finitely generated over k and $\mathcal{M}(R) \stackrel{\text{def}}{=} \mathcal{M}(\operatorname{Spec} R)$. The equivalence relations on the sets $\mathcal{M}(R)$ define an equivalence relation on $\mathcal{M}(\Omega)$. A point t of a k-variety T with coordinates in Ω, i.e., a k-morphism $\operatorname{Spec}\Omega \to T$, defines a map $m \mapsto m_t: \mathcal{M}(T) \to \mathcal{M}(\Omega)$. A ***solution*** (S,α) ***to the moduli problem*** is an algebraic variety S over k together and a bijection $\alpha: \mathcal{M}(\Omega)/\sim \to S(\Omega)$ with the following properties:

(2.3.1) for every variety T over k and $m \in \mathcal{M}(T)$, there exists a morphism $\beta_m: T \to S$ such that $\beta_m(t) = \alpha(m_t)$ for all $t \in T(\Omega)$;

(2.3.2) for some open covering (S_i) of S, there exist elements $m_i \in \mathcal{M}(S_i)$ such that $\alpha((m_i)_t) = t$ for all $t \in S(\Omega)$.

The map β_m in (2.3.1) is uniquely determined, and the conditions determine the pair (S,α) uniquely up to a unique isomorphism.

Moduli of Hodge structures

Since our approach to the moduli of motives is through their Hodge structures, it is natural to begin by considering the problem of realizing a rational polarizable Hodge structure as a member of a universal family. Experience from abelian varieties suggests that we should study the moduli, not of Hodge structures, but of polarized Hodge structures. Also that, in order to obtain more general results, we should endow the Hodge structures with additional structure, for example with an endomorphism ring or, more generally, a family of Hodge tensors.

Thus let (V,h_0) be a polarizable rational Hodge structure, which, for simplicity, we take to be of pure weight m. Let $\mathfrak{t}' = (t_i')_{i\in I}$ be a family of Hodge tensors of (V,h_0), i.e., elements of type $(0,0)$ in $V^{\otimes r_i} \otimes V^{\vee\otimes s_i}(m_i)$ for some r_i, s_i, m_i with $mr_i - ms_i - 2m_i = 0$. We assume that $\mathfrak{t}'$ contains a tensor $t_0' \in V^{\vee\otimes 2}(-m)$ that is a polarization for (V,h_0), and we write $t_0' = \psi \otimes (2\pi i)^m$. Thus ψ is a nondegenerate bilinear pairing

$$\psi\colon V \times V \to \mathbb{Q}$$

such that

(2.4.1) ψ is symmetric or skew-symmetric according as m is even or odd;

(2.4.2) $\psi(V^{p,q}, V^{r,s}) = 0$ if $p+r \neq m$; equivalently, $\psi(F^p, F^{m-p+1}) = 0$ for all p, where $F = F_{h_0}$ (the Hodge filtration of h_0);

(2.4.3) $\psi(v, h_0(i)v) > 0$, if $v \in V(\mathbb{R})$, $v \neq 0$; equivalently, $i^{p-q}\psi(v,\bar{v}) > 0$ if $v \in F^p \cap \overline{F^q}$, $v \neq 0$.

Let G' be the subgroup of $\mathrm{GL}(V) \times \mathbb{G}_m$ fixing the t_i'. The action of G' on $\mathbb{Q}(1)$ defines a homomorphism $t\colon G' \to \mathbb{G}_m$, and we let $G = \mathrm{Ker}(t)$. Write $t_i' = t_i \otimes (2\pi i)^{m_i}$ with $t_i \in V^{\otimes r_i} \otimes V^{\vee\otimes s_i}$, and let $\mathfrak{t} = (t_i)_{i\in I}$. Then t_i is of type (m_i, m_i), and G is the subgroup of $\mathrm{GL}(V)$ fixing the t_i, i.e., for all $\mathbb{Q}$-algebras R,

$$G(R) = \{\alpha \in \mathrm{GL}(V(R)) \mid (\alpha^{\otimes r_i} \otimes \check{\alpha}^{\otimes s_i})(t_i) = t_i, \text{ all } i \in I\}.$$

Note that $h_0' \overset{\text{def}}{=} (h_0, \mathrm{Nm})$ maps into $G'_{\mathbb{R}}$, and $u_0 = h_0|U^1$ maps into $G_{\mathbb{R}}$. In particular, $C \overset{\text{def}}{=} h_0(i) \in G(\mathbb{R})$, and ψ is a C-polarization of the representation of G on $V(\mathbb{R})$. Therefore (see (1.5)) G is reductive.

EXAMPLE 2.5. There are two cases of particular interest.

(a) The family $\mathfrak{t}'$ contains all the Hodge tensors of (V,h_0). In this case G' is the extended Mumford-Tate group of (V,h_0), and we call G the ***special Mumford-Tate group*** of (V,h_0).

(b) The family $\mathfrak{t}$ consists only of $t_0 = \psi$, so that G is the subgroup of $\mathrm{GL}(V)$ fixing ψ and $G' = G \cdot \mathbb{G}_m$. This case is studied in Griffiths 1968, 1970; Schmid 1973; El Zein 1991, Chapter 7.

Let $\mathcal{F}^\vee$ be the set of filtrations F on $V(\mathbb{C})$ such that

$$\dim F^p = \dim F^p_{h_0} \text{ all } p.$$

The group $\mathrm{GL}(V(\mathbb{C}))$ acts transitively on $\mathcal{F}^\vee$, and the subgroup P stabilizing F_{h_0} is parabolic.[14] Hence the bijection

$$\mathrm{GL}(V)/P \longrightarrow \mathcal{F}^\vee, \quad gP \mapsto gF_{h_0}$$

[14] Let $\xi\colon G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation of a reductive group G over a field k of characteristic zero, and let F be a filtration on V. Suppose that there exists a cocharacter μ of G splitting the filtration, i.e., such that $F^pV = \bigoplus_{i\geq p} V^i$, $V^i = \{v \in V \mid \xi\mu(x)v = x^i v, \forall x\}$. The subgroup P of G of elements preserving F is parabolic with unipotent radical the subgroup U of elements acting as the identity map on $\bigoplus F^iV/F^{i+1}V$. The cocharacter μ defines a filtration on every representation of G, in particular on the adjoint representation of G on $\mathfrak{g}$, and $\mathrm{Lie}(P) = F^0\mathfrak{g}$, $\mathrm{Lie}(U) = F^1\mathfrak{g}$. See Saavedra 1972, IV.2.2.5.

realizes $\mathcal{F}^{\vee}$ as a smooth projective algebraic variety over $\mathbb{C}$. The subset $\mathcal{F}$ of $\mathcal{F}^{\vee}$ of those filtrations F such that

$$V(\mathbb{C}) = F^p \oplus \bar{F}^{m-p+1} \text{ for each } p,$$

is open in $\mathcal{F}^{\vee}$ (for the complex topology). It parametrizes exactly the Hodge structures on $V(\mathbb{R})$ whose Hodge numbers are the same as those of (V, h_0).

Consider the set of filtrations F in $\mathcal{F}^{\vee}$ such that, for all $i \in I$, $t_i \in F^{m_i}$. It is a closed algebraic subvariety of $\mathcal{F}^{\vee}$ stable under the action of $G(\mathbb{C})$, and we let $X^{\vee}$ denote the orbit containing F_{h_0}. Then $X^{\vee}$ is the quotient of $G(\mathbb{C})$ by a parabolic subgroup, and so it is a smooth projective variety over $\mathbb{C}$. An $F \in X^{\vee}$ satisfying (2.4.3) automatically defines a Hodge structure (see the references in (2.5b)), and so lies in $\mathcal{F}$. The set of such F's is open in $X^{\vee}$ and stable under $G(\mathbb{R})$, and we write X for the orbit containing F_{h_0}.

We now regard X as a complex manifold rather than a set of filtrations, and we write F_x for the filtration, and h_x, u_x, and μ_x for the homomorphisms, corresponding to $x \in X$.

REMARK 2.6. (a) When $\mathfrak{t}$ consists only of t_0, $X^{\vee}$ contains all filtrations such that $t_0' \in F^0$, and X contains all $F \in X^{\vee}$ satisfying (2.4.3). Thus in this case, X contains all the filtrations on $V(\mathbb{C})$ defining Hodge structures for which $(2\pi i)^m \psi$ is a polarization (see the references in (2.5b)).

(b) Let $x \in X^{\vee} \cap \mathcal{F}$, and let F_x be the corresponding filtration on $V(\mathbb{C})$. The cocharacter $\mu_x \colon \mathbb{G}_m \to \mathrm{GL}(V(\mathbb{C}))$ has image in $G \cdot Z$, where $Z = \mathbb{G}_m = Z(\mathrm{GL}(V(\mathbb{C}))$, and so μ_x defines a filtration on $\mathfrak{g}_{\mathbb{C}}$. The stabilizer of F_x in G has Lie algebra $F^0\mathfrak{g}_{\mathbb{C}}$, and so

$$\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \xrightarrow{\simeq} \mathrm{Tgt}_x X^{\vee}.$$

Now assume $x \in X$. The stabilizer B of x in $G_{\mathbb{R}}$ has Lie algebra $\mathfrak{g}_{\mathbb{R}} \cap F^0\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{0,0}$, and so

$$\mathfrak{g}_{\mathbb{R}}/(\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{0,0}) \xrightarrow{\simeq} \mathrm{Tgt}_x X.$$

Note that

$$\dim \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} = \dim \mathfrak{g}_{\mathbb{R}}/(\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{0,0})$$

(as real vector spaces).

Choose a lattice $V(\mathbb{Z})$ in V, and let

$$\Gamma(N) = \{g \in G(\mathbb{Q}) \mid gV(\mathbb{Z}) = V(\mathbb{Z}), \quad g = \mathrm{id} \text{ on } V(\mathbb{Z})/NV(\mathbb{Z})\}.$$

Consider a triple $(W, \mathfrak{s}, \eta)$ consisting of an integral Hodge structure $W = (W(\mathbb{Z}), h)$, a family of tensors $\mathfrak{s} = (s_i)_{i \in I}$ of $W(\mathbb{Q}) \stackrel{\text{def}}{=} W(\mathbb{Z}) \otimes \mathbb{Q}$, and an isomorphism

$$\eta \colon V(\mathbb{Z})/NV(\mathbb{Z}) \longrightarrow W(\mathbb{Z})/NW(\mathbb{Z}).$$

We call η a ***level-N-structure*** on W. Let $\mathcal{H}(\mathbb{C})$ be the set of such triples for which there exists an isomorphism $\beta \colon W(\mathbb{Q}) \to V$ satisfying the following conditions:

(2.7.1) for some $x \in X$, β is a morphism of rational Hodge structures

$$(W(\mathbb{Q}), h) \longrightarrow (V, h_x);$$

(2.7.2) for all $i \in I$, $\beta(s_i) = t_i$;

(2.7.3) $\beta(W(\mathbb{Z})) = V(\mathbb{Z})$ and the composite

$$V(\mathbb{Z})/NV(\mathbb{Z}) \xrightarrow{\eta} W(\mathbb{Z})/NW(\mathbb{Z}) \xrightarrow{\beta(N)} V(\mathbb{Z})/NV(\mathbb{Z})$$

is the identity map.

Note that the conditions imply that s_i is an element of type (m_i, m_i) in $W(\mathbb{Q})^{\otimes r_i} \otimes W(\mathbb{Q})^{\vee \otimes s_i}$. An ***isomorphism*** from one such system $(W, \mathfrak{s}, \eta)$ to a second $(W', \mathfrak{s}', \eta')$ is an isomorphism of integral Hodge structures $\gamma\colon (W(\mathbb{Z}), h) \to (W'(\mathbb{Z}), h')$ such that $\gamma(s_i) = s_i'$ for all $i \in I$ and $\gamma(N) \circ \eta = \eta'$.

Let $(W, \mathfrak{s}, \eta)$ be such a system, and choose an isomorphism $\beta\colon W(\mathbb{Q}) \to V$ satisfying the conditions (2.7). A second such isomorphism is of the form $g \circ \beta$ for some $g \in \mathrm{GL}(V)$. But (2.7.2) implies that $g(t_i) = t_i$ for all i, and so $g \in G(\mathbb{Q})$. Also (2.7.3) implies that $gV(\mathbb{Z}) = V(\mathbb{Z})$ and that g acts as the identity map on $V(\mathbb{Z})/NV(\mathbb{Z})$, and so $g \in \Gamma(N)$. Therefore, when we write $\mathrm{ad}\,\beta \circ h = h_x$, the orbit of x in $\Gamma(N)\backslash X$ is independent of the choice of β.

PROPOSITION 2.8. *The map $(W, \mathfrak{s}, \eta) \mapsto [x]$ just defined gives a bijection*

$$\alpha\colon \mathcal{H}(\mathbb{C})/\approx \; \to \Gamma(N)\backslash X.$$

PROOF. Let $(W', \mathfrak{s}', \eta')$ be a second system. If $\gamma\colon (W', \mathfrak{s}', \eta') \to (W, \mathfrak{s}, \eta)$ is an isomorphism of triples and $\beta\colon W(\mathbb{Q}) \to V(\mathbb{Q})$ is an isomorphism of vector spaces satisfying (2.7), then $\beta \circ \gamma$ satisfies (2.7) for $(W', \mathfrak{s}', \eta')$, and it follows that $(W', \mathfrak{s}', \eta')$ maps to the same element of $\Gamma(N)\backslash X$ as $(W, \mathfrak{s}, \eta)$. Conversely, if $(W, \mathfrak{s}, \eta)$ and $(W', \mathfrak{s}', \eta')$ map to the same class $[x]$, then we can choose the maps β and β' so that the triples map to the same element of X; now $\gamma \overset{\text{def}}{=} \beta^{-1} \circ \beta'$ is an isomorphism

$$(W', \mathfrak{s}', \eta') \to (W, \mathfrak{s}, \eta).$$

Finally, if $x \in X$, then $((V(\mathbb{Z}), h_x), \mathfrak{t}, \mathrm{id})$ maps to $[x]$. ■

We next wish to endow $\Gamma(N)\backslash X$ with the structure of a complex manifold.

LEMMA 2.9. *(a) The group $\Gamma(N)$ acts properly discontinuously*[15] *on X.*

(b) For N sufficiently divisible, $\Gamma(N)$ is torsion-free.

PROOF. (a) According to a standard criterion (see Miyake 1989, 1.5.2), it suffices to check that the stabilizers in $G(\mathbb{R})$ of the elements of X are compact and that $\Gamma(N)$ is a discrete subgroup of $G(\mathbb{R})$, but the stabilizer of $h \in X$ is compact because it fixes the positive definite symmetric form

$$(v, w) \mapsto \psi(v, Cw), \quad C = h(i),$$

and $\Gamma(N)$ is discrete because it is a congruence subgroup.

(b) If N is sufficiently divisible, $\Gamma(N)$ will be neat, and hence torsion-free (see Borel 1969, §17). ■

PROPOSITION 2.10. *If $\Gamma(N)$ is torsion-free, then $\Gamma(N)\backslash X$ has a unique structure of a complex manifold such that the quotient map $X \to \Gamma(N)\backslash X$ is a local isomorphism.*

[15] Recall that this means that for any pair of points x_1 and x_2 of X, there exist neighbourhoods U_1 and U_2 of x_1 and x_2 such that $\{g \in \Gamma(N) \mid gU_1 \cap U_2 \neq \varnothing\}$ is finite.

PROOF. If $\Gamma(N)$ is torsion-free, then the map $X \to \Gamma(N)\backslash X$ is a local homeomorphism.

When N is sufficiently divisible that $\Gamma(N)$ is neat, we write $S(N)$ for $\Gamma(N)\backslash X$ regarded as a complex manifold.

An ***integral structure*** on a holomorphic family of rational Hodge structures $(\mathbb{V}, F, W)$ is a local system of $\mathbb{Z}$-modules $\mathbb{V}(\mathbb{Z}) \subset \mathbb{V}$ such that $\mathbb{V}(\mathbb{Z}) \otimes \mathbb{Q} = \mathbb{V}$ and $\mathbb{V}(\mathbb{Z}) = \bigoplus \mathbb{V}(\mathbb{Z}) \cap W_m$. A holomorphic family of rational Hodge structures together with an integral structure will be referred to as a ***holomorphic family of integral Hodge structures***.

Let T be a complex manifold, and consider triples $m = (\mathbb{W}, \mathfrak{s}, \eta)$ consisting of a holomorphic family of integral Hodge structures $\mathbb{W} = (\mathbb{W}(\mathbb{Z}), F, W)$ on T, a family of global tensors $\mathfrak{s}$ of $\mathbb{W}(\mathbb{Q}) \overset{\text{def}}{=} \mathbb{W}(\mathbb{Z}) \otimes \mathbb{Q}$ indexed by I, and an isomorphism η from the constant sheaf $(V(\mathbb{Z})/NV(\mathbb{Z}))_T$ on T to $\mathbb{W}(\mathbb{Z})/N\mathbb{W}(\mathbb{Z})$. We define $\mathcal{H}(T)$ to be the set of such triples m with the property that $m_t \in \mathcal{H}(\mathbb{C})$ for all $t \in T$. An ***isomorphism***

$$(\mathbb{W}, \mathfrak{s}, \eta) \longrightarrow (\mathbb{W}', \mathfrak{s}', \eta')$$

is an isomorphism of integral Hodge structures $\mathbb{W} \to \mathbb{W}'$ carrying carrying $\mathfrak{s}$ and η into $\mathfrak{s}'$ and η'. With the obvious notion of pull-back, the pair $(\mathcal{H}, \approx)$ becomes a moduli problem on the category of complex manifolds and holomorphic maps.

EXAMPLE 2.11. On X there is a holomorphic family of integral Hodge structures of weight m whose underlying local system of $\mathbb{Z}$-modules is the constant system defined by $V(\mathbb{Z})$ and which is such that the filtration at a point x is the Hodge filtration of h_x.

If N is sufficiently divisible, then $\Gamma(N)$ is the group of covering transformations of X over $S(N)$, and we obtain a holomorphic family of integral Hodge structures $\mathbb{V}(N)$ on $S(N)$: for $o = h_0$, the local system of $\mathbb{Z}$-modules underlying $\mathbb{V}(N)$ corresponds to the representation of $\Gamma(N) = \pi_1(S(N), o)$ on $V(\mathbb{Z})$. Because t_i is fixed by $\Gamma(N)$, it defines a global tensor of $\mathbb{V}(N)$, and because $\Gamma(N)$ acts trivially on $V(\mathbb{Z})/NV(\mathbb{Z})$ there is a canonical isomorphism $\eta\colon (V(\mathbb{Z})/NV(\mathbb{Z}))_{S(N)} \to \mathbb{V}(N)/NV(N)$. The system $m_0 = (\mathbb{V}(N), \mathfrak{t}, \eta) \in \mathcal{H}(S(N))$.

THEOREM 2.12. *For N sufficiently divisible, the pair $(S(N), \alpha)$ is a fine solution to the moduli problem $(\mathcal{H}, \approx)$ (in the category of complex manifolds and holomorphic maps).*

PROOF. Let $m = (\mathbb{W}, \mathfrak{s}, \eta) \in \mathcal{H}(T)$ for some complex manifold T. We have to prove that the map

$$\varphi_m\colon T \longrightarrow S(N), \quad t \mapsto \alpha(m_t),$$

is holomorphic. Let $t_0 \in T$. Choose an open neighbourhood U of t_0 over which $\mathbb{W}$ is trivial, and fix an isomorphism $\mathbb{W}|U \to V_U$ (constant local system on U). The map $t \mapsto F_t\colon T \to \mathcal{F}$ is holomorphic, and its image is contained in X. Hence it is holomorphic as a map into X, and so the composite $U \to X \to S(N)$ is holomorphic. Obviously, $\varphi_m^* m_0 \approx m$, and φ_m is the unique morphism with this property, and so $(S(N), [m_0])$ represents the functor $\mathcal{H}/{\approx}$.

REMARK 2.13. In the above, we should of course allow (V, h_0) to have more than one weight, but this complicates the exposition. Once one is willing to work in that generality, it is natural to replace V with $V \oplus \mathbb{Q}(1)$ to ensure that $\mathbb{Q}(1)$ is in the tensor category generated by (V, h_0).

From now on, we assume that we are in Case (2.5a). It follows from (1.38), that there is no essential loss of generality in doing this.

Questions

If (V,h_0) is the Betti realization of a motive, then is $S(N)$ a moduli variety for motives? Four problems present themselves.

(2.14.1) Does $S(N)$ have a (unique) structure of an algebraic variety compatible with its complex structure?

(2.14.2) Is every Hodge structure (V,h), $h \in X$, motivic, i.e., the Betti realization of motive?

(2.14.3) Assuming that (2.14.1) and (2.14.2) have positive answers, is the canonical family $\mathbb{V}(N)$ of Hodge structures on $S(N)$ defined in (2.11) the Betti realization of a "family of motives" (whatever that may be)?

(2.14.4) Is $S(N)$ a moduli variety for motives? Since $S(N)$ parametrizes Hodge structures with additional structure, and we are asking that it parametrize motives, this implies that the motives in the family are determined (up to isomorphism) by their Betti realizations.

We shall see shortly that, in order for (2.14.3) to be true, it is necessary that (V,h_0) be a special Hodge structure. Remarkably, when we assume this and that Hypothesis 1.13 holds in families, then all statements become true.

Variations of Hodge structures

Let S be a connected complex manifold. Recall that a ***connection*** on a holomorphic vector bundle $\mathcal{V}$ is a $\mathbb{C}$-linear homomorphism

$$\nabla\colon \mathcal{V} \longrightarrow \Omega^1_S \otimes \mathcal{V}$$

satisfying the Leibnitz identity

$$\nabla(fv) = df \otimes v + f \cdot \nabla v,$$

for f and v local sections of $\mathcal{O}$ and $\mathcal{V}$. A connection is ***flat*** if its curvature tensor is zero. A local section of $\mathcal{V}$ is ***horizontal*** if $\nabla v = 0$, and we write $\mathcal{V}^\nabla$ for the sheaf of horizontal sections. The functor $(\mathcal{V},\nabla) \rightsquigarrow \mathcal{V}^\nabla$ is an equivalence from the category of vector bundles with flat connections to that of complex local systems on S.

DEFINITION 2.15. A holomorphic family of rational Hodge structures $(\mathbb{V},F,W)$ on a complex manifold is a ***variation of rational Hodge structures*** if

(2.15.1) $(\mathbb{V},F,W)$ admits an integral structure;

(2.15.2) (axiom of transversality): $\nabla(F^p\mathcal{V}) \subset \Omega^1_S \otimes F^{p-1}\mathcal{V}$.

A ***polarization*** of a variation of Hodge structures $\mathbb{V}$ of weight m is a morphism of local systems $\mathbb{V} \otimes V \to \mathbb{Q}(-m)$ that at each point s of S defines a polarization of the Hodge structure $\mathbb{V}_s$.

PROPOSITION 2.16. *The category $\mathsf{Hdg}_{\mathbb{Q}}(S)$ of polarizable variations of rational Hodge structures on a connected complex manifold S is a semisimple Tannakian category over $\mathbb{Q}$.*

PROOF. It is obvious that $\mathsf{Hdg}_{\mathbb{Q}}(S)$ is closed under the formation of direct summands, direct sums, and tensor products; moreover it contains the constant variations $\mathbb{Q}(m)$. Therefore we can apply Deligne 1971a, 4.2.3, to deduce that $\mathsf{Hdg}_{\mathbb{Q}}(S)$ is a semisimple abelian subcategory of the category of continuous families of Hodge structures, and that it is closed under the formation of duals. For every point $o \in S$, $(\mathbb{V}, F, W) \rightsquigarrow V_o$ is a fibre functor, and so $\mathsf{Hdg}_{\mathbb{Q}}(S)$ is a Tannakian category. The identity object is $\mathbb{Q}(0)$, and $\mathrm{End}(\mathbb{Q}(0)) = \mathbb{Q}$. ∎

THEOREM 2.17. *Let $\pi: Y \to S$ be a smooth projective map of algebraic varieties over $\mathbb{C}$; for all r, $R^r\pi_*\mathbb{Q}$ is a polarizable variation of rational Hodge structures on S of weight r.*

PROOF. This is a fundamental result in Griffiths's theory. For proofs see Griffiths 1968 and Deligne 1971c. ∎

The last two results show that the Betti realization of a family of motives must be a polarizable variation of Hodge structures. It is therefore natural to require that $\mathbb{V}(N)$ be a variation of Hodge structures on $S(N)$, or equivalently, that (V, F_x) be a variation of Hodge structures on X. There is a simple criterion for this.

PROPOSITION 2.18. *The following statements are equivalent:*

(a) *the family (V, F_x) is a variation of Hodge structures on X;*
(b) *for all $x \in X$, (G', h'_x) satisfies (SV1);*
(c) *for all $x \in X$, the Hodge structure (V, h_x) is special;*
(d) *the Hodge structure (V, h_0) is special.*

PROOF. (a) $\iff$ (b). Consider the inclusion map $\varphi: X \hookrightarrow X^\vee$. The map on the tangent spaces at a point x of X is

$$(d\varphi)_x : \mathrm{Tgt}_x(X) = \mathfrak{g}_{\mathbb{R}}/(\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{0,0}) \xrightarrow{\simeq} \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}, \quad \mathfrak{g} = \mathrm{Lie}(G),$$

(see 2.6). The axiom of tranversality says that the image of $(d\varphi)_x$ is contained in $F_x^{-1}\mathfrak{g}_{\mathbb{C}}/F_x^0\mathfrak{g}_{\mathbb{C}}$ for all x, i.e., that $\mathfrak{g}_{\mathbb{C}} = F_x^{-1}\mathfrak{g}_{\mathbb{C}}$. But

$$\mathfrak{g}_{\mathbb{C}} = F_x^{-1}\mathfrak{g}_{\mathbb{C}} \iff \mathfrak{g} \text{ is of type } \{(-1,1),(0,0),(1,-1)\} \iff (G', h'_x) \text{ satisfies (SV1).}$$

(b) $\implies$ (c). The extended Mumford-Tate group of (V, h_x) is a subgroup of G'.

(c) $\implies$ (d). Obvious.

(d) $\implies$ (b). Because of our assumption that we are in case (2.5a), G' is the extended Mumford-Tate group of (V, h_0), and so (d) says that (G', h'_0) satisfies (SV1). To deduce (b), set $x = gx_0$, $g \in G(\mathbb{R})$, and note that $\mathrm{ad}(g)$ is an isomorphism $(G', h'_0) \to (G', h'_x)$. ∎

Before providing answers to the Questions 2.14, we review some of the fundamental results concerning variations of Hodge structures.

THEOREM 2.19. *Let $\mathbb{V}$ be a polarizable variation of Hodge structures on a smooth quasi-projective algebraic variety S over $\mathbb{C}$. Then the vector bundle $\mathcal{V} \stackrel{\text{def}}{=} \mathbb{V} \otimes \mathcal{O}_S^{\mathrm{an}}$ carries a unique algebraic structure such that the connection ∇ becomes algebraic and such that ∇ has regular singular points at infinity relative to any smooth compactification of M. With respect to this structure, the subbundles $F^r \subset \mathcal{V}$ are algebraic.*

PROOF. See Schmid 1973, 4.13, where the result is credited to Griffiths. ∎

REMARK 2.20. Let $\pi: Y \to S$ be a smooth projective morphism of algebraic varieties over a field k. The vector bundle $R^r\pi_*\mathcal{O}_Y$ has a canonical (de Rham) filtration F_{dR} arising from the identification $R^r\pi_*\mathcal{O}_Y = \mathbb{R}^r\pi_*\Omega^{\cdot}_{Y/S}$ and a flat (Gauss-Manin) connection ∇. When $k = \mathbb{C}$ these structures agree with those defined by (2.17) and (2.19).

THEOREM 2.21 (THEOREM OF THE FIXED PART). *Let S be a smooth connected algebraic variety over $\mathbb{C}$, and let $\mathbb{V}$ be a polarizable variation of Hodge structures on S. The largest constant local system $\mathbb{V}^f \subset V$ is a constant variation of Hodge substructures of $\mathbb{V}$.*

PROOF. When the base space is a compact complex manifold, this is proved in Griffiths 1970, §7, and in the general case it is proved in Schmid 1973, 7.22. See also 4.1.2 and footnote p. 45 of Deligne 1971a. ■

Consequently, a global section of $\mathbb{V}$ (in fact, even of $\mathbb{V} \otimes \mathbb{C}$) that is of type (p,q) at one point, is of type (p,q) at every point. When we apply this to an endomorphism of $\mathbb{V}$, we obtain the following result.

COROLLARY 2.22. *Let $(\mathbb{V}, F)$ and $(\mathbb{V}', F')$ be two polarizable variations of Hodge structures on a smooth connected algebraic variety S. An isomorphism $\alpha: \mathbb{V} \to \mathbb{V}'$ of local systems such that $\alpha(o)$ is an isomorphism of Hodge structures for some $o \in S(\mathbb{C})$ is an isomorphism of variations of Hodge structures.*

Let S be a smooth connected algebraic variety over $\mathbb{C}$. The corollary implies that, given a representation of $\pi_1(S,o)$ on a finite-dimensional rational vector space V and a polarized Hodge structure on V, there is at most one way of extending these data to a polarizable variation of Hodge structures on S.

Algebraicity of $S(N)$

Recall that a ***bounded domain*** in $\mathbb{C}^g$ is a bounded open connected subset of $\mathbb{C}^g$, and that a ***hermitian manifold*** is a complex manifold together with a holomorphic family of positive-definite hermitian forms on its tangent spaces. A bounded domain or hermitian manifold D is said to be ***symmetric*** if each point of D is an isolated fixed point of an involution of D. A complex manifold isomorphic to a symmetric bounded domain is called a ***symmetric hermitian domain***. Since the Bergmann metric provides a symmetric bounded domain with a hermitian structure invariant under all automorphisms of the domain, every symmetric hermitian domain has a canonical hermitian structure with respect to which it is symmetric.

PROPOSITION 2.23. *If (V, h_0) is special, then every connected component of X is a symmetric hermitian domain.*

PROOF. Identify a connected component X^+ of X with a $G'(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G'_{\mathbb{R}}$, and apply Deligne 1979, 1.1.17. ■

THEOREM 2.24. *Let X^+ be a symmetric hermitian domain, and let Γ be a torsion-free arithmetic subgroup of $\mathrm{Aut}(X^+)$. Then $S(\Gamma) \overset{\text{def}}{=} \Gamma \backslash X^+$ has a unique structure of an algebraic variety with the following property:*

2.24.1 every holomorphic map $S \to S(\Gamma)$ from a smooth complex algebraic variety to $S(\Gamma)$ is a morphism of algebraic varieties.

PROOF. The main theorem of Baily and Borel 1966 shows that $S(\Gamma)$ has a canonical algebraic structure compatible with its complex structure. That the structure has the property (2.24.1) is proved in Borel 1972, 3.10. It is obvious that this property determines the algebraic structure uniquely. ■

THEOREM 2.25. *If (V,h_0) is special, then $\mathbb{V}(N)$ is a polarizable variation of Hodge structures on $S(N)$, and $S(N)$ has a unique algebraic structure compatible with its complex structure. Conversely, if $\mathbb{V}(N)$ is a variation of Hodge structures, then (V,h_0) is special.*

PROOF. Combine (2.18), (2.23), and (2.24). ∎

The motivicity of (V,h_x)

PROPOSITION 2.26. *If (V,h_0) is the Betti realization of an abelian motive, then so also is (V,h_x) for all $x \in X$.*

PROOF. We saw in (2.18) that if (V,h_0) is a special Hodge structure, then so also is (V,h_x) for all $x \in X$.

Let (G_0,h_0) and (G_x,h_x) be the Mumford-Tate groups of (V,h_0) and (V,h_x) respectively (thus $G = \mathrm{Ker}(G_0 \to \mathbb{G}_m)$). If $h_x = \mathrm{ad}\, g \circ h_0$ with $g \in G_0(\mathbb{Q})$, then $G_x = G$, and it follows from (1.35) that (V,h_x) is the Betti realization of an abelian motive. The real approximation theorem states that $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$, and so we can assume that $h_x = \mathrm{ad}\, g \circ h_0$ with $g \in G_0(\mathbb{R})^+$.

By assumption, (V,h_0) lies in the tensor category generated by the Betti realization of an abelian variety A. Let (G_1,h_1) be the Mumford-Tate group of A. We have a diagram

$$(G_0,h_0) \twoheadleftarrow (G_1,h_1) \hookrightarrow (G(\psi),X(\psi))$$

with $G(\psi)$ the group of symplectic similitudes defined by $H_1(A)$ and a Riemann form. Lift g to an element $g_1 \in G_1(\mathbb{R})^+$. Then we have a diagram:

$$(G_0,h_x) \twoheadleftarrow (G_1,\mathrm{ad}(g)\circ h_1) \hookrightarrow (G(\psi),X(\psi)).$$

If G_{x1} denotes the inverse image of G_x in G_1, then $(G_{x1},\mathrm{ad}(g)\circ h_1)$ is the Mumford-Tate group of an abelian variety, and it has (G_x,h_x) as a quotient. It follows that (V,h_x) is the Betti realization of an abelian motive. ∎

REMARK 2.27. Assume Hypothesis 1.13. If (V,h_0) is special, then for all $x \in X$, the Hodge structure (V,h_x) is the Betti realization of a special motive.

Motivic variations of Hodge structures

By a ***global tensor*** of a local system of $\mathbb{Q}$-vector spaces $\mathbb{V}$ on a complex manifold S, we mean an element of $\Gamma(S,\mathbb{V}^{\otimes r} \otimes \mathbb{V}^{\vee\otimes s} \otimes \mathbb{Q}(m))$ for some $r,s \in \mathbb{N}$, $m \in \mathbb{Z}$.

DEFINITION 2.28. Let $\pi\colon Y \to S$ be a projective smooth morphism of smooth varieties over $\mathbb{C}$. A global tensor t of $\mathcal{H}_B(Y/S) \overset{\text{def}}{=} \bigoplus_i R^i\pi_*\mathbb{Q}$ is an ***absolute Hodge tensor*** of Y/S if, for all $s \in S(\mathbb{C})$, t_s is an absolute Hodge tensor of Y_s. A sum of absolute Hodge tensors will also be called an absolute Hodge tensor.

REMARK 2.29. When S is connected, a global tensor t of $\mathcal{H}_B(Y/S)$ is an absolute Hodge tensor of Y/S if t_s is an absolute Hodge tensor on Y_s for a single s (Deligne 1982, 2.12, 2.14).

DEFINITION 2.30. Let S be a smooth variety over $\mathbb{C}$.

(a) (Deligne 1971, 4.2.4). A variation of Hodge structures $\mathbb{V}$ on S is ***algebraic*** if there exists a dense open subset U of S, an integer m, and a projective smooth morphism $\pi\colon Y \to U$ such that $\mathbb{V}|U$ is a direct summand of $\mathcal{H}_B(Y/U)(m)$.

(b) If in (a) the projector realizing $\mathbb{V}|U$ as a direct summand is an absolute Hodge tensor, then $\mathbb{V}$ will be said to be ***motivic.***

(c) If in (a) Y is an abelian scheme over U, then $\mathbb{V}$ will be said to be ***abelian-motivic.***

Note that the projector implicit in (a) of the definition is automatically a Hodge tensor at every point of U, and so, when Y is an abelian scheme, the main theorem of (Deligne 1982) implies that it is an absolute Hodge tensor. Thus

$$\text{abelian-motivic} \implies \text{motivic} \implies \text{algebraic}.$$

In general, when $\pi\colon Y \to S$ is a projective smooth morphism and $p \in \mathrm{End}(\mathcal{H}_B(Y/S))$ is both a projector and an absolute Hodge tensor, we write $\mathcal{H}_B(Y/S, p, m)$ for $\mathrm{Im}(p) \otimes \mathbb{Q}(m)$.

LEMMA 2.31. *(a) A direct sum, or tensor product of algebraic (resp. motivic, abelian-motivic) variations of Hodge structures is algebraic (resp. motivic, abelian-motivic); the constant variation of Hodge structures $\mathbb{Q}(m)$ is abelian-motivic.*

(b) A direct summand of an algebraic (resp. abelian-motivic) variation of Hodge structures is algebraic (resp. abelian-motivic).

(c) Every algebraic variation of Hodge structures is polarizable.

PROOF. The statements concerning algebraic variations are proved in (Deligne 1971a, 4.2.5). Similar proofs give the statements concerning abelian-motivic or motivic variations. ■

PROPOSITION 2.32. *The category of abelian-motivic variations of Hodge structures on S is a semisimple abelian category; it is a tensor subcategory of the semisimple Tannakian category of polarizable variations of Hodge structures on S. If S is connected, then for every $o \in S(\mathbb{C})$, $\mathbb{V} \rightsquigarrow V_o$ is a fibre functor.*

PROOF. We can again apply Deligne 1971a, 4.2.3. ■

LEMMA 2.33. *A variation of Hodge structures $\mathbb{V}$ on S is abelian-motivic if there is a smooth dominant morphism $f\colon S' \to S$ of finite-type such that $f^*\mathbb{V}$ is abelian-motivic.*

PROOF. The image of f is a dense open subscheme U of S, and there exists a surjective étale morphism $h\colon U' \to U$ and a morphism $g\colon U' \to S'$ such that $h = f \circ g$ (see Grothendieck 1964/67, 17.16.3):

$$\begin{tikzcd} S' \arrow[d, "f"'] & U' \arrow[l, "g"'] \arrow[dl, "h", "\text{étale}"'] \\ S \overset{\text{open}}{\supset} U & \end{tikzcd}$$

Hence $h^*\mathbb{V}$ is abelian-motivic, say $h^*\mathbb{V} = \mathcal{H}_B(Y/U', p, m)$. After replacing U with an open subset, the map $h\colon U' \to U$ will be finite. It is clear that $h_* h^*\mathbb{V}$ is abelian-motivic, and $\mathbb{V}|U$ is a subobject, hence direct summand, of it, and so we can apply (2.31b). ■

The motivicity of $\mathbb{V}(N)$

THEOREM 2.34. *If* (V, h_0) *is the Betti realization of an abelian motive, then the variation of Hodge structures* $\mathbb{V}(N)$ *on* $S(N)$ *defined in (2.11) is abelian-motivic.*

PROOF. Suppose first that (V, h_0) is the Betti realization of an abelian variety. In this case $\mathbb{V}(N)$ is a polarizable variation of Hodge structures of type $\{(-1,0),(0,-1)\}$, and so $\mathbb{V}(N) = \mathcal{H}_B(Y/S(N))$ for some abelian scheme Y (see Deligne 1971a, 4.4.3). Moreover, if $G \to \mathrm{GL}(W)$ is a second representation of G and $\mathbb{W}$ is the corresponding variation of Hodge structures on $S(N)$, then, because W lies in the tensor category generated by V, $\mathbb{W}$ will lie in the tensor category generated by $\mathbb{V}$, and so (2.32) shows it to be abelian-motivic.

Now consider the general case. Because (V, h_0) is the Betti realization of an abelian motive, there is a surjective homomorphism $(G_1, u_1) \to (G, u_0)$ with (G_1, u_1) the special Mumford-Tate group of an abelian variety. Correspondingly, we have a smooth morphism

$$f\colon \Gamma_1(N)\backslash X_1 \to \Gamma(N)\backslash X.$$

The pull-back $f^*\mathbb{V}(N)$ of $\mathbb{V}(N)$ to $\Gamma_1(N)\backslash X_1$ is the variation of Hodge structures defined by the representation $G_1 \to G \hookrightarrow \mathrm{GL}(V)$, and is therefore abelian-motivic. Now apply (2.33). ■

SPECIAL HODGE STRUCTURES

Hypothesis 1.13 asserts that every special Hodge structure is of the form $H_B(Y, p, m)$ for some projective smooth variety Y over $\mathbb{C}$, projector p that is an absolute Hodge tensor, and integer m. The next hypothesis asserts that this holds in families.

HYPOTHESIS 2.35. *Let* (V, h_0) *be a special Hodge structure, and let* o *be the point* $\Gamma(N)\cdot h_0$ *in* $S(N)$ *(*N *sufficiently divisible). There exists an open neighbourhood* U *of* o*, a projective smooth morphism* $\pi\colon Y \to U$*, a projector* p *that is an absolute Hodge tensor, and an integer* m *such that* $\mathbb{V}|U = \mathcal{H}_B(Y/U, p, m)$*.*

Let M be a motive over $\mathbb{C}$. We say that ***all the Hodge tensors of*** M ***are absolutely Hodge*** if the functor

$$\omega_B\colon \mathsf{Mot}(\mathbb{C}) \longrightarrow \mathsf{Hdg}_{\mathbb{Q}}$$

becomes fully faithful when restricted to the tensor subcategory generated by M and $\mathbb{Q}(1)$.

PROPOSITION 2.36. *Let* (V, h_0) *be special. If Hypothesis 2.35 holds for* (V, h_0)*, then* (V, h_0) *is the Betti realization of a motive* M*, and all Hodge tensors on* M *are absolutely Hodge.*

PROOF. A Hodge tensor t of (V, h_0) defines a global Hodge tensor of $\mathbb{V}(N)$. Let U be a neighbourhood of o as in (2.35). There will be a point $x \in U$ such $\mathrm{Im}(h_x) \subset T_{\mathbb{R}}$ for some torus $T \subset G$, and the pair (V, h_x) will be the Betti realization of a CM-motive. Hence t_x is an absolute Hodge tensor, and, as we noted in (2.29), this implies that t_u is an absolute Hodge tensor for all $u \in U$. In particular, $t = t_0$ is an absolute Hodge tensor. ■

Hypothesis 2.35 is definitely a stronger statement than Hypothesis 1.13, but any proof of 1.13 is likely also to yield a proof of 2.35. Note that 2.35 implies that

$$\omega_B\colon \mathsf{Mot}^{\mathrm{sp}}(\mathbb{C}) \to \mathsf{Hdg}_{\mathbb{Q}}$$

is fully faithful, where $\mathsf{Mot}^{\mathrm{sp}}(\mathbb{C})$ is the category of special motives. Hence every special Hodge structure will be the Betti realization of a *unique* special motive (unique up to a unique isomorphism).

Moduli of motives

We now define the category of motives for absolute Hodge tensors over any smooth variety S in characteristic zero. Our definition is suggested by the following theorem: let S be a smooth connected variety with generic point η over a field of characteristic zero; the functor $A \rightsquigarrow A_\eta$ from the category of abelian schemes over S to the category of abelian varieties over η is fully faithful, with essential image the category of abelian varieties B such that the action of $\pi_1(\eta,\overline{\eta})$ on $H^1(B_{\mathrm{et}},\mathbb{Q}_\ell)$ factors through $\pi_1(S,\overline{\eta})$. Here $\overline{\eta}$ is the spectrum of an algebraically closed field containing $\kappa(\eta)$. (This theorem is a consequence of the theory of Néron models and of a theorem of Chai and Faltings[16]—see Milne 1992, 2.13, for a discussion of it.)

DEFINITION 2.37. A ***motive*** M over a smooth connected k-variety S is a motive M_η over the generic point η of S such that the action of $\pi_1(\eta,\overline{\eta})$ on $\omega_f(M_\eta)$ factors through $\pi_1(S,\overline{\eta})$. If M_η is an abelian motive, then we call M an ***abelian motive*** over S. (Here ω_f is the fibre functor, defined by étale cohomology, taking values in $\mathbb{A}_f$-modules.)

Let M be a motive over S. For some m, $M_\eta(-m)$ will be an effective motive, and hence a direct summand of a motive $h(Y)$, where Y is a smooth projective variety over the field $\kappa(\eta)$. Let p be the absolute Hodge tensor for Y projecting $h(Y)(m)$ onto M_η. For some open subset U of S, Y will extend to a smooth projective scheme Y_U over U, and p will extend to a global tensor p_U for the de Rham and étale cohomologies of Y_U/U. We then say that (Y_U, p_U, m) represents M over U, and we write $M|U = h(Y_U, p_U, m)$.

PROPOSITION 2.38. *For every smooth connected variety S over a field k of characteristic zero, the category $\mathsf{Mot}(S)$ of motives over S is a Tannakian category over $\mathbb{Q}$, and the category $\mathsf{Mot}^{\mathrm{ab}}(S)$ of abelian motives over S is a Tannakian subcategory of $\mathsf{Mot}(S)$. There is an exact tensor functor from $\mathsf{Mot}(S)$ to the Tannakian category of local systems of $\mathbb{A}_f$-modules on S_{et}.*

PROOF. Since $\mathsf{Mot}(\eta)$ is a Tannakian category, it suffices to prove that $\mathsf{Mot}(S)$ is a Tannakian subcategory of $\mathsf{Mot}(\eta)$, but this is obvious. Similarly $\mathsf{Mot}^{\mathrm{ab}}(S)$ is a Tannakian subcategory of $\mathsf{Mot}(S)$ and of $\mathsf{Mot}^{\mathrm{ab}}(\eta)$. To give a local system of $\mathbb{A}_f$-modules on S is the same as to give a continuous representation of $\pi_1(S,\overline{\eta})$ on a finite-dimensional $\mathbb{A}_f$-module, and, by assumption, the representation of $\pi_1(\eta,\overline{\eta})$ on $\omega_f(M_\eta)$ factors through $\pi_1(S,\overline{\eta})$. ∎

DEFINITION 2.39. An ***integral structure*** on a motive M is the choice of a local system of torsion-free $\widehat{\mathbb{Z}}$-modules $M(\widehat{\mathbb{Z}})$ such that $M(\widehat{\mathbb{Z}})\otimes\mathbb{Q} = M_f$. A motive together with an integral structure is an ***integral motive.***

As we have just seen, almost by definition, a motive over a smooth variety S defines a local system of $\mathbb{A}_f$-modules. Less obvious is that, when the ground field is $\mathbb{C}$, a motive defines a polarizable variation of Hodge structures on S.

[16] The "theorem" in question is the notorious Theorem 6.7 of Chapter V of Faltings and Chai 1990 and its Corollary 6.8, which are true in characteristic zero (this is all we need here), but not in general.

THEOREM 2.40. *Let M be a motive over a smooth algebraic variety S over $\mathbb{C}$. There exists a unique polarizable variation of Hodge structures $\mathcal{H}_B(M)$ on S with the following property: if (Y,p,m) represents M over the open subset U of S, then $\mathcal{H}_B(M) = \mathcal{H}_B(Y/U,p,m)$.*

PROOF. Let (Y,p,m) represent M over U, and let $\mathbb{V} = \mathcal{H}_B(Y/U)$. Choose a point $u \in U$. The action of $\pi_1(U,u)$ on $\mathbb{V}_u$ factors through $\pi_1(S,u)$, and so $\mathbb{V}$ extends (uniquely) to a local system of $\mathbb{Q}$-vector spaces on S, which we still denote $\mathbb{V}$. Thus we have a local system of $\mathbb{Q}$-vector spaces $\mathbb{V}$ on S, and the structure of polarizable variation of Hodge structures on $\mathbb{V}|U$. The next lemma shows that this structure extends uniquely to $\mathbb{V}$, which completes the proof. ∎

LEMMA 2.41. *Let S be a smooth algebraic variety over $\mathbb{C}$. Let $\mathbb{V}$ be a local system of $\mathbb{Q}$-vector spaces on S, and let ψ be a bilinear form on $\mathbb{V}$. Suppose that there is a Zariski open subset U of S and a filtration F on $\mathcal{O}_U \otimes \mathbb{V}$ such that $(\mathbb{V}|U, F, \psi|U)$ is a polarized variation of Hodge structures on U of some weight m. Then F extends uniquely to a filtration on $\mathbb{V} \otimes \mathcal{O}_V$ such that $(\mathbb{V}, F, \psi)$ is a polarized variation of Hodge structures on S.*

PROOF. There exists a Zariski open subvariety S' of S containing U and such that $S \smallsetminus S'$ has codimension ≥ 2 and $S' \smallsetminus U$ is smooth of pure codimension 1 (i.e., a smooth divisor). Thus it suffices to consider two cases: $S \smallsetminus U$ is a smooth divisor; $S \smallsetminus U$ has codimension 2. In the first case, the Hodge structure on $\mathbb{V}|\mathbb{U}$ will in general degenerate into a mixed Hodge structure on the boundary, but the description of the weight filtration in terms of the action of the local monodromy group shows that it is trivial (i.e., the mixed Hodge structure is pure) when the local monodromy group acts trivially. See (Schmid 1973, 4.12; Cattani et al 1986). For the second case, let D be the classifying space for polarized Hodge structures of the same type as (V, h_u, ψ), $u \in U$. Then $\mathbb{V}|\mathbb{U}$ defines a horizontal, locally liftable holomorphic mapping $U \to \Gamma \backslash D$, which (Griffiths and Schmid 1969, 9.8) shows extends to all of S (because D is "negatively curved in the horizontal direction"). From the extended map we obtain an extension of the variation of Hodge structures to S. ∎

PROPOSITION 2.42. *Let S be a smooth connected scheme over $\mathbb{C}$. The functor*

$$\mathcal{H}_B\colon \mathsf{Mot}^{\mathrm{ab}}(S) \to \mathsf{Hdg}_{\mathbb{Q}}(S)$$

defined in (2.41) is fully faithful with essential image the category of abelian-motivic variations of Hodge structures on S.

PROOF. Recall the following result (Deligne 1971a, 4.4.3): there is an equivalence between the category of abelian schemes on S and the category of torsion-free integral polarizable variations of Hodge structures on S of type $\{(-1,0),(0,-1)\}$. When we apply this to an open subvariety U of S, we find that there is an equivalence between the category of abelian motives on S whose restriction to U can be realized as a Tate twist of a factor of the motive of an abelian scheme over U and the category of abelian-motivic variations of Hodge structures on S whose restriction to U can be represented in the form $\mathcal{H}_B(Y/U,p,m)$ with Y an abelian scheme over U. Now take the union over all U. ∎

Let T be a smooth variety over $\mathbb{C}$, and consider triples $(M,\mathfrak{s},\eta)$ consisting of an abelian motive M over T, a family $\mathfrak{s}$ of Hodge tensors of M indexed by I, and a level-N structure on $\mathcal{H}_B(M)$. We define $\mathcal{M}(T)$ to be the set of triples $(M,\mathfrak{s},\eta)$ such that $(\mathcal{H}_B(M),\mathfrak{s},\eta) \in \mathcal{H}(T)$.

With the obvious notion of pull-back and isomorphism, we obtain a moduli problem $(\mathcal{M}, \approx)$ on the category of smooth varieties over $\mathbb{C}$.

Now assume that (V, h_0) is the Betti realization of an abelian motive. It follows from (2.26) and (2.8) that the elements of $\mathcal{H}(\mathbb{C})$ are also Betti realizations of abelian motives, and hence that H_B defines a bijection

$$H_B \colon \mathcal{M}(\mathbb{C})/\approx \to \mathcal{H}(\mathbb{C})/\approx .$$

THEOREM 2.43. *If (V, h_0) is the Betti realization of an abelian motive, then the pair $(S(N), \alpha \circ H_B)$ is a fine solution to the moduli problem $(\mathcal{M}, \approx)$.*

PROOF. Let T be a smooth variety over $\mathbb{C}$. It follows from (2.42) that

$$\mathcal{H}_B \colon \mathcal{M}(T)/\approx \to \mathcal{H}(T)/\approx$$

is injective, and from (2.34), (2.12), and (2.42) that it is surjective. Hence the theorem follows from (2.12). ■

REMARK 2.44. Theorem 2.43 realizes a vast family of varieties as moduli varieties. Except for the moduli varieties of abelian varieties, the only example I know of where this has been exploited is that in which the initial Hodge structure is the second cohomology group of a K3-surface (see for example Deligne 1972).

REMARK 2.45. The moduli problems for Hodge structures and motives have been defined above only for the category of smooth varieties over $\mathbb{C}$ (see 2.15 and 2.37). Ching-Li Chai has suggested to me that they can be defined for all schemes over $\mathbb{C}$ by replacing the connections with Grothendieck's notion of a stratification (Grothendieck 1968). A ***stratification*** on an $\mathcal{O}_S$-module $\mathcal{V}$ is an isomorphism

$$\varphi \colon p_1^* \mathcal{V} \longrightarrow p_2^* \mathcal{V},$$

where p_1, p_2 are the projections $\hat{\Delta} \rightrightarrows S$ from the formal completion of the diagonal in $S \times S$ to S, satisfying the cocycle condition $p_{31}^*(\varphi) = p_{32}^*(\varphi) p_{21}^*(\varphi)$. Each of $p_1^* \mathcal{V}$ and $p_2^* \mathcal{V}$ has the product filtration, that on $\mathcal{O}_{\hat{\Delta}}$ being given by the defining ideal and its powers, and in this context Griffiths transversality (2.15.2) becomes the condition that φ preserves the filtrations. It should be possible to prove that $S(N)$ solves the moduli problems on the category of all schemes by using the methods of Artin. A similar remark applies to Theorem 3.31 below.

3 Shimura Varieties as Moduli Varieties

In the last two sections we saw that the study of Mumford-Tate groups and the moduli varieties of motives leads to the consideration of pairs (G, h) satisfying certain conditions (SV). In this section, we reverse the process: starting with a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ satisfying conditions (SV), we construct a pro-variety $\mathrm{Sh}(G, X)$, the Shimura variety defined by (G, X), and show that in many cases $\mathrm{Sh}(G, X)$ can be realized as a moduli variety for motives over a number field. Then we give some applications of this result.

Review of Shimura varieties

Let G be a connected reductive group over $\mathbb{Q}$, and let X be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon\mathbb{S}\to G_{\mathbb{R}}$ satisfying the conditions (SV1), (SV2), and

(SV3) G^{ad} has no $\mathbb{Q}$-factor that is anisotropic over $\mathbb{R}$.

Note that it suffices to verify the conditions for a single h, because the map $\mathrm{ad}(g)\colon G_{\mathbb{R}}\to G_{\mathbb{R}}$ is an isomorphism carrying h into $\mathrm{ad}(g)\circ h$. The condition (SV1) implies that the image of w_h is contained in $Z(G)$, and hence is independent of $h\in X$—we denote it by w_X and call it the ***weight.*** It is convenient to impose the following condition on the group G:

(SV6) [17]the torus $Z(G)^{\circ}$ splits over a CM-field.

Recall that a ***CM-field*** is a finite extension E of $\mathbb{Q}$ admitting a nontrivial involution ι_E such that $\rho(\iota_E z)=\iota\rho(z)$ for all embeddings $\rho\colon E\hookrightarrow\mathbb{C}$. If $X^*(Z^{\circ})$ denotes the group of characters of Z° defined over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, then Z° splits over a CM-field if and only if, for all $\tau\in\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\iota\tau$ and $\tau\iota$ have the same action on $X^*(Z^{\circ})$. Since Z° and G^{ab} are isogenous tori, the condition is equivalent to G^{ab} splitting over a CM-field, and in the presence of (SV2), it implies that G itself splits over a CM-field.

The condition on G^{ad} implies that the strong approximation theorem holds in the following form: the group $G^{\mathrm{sc}}(\mathbb{Q})$ is dense in $G^{\mathrm{sc}}(\mathbb{A}_f)$. This simplifies the theory, but unfortunately eliminates zero-dimensional Shimura varieties except for those defined by tori.

Fix a pair (G,X) satisfying the conditions (SV1,2,3,6). Then X has a unique complex structure for which the Hodge filtrations F_h on $V\otimes\mathbb{C}$ vary holomorphically (Deligne 1979, 1.1.14; cf. §2). Moreover, X has only finitely many connected components, and each is a symmetric hermitian domain (Deligne 1979, 1.1.17; cf. 2.23). As before, I write x for an element of X, and h_x, μ_x for the corresponding homomorphisms.

If K is a compact open subgroup of $G(\mathbb{A}_f)$, then $\Gamma(K)\overset{\mathrm{def}}{=}G(\mathbb{Q})\cap K$ is (by definition) a congruence subgroup of $G(\mathbb{Q})$, and hence its image $\Gamma^{\mathrm{ad}}(K)$ in $G^{\mathrm{ad}}(\mathbb{Q})$ is an arithmetic group. For K sufficiently small, $\Gamma(K)$ is contained in $G(\mathbb{R})^+$ (Deligne 1979, 2.0.14) and $\Gamma^{\mathrm{ad}}(K)$ is torsion-free (cf. 2.9).

For K a compact open subgroup of $G(\mathbb{A}_f)$, define

$$\mathrm{Sh}_K(G,X)=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K,$$

where $G(\mathbb{Q})$ and K act on $X\times G(\mathbb{A}_f)$ according to the rule

$$q(x,a)k=(qx,qak),\quad q\in G(\mathbb{Q}),\quad x\in X,\quad a\in G(\mathbb{A}_f),\quad k\in K.$$

Let $G(\mathbb{Q})_+$ be the subgroup of $G(\mathbb{Q})$ of elements mapping into $G^{\mathrm{ad}}(\mathbb{R})^+$; it is the stabilizer in $G(\mathbb{Q})$ of any connected component X^+ of X (Deligne 1979, 1.2.7). Let $\mathcal{C}$ be a set of representatives for $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$—the strong approximation theorem implies that it is finite. For K sufficiently small, the map

$$\bigsqcup_{c\in\mathcal{C}}\Gamma^{\mathrm{ad}}(cKc^{-1})\backslash X^+\longrightarrow G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K$$

sending an element $[x]\in\Gamma^{\mathrm{ad}}(cKc^{-1})\backslash X^+$ to $[x,c]$ is a homeomorphism. Therefore (see 2.24) $\mathrm{Sh}_K(G,X)$ has a unique algebraic structure compatible with its complex structure;

[17]Added 2017. I have changed the numbering to agree with that in my article *Introduction to Shimura varieties* (2005).

moreover, for every smooth algebraic variety T over $\mathbb{C}$, every holomorphic map $T(\mathbb{C}) \to \mathrm{Sh}_K(G,X)$ is algebraic.

From now on, we regard $\mathrm{Sh}_K(G,X)$ as an algebraic variety.

When we vary K among (small) compact open subgroups of $G(\mathbb{A}_f)$, we obtain a filtered projective system of algebraic varieties

$$\mathrm{Sh}(G,X) = (\mathrm{Sh}_K(G,X))_K.$$

The group $G(\mathbb{A}_f)$ acts continuously on this system in the sense of (Deligne 1979, 2.7.1), and $\mathrm{Sh}(G,X)$, together with this action, is called the ***Shimura variety*** defined by (G,X).

Alternatively (and equivalently), we can define $\mathrm{Sh}(G,X)$ to be the projective limit,

$$\mathrm{Sh}(G,X) = \varprojlim \mathrm{Sh}_K(G,X) \qquad \text{(scheme, not of finite type, over } \mathbb{C}\text{)},$$

together with the action of $G(\mathbb{A}_f)$. The variety $\mathrm{Sh}_K(G,X)$ can be recovered as the quotient $\mathrm{Sh}(G,X)/K$ of $\mathrm{Sh}(G,X)$.

Recall (Deligne 1979, 2.1.10; see also 4.11 below) that

$$\mathrm{Sh}(G,X)(\mathbb{C}) \overset{\text{def}}{=} \varprojlim \mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{Q})^-$$

where $Z = Z(G)$ and $Z(\mathbb{Q})^-$ is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$. An element $g \in G(\mathbb{A}_f)$ acts on $\mathrm{Sh}(G,X)(\mathbb{C})$ as follows:

$$[x,a]g = [x,ag], \quad x \in X, \quad a \in G(\mathbb{A}_f).$$

Because Z° splits over a CM-field, the largest split subtorus of $Z_\mathbb{R}$ is defined over $\mathbb{Q}$. When this subtorus is also split over $\mathbb{Q}$, $Z(\mathbb{Q})$ is closed in $Z(\mathbb{A}_f)$ and we have

$$\mathrm{Sh}(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f).$$

EXAMPLE 3.1. Let (G,h) be the Mumford-Tate group of a special Hodge structure (V,h). We saw in (1.6) that h satisfies (SV2*) and, ***a fortiori***, (SV2) and that the weight w_h is defined over $\mathbb{Q}$. If H is a factor of G^{ad} such that $H_\mathbb{R}$ is anisotropic, then (SV2) implies that the composite $\mathbb{S} \xrightarrow{h} G_\mathbb{R} \longrightarrow H_\mathbb{R}$ is trivial, and since h generates H, H itself must be trivial. The pair $(G^{\mathrm{ab}}, h^{\mathrm{ab}})$ is the Mumford-Tate group of a Hodge structure of CM-type, and therefore G^{ab} splits over a CM-field. Thus G satisfies (SV3,6), and, by assumption, (G,h) satisfies (SV1). Therefore, when we define X to be the $G(\mathbb{R})$-conjugacy class of h, (G,X) satisfies the conditions to define a Shimura variety. Choose a lattice $V(\mathbb{Z})$ in V, and let

$$K(N) = \{g \in G(\mathbb{A}_f) \mid gV(\widehat{\mathbb{Z}}) = V(\widehat{\mathbb{Z}}), \quad g = \mathrm{id} \text{ on } V(\widehat{\mathbb{Z}})/NV(\widehat{\mathbb{Z}})\}.$$

Then the variety $S(N)$ attached to (V,h) in §2 is an open and closed subvariety of $\mathrm{Sh}_{K(N)}(G,X)$.

DEFINITION 3.2. If G^{der} satisfies the condition (1.25), then the Shimura variety $\mathrm{Sh}(G,X)$ is said to be of ***abelian type***.[18]

[18]This is precisely the class of Shimura varieties for which Deligne proved the existence of canonical models in his Corvallis article (Deligne 1979, 2.7.21). The name was coined by Shih and the author (Milne and Shih 1982) because, at the time, they seemed to be exactly the Shimura varieties that were approachable by methods involving the moduli of abelian varieties. Below we shall see a much more compelling justification for the name: among the Shimura varieties whose weight is defined over $\mathbb{Q}$, they are the varieties that are moduli varieties for abelian motives.

REMARK 3.3. Let (G,h) be the Mumford-Tate group of an abelian motive, and let X be the $G(\mathbb{R})$-conjugacy class of h; then $\mathrm{Sh}(G,X)$ is a Shimura variety of abelian type, and w_X is defined over $\mathbb{Q}$. Conversely, let $\mathrm{Sh}(G,X)$ be a Shimura variety of abelian type whose weight is defined over $\mathbb{Q}$; if G' is the subgroup of G generated by the elements of X, then there is an $h \in X$ such that (G',h) is the Mumford-Tate group of an abelian motive.

CANONICAL MODELS

We recall the notion of the canonical model of a Shimura variety. Let T be a torus over $\mathbb{Q}$ that splits over a CM-field, and let $\mu \in X_*(T)$ (group of cocharacters defined over $\mathbb{Q}^{\mathrm{al}} \subset \mathbb{C}$). The pair (T,h), $h(z) = \mu(z) \cdot \overline{\mu(z)}$, defines a Shimura variety. Let $E = E(T,h) \subset \mathbb{Q}^{\mathrm{al}}$ be the field of definition of μ, let E^{ab} be the maximal abelian extension of E (in $\mathbb{Q}^{\mathrm{al}}$), and let rec_E be the Artin reciprocity map $\mathbb{A}_E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$. On applying the restriction of scalars functor $\mathrm{Res}_{E/\mathbb{Q}}$ to the homomorphism $\mu\colon \mathbb{G}_{mE} \to T_E$ and composing with the norm map, we obtain a homomorphism

$$N_h\colon \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \xrightarrow{\mathrm{Res}(\mu)} \mathrm{Res}_{E/\mathbb{Q}} T_E \xrightarrow{\mathrm{Nm}} T.$$

For every $\mathbb{Q}$-algebra R, this gives a homomorphism

$$N_h(R)\colon (E \otimes R)^\times \longrightarrow T(R).$$

Let $T(\mathbb{Q})^-$ be the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$. The ***reciprocity map***[19]

$$r(T,h)\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \longrightarrow T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

is defined as follows: let $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, and let $s \in \mathbb{A}_E^\times$ be such that $\mathrm{rec}_E(s) = \tau$; write $s = s_\infty \cdot s_f$ with $s_\infty \in (E \otimes \mathbb{R})^\times$ and $s_f \in (E \otimes \mathbb{A}_f)^\times$; then $r(T,x)(\tau) = N_h(s_f)$ (mod $T(\mathbb{Q})^-$).

Now consider a Shimura variety $\mathrm{Sh}(G,X)$. The ***reflex field*** $E(G,X)$ of $\mathrm{Sh}(G,X)$ is the field of definition (in $\mathbb{C}$) of the $G(\mathbb{C})$-conjugacy class of homomorphisms $\mu\colon \mathbb{G}_m \to G_{\mathbb{C}}$ containing μ_h, $h \in X$. A ***special pair*** (T,h) in (G,X) is a torus $T \subset G$ together with an $h \in X$ such that h factors through $T_{\mathbb{R}}$. Clearly $E(T,h) \supset E(G,X)$.

By a ***model*** of $\mathrm{Sh}(G,X)$ over a subfield k of $\mathbb{C}$, we mean a scheme S over k endowed with an action of $G(\mathbb{A}_f)$ (defined over k) and a $G(\mathbb{A}_f)$-equivariant isomorphism $\mathrm{Sh}(G,X) \to S \otimes_k \mathbb{C}$. We use this isomorphism to identify $\mathrm{Sh}(G,X)(\mathbb{C})$ with $S(\mathbb{C})$.

DEFINITION 3.4. A model of $\mathrm{Sh}(G,X)$ over a number field E, $\mathbb{C} \supset E \supset E(G,X)$, is said to be ***canonical*** if it has the following property: for all special pairs $(T,h) \subset (G,X)$ and elements $a \in G(\mathbb{A}_f)$, the point $[h,a]$ is rational over $E(h)^{\mathrm{ab}}$ and $\tau \in \mathrm{Gal}(E(h)^{\mathrm{ab}}/E(h))$ acts on $[h,a]$ according to the rule:

$$\tau[h,a] = [h, r(\tau)\cdot a], \quad \text{where } r = r(T,h); \tag{3.4.1}$$

here $E(h) = E \cdot E(T,h)$.

PROPOSITION 3.5. *Let $f\colon G \to G'$ be a homomorphism mapping X into X', and suppose that $\mathrm{Sh}(G,X)$ and $\mathrm{Sh}(G',X')$ have canonical models over E. Then the morphism*

$$[x,g] \mapsto [f(x), f(g)]\colon \mathrm{Sh}(G,X) \to \mathrm{Sh}(G',X')$$

is defined over E.

[19] For an explanation of the sign, which differs from that in Deligne 1979, see Milne 1992, 1.10.

PROOF. See Deligne 1971b, 5.4. ■

COROLLARY 3.6. *If it exists, the canonical model of* $\mathrm{Sh}(G,X)$ *over* E *is uniquely determined up to a unique isomorphism.*

PROOF. Apply the proposition with f the identity map $G \to G$. ■

In his report on Shimura's work, Deligne (1971b) proves that Shimura varieties that are moduli varieties for abelian varieties have canonical models over their reflex fields, and he deduces a similar result for one class of Shimura varieties whose members are not moduli varieties and, in fact, do not have weight defined over $\mathbb{Q}$ ("les modèles étranges", ibid. §6). In the next subsection, we prove that Shimura varieties of abelian type with rational weight are moduli varieties for abelian ***motives***. This allows us in the following subsection to prove, using the methods of Deligne's article, that all Shimura varieties of abelian type have canonical models over their reflex fields[20].

Shimura varieties as moduli varieties over $\mathbb{C}$

Throughout this subsection, $\mathrm{Sh}(G,X)$ is a Shimura variety whose weight w_X is defined over $\mathbb{Q}$. For simplicity, we assume that there is given a homomorphism $t\colon G \to \mathbb{G}_m = \mathrm{GL}(\mathbb{Q}(1))$ such that $t \circ w_X = -2$. Then $t_{\mathbb{R}} \circ h_x$ defines on $\mathbb{Q}(1)$ its usual Hodge structure for all $x \in X$.

The realization of $\mathrm{Sh}(G,X)$ as a moduli variety depends on the choice of a faithful representation $\xi\colon G \hookrightarrow \mathrm{GL}(V)$ of G. We fix such a ξ and identify G with a subgroup of $\mathrm{GL}(V)$. There will be a family of tensors $\mathfrak{t} = (t_i)_{i \in I}$, $t_i \in V^{\otimes r_i} \otimes V^{\vee \otimes s_i}$ some r_i, s_i, such that, for every $\mathbb{Q}$-algebra R,

$$G(R) = \{g \in \mathrm{GL}(V \otimes R) \mid g t_i = t_i, \quad \text{all } i \in I\}.$$

For all $x \in X$, t_i will be fixed by $h_x(\mathbb{S})$, and so t_i is a Hodge tensor for the rational Hodge structure (V, h_x). For some r and s, the representation t of G on $\mathbb{Q}(1)$ will be a direct summand of the representation of G on $V^{\otimes r} \otimes V^{\vee \otimes s}$ defined by ξ. Thus it makes sense to add the requirement that there is an element $0 \in I$ such that t_0 or $-t_0$ is a polarization for (V, h_x), all $x \in X$.

Fix a (small) compact open subgroup K of $G(\mathbb{A}_f)$, and let (W,h) be a rational Hodge structure. Then K acts on the space of $\mathbb{A}_f$-linear isomorphisms $V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ on the right, and an orbit for the action is called a ***K-level structure*** on (W,h).

EXAMPLE 3.7. Choose lattices $V(\mathbb{Z})$ and $W(\mathbb{Z})$ in V and W, and define $K(N)$ as in (3.1). To give a $K(N)$-level structure on W is the same as to give a level N-structure in the sense of §2.

Consider triples $(W, \mathfrak{s}, [\eta])$ consisting of a rational Hodge structure $W = (W,h)$, a family $\mathfrak{s}$ of Hodge cycles indexed by I, and a K-level structure $[\eta]$ on (W,h). We define $\mathcal{H}_K(G,X,\xi)$ to be the set of such triples satisfying the following conditions:

(3.8.1) there exists an isomorphism of $\mathbb{Q}$-vector spaces $\beta\colon W \to V$ mapping each s_i to t_i and sending h to h_x, some $x \in X$;

[20] As noted in a previous footnote, the existence of canonical models for Shimura varieties of abelian type was proved in Deligne 1979, but by less explicit methods involving connected Shimura varieties. The result was extended to all Shimura varieties in Milne 1983 and in Borovoi 1983/4, 1987.

(3.8.2) for one (hence every) η representing the level structure, η maps each t_i to s_i.

An isomorphism from one such triple $(W,\mathfrak{s},[\eta])$ to a second $(W',\mathfrak{s}',[\eta'])$ is an isomorphism $\gamma\colon(W,h)\to(W',h')$ of rational Hodge structures mapping each s_i to s_i' and such that $[\alpha\circ\eta]=[\eta']$. An element $g\in G(\mathbb{A}_f)$ defines a map

$$(W,\mathfrak{s},[\eta])\mapsto(W,\mathfrak{s},[\eta\circ g])\colon\mathcal{H}_K(G,X,\xi)\longrightarrow\mathcal{H}_{g^{-1}Kg}(G,X,\xi).$$

Let $(W,\mathfrak{s},[\eta])$ be an element of $\mathcal{H}_K(G,X,\xi)$. Choose an isomorphism $\beta\colon W\to V$ satisfying (3.8.1), so that β sends h to h_x some $x\in X$. The composite

$$V(\mathbb{A}_f)\xrightarrow{\eta}W(\mathbb{A}_f)\xrightarrow{\beta}V(\mathbb{A}_f),\quad \eta\in[\eta],$$

sends each t_i to t_i, and it is therefore multiplication by an element $g\in G(\mathbb{A}_f)$, well-defined up to multiplication on the right by an element of K (corresponding to a different choice of the representative η of the level structure). Since every other choice of β is of the form $q\circ\beta$ for some $q\in G(\mathbb{Q})$, $[x,g]$ is a well-defined element of $G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K=\mathrm{Sh}_K(G,X)(\mathbb{C})$.

PROPOSITION 3.9. *The above construction defines a bijection*

$$\alpha_K\colon\mathcal{H}_K(G,X,\xi)/{\approx}\longrightarrow\mathrm{Sh}_K(G,X)(\mathbb{C}).$$

The maps α_K are compatible with the action of $G(\mathbb{A}_f)$, in the sense that, for every $g\in G(\mathbb{A}_f)$, there is a commutative diagram

$$\begin{array}{ccc}\mathcal{H}_K(G,X,\xi)/{\approx} & \xrightarrow{\alpha_K} & \mathrm{Sh}_K(G,X)\\ \big\downarrow g & & \big\downarrow g\\ \mathcal{H}_{g^{-1}Kg}(G,X,\xi)/{\approx} & \xrightarrow{\alpha_{g^{-1}Kg}} & \mathrm{Sh}_{g^{-1}Kg}(G,X).\end{array}$$

PROOF. Straightforward, and essentially the same as that of (2.8). ∎

We now fix (G,X) and ξ and drop them from the notation.

Let $\mathbb{V}$ be a variation of Hodge structures on a smooth algebraic variety T over $\mathbb{C}$. Because in our definition (2.15) we required $\mathbb{V}$ to admit an integal structure, there is a well-defined local system of $\mathbb{A}_f$-modules $\mathbb{W}$ for the étale topology on T such that, for every connected component T° of T and any $o\in T^\circ(\mathbb{C})$, $\mathbb{W}_o=V_o\otimes_{\mathbb{Q}}\mathbb{A}_f$ as $\pi_1(T^\circ(\mathbb{C}),o)$-modules. We denote this étale sheaf by $\mathbb{V}(\mathbb{A}_f)$.

A ***K-level structure*** on a local system $\mathbb{W}(\mathbb{A}_f)$ of $\mathbb{A}_f$-modules on T_{et} is a K-equivalence class of isomorphisms $\eta\colon V(\mathbb{A}_f)_T\to\mathbb{W}(\mathbb{A}_f)$ on T. Here $V(\mathbb{A}_f)_T$ is the constant local system defined by the $\mathbb{A}_f$-module $V(\mathbb{A}_f)$. The class $[\eta]$ is required to be defined on T_{et}, not its individual members, which may only be defined on the universal covering of T. If T is connected and $o\in T$, then to give a K-level structure on $\mathbb{W}(\mathbb{A}_f)$ is the same as to give K-level structure on $\mathbb{W}_o$ that is stable under the action of $\pi_1(T_{\text{et}},o)$ (algebraic fundamental group).

Consider triples $(\mathbb{W},\mathfrak{s},[\eta])$ consisting of a polarizable variation of Hodge structures $\mathbb{W}$ on T, a family of global Hodge tensors $\mathfrak{s}$ of $\mathbb{W}$ indexed by I, and a K-level structure $[\eta]$ on $\mathbb{W}(\mathbb{A}_f)$. We define $\mathcal{H}_K(T)$ to be the set of such triples having the property that, for all $t\in T$, $(\mathbb{V}_t,\mathfrak{s}_t,[\eta_t])$ lies in $\mathcal{H}_K(\mathbb{C})$. With the obvious notions of isomorphism and pull-back, $\mathcal{H}_K$ is a moduli problem on the category of smooth algebraic varieties over $\mathbb{C}$, and $\mathcal{H}_K(\text{point})=\mathcal{H}_K(G,X,\xi)$.

PROPOSITION 3.10. *With the above notations,* $(\mathrm{Sh}_K(G,X),\alpha_K)$ *is a solution to the moduli problem* $(\mathcal{H}_K,\approx)$.

PROOF. Let $m \in \mathcal{H}_K(T)$. The same argument as in the proof of (2.12) shows that $m \mapsto \alpha(m_t)\colon T \to \mathrm{Sh}_K(G,X)(\mathbb{C})$ is holomorphic, and hence is a morphism of algebraic varieties. As in (2.11), on each connected component $S_c = \Gamma^{\mathrm{ad}}(cKc^{-1})\backslash X^+$ of $\mathrm{Sh}_K(G,X)(\mathbb{C})$, there is an $m_c \in \mathcal{H}_K(S_c)$ such that $\alpha((m_c)_s) = s$ for all $s \in S_c(\mathbb{C})$, and so $(\mathrm{Sh}_K(G,X),\alpha_K)$ satisfies the conditions (2.1). ∎

REMARK 3.11. If the largest $\mathbb{R}$-split subtorus of Z is already split over $\mathbb{Q}$, then $Z(\mathbb{Q})$ is closed in $Z(\mathbb{A}_f)$, and the moduli problem is fine. More precisely, there is an element $m_0 \in \mathcal{H}(\mathrm{Sh}_K(G,X))$ such that $(\mathrm{Sh}_K(G,X),[m_0])$ represents the functor $\mathcal{H}_K/\approx$.

The next proposition and theorem show that $\mathrm{Sh}(G,X)$ is a moduli variety for abelian motives if and only if it is of abelian type. (Recall that we are assuming that w_X is defined over $\mathbb{Q}$.)

PROPOSITION 3.12. *The elements of* $\mathcal{H}_K(\mathbb{C})$ *are the Betti realizations of abelian motives if and only if* $\mathrm{Sh}(G,X)$ *is of abelian type. When this is the case, then, for each connected component* S_c *of* $\mathrm{Sh}(G,X)$, *the element* $m_c \in \mathcal{H}_K(S_c)$ *defined in the proof of (3.10) is abelian-motivic.*

PROOF. Let G' be the $\mathbb{Q}$-subgroup of G generated by $\{h_x \mid x \in X\}$. The second part of condition (SV3,6) implies that $\{\mathrm{ad}\circ h_x \mid x \in X\}$ generates G^{ad}, and therefore $G'/G' \cap Z(G) = G^{\mathrm{ad}}$. Hence G'^{der} is of finite index in G^{der}, and, being connected, the two groups are equal. Proposition 1.38 applied to the holomorphic family of Hodge structures (V,h_x) on a connected component X^+ of X shows that G' is the Mumford-Tate group of (V,h_o) for some $o \in X^+$. If (V,h_o) is the Betti realization of an abelian motive, then (1.27) shows that G'^{der} satisfies (1.25), and therefore that $\mathrm{Sh}(G,X)$ is of abelian type. Conversely, if $\mathrm{Sh}(G,X)$ is of abelian type, then (1.27) shows that (V,h_o) is the Betti realization of an abelian motive, and the same argument as in the proof of (2.26) then shows that the same is true of (V,h_x) for every $x \in X$. The proof of the last statement is the same as that of (2.34). ∎

For any smooth variety T over $\mathbb{C}$, define $\mathcal{M}_K(T)$ to be the set of triples $(M,\mathfrak{s},[\eta])$ consisting of an abelian motive M over T, a family of tensors $\mathfrak{s}$ of M indexed by I, and a level K-structure $[\eta]$ on M, such that the Betti realization of the triple lies in $\mathcal{H}_K(T)$. With the obvious notions of pull-back and isomorphism, $\mathcal{M}_K$ becomes a moduli problem on the category of smooth algebraic varieties over $\mathbb{C}$.

THEOREM 3.13. *Let* $\mathrm{Sh}(G,X)$ *be a Shimura variety of abelian type whose weight is defined over* $\mathbb{Q}$ *and for which there exists a homomorphism* $t\colon G \to \mathbb{G}_m$ *such that* $t \circ w_X = -2$. *For every representation* $\xi\colon G \hookrightarrow \mathrm{GL}(V)$ *possessing a fixed tensor* t_0 *such that* $\pm t_0$ *is a polarization of* $(V,\xi_{\mathbb{R}}\circ h_x)$ *for all* $x \in X$, $(Sh_K(G,X),\alpha_K)$ *is a solution of the moduli problem* $(\mathcal{M}_K,\approx)$. *When* $Z(\mathbb{Q})$ *is discrete in* $Z(\mathbb{A}_f)$, *there is an element* $m_0 \in \mathcal{M}_K(\mathrm{Sh}_K(G,X))$ *such that* $(\mathrm{Sh}(G,X),[m_0])$ *represents the functor* $\mathcal{M}_K/\approx$.

PROOF. Propositions 2.42 and 3.12 show that the map sending an element of $\mathcal{M}_K(T)$ to its Betti realization defines an isomorphism of moduli problems $(\mathcal{M}_K,\approx) \to (\mathcal{H}_K,\approx)$. Thus the theorem follows from (3.10) and (3.11). ∎

REMARK 3.14. When we assume Hypothesis 2.35, the same arguments show that every Shimura variety whose weight is defined over $\mathbb{Q}$ is a moduli variety for special motives.

Canonical models of Shimura varieties of abelian type

In this subsection, we prove the existence of canonical models for Shimura varieties of abelian type. For those whose weight is defined over $\mathbb{Q}$, we realize the canonical model as a moduli variety.

MOTIVES OF CM-TYPE

Let M be a motive of CM-type over $\mathbb{C}$, and let (T,h) be the extended Mumford-Tate group of M. Then $T(\mathbb{Q})$ acts on $H_B(M)$, and $T(\mathbb{A}_f)$ acts on $\omega_f(M)$. Associated with the pair (T,h), we have the reciprocity map

$$r(T,h)\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \longrightarrow T(\mathbb{A}_f)/T(\mathbb{Q}).$$

As was noted in (Milne 1994, 4.7), we can regard M and any Hodge tensor on it as being defined over $\mathbb{Q}^{\mathrm{al}}$.

THEOREM 3.15. *Let $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E(T,h))$. For every representative $\tilde{r}(\tau) \in T(\mathbb{A}_f)$ of $r(T,h)(\tau)$, there exists a unique isomorphism $\gamma\colon M \to \tau M$ such that*

(a) *for all Hodge tensors s on M, $\gamma(s) = \tau s$;*

(b) *for all $v \in \omega_f(M)$, $\tau v = \gamma_f(\tilde{r}(\tau)v)$.*

PROOF. When M is an abelian variety and the Hodge tensors are endomorphisms or a polarization, this theorem essentially goes back to Shimura and Taniyama 1961. For a discussion of a stronger result (due to Deligne and Langlands) see Milne 1990, I.5. ∎

GENERALIZED SIEGEL MODULAR VARIETIES

We first treat the varieties that play the same role for abelian motives that the Siegel modular varieties play for abelian varieties.

Let M be an abelian motive, and let N be the direct sum of M with the Tate motive. Let $H_B(N) = (V, h_0)$ and let t_0 be a polarization of M, which we can identify with a polarization of $H_B(M)$. Define G to be the subgroup of $\mathrm{GL}(V)$ of elements g such that

(3.16.1) g centralizes w_{h_0};

(3.16.2) g preserves the decomposition $V = H_B(M) \oplus \mathbb{Q}(1)$;

(3.16.3) $gt_0 = t_0$.

The second condition implies that $G \subset \mathrm{GL}(H_B(M)) \times \mathbb{G}_m$, and we write t for the projection of G onto $\mathbb{G}_m$. In (3.16.3), G is to be understood as acting on $\mathbb{Q}(1)$ through t. Let X be the $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to G_{\mathbb{R}}$ containing h_0. Then (cf. 2.6a) X consists of all Hodge structures h on V such that:

(3.17.1) the weight gradation $V = \bigoplus_{m\in\mathbb{Z}} V_m$ defined by h is the same as that of h_0;

(3.17.2) for each $m \in \mathbb{Z}$, the Hodge structure on V_m defined by h has the same Hodge numbers as that of h_0;

(3.17.3) the projection of V onto $\mathbb{Q}(1)$ is a morphism of Hodge structures relative to h, and $\pm t_0$ is a polarization of $\mathrm{Ker}(V \to \mathbb{Q}(1))$.

The pair (G,X) satisfies the conditions (SV1,2*,3,4,6). In particular, it defines a Shimura variety $\mathrm{Sh}(G,X)$.

REMARK 3.18. Let $W = H_B(M)$, and let $W = \bigoplus_{m\in\mathbb{Z}} W_m$ be its weight gradation. Under the decomposition, t_0 corresponds to a family (t_m) with t_m a polarization of W_m. Let $\psi_m = (2\pi i)^m t_m$. Then ψ_m is a pairing $V_m \times V_m \to \mathbb{Q}$ satisfying the conditions (2.4), and $\operatorname{Ker}(G \to \mathbb{G}_m)$ is a product of groups G_m, $m \in \mathbb{Z}$, with

$$G_m(\mathbb{Q}) = \{g \in \mathrm{GL}(W_m) \mid g\psi_m = \psi_m\}.$$

In particular, $\operatorname{Ker}(G \to \mathbb{G}_m)$ is a product of symplectic and orthogonal groups.

EXAMPLE 3.19. Let A be an abelian variety over $\mathbb{C}$, and let t_0 be a polarization of A. Identify t_0 with a polarization of $W \overset{\text{def}}{=} H_B(A) \overset{\text{def}}{=} H_1(A,\mathbb{Q})$, and let $\psi = (2\pi i)^{-1} t_0$. The projection $G \to \mathrm{GL}(W)$ identifies G with the group of symplectic similitudes of the symplectic space (W,ψ), and X is the Siegel double space of all real Hodge structures of type $\{(-1,0),(0,-1)\}$ on W for which $\pm t_0$ is a polarization. Thus $\mathrm{Sh}(G,X)$ is the Siegel modular variety.

LEMMA 3.20. *The reflex field* $E(G,X) = \mathbb{Q}$.

PROOF. Let $V = \bigoplus_{m\in\mathbb{Z}} V_m$ be the weight gradation of h_0 and let $V(\mathbb{C}) = V^{p,q}$ be the Hodge decomposition of h_0. The $G(\mathbb{C})$-conjugacy class of μ_{h_0} can be identified with the set of gradations $V(\mathbb{C}) = \bigoplus_p V^p$ of $V(\mathbb{C})$ such that

$$\begin{aligned}
&V_m(\mathbb{C}) = \bigoplus\nolimits_p V_m(\mathbb{C}) \cap V^p \\
&\dim(V_m(\mathbb{C}) \cap V^p) = \dim V^{p,m-p}, \text{ and} \\
&\psi(V_m(\mathbb{C}) \cap V^p, V_m(\mathbb{C}) \cap V^{p'}) = 0, \quad p + p' \neq m.
\end{aligned}$$

Since both the weight gradation and ψ are defined over $\mathbb{Q}$, so is this set, which shows that $E(G,X) = \mathbb{Q}$. ∎

THEOREM 3.21. *The Shimura variety* $\mathrm{Sh}(G,X)$ *has a canonical model over* $\mathbb{Q}$.

The proof will occupy the rest of this subsubsection. The centre of G is $\mathbb{G}_m$, and $\mathbb{Q}^\times$ is discrete in $\mathbb{A}_f^\times$. Hence (see 3.13), there is an element $m_0 \in \mathcal{M}(\mathrm{Sh}_K(G,X))$ with the following property: for every $m \in \mathcal{M}_K(T)$, there is a unique morphism $\gamma\colon T \to \mathrm{Sh}_K(G,X)$ such that $\gamma^* m_0 \approx m$. For every automorphism τ of $\mathrm{Aut}(\mathbb{C})$, $\tau m_0 \in \mathcal{M}(\tau\,\mathrm{Sh}(G,X))$, and so there is a unique morphism

$$\gamma_\tau\colon \tau\,\mathrm{Sh}(G,X) \longrightarrow \mathrm{Sh}(G,X)$$

such that $\gamma_\tau^* m_0 \approx \tau m_0$.

LEMMA 3.22. *For all* $\sigma,\tau \in \mathrm{Aut}(\mathbb{C})$, $\gamma_\sigma \circ \sigma\gamma_\tau = \gamma_{\sigma\tau}$.

PROOF. The composite $\gamma_\sigma \circ \sigma\gamma_\tau$ is a map $\sigma\tau\,\mathrm{Sh}(G,X) \to \mathrm{Sh}(G,X)$ with the property that

$$(\gamma_\sigma \circ \sigma\gamma_\tau)^* m_0 = (\sigma\gamma_\tau)^* \circ \gamma_\sigma^* m_0 \approx (\sigma\gamma_\tau)^*(\sigma m_0) = \sigma(\gamma_\tau^* m_0) \approx \sigma\tau m_0.$$

It therefore equals $\gamma_{\sigma\tau}$. ∎

We shall need to use some descent theory. Let V be a variety over an algebraically closed field Ω of characteristic zero, and let k be a subfield of Ω. A family of isomorphisms $\gamma_\sigma\colon\sigma V \to V$ indexed by the elements σ of $\mathrm{Aut}(\Omega/k)$ is a ***descent system*** if $\gamma_\sigma\circ\sigma\gamma_\tau = \gamma_{\sigma\tau}$ for all σ,τ. A model $(V_0,\gamma\colon V_{0,\Omega}\to V)$ of V over k ***splits*** $(\gamma_\sigma)_\sigma$ if $\gamma_\sigma = \gamma\circ(\sigma\gamma)^{-1}$ for all σ; such a model (V_0,γ) is uniquely determined up to a unique k-isomorphism. A descent system is ***effective*** if it is split by some model over k.

PROPOSITION 3.23. [21] *Assume that Ω has infinite transcendence degree over k. A descent system $(\gamma_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$ on a quasiprojective variety V over Ω is effective if, for some subfield L of Ω finitely generated over k, the descent system $(\gamma_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/L)}$ is effective.*

PROOF. This is a restatement of a theorem of Weil (1956); see Milne 1999, 1.1. ∎

The proposition has the following corollary: let Ω, k, and V be as in the theorem; a descent system $(\gamma_\sigma)_{\sigma\in\mathrm{Aut}(\Omega/k)}$ is effective if there exists a finite set Σ of points in $V(\Omega)$ such that

(a) the only automorphism of V fixing all $P\in\Sigma$ is the identity map, and
(b) there exists a subfield L of Ω finitely generated over k such that $f_\sigma(\sigma P) = P$ for all $P\in\Sigma$ and all $\sigma\in\mathrm{Aut}(\Omega/L)$,

(Milne 1999, 1.2).

Consider a Shimura variety $\mathrm{Sh}(G,X)$ over $\mathbb{C}$, and let $E(G,X)$ be its reflex field. We say that a descent system $(\gamma_\sigma)_\sigma$ for $\mathrm{Sh}(G,X)$ over $E(G,X)$ is ***canonical*** if the maps γ_σ are $G(\mathbb{A}_f)$-equivariant and

$$\gamma_\sigma(\sigma[x,a]) = [x, r_x(\sigma|E(x)^{\mathrm{ab}}\cdot a]$$

whenever x is a special point of X, σ is an automorphism of $\mathbb{C}$ over $E(x)$, and $a\in G(\mathbb{A}_f)$. On applying the corollary to a suitable finite set of special points, one sees that every canonical descent system is effective; moreover, the resulting model over $E(G,X)$ is canonical.

We now prove Theorem 3.21. We have to show that the descent system given by Lemma 3.22 is canonical. Obviously, the maps γ_σ commute with the action of $G(\mathbb{A}_f)$. Let

$$\alpha\colon\mathcal{M}_K(\mathbb{C})\longrightarrow S_K(\mathbb{C})$$

be the given bijection. Let x be a special point in X, and let $(M,\mathfrak{s},[\eta])$ map to $[x,1]$ under α. Recall that this means that there exists an isomorphism

$$\beta\colon H_B(M)\longrightarrow(V,h_x)$$

of rational Hodge structures such that $\beta(s_i)=t_i$ for all $i\in I$ and $\eta\circ\beta_f=\mathrm{id}$. Such a β defines an isomorphism of the Mumford-Tate group T of M with the Mumford-Tate group of (V,h_x), which (by definition) is commutative, and so M is of CM-type. Let $\tau\in\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and extend it to an automorphism of $\mathbb{C}$. Then

$$\tau[x,1] = \tau\alpha(M,\mathfrak{s},[\eta]) = \alpha(\tau M,(\tau s_i)_i,[\tau\circ\eta]).$$

According to (3.15), there is an isomorphism $\gamma\colon M\to\tau M$ such that $\gamma(s)=\tau s$ for all Hodge cycles s on M and $\tau v = \gamma_f(\tilde{r}(\tau)v)$ for all $v\in\omega_f(M)$. Define $\beta'\colon H_B(\tau M)\to V$ to be

[21] Added. This corrects the original statement, which omitted the "continuity" condition.

$\beta \circ H_B(\gamma^{-1})$. Then β' is a morphism of Hodge structures $H_B(\tau M) \to (V, h_x)$, that maps τs_i to t_i for each i, and has the property that $\beta'_f \circ (\tau \circ \eta) = \tilde{r}(\tau)$. Hence

$$\alpha(\tau M, (\tau s_i)_i, [\tau \circ \eta]) = [x, \tilde{r}(\tau)],$$

where $\tilde{r}(\tau)$ represents $r(T, h_x)(\tau)$.

SHIMURA VARIETIES WHOSE WEIGHT IS DEFINED OVER $\mathbb{Q}$

We next construct the canonical model of a Shimura variety of abelian type whose weight is defined over $\mathbb{Q}$. The following proposition from Deligne 1971b will be useful.

PROPOSITION 3.24. *Let $f: G \hookrightarrow G'$ be an injective homomorphism sending X into X'. The map*

$$[x, g] \mapsto [f(x), f(g)]: \mathrm{Sh}(G, X) \longrightarrow \mathrm{Sh}(G', X')$$

is a closed immersion. If $\mathrm{Sh}(G', X')$ has a canonical model over a number field $E \supset E(G', X')$, then the image of $\mathrm{Sh}(G, X)$ in $\mathrm{Sh}(G', X')$ is defined over $E \cdot E(G, X)$ and is a canonical model.

PROOF. That the map is a closed immersion is 1.15.1 of Deligne 1971b. Identify $\mathrm{Sh}(G, X)$ with its image. Let (T, x) be a special pair in (G, X). For every τ fixing $E \cdot E(T, x)$ and $a \in G(\mathbb{A}_f)$, we have $\tau[x, a] = [x, a \cdot r(T, h_x)(\tau)] \in \mathrm{Sh}(G, X)(\mathbb{C})$. We now apply the following two lemmas. The first shows that $\tau\,\mathrm{Sh}(G, X) = \mathrm{Sh}(G, X)$ for any τ fixing $E \cdot E(T, x)$, and the second shows that such elements generate $\mathrm{Aut}(\mathbb{C}/E \cdot E(G, X))$. ∎

LEMMA 3.25. *For every special point x of X,*

$$\{[x, a] \mid a \in G(\mathbb{A}_f)\}$$

is Zariski dense in $\mathrm{Sh}(G, X)$.

PROOF. See Deligne 1971b, 5.2. ∎

LEMMA 3.26. *Let (G, X) be a pair satisfying (SV1,2,3,6). For every finite extension E' of $E(G, X)$, there exists a special pair $(T, x) \subset (G, X)$ such that E' and $E(T, x)$ are linearly disjoint over $E(G, X)$.*

PROOF. See Deligne 1971b, 5.1. ∎

THEOREM 3.27. *Every Shimura variety of abelian type whose weight is defined over $\mathbb{Q}$ has a canonical model over its reflex field.*

PROOF. From (3.21) and (3.24) we obtain the following criterion: a Shimura variety $\mathrm{Sh}(G, X)$ has a canonical model over its reflex field if there exists an inclusion $(G, X) \hookrightarrow (G', X')$ with (G', X') the pair attached (as in (3.16)) to a polarized abelian motive (M, t_0).

Let $\mathrm{Sh}(G, X)$ be a Shimura variety of abelian type with weight defined over $\mathbb{Q}$. Assume that G is generated as a group over $\mathbb{Q}$ by $\{h_x \mid x \in X\}$, and choose a faithful representation $\xi: G \hookrightarrow \mathrm{GL}(V)$. Then it follows from (1.38) that G is the Mumford-Tate group of (V, h_o) for some $o \in X$. Since $\mathrm{Sh}(G, X)$ is of abelian type, (V, h_o) is the Betti realization of an abelian motive M (see 3.12), and it is clear that $\mathrm{Sh}(G,X)$ satisfies the above criterion relative to M and any polarization t_0 of M.

Now drop the hypothesis that G is generated by $\{h_x\}$. The composite

$$\mathbb{S} \xrightarrow{h_x} G_{\mathbb{R}} \longrightarrow G_{\mathbb{R}}^{\mathrm{ab}}$$

is independent of x. Denote it by h_X, and let H be the $\mathbb{Q}$-subtorus of G^{ab} generated by h_X. Let G' be the inverse image of H in G, and let X' be a $G'(\mathbb{R})$-conjugacy class of maps $\mathbb{S} \to G'_{\mathbb{R}}$ such that the inclusion $G' \hookrightarrow G$ maps X' into X. Then $E(G,X) = E(G',X')$, $\mathrm{Sh}(G,X) = \mathrm{Sh}(G',X') \cdot G(\mathbb{A}_f)$, and, after possibly enlarging $Z(G)$ so that $H^1(\mathbb{Q}, Z(G)) = 0$ (this is permissible by 3.24)

$$s \cdot g = s', \quad s, s' \in \mathrm{Sh}(G',X'), \quad g \in G(\mathbb{A}_f) \quad \Longrightarrow \quad g \in G'(\mathbb{A}_f) \cdot Z(G)(\mathbb{Q})^{-}.$$

For $\tau \in \mathrm{Aut}(\mathbb{C}/E(G,X))$, let $\gamma_\tau \colon \tau\,\mathrm{Sh}(G',X') \to \mathrm{Sh}(G',X')$ be the map defined by the canonical model of $\mathrm{Sh}(G',X')$. Then γ_τ has a unique extension to a $G(\mathbb{A}_f)$-equivariant map $\gamma'_\tau \colon \tau\,\mathrm{Sh}(G,X) \to \mathrm{Sh}(G,X)$, namely,

$$\gamma'_\tau(s \cdot g) \stackrel{\text{def}}{=} \gamma_\tau(s) \cdot g, \quad s \in \mathrm{Sh}(G',X'), \quad g \in G(\mathbb{A}_f),$$

Obviously the γ'_τs satisfy the cocycle condition, and so define a model of $\mathrm{Sh}(G,X)$ over $E(G,X)$, which is canonical. ■

Let M be a motive over a field k of characteristic zero. Choose an algebraic closure k^{al} of k, and let $\omega_f(M)$ be the restricted product (over ℓ) of the ℓ-adic étale cohomology groups of $M \otimes k^{\mathrm{al}}$. For every compact open subgroup K of $G(\mathbb{A}_f)$, a ***level K-structure*** on M is a K-orbit $[\eta]$ of isomorphisms $\eta \colon V(\mathbb{A}_f) \to \omega_f(M)$ that is stable under the action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$, i.e., such that

$$\eta_0 \in [\eta], \quad \tau \in \mathrm{Gal}(k^{\mathrm{al}}/k) \implies \tau \circ \eta_0 \in [\eta].$$

Let $\mathrm{Sh}(G,X)$ be a Shimura variety of abelian type whose weight is defined over $\mathbb{Q}$. Assume that there is given a homomorphism $t \colon G \to \mathbb{G}_m$ such that $t \circ w_X = -2$, a faithful representation $\xi \colon G \hookrightarrow \mathrm{GL}(V)$ of G, and a tensor t_0 for V fixed by G and such that $\pm t_0$ is a polarization of (V, h_x) for all $x \in X$. Choose a point $o \in X$. Then $((V, h_o), t_0)$ is the Betti realization of a polarized abelian motive M, and if (G', X') is the pair associated with M (as in 3.16), then $(G,X) \subset (G',X')$.

For any field $k \supset E(G,X)$, we define $\mathcal{M}_K(k)$ to be the set of triples $(M, \mathfrak{s}, [\eta])$ consisting of an abelian motive M over k, a set $\mathfrak{s}$ of Hodge cycles on M, and a level K-structure $[\eta]$ on M satisfying the following conditions:

(3.28.1) for every $E(G,X)$-homomorphism $\tau : k \hookrightarrow \mathbb{C}$, there is an isomorphism $\beta : H_B(\tau M) \to V$ sending each s_i to t_i and h_M to an element of X;

(3.28.2) one (hence every) representative η of the level structure maps each s_i to t_i.

LEMMA 3.29. *If (3.28.1) holds for one $E(G,X)$-homomorphism τ, then it holds for all.*

PROOF. Suppose (3.28.1) holds for one embedding τ. Then $\tau(M, \mathfrak{s}, [\eta])$ defines a point P of $\mathcal{M}_K(G,X)(\mathbb{C})$. There exists a compact open subgroup K' of $G'(\mathbb{A}_f)$ such that $(G,X) \hookrightarrow (G',X')$ induces a closed immersion

$$\mathrm{Sh}_K(G,X) \hookrightarrow \mathrm{Sh}_{K'}(G',X')$$

and, by construction, this is defined over $E(G,X)$. The point P in $\mathrm{Sh}_{K'}(G',X')$ is rational over k, and lies in $\mathrm{Sh}_K(G,X)(\mathbb{C})$. If we replace τ by its composite τ' with an element of $\mathrm{Aut}(\mathbb{C}/E(G,X))$, the point remains in $\mathrm{Sh}_K(G,X)(\mathbb{C})$, and this implies that $\tau'(M,\mathfrak{s},[\eta]) \in \mathcal{M}_K(G,X)(\mathbb{C})$. ■

REMARK 3.30. In essence, Lemma 3.29 says that the moduli problem $\mathcal{M}$ is defined over $E(G,X)$. It is possible to prove this directly, i.e., without using the existence of canonical models, but the argument then is more complicated.

Lemma 3.29 allows us to define a moduli problem on smooth algebraic varieties T over $E(G,X)$. Let $\mathcal{M}_K(T)$ be the set of triples $(M,\mathfrak{s},[\eta])$ consisting of an abelian motive M over T, a family of Hodge tensors $\mathfrak{s}$ on M indexed by I, and a level K-structure on M such that, for all $\mathbb{C}$-valued points t of T, $(M,\mathfrak{s},[\eta])_t \in \mathcal{M}_K(\mathbb{C})$. With the obvious notion pull-back and isomorphism, this becomes a moduli problem on the category of smooth algebraic varieties over $E(G,X)$.

THEOREM 3.31. *The pair ($\mathrm{Sh}_K(G,X),\alpha_K$), where $\mathrm{Sh}(G,X)$ here denotes the canonical model of the Shimura variety, is a solution to the moduli problem over $E(G,X)$.*

PROOF. First note that it follows from Lemma 3.29 that $\mathcal{M}_K(\mathbb{C}) = \varinjlim \mathcal{M}_K(R)$, where the limit is over the subalgebras R of Ω that are finitely generated over $E(G,X)$. From our construction of the canonical model, it is clear that the map $\beta_m\colon T_{\mathbb{C}} \to \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ corresponding to an element $m \in \mathcal{M}_K(T)$ is defined over $E(G,X)$. Finally, an obvious extension of (3.23) to motives, provides us with elements m_i defined on some étale covering of $\mathrm{Sh}_K(G,X)$ and satisfying (2.3.2). ■

APPLICATION 3.32. In the case that the reflex field $E(G,X)$ is real, it is possible to use (3.31) to give a description of the action of complex conjugation on $\mathrm{Sh}(G,X)(\mathbb{C})$—see Milne and Shih 1981.

THE GENERAL CASE

THEOREM 3.33. *Every Shimura variety of abelian type admits a canonical model over its reflex field.*

PROOF. Consider a Shimura variety $\mathrm{Sh}(G,X)$ of abelian type. The weight map w_X is a homomorphism $\mathbb{G}_m \to Z^\circ \overset{\text{def}}{=} Z(G)^\circ$.

Write $(G^{\mathrm{ad}},X^{\mathrm{ad}}) = \prod(G_i,X_i)$, where the G_i are the simple factors of G^{ad}, and X_i is the projection of X onto G_i. Then $E(G,X)$ is the composite of the fields $E(G_i,X_i)$ and $E(G^{\mathrm{ab}},h_X)$, and each field $E(G_i,X_i)$ is either totally real or a CM-field (Deligne 1971b, 3.8). Because of our assumption (SV3,6), $E(G^{\mathrm{ab}},h_X)$ is a subfield of a CM-field, and so $E(G,X)$ is a subfield of a CM-field. The weight w_X is defined over $E(G,X)$ and is invariant under ι; it is therefore defined over a totally real subfield F of $E(G,X)$. Choose a quadratic imaginary extension E of $\mathbb{Q}$ in $\mathbb{C}$, and let $Z_* = \mathrm{Res}_{E/\mathbb{Q}}(Z^\circ)_E$. Define

$$\varepsilon_0 = \mathrm{Res}_{EF/F}(w_X^{-1}) : \mathrm{Res}_{EF/F}\,\mathbb{G}_m \longrightarrow \mathrm{Res}_{E/F}\,Z_E^0 = (Z_*)_F.$$

Then ε_0 is defined over F, and over $\mathbb{R}$ it can be identified with a homomorphism $\varepsilon \overset{\text{def}}{=} \varepsilon_{0,\mathbb{R}}\colon \mathbb{S} \to Z_{*,\mathbb{R}}$. The weight of ε is w_X.

There are natural inclusions $Z^{\circ} \hookrightarrow G$, $Z^{\circ} \hookrightarrow Z_*$, and we define G_* to be the quotient:

$$G_* = G \times Z_*/Z^{\circ} \qquad \text{(diagonal embedding of } Z^{\circ}\text{).}$$

Let $h_0 \in X$. The composite

$$\mathbb{S} \xrightarrow{(h_0,\varepsilon)} G_{\mathbb{R}} \times Z_{*\mathbb{R}} \longrightarrow G_{*\mathbb{R}}$$

has weight zero, and we define X_* to be its $G_*(\mathbb{R})$-conjugacy class. Clearly $\mathrm{Sh}(G_*, X_*)$ is of abelian type, and so (3.27) shows that it has a canonical model over its reflex field $E(G_*, X_*)$. Moreover

$$E(G_*, X_*) \cdot E = E(G, X) \cdot E.$$

Let ε' be the composite

$$\mathbb{S} \xrightarrow{(1,\varepsilon)} G_{\mathbb{R}} \times Z_{*\mathbb{R}} \longrightarrow G_{*\mathbb{R}},$$

and let

$$X_* \cdot \varepsilon'^{-1} = \{h \cdot \varepsilon'^{-1} \mid h \in X_*\}.$$

On applying Lemma 3.34 below to $\mathrm{Sh}(G_*, X_*)$, we find that $\mathrm{Sh}(G_*, X_* \cdot \varepsilon'^{-1})$ has a canonical model over $E \cdot E(G_*, X_*)$. The inclusion $G \hookrightarrow G_*$ maps X into $X_* \cdot \varepsilon'^{-1}$, and so we can apply Lemma 3.24 to show that $\mathrm{Sh}(G, X)$ has a canonical model over $E \cdot E(G_*, X_*) = E \cdot E(G, X)$. Since E is an arbitrary quadratic imaginary extension of $\mathbb{Q}$, we can apply Lemma 3.35 below to show that $\mathrm{Sh}(G, X)$ has a canonical model over $E(G, X)$. ∎

LEMMA 3.34. *Let $\mathrm{Sh}(G, X)$ be a Shimura variety, and let ε be a homomorphism $\mathbb{S} \to Z(G)_{\mathbb{R}}$. If $\mathrm{Sh}(G, X)$ admits a canonical model over a field E containing $E(G, X) \cdot E(Z^0, \varepsilon)$, then so also does $\mathrm{Sh}(G, X \cdot \varepsilon)$.*

PROOF. We have a morphism

$$[h, g], [\varepsilon, z] \mapsto [h \cdot \varepsilon, gz] \colon \mathrm{Sh}(G, X) \times \mathrm{Sh}(Z^0, \varepsilon) \longrightarrow \mathrm{Sh}(G, X \cdot \varepsilon).$$

Let $d \colon Z^0 \to G \times Z^0$ be the homomorphism $z \mapsto (z, z^{-1})$. This defines an action of $Z^0(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X) \times \mathrm{Sh}(Z^0, \varepsilon)$ with quotient $\mathrm{Sh}(G, X \cdot \varepsilon)$. The quotient of the product of the canonical models of $\mathrm{Sh}(G, X)$ and $\mathrm{Sh}(Z^0, \varepsilon)$ by $Z^0(\mathbb{A}_f)$ is a canonical model for $\mathrm{Sh}(G, X \cdot \varepsilon)$. See Deligne 1971b, 5.11. ∎

LEMMA 3.35. *Let $\{E_i\}$ be a family of finite extensions of $E(G, X)$ whose intersection is $E(G, X)$. If $\mathrm{Sh}(G, X)$ has a canonical model $\mathrm{Sh}(G, X)_{E_i}$ over each E_i, then it has a unique model $\mathrm{Sh}(G, X)_E$ over $E(G, X)$ such that, for all i, the model $\mathrm{Sh}(G, X)_E \otimes E_i$ over E_i is isomorphic to $\mathrm{Sh}(G, X)_{E_i}$. If further the family $\{E_i\}$ has the property that $\bigcap E_i F = F$ for any finite extension F of $E(G, X)$, then the model $\mathrm{Sh}(G, X)_E$ is canonical.*

PROOF. The first statement is proved in Deligne 1971b, 5.10. To show that the model satisfies (3.4.1) for a specific special pair (T, h), use that $\bigcap E_i \cdot E(T, h) = E(T, h)$. ∎

REMARK 3.36. Let $F \subset E(G, X)$ be the field of definition of w_X. Corresponding to any quadratic imaginary extension E of $\mathbb{Q}$, we have map

$$\mathrm{Sh}(G, X) \times \mathrm{Sh}(Z_*, \varepsilon) \longrightarrow \mathrm{Sh}(G_*, X_*)$$

rational over $E \cdot E(G, X)$. Sometimes this can be interpreted as the map sending two Hodge structures to their tensor product (see Deligne 1971b 6.6 for an example of this), and it can be used to obtain information about the points of $\mathrm{Sh}(G, X)$.

REMARK 3.37. The existence of a canonical model over the reflex field shows that, for any automorphism τ of $\mathbb{C}$ fixing $E(G,X)$, there is a canonical isomorphism $\tau\,\mathrm{Sh}(G,X) \to \mathrm{Sh}(G,X)$. Langlands (1979) conjectured that for *every* automorphism τ of $\mathbb{C}$, there is a canonical isomorphism $\tau\,\mathrm{Sh}(G,X) \to \mathrm{Sh}(G',X')$ for a suitable pair (G',X') defined explicitly in terms of (G,X), τ, and a special point of x. For Shimura varieties of abelian type, this conjecture can be proved using similar techniques to the above.

Applications

In many respects, Theorem 3.31 allows us treat Shimura varieties of abelian type as easily as Shimura varieties of PEL-type, at least in characteristic zero. To illustrate this, I list some applications.

CONSEQUENCES OF THE TATE CONJECTURE

The Tate conjectures (Tate 1994, T(X), E(X)) imply the following statement:

(3.38) For all motives M and N over a field k finitely generated over $\mathbb{Q}$ and primes ℓ, the homomorphism

$$\mathrm{Hom}(M,N)\otimes\mathbb{Q}_\ell \longrightarrow \mathrm{Hom}(\omega_\ell(M),\omega_\ell(N))^{\Gamma}, \quad \Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k),$$

is bijective.

Faltings (1983) proved (3.38) for motives of the form $h_1(A)$, A an abelian variety. Consequently, it is also true for direct factors of such motives.[22] Silverberg (1992, 1993) investigated the consequences of (3.38) for one class of Shimura varieties (essentially that described in (4.31) below). Here we explain its consequences for a Shimura variety $\mathrm{Sh}(G,X)$ of abelian type whose weight is defined over $\mathbb{Q}$ and such that $Z(\mathbb{Q})$ is closed in $Z(\mathbb{A}_f)$.

To a point $x \in X$, Shimura attaches an adèlic representation ρ_x (Shimura 1970, 7.2, 7.3, 7.6, 7.8). When we choose a faithful representation (V,ξ) of G as above, then $\xi_f \circ \rho_x$ becomes the Galois representation on $\omega_f(M_x)$ for M_x the motive attached to $[x,1] \in \mathrm{Sh}(G,X)$. If (3.38) holds for the motives in the family parametrized by $\mathrm{Sh}(G,X)$, then we can read off the following result:

(3.39) for $x,y \in X$ and $\alpha \in G(\mathbb{Q})$,

$$\alpha x = y \iff \mathrm{ad}\,\alpha \circ \rho_x = \rho_y.$$

Let $I(x) = \mathrm{Aut}(N_x)$, where $N_x = (M_x, \mathfrak{s}_x)$ is the motive together with tensor structure attached to the point [x,1] and a faithful representation of G. Assume that $I(x)$ satisfies the Hasse principle for H^1, i.e.,

$$H^1(\mathbb{Q}, I(x)) \longrightarrow \prod\nolimits_\ell H^1(\mathbb{Q}_\ell, I(x))$$

is injective. Then (3.38) implies the following statement:

(3.40) if there exists an $\alpha_f \in G(\mathbb{A}_f)$ such that $\mathrm{ad}(\alpha_f)\circ\rho_x = \rho_y$, then there exists an $\alpha_0 \in G(\mathbb{Q})$ such that $\alpha_0 x = y$.

[22] Of course, if the Tate conjecture were known for abelian varieties, then the map would be bijective for all abelian motives, but that is not what Faltings proves.

In geometric terms, the Tate conjecture implies that the $I(x)_{\mathbb{A}_f}$-torsor

$$\mathrm{Hom}(\omega_f(N_x), \omega_f(N_y))$$

is obtained from the $I(x)$-torsor $\mathrm{Hom}(N_x, N_y)$ by the base change $\mathbb{Q} \to \mathbb{A}_f$. The hypothesis in (3.40) implies that the first torsor is trivial. The Hasse principal then implies that the second torsor is trivial, which implies the conclusion of (3.40).

SYSTEMS OF REALIZATIONS

Let S be a smooth scheme over a number field E. A ***system of realizations*** on S is given by the following data:

(3.41.1) For each embedding $\tau\colon E \hookrightarrow \mathbb{C}$, a local system of $\mathbb{Q}$-vector spaces on $(\tau S)(\mathbb{C})$.

(3.41.2) A vector bundle H_{dR} on S endowed with a flat connection ∇ and a descending (Hodge) filtration F by subbundles. The connection is required to satisfy the axiom of transversality (2.15.2) and to have regular singularities at infinity.

(3.41.3) A local system of $\mathbb{A}_f$-modules H_f on S_{et}.

(3.41.4) Comparison isomorphisms relating the above data.

(3.41.5) Weight gradations on each of H_τ, H_{dR}, and H_f which are respected by the comparison isomorphisms.

(3.41.6) An involution $F_\infty\colon \bigoplus H_\tau \to \bigoplus H_\tau$ (Frobenius map at infinity) respecting the weight gradation.

The data are required to satisfy certain conditions, for example, the "Betti realization" H_τ, endowed with the weight gradation and the filtration provided, via the comparison isomorphism, by that on H_{dR}, is a variation of Hodge structures. See Deligne 1989, §1, for more details.

Consider a Shimura variety of abelian type, and let G^c be the largest quotient of G such that, for any $h \in X$, (G^c, h) satisfies (SV2*) and has weight defined over $\mathbb{Q}$ (G^c is the quotient of G by a subgroup of its centre). Then there is a canonical tensor functor from $\mathsf{Rep}_{\mathbb{Q}}(G^c)$ to the category of systems of realizations on the canonical model of $\mathrm{Sh}(G,X)$. In fact, every rational representation of G^c defines a family of abelian motives on $\mathrm{Sh}(G,X)$, and every family of motives defines a system of realizations.

AUTOMORPHIC VECTOR BUNDLES

Automorphic vector bundles are those vector bundles on Shimura varieties whose sections are holomorphic automorphic forms (in the classical sense). It is known that automorphic vector bundles have canonical models over number fields, and hence that it makes sense to speak of an automorphic form being defined over such a field. In Milne 1990, III, a heuristic explanation of this statement is given in terms of motives. For Shimura varieties of abelian type, the explanation is now a proof.

THE BOUNDARIES OF SHIMURA VARIETIES

The study of the boundary of a moduli variety for abelian varieties (with additional structure) is equivalent to the study of the degeneration of the abelian varieties—see Namikawa 1980 for for Siegel modular varieties over $\mathbb{C}$, Faltings and Chai 1990 for Siegel modular varieties over $\mathbb{Z}$, and Brylinski 1983 for Shimura varieties of Hodge type over $\mathbb{Q}$. Theorem 3.31

allows us to treat Shimura varieties of abelian type in the same fashion. I hope to return to this in a future work.

4 The Points on a Shimura Variety Modulo a Prime of Good Reduction

Let $\mathrm{Sh}(G,X)$ be a Shimura variety of abelian type whose weight is defined over $\mathbb{Q}$. In the last section, we obtained a motivic description of the points of $\mathrm{Sh}(G,X)$ with coordinates in any field containing $E(G,X)$. In this section, we show that, if one assumes the existence of a good theory of abelian motives in mixed characteristic, then the description extends to the points of $\mathrm{Sh}(G,X)$ in finite fields, and we thereby obtain a heuristic derivation of the conjecture of Langlands and Rapoport 1987.

Statement of the problem

As we saw in §3, starting from a connected reductive group G over $\mathbb{Q}$, a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$ satisfying the conditions (SV1,2,3,6), and a compact open subgroup K of $G(\mathbb{A}_f)$, we obtain a variety $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$. The reflex field $E(G,X)$ is a number field (contained in $\mathbb{C}$) that is defined purely in terms of G and X, and $\mathrm{Sh}_K(G,X)$ has a canonical model over $E(G,X)$. Let v be a prime of E lying over a finite prime p of $\mathbb{Q}$, and let E_v be the completion of $E(G,X)$ at v. Assume that $\mathrm{Sh}_K(G,X)$ has good reduction at v, i.e., that there is a smooth scheme $\mathrm{Sh}_K(G,X)_v$ over the ring of integers $\mathcal{O}_v$ in E_v whose generic fibre is $\mathrm{Sh}_K(G,X)_{E_v}$. The problem then is to describe the sets

$$\mathrm{Sh}_K(G,X)_v(k)$$

for k a finite field containing the residue field $\kappa(v)$ at v, or, equivalently, to describe the set

$$\mathrm{Sh}_K(G,X)_v(\mathbb{F}),\quad \mathbb{F}=k^{\mathrm{al}},$$

together with the action of the Frobenius element of $\mathrm{Gal}(\mathbb{F}/\kappa(v))$. For the applications, we shall also need to know how $G(\mathbb{A}_f)$ acts on the sets. Note that the problem is well-posed only if $\mathrm{Sh}_K(G,X)$ has a *canonical* smooth model over $\mathcal{O}_v$.

The next example illustrates the fact that, unless the component of K at p is maximal, we cannot expect $\mathrm{Sh}_K(G,X)$ to have good reduction at primes lying over p.

EXAMPLE 4.1. Let $G=\mathrm{GL}_2$, and let X be the conjugacy class of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$ containing the map $a+bi\mapsto\left(\begin{smallmatrix}a&-b\\b&a\end{smallmatrix}\right)$. Then $\mathrm{Sh}_{K(N)}(G,X)$ is the moduli variety for elliptic curves with level N structure, and it is known that this variety has good reduction at p if and only if p does not divide N (Deligne and Rapoport 1973).

Thus we should assume that K is of the form $K^p\cdot K_p$ with K_p a *maximal* compact open subgroup of $G(\mathbb{Q}_p)$, and K^p a compact open subgroup of $G(\mathbb{A}_f^p)$. However, the next example shows that even this is not sufficient to ensure that $\mathrm{Sh}_K(G,X)$ has good reduction at v.

EXAMPLE 4.2. Let B be a quaternion algebra over $\mathbb{Q}$ that splits at the real prime. Let $G=\mathrm{GL}_1(B)$. Then $G_{\mathbb{R}}\approx\mathrm{GL}_2$, and we can define X as in the last example. Let $K=K^p\cdot K_p$, where K_p is a maximal compact subgroup of $G(\mathbb{Q}_p)$. Then $\mathrm{Sh}_K(G,X)$ has good reduction at p if and only if p does not divide the discriminant of B.

Langlands (1976, p. 411) suggested that $\mathrm{Sh}_K(G,X)$ should have good reduction at $v|p$ if $K = K^p \cdot K_p$ with K_p a hyperspecial subgroup [23] of $G(\mathbb{Q}_p)$. Recall (Tits 1979, 3.8.1) that a subgroup K_p of $G(\mathbb{Q}_p)$ is said to be ***hyperspecial*** if there is a smooth group scheme G_p over $\mathbb{Z}_p$ such that

(4.3.1) $G_p(\mathbb{Z}_p) = K_p$;

(4.3.2) the reduction of G_p modulo p is a connected reductive group over $\mathbb{F}_p$.

Since algebraic groups over finite fields are quasi-split, a necessary condition that there exist a hyperspecial subgroup of $G(\mathbb{Q}_p)$ is that G be quasi-split over $\mathbb{Q}_p$ and split over an unramified extension of $\mathbb{Q}_p$, i.e., that G be ***unramified*** at p. Conversely, Tits 1979, p. 36, shows that this condition is sufficient. Consequently hyperspecial subgroups exist in $G(\mathbb{Q}_p)$ for almost all p.

EXAMPLE 4.4. Let (V,ψ) be a symplectic space over $\mathbb{Q}$, and let $G = G(\psi)$ be the group of symplectic similitudes. A hyperspecial subgroup of $G(\mathbb{Q}_p)$ is the stabilizer of a lattice $\Lambda \subset V(\mathbb{Q}_p)$ such that ψ (or some multiple of ψ) restricts to a $\mathbb{Z}_p$-valued form on Λ with determinant a p-adic unit.

In the following, we fix a hyperspecial subgroup K_p, and we again write G for the smooth group scheme over $\mathbb{Z}_p$ such that $G(\mathbb{Z}_p) = K_p$. Thus $G(R)$ is defined whenever R is a $\mathbb{Q}$-algebra or a $\mathbb{Z}_p$-algebra.

We may as well pass to the limit over the compact open subgroups K^p of $G(\mathbb{A}_f^p)$, and write

$$\mathrm{Sh}_p(G,X) = \varprojlim_{K^p} \mathrm{Sh}_{K^p \cdot K_p}(G,X).$$

Assume that $\mathrm{Sh}_p(G,X)$ has a smooth model over $\mathcal{O}_v$ (see below), and denote it by $\mathrm{Sh}_p(G,X)_v$. By definition, the action of $G(\mathbb{A}_f^p)$ extends to $\mathrm{Sh}_p(G,X)_v$, and so we obtain a set

$$\mathrm{Sh}_p(\mathbb{F}) = \mathrm{Sh}_p(G,X)_v(\mathbb{F})$$

together with commuting actions of $G(\mathbb{A}_f^p)$ and the geometric Frobenius element[24] $\Phi \in \mathrm{Gal}(\mathbb{F}/\kappa(v))$. The problem discussed in this section is that of determining the isomorphism class of the system $(\mathrm{Sh}_p(\mathbb{F}), \times, \Phi)$ consisting of the set $\mathrm{Sh}_p(\mathbb{F})$ together with the action "$\times$" of $G(\mathbb{A}_f^p)$ and the action of Φ.

The building

There is a more natural definition of hyperspecial subgroups in terms of the building $\mathcal{B}(G,F)$ that Bruhat and Tits attach to a reductive group G over a local field F (Tits 1979). This is a set with a left action of $G(F)$, certain of whose vertices are said to be ***hyperspecial***, and a subgroup K_p of $G(F)$ is hyperspecial if it is the stabilizer of such a vertex. The construction of the building commutes with the formation of unramified extensions of F.

The building $\mathcal{B}(G,F)$ is a union of apartments, and the apartments are in one-to-one correspondence with the maximal F-split subtori S of G. Let p_0 be a hyperspecial vertex of the apartment of S, and let $G(\mathcal{O}_F)$ be its stabilizer. Assume that G is split over F. Because S is split, it has a canonical $\mathcal{O}_F$-structure such that $S(R) = \mathrm{Hom}(X^*(S), R^\times)$ for every $\mathcal{O}_F$-algebra R. Our hypotheses imply that

[23] Conversations with Blasius, Chai, and Prasad suggest that this condition can be weakened.

[24] This is the element $x \mapsto x^{q^{-1}}$.

(4.5.1) $S(\mathcal{O}_F) \subset G(\mathcal{O}_F)$;

(4.5.2) $N(F) \subset G(\mathcal{O}_F) \cdot S(F)$, where N is the normalizer of S;

(4.5.3) $G(F) = G(\mathcal{O}_F) \cdot S(B) \cdot G(\mathcal{O}_F)$.

The equality (4.5.3) is the Cartan decomposition (Tits 1979, 3.3.3). The remaining two statements imply that the Weyl group has a set of representatives in $G(\mathcal{O}_F)$.

Now return to the situation of the previous subsection, so that G is a reductive group over $\mathbb{Q}$. For any field $F \supset \mathbb{Q}$, write $\mathcal{C}(F)$ for the set of $G(F)$-conjugacy classes of homomorphisms $\mathbb{G}_m \to G_F$. Note that a map $F \to F'$ defines a map $\mathcal{C}(F) \to \mathcal{C}(F')$; in particular, when F' is Galois over F, $\mathrm{Gal}(F'/F)$ acts on $\mathcal{C}(F')$.

PROPOSITION 4.6. *(a) For any maximal F-split torus S in G_F, with F-Weyl group Ω, the map $X_*(S)/\Omega \to \mathcal{C}(F)$ is bijective.*

(b) If G is quasi-split over F, then $\mathcal{C}(F) = \mathcal{C}(F^{\mathrm{al}})^{\mathrm{Gal}(F^{\mathrm{al}}/F)}$.

(c) If $F \subset F'$ are algebraically closed fields, then $\mathcal{C}(F) \to \mathcal{C}(F')$ is a bijection.

(d) If G is split over F, then $\mathcal{C}(F) \to \mathcal{C}(F^{\mathrm{al}})$ is a bijection.

PROOF. [25](a) Because S is split, $X_*(S) = \mathrm{Hom}_F(\mathbb{G}_m, S)$. The surjectivity of $X_*(S) \to \mathcal{C}(F)$ follows from the fact that any two maximal F-split tori in G_F are conjugate under $G(F)$ (Milne 2017, 25.10). To prove the injectivity, let $\mu_1, \mu_2 \in X_*(S)$ be such that $\mu_2 = \mathrm{ad}(g) \circ \mu_1$ for some $g \in G(F)$. Then $\mathrm{ad}(g)(S)$ and S are both maximal F-split tori in the centralizer M of $\mu_2(\mathbb{G}_m)$ in G, which is reductive (ibid. 17.59), and so there exists an $m \in M(F)$ such that $\mathrm{ad}(mg)S = S$. Now $\mathrm{ad}(mg) \circ \mu_1 = \mu_2$. As mg is an element of $G(F)$ normalizing S, it follows that μ_1 and μ_2 lie in the same Ω_F-orbit.

(b) Because G_F is quasi-split, the centralizer T of S in G is a maximal torus in G_F. After (a), it remains to show that the map

$$X_*(S)/\Omega_F \to (X_*(T)/\Omega)^{\mathrm{Gal}(F^{\mathrm{al}}/F)}$$

is an isomorphism, where Ω is the absolute Weyl group of T in G. Let B be a Borel subgroup of G_F containing T, and let C denote the B-positive Weyl chamber of $X_*(T) \otimes \mathbb{R}$. Recall that Ω acts simply transitively on the set of Weyl chambers. Because B is defined over F, C is stable under $\mathrm{Gal}(F^{\mathrm{al}}/F)$.

We first show that the map is surjective. Let μ be an element of $X_*(T)$ whose Ω-orbit is stable under $\mathrm{Gal}(F^{\mathrm{al}}/F)$. We may suppose that $\mu \in \bar{C}$. Then $\sigma\mu \in \bar{C}$ for all $\sigma \in \mathrm{Gal}(F^{\mathrm{al}}/F)$. As $\sigma\mu$ and μ are in the same Ω-orbit, this implies that $\sigma\mu = \mu$ (because $\bar{C}$ is a fundamental domain for the action of Ω on $X_*(T) \otimes \mathbb{R}$). Therefore $\mu \in X_*(T)^{\mathrm{Gal}(F^{\mathrm{al}}/F)} = X_*(S)$.

We now show that the map is injective. Let $\mu_1, \mu_2 \in X_*(S)$ be such that $\mu_2 = w \cdot \mu_1$ for some $w \in \Omega$. Let $C_0 = C \cap (X_*(S) \otimes \mathbb{R})$. Then C_0 is a Weyl chamber in $X_*(S) \otimes \mathbb{R}$, and we may suppose that $\mu_1, \mu_2 \in \bar{C}_0$. Then $\mu_1, \mu_2 \in \bar{C}$, and so $w = 1$ (as before).

The remaining statements follow from the first two. ■

Let $\mathfrak{c}(X)$ denote the $G(\mathbb{C})$-conjugacy class of cocharacters of $G_{\mathbb{C}}$ containing μ_x for $x \in X$. According to (c) of the proposition, $\mathfrak{c}(X)$ corresponds to an element $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ of $\mathcal{C}(\mathbb{Q}^{\mathrm{al}})$. The group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts on $\mathcal{C}(\mathbb{Q}^{\mathrm{al}})$, and, by definition, the fixed field of the stabilizer of $\mathfrak{c}(X)_{\mathbb{Q}^{\mathrm{al}}}$ in $\mathcal{C}(\mathbb{Q}^{\mathrm{al}})$ is the reflex field $E = E(G, X)$ of $\mathrm{Sh}(G, X)$.

Consequently E_v is the subfield of E_v^{al} fixed by the stabilizer of $\mathfrak{c}(X)_{E_v^{\mathrm{al}}}$. Because G splits over $\mathbb{Q}_p^{\mathrm{un}}$, (d) of the proposition shows $E_v \subset \mathbb{Q}_p^{\mathrm{un}}$, and because G is quasi-split over

[25] Added the proof to the original.

$\mathbb{Q}_p$, (b) shows that there exists a cocharacter of G defined over E_v representing $\mathfrak{c}(X)_{E_v^{\mathrm{al}}}$. Hence we have the following result.

COROLLARY 4.7. *(a) The prime v is unramified over p.*

(b) Let S be a maximal split torus of $G_{E_v^{\mathrm{un}}}$ whose apartment contains the hyperspecial vertex fixed by K_p. Then $\mathfrak{c}(X)_{E_v^{\mathrm{al}}}$ is represented by a cocharacter μ_0 of S.

NOTATION 4.8. We write B for the completion of E_v^{un}, W for the ring of integers in B, and $\mathbb{F}$ for the residue field of W. Thus $\mathbb{F}$ is an algebraic closure of $\kappa(v)$, W is the ring of Witt vectors over $\mathbb{F}$, and B is the field of fractions of W.

Let $\mathbb{C}_p$ be the completion of an algebraic closure of B, and extend the inclusion $E \hookrightarrow B$ to an isomorphism $\mathbb{C} \to \mathbb{C}_p$. We use this isomorphism to identify B with a subfield of $\mathbb{C}$, and hence to define a fibre functor $\omega_B \colon \mathsf{Mot}(B) \to \mathsf{Vec}(\mathbb{Q})$.

We fix a choice of a maximal $\mathbb{Q}_p$-split torus S_0 of $G_{\mathbb{Q}_p}$ whose apartment contains the hyperspecial point fixed by K_p, a maximal B-split torus S of G_B containing S_0, and a cocharacter μ_0 of S representing $\mathfrak{c}(X)_B$.

The points with coordinates in $\mathbb{C}$

By definition

$$\mathrm{Sh}_p(\mathbb{C}) = \varprojlim_{K^p} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^p \cdot K_p.$$

Set $G(\mathbb{Z}_{(p)}) = G(\mathbb{Q}) \cap K_p$ and $Z(\mathbb{Z}_{(p)}) = Z(\mathbb{Q}) \cap K_p$, where Z is the centre of G.

LEMMA 4.9. *Let G be a connected reductive group over $\mathbb{Q}$. For every hyperspecial subgroup K_p of $G(\mathbb{Q}_p)$, $G(\mathbb{Q}_p) = G(\mathbb{Q}) \cdot K_p$.*

PROOF. As we noted above, the existence of a hyperspecial subgroup implies that $G_{\mathbb{Q}_p}$ splits over an unramified extension of $\mathbb{Q}_p$. Because K_p is open in $G(\mathbb{Q}_p)$, the lemma follows from the next result. ∎

LEMMA 4.10. *Let G be a reductive group over $\mathbb{Q}$ that splits over an unramified extension of $\mathbb{Q}_p$. Then $G(\mathbb{Q})$ is dense in $G(\mathbb{Q}_p)$.*

PROOF. The hypothesis says that G acquires a split maximal torus T over an unramified extension of $\mathbb{Q}_p$, and it is known (see Tits 1979, p. 36) that T can be chosen to be defined over $\mathbb{Q}_p$. According to Harder 1966, 5.5.3, there will exist a torus $T_0 \subset G$ such that $T_{0\mathbb{Q}_p}$ is conjugate to T (by an element of $G(\mathbb{Q}_p)$). Let E be the smallest extension of $\mathbb{Q}$ over which T_0 splits. It is Galois over $\mathbb{Q}$, and the decomposition group at a prime lying over p is the Galois group of the smallest extension of $\mathbb{Q}_p$ splitting T, which is unramified. This group is therefore cyclic, and we can apply Sansuc 1981, 3.5ii, to conclude that $G(\mathbb{Q})$ is dense in $G(\mathbb{Q}_p)$. ∎

PROPOSITION 4.11. *We have*

$$\mathrm{Sh}_p(\mathbb{C}) = G(\mathbb{Z}_{(p)}) \Bigg\backslash \left(X \times \frac{G(\mathbb{A}_f^p)}{Z^p} \right),$$

where Z^p is the closure of $Z(\mathbb{Z}_{(p)})$ in $Z(\mathbb{A}_f^p)$.

PROOF. It follows from (4.9) that the natural map

$$G(\mathbb{Z}_{(p)})\backslash X \times \frac{G(\mathbb{A}_f^p)}{K^p} \longrightarrow G(\mathbb{Q})\backslash X \times \frac{G(\mathbb{A}_f^p)}{K^p} \times \frac{G(\mathbb{Q}_p)}{K_p}.$$

is a homeomorphism. For K^p open and compact in $G(\mathbb{A}_f^p)$,

$$G(\mathbb{Z}_{(p)})\backslash X \times \frac{G(\mathbb{A}_f^p)}{K^p} = \frac{G(\mathbb{Z}_{(p)})}{Z(\mathbb{Z}_{(p)})} \Bigg\backslash X \times \frac{G(\mathbb{A}_f^p)}{K^p \cdot Z(\mathbb{Z}_{(p)})} = \frac{G(\mathbb{Z}_{(p)})}{Z(\mathbb{Z}_{(p)})} \Bigg\backslash X \times \frac{G(\mathbb{A}_f^p)}{K^p \cdot Z^p}.$$

Write Γ for the discrete group $G(\mathbb{Z}_{(p)}) \cap K^p$. The image of this in $G^{\mathrm{ad}}(\mathbb{Q})$ acts properly discontinuously on X (cf. 2.9), and it follows easily that $G(\mathbb{Z}_{(p)})/Z(\mathbb{Z}_{(p)})$ acts properly on $X \times \left(G(\mathbb{A}_f^p)/Z^p\right)$ (see Bourbaki 1960, III, 4.4, Proposition 7). Hence the quotient space $G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p)/Z^p$ is separated (ibid. III.4.2, Proposition 3). Now we can apply (ibid. III, 7.2, Corollary 1) to the compact groups $K^p \cdot Z^p$ acting on the (fixed) separated space $E = G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p)/Z^p$ to conclude that

$$E/\varprojlim(K^p \cdot Z^p) = \varprojlim E/K^p \cdot Z^p.$$

But $\varprojlim(K^p \cdot Z^p) = \cap K^p \cdot Z^p = Z^p$. ■

COROLLARY 4.12. *We have*

$$\mathrm{Sh}_p(\mathbb{C}) = G(\mathbb{Q}) \Bigg\backslash \left(X \times \frac{G(\mathbb{A}_f^p)}{Z^p} \times \frac{G(\mathbb{Q}_p)}{K_p}\right).$$

PROOF. Again (4.9) implies that the natural map

$$G(\mathbb{Z}_{(p)})\backslash X \times \frac{G(\mathbb{A}_f^p)}{Z^p} \longrightarrow G(\mathbb{Q})\backslash X \times \frac{G(\mathbb{A}_f^p)}{Z^p} \times \frac{G(\mathbb{Q}_p)}{K_p}.$$

is a homeomorphism. ■

For $x \in X$, let $I(x)$ be the stabilizer of x in $G(\mathbb{Q})$. Set

$$S(x) = I(x)\backslash X^p(x) \times X_p(x)$$

with $X^p(x) = G(\mathbb{A}_f^p)/Z^p$ and $X_p(x) = G(\mathbb{Q}_p)/K_p$.

COROLLARY 4.13. *There is a canonical bijection*

$$\coprod S(x) \to \mathrm{Sh}_p(\mathbb{C}),$$

where the left hand side runs over a set of representatives for $G(\mathbb{Q})\backslash X$.

PROOF. Trivial. ■

We interpret the decomposition in (4.13) motivically. For this we need to return to the situation of (3.13), namely, we suppose that $\mathrm{Sh}(G,X)$ is of abelian type, that w_X is defined over $\mathbb{Q}$, and that there is a homomorphism $t\colon G \to \mathbb{G}_m$ such that $t \circ w_X = -2$. Moreover, we choose a faithful representation $\xi\colon G \hookrightarrow \mathrm{GL}(V)$ and a family of tensors $\mathfrak{t} = (t_i)_{i\in I}$ such that G is the subgroup of $\mathrm{GL}(V)$ fixing the t_i. We assume that for some $0 \in I$, $\pm t_0$ is a polarization of $(V, \xi\circ h_x)$, all $x \in X$. Because K_p is a maximal compact subgroup, there exists a lattice $V(\mathbb{Z}_p)$ in $V(\mathbb{Q}_p)$ whose stabilizer is K_p.

Call a pair $N = (M,\mathfrak{s})$ consisting of an abelian motive and a family of tensors ***admissible*** if there exists an isomorphism $\beta\colon \omega_B(M) \to V$ mapping each s_i to t_i and sending h_M to h_x, some $x \in X$. Given such a pair, define $X^p(N)$ to be the set of isomorphisms

$$\eta\colon V(\mathbb{A}_f^p) \to \omega_f^p(M)$$

modulo Z^p-equivalence mapping each t_i to s_i, and let $X_p(N)$ be the set of lattices Λ_p in $\omega_p(M)$ for which there exists an isomorphism $V(\mathbb{Q}_p) \to \omega_p(M)$ mapping each t_i to s_i and $V(\mathbb{Z}_p)$ onto Λ_p. Here ω_ℓ is the ℓ-adic étale fibre functor, and ω_f^p is the restricted product of the ω_ℓ for $\ell \neq p$. Let

$$S(N) = I(N)\backslash X^p(N) \times X_p(N), \quad I(N) = \mathrm{Aut}(N).$$

The group $G(\mathbb{A}_f^p)$ acts on $S(N)$ according to the following rule:

$$[\eta, \Lambda_p]g = [\eta\circ g, \Lambda_p].$$

COROLLARY 4.14. *There is a canonical equivariant bijection*

$$\coprod S(N) \to \mathrm{Sh}_p(\mathbb{C}),$$

where the disjoint union is over a set of representatives for the isomorphism classes of admissible pairs $N = (M,\mathfrak{s})$.

PROOF. Let N be an admissible pair, and choose a β satisfying the above condition. The $G(\mathbb{Q})$-orbit of the element $x \in X$ corresponding to h_M is independent of the choice of β, and in this way we obtain a one-to-one correspondence between the isomorphism classes of admissible pairs and the set $G(\mathbb{Q})\backslash X$ (cf. 3.13). The choice of a β determines an isomorphism $I(N) \to I(x)$ and equivariant bijections

$$X^p(N) \to X^p(x), \quad \eta \mapsto \beta\circ\eta,$$
$$X_p(N) \to X_p(x), \quad \Lambda_p \mapsto [g] \text{ if } \beta(\Lambda_p) = gV(\mathbb{Z}_p),$$

and hence an equivariant bijection

$$S(N) \longrightarrow S(x).$$

∎

We interpret the decomposition in (4.14) in terms of homomorphisms from the motivic Galois group G_{Mab} to G. (Recall that $G_{\mathrm{Mab}} = \mathit{Aut}^{\otimes}(\omega_B)$, where ω_B is the Betti fibre functor on the category of abelian motives.)

If H and G are algebraic groups over a field k, and φ and φ' are homomorphisms $H \to G$, we set jsm

$$\mathrm{Isom}(\varphi,\varphi') = \{g \in G(k) \mid \mathrm{ad}(g)\circ\varphi = \varphi'\},$$
$$\mathrm{Aut}(\varphi) = \mathrm{Isom}(\varphi,\varphi).$$

This notation is justified by noting that $\mathrm{Isom}(\varphi,\varphi')$ is the set of isomorphisms from φ to φ' regarded as functors of groupoids. [26]

Call a homomorphism $\varphi\colon G_{\mathrm{Mab}} \to G$ ***admissible*** if $\varphi_{\mathbb{R}} \circ h_{\mathrm{Mab}} \in X$. For such a homomorphism, set

$$I(\varphi) = \mathrm{Aut}(\varphi).$$

Let ζ_ℓ be the inclusion $e \hookrightarrow (G_{\mathrm{Mab}})_{\mathbb{Q}_\ell}$, where e is the one-element group scheme, and let ξ_ℓ be the inclusion $e \hookrightarrow G_{\mathbb{Q}_\ell}$. Let $\varphi(\ell)$ be the homomorphism obtained from φ by the base change $\mathbb{Q} \to \mathbb{Q}_\ell$. Define

$$X_\ell(\varphi) = \mathrm{Isom}(\xi_\ell, \zeta_\ell \circ \varphi(\ell)) = \{g \in G(\mathbb{Q}_\ell) \mid \mathrm{ad}\, g \circ \xi_\ell = \varphi(\ell) \circ \zeta_\ell\}.$$

Then $I(\varphi)$ acts on $X_\ell(\varphi)$ on the left, and $G(\mathbb{Q}_\ell)$ acts on it on the right and makes it into a principal homogeneous space. Choose a $\mathbb{Z}$-structure on G, and let $X'_\ell(\varphi)$ be the subset of $X_\ell(\varphi)$ of integral elements. Define $X^p(\varphi)$ to be the restricted product of the $X_\ell(\varphi)$, $\ell \neq p$, relative to the subsets $X'_\ell(\varphi)$. It is independent of the choice of the $\mathbb{Z}$-structure, and it is a principal homogeneous space for the group $G(\mathbb{A}_f^p)$. Define

$$X_p(\varphi) = G(\mathbb{Q}_p)/G(\mathbb{Z}_p),$$

and let

$$S(\varphi) = I(\varphi)\backslash(X^p(\varphi)/Z^p) \times X_p(\varphi).$$

COROLLARY 4.15. *There is a canonical bijection*

$$\coprod S(\varphi) \to \mathrm{Sh}_p(\mathbb{C}),$$

where φ runs over the isomorphism classes of admissible homomorphisms $G_{\mathrm{Mab}} \to G$.

PROOF. The Betti functor ω_B identifies the category of abelian motives with the category of representations of G_{Mab}. Choose a representation (V,ξ) and tensors t_i as in the discussion preceding (4.14). An admissible homomorphism φ then defines an admissible pair $N(\varphi)$, and there is a canonical isomorphism $S(\varphi) \to S(N(\varphi))$. Since the map $\varphi \mapsto N(\varphi)$ defines a bijection from the set of isomorphism classes of admissible homomorphisms to the set of isomorphism classes of admissible pairs, (4.15) follows from (4.14). ■

Of course, (4.14) and (4.15) are clumsy compared to the description (4.11) of $\mathrm{Sh}_p(\mathbb{C})$ as a single set of double cosets. However, they are the descriptions that will persist into characteristic p.

The points of $\mathrm{Sh}_p(G,X)$ *with coordinates in* B

Since B is not algebraically closed, in order to have a good description of the points we should assume that the moduli problem is fine. In the present context, this amounts to assuming that $Z(\mathbb{Z}_{(p)})$ is closed in $Z(\mathbb{A}_f^p)$. Then (cf. 3.28, 4.14) the points of $\mathrm{Sh}_p(G,X)$ with coordinates in $B \overset{\mathrm{def}}{=} B(\mathbb{F})$ are in one-to-one correspondence with the isomorphism classes of quadruples $(M,\mathfrak{s},\eta^p,\Lambda_p)$ satisfying the following conditions:

[26] Let H and G be (abstract) groups, and regard them as groupoids in sets, i.e., as categories with a single object and with G and H as the sets of morphisms. A homomorphism $\varphi\colon H \to G$ of groups can be regarded as a functor $H \to G$, and if φ' is a second homomorphism (functor), then to give a morphism of functors $\varphi \to \varphi'$ is to give an element $g \in G$ such that $\varphi' = \mathrm{ad}(g) \circ \varphi$.

(4.16.1) M is an abelian motive over B and $\mathfrak{s} = (s_i)_{i \in I}$ is a family of tensors on M for which there exists an isomorphism

$$\beta: \omega_B(M) \longrightarrow V(\mathbb{Q})$$

mapping each s_i to t_i and h_M to h_x, some $x \in X$. (Here h_M defines the Hodge structure on $\omega_B(M)$.)

(4.16.2) η^p is an isomorphism

$$\eta^p: V(\mathbb{A}_f^p) \longrightarrow \omega_f^p(M)$$

which maps each t_i to s_i and which is invariant under the action of $\mathrm{Gal}(\overline{B}/B)$.

(4.16.3) Λ_p is a $\mathbb{Z}_p$-lattice in $\omega_p(M)$, invariant under the action of $\mathrm{Gal}(\overline{B}/B)$, for which there exists an isomorphism

$$\eta_p: V(\mathbb{Q}_p) \longrightarrow \omega_p(M)$$

which maps each t_i to s_i and maps $V(\mathbb{Z}_p)$ onto Λ_p.

REMARK 4.17. (a) The condition (4.16.1) is independent of the choice of the isomorphism $\mathbb{C} \to \mathbb{C}_p$ (extending the embedding $E \hookrightarrow B$) because of (3.29).

(b) To say that η^p is invariant under the action of $\mathrm{Gal}(\overline{B}/B)$ simply means that $\mathrm{Gal}(\overline{B}/B)$ acts trivially on $\omega_f^p(M)$.

(c) Giving Λ_p is equivalent to giving a K_p-equivalence class of isomorphisms $\eta_p: V(\mathbb{Q}_p) \to \omega_p(M)$ such that each η_p maps each t_i to s_i and such that the class is stable under the action of $\mathrm{Gal}(\overline{B}/B)$.

REMARK 4.18. Fix a quadruple $(M, \mathfrak{s}, \eta^p, \Lambda_p)$, and an isomorphism η_p as in (4.16.3). Use η_p to transfer the action of G on $V(\mathbb{Q}_p)$ to $\omega_p(M)$. If we assume that the p-adic realizations of the s_i are fixed by the action of $\mathrm{Gal}(\overline{B}/B)$, then the action of $\mathrm{Gal}(\overline{B}/B)$ on Λ_p defines a homomorphism $\mathrm{Gal}(\overline{B}/B) \to G(\mathbb{Z}_p)$. Let Q be its image. Every Λ satisfying (4.16.3) is of the form $g\Lambda_p$ for some $g \in G(\mathbb{Q}_p)$, and there is a one-to-one correspondence

$$\{\Lambda \text{ satisfying (4.16.3)}\} \leftrightarrow \{g \in G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \mid g^{-1}Qg \subset G(\mathbb{Z}_p)\}.$$

We want to pass from the points of $\mathrm{Sh}_p(G, X)$ with coordinates in B to its points in $\mathbb{F}$, via its points in W, but the p-adic étale fibre functor does not persist into characteristic p. Thus we need to re-interpret the data in terms of the de Rham, or crystalline, fibre functor. First we review some of the theory of p-adic cohomology.

Review of p-adic cohomology

Let k be a perfect field, let $W(k)$ be the ring of Witt vectors over k, and let $B(k)$ be the field of fractions of $W(k)$. The absolute Frobenius automorphism $x \mapsto x^p$ and its liftings to $W(k)$ and $B(k)$ are denoted by σ. A ***crystal*** over k is a free finitely generated $W(k)$-module N together with an injective σ-linear map $\phi: N \to N$; an ***isocrystal*** over k is a finite-dimensional $B(k)$-vector space N together with a bijective σ-linear map $\phi: N \to N$. For example, if X is a smooth projective variety over k, then $H^i_{\mathrm{crys}}(X)/\{\text{torsion}\}$ is a crystal over k, and $H^i_{\mathrm{crys}}(X) \otimes_{W(k)} B(k)$ is an isocrystal. The isocrystal with underlying space $B(k)$ and with $\varphi = p^{-1}\,\mathrm{id}$ is called the ***Tate isocrystal.***

The category of k-isocrystals is a nonneutral Tannakian category over $\mathbb{Q}_p$. The dual of an isocrystal N is obtained as follows: regard ϕ as a $B(k)$-linear map ${}^{\sigma}N \to N$, where ${}^{\sigma}N = N \otimes_{B(k),\sigma} B(k)$; form the $B(k)$-linear dual $\phi^t : N^\vee \to ({}^{\sigma}N)^\vee = {}^{\sigma}(N^\vee)$, and take ϕ on $N^\vee$ to be $\phi^\vee \overset{\text{def}}{=} (\phi^t)^{-1} : {}^{\sigma}N^\vee \to N^\vee$.

Let R be a complete discrete valuation ring of characteristic zero and residue field k, and let K be the field of fractions of R. Identify $W(k)$ and $B(k)$ with subrings of R and K. A ***filtered Dieudonné K-module*** is an isocrystal (N,ϕ) over k together with a finite decreasing filtration Fil on $N \otimes_{B(k)} K$ such that $\mathrm{Fil}^i(N \otimes K) = N \otimes K$ for i sufficiently small and $\mathrm{Fil}^i(N \otimes K) = 0$ for i sufficiently large. We shall be mainly concerned with filtered Dieudonné $B(k)$-modules, and we usually drop the "Dieudonné". A filtered $B(k)$-module N is said to be ***weakly admissible*** if it contains a lattice Λ such that

$$\sum p^{-i}\phi(\mathrm{Fil}^i N \cap \Lambda) = \Lambda,$$

and a lattice Λ with this property is said to be ***strongly divisible***. The weakly admissible filtered $B(k)$-modules form a neutral Tannakian category with coefficients in $\mathbb{Q}_p$ (see Fontaine 1979 and Lafaille 1980).

Let X be a smooth proper scheme over R. Then (Berthelot and Ogus 1983) there is a canonical isomorphism

$$H^i_{\mathrm{dR}}(X) \otimes_R K \xrightarrow{\simeq} H^i_{\mathrm{crys}}(X_k) \otimes_{W(k)} K, \quad X_k = X \times_{\mathrm{Spec}\,R} \mathrm{Spec}\,k.$$

Therefore the Hodge filtration on $H^i_{\mathrm{dR}}(X)$ defines on $H^i_{\mathrm{crys}}(X_k) \otimes B(k)$ the structure of a filtered Dieudonné K-module. For a smooth proper scheme X over W, one even has a canonical isomorphism $H^i_{\mathrm{dR}}(X) \simeq H^i_{\mathrm{crys}}(X_k)$ (Berthelot and Ogus 1978, 7.26).

If X is a smooth proper scheme over $B(k)$ having good reduction, i.e., extending to a smooth proper scheme $\mathcal{X}$ over W, then the σ-linear map on $H^i_{\mathrm{dR}}(X)$ induced by that on $H^i_{\mathrm{crys}}(\mathcal{X}_k)$ via the isomorphisms

$$H^i_{\mathrm{dR}}(X) \simeq H^i_{\mathrm{dR}}(\mathcal{X}) \otimes_{W(k)} B(k) \simeq H^i_{\mathrm{crys}}(\mathcal{X}_k) \otimes_{W(k)} B(k),$$

is independent of the choice of $\mathcal{X}$ (Gillet and Messing, 1987). Therefore $H^i_{\mathrm{dR}}(X)$ has a canonical structure of a filtered Dieudonné $B(k)$-module.

By a ***p-adic representation*** of $\mathrm{Gal}(B(k)^{\mathrm{al}}/B(k))$ we mean a continuous representation of $\mathrm{Gal}(B(k)^{\mathrm{al}}/B(k))$ on a finite-dimensional $\mathbb{Q}_p$-vector space. Let B_{crys} be the topological $B(k)$-algebra defined in Fontaine 1983; it has a continuous action of $\mathrm{Gal}(B(k)^{\mathrm{al}}/B(k))$ and a decreasing filtration. When N is a weakly admissible filtered $B(k)$-module, we set

$$\mathbb{V}(N) = \{x \in \mathrm{Fil}^0(B_{\mathrm{crys}} \otimes_{B(k)} N) \mid \phi(x) = x\}.$$

This is a p-adic representation of $\mathrm{Gal}(B(k)^{\mathrm{al}}/B(k))$ with dimension at most that of N; if $\dim_{\mathbb{Q}_l} \mathbb{V}(N) = \dim_{B(k)} N$, then N is said to be ***admissible.***

When V is a p-adic representation, we set

$$\mathbb{D}(V) = (B(k)_{\mathrm{crys}} \otimes_{\mathbb{Q}_l} V)^{\mathrm{Gal}(B(k)^{\mathrm{al}}/B(k))}.$$

This is a filtered $B(k)$-module of dimension at most that of V; if $\dim_{B(k)} \mathbb{D}(V) = \dim_{\mathbb{Q}_l} V$, then V is said to be ***crystalline***.

PROPOSITION 4.19. *The functor $\mathbb{D}$ defines a $\otimes$-equivalence from the category of crystalline p-adic representations to that of admissible filtered Dieudonné modules, with quasi-inverse $\mathbb{V}$.*

PROOF. See Fontaine 1979, 1983, and Fontaine and Laffaille 1982. ∎

THEOREM 4.20. *Let X be a nonsingular projective variety over $B(k)$ with good reduction; then $H^i_{\mathrm{dR}}(X) \otimes_{W(k)} B(k)$ is an admissible filtered $B(k)$-module, $H^i_{\mathrm{et}}(X) \stackrel{\mathrm{def}}{=} H^i_{\mathrm{et}}(X_{B^{\mathrm{al}}}, \mathbb{Q}_p)$ is a crystalline p-adic representation, and there are canonical isomorphisms*

$$\mathbb{V}(H^i_{\mathrm{crys}}(X) \otimes_{W(k)} B(k)) \simeq H^i_{\mathrm{et}}(X), \qquad \mathbb{D}(H^i_{\mathrm{et}}(X)) \simeq H^i_{\mathrm{crys}}(X).$$

PROOF. See Fontaine and Messing 1987 and Faltings 1989. ∎

REMARK 4.21. (a) Let Λ be a lattice in a filtered $B(k)$-module N, and suppose $\mu: \mathbb{G}_m \to \mathrm{GL}(\Lambda)$ splits the filtration on Λ, i.e., if we set

$$N^i = \{n \in N \mid \mu(x)n = x^i n \text{ all } x \in B(k)^\times\},$$

then

$$\mathrm{Fil}^p N = \bigoplus\nolimits_{i \geq p} N^i, \quad \Lambda = \bigoplus\nolimits_i \Lambda \cap N^i.$$

The condition for Λ to be strongly divisible then becomes

$$\phi\Lambda = \mu(p)\Lambda. \tag{4.21.1}$$

(b) Let N be a weakly admissible filtered $B(k)$-module. In Wintenberger 1984 it is shown that there is a ***canonical*** splitting $\mu_W: \mathbb{G}_m \to \mathrm{GL}(N)$ of the filtration on N. When k is algebraically closed, a lattice Λ in N is strongly divisible if and only if μ_W splits the filtration on Λ and (4.21.1) holds.

The points of $\mathrm{Sh}_p(G, X)$ *with coordinates in* B*: crystalline interpretation*

The points of $\mathrm{Sh}_p(G, X)$ with coordinates in $B \stackrel{\mathrm{def}}{=} B(\mathbb{F})$ are in one-to-one correspondence with the set of isomorphism classes of of quadruples $(M, \mathfrak{s}, \eta^p, \Lambda_{\mathrm{crys}})$, where $(M, \mathfrak{s})$ is as in (4.16.1), η^p is as in (4.16.2), and

(4.22) Λ_{crys} is a strongly divisible lattice in $\omega_{\mathrm{dR}}(M)$ for which there exists an isomorphism

$$\eta_{\mathrm{dR}}: V(B) \longrightarrow \omega_{\mathrm{dR}}(M)$$

which maps each t_i to s_i, maps $V(W)$ onto Λ_{crys}, and makes the filtration $\mathrm{Filt}(\mu_0^{-1})$ correspond to the Hodge filtration on $\omega_{\mathrm{dR}}(M)$.

EXPLANATION 4.23. We give a *heuristic* explanation of how to pass from (4.16.3) to (4.22).

Let $\Gamma = \mathrm{Gal}(\bar{B}/B)$. We noted above that $\mathbb{D}$ and $\mathbb{V}$ define a $\otimes$-equivalence between the category of crystalline representations of Γ and that of admissible filtered Dieudonné $B(k)$-modules. In Fontaine and Laffaille 1982, it is shown that a weakly admissible filtered $B(k)$-module is admissible if the length of its filtration is $< p-1$. Hence $\mathbb{V}$ defines a $\otimes$-equivalence from the category of (weakly) admissible filtered $B(k)$-modules generated by those with length $< p-1$ to a category crystalline representations of Γ. It is known that this equivalence underlies an equivalence between a category of strongly divisible lattices and a category of Γ-stable lattices in crystalline representations. See Fontaine 1990, 2.3.

Let $\sigma = \mu_W(p^{-1}) \circ \phi$. Then σ defines a $\mathbb{Q}_p$-structure on any weakly admissible filtered $B(\mathbb{F})$-module i.e., if we set

$$N^{\sigma=1} = \{x \in N \mid \sigma x = x\}$$

then $N^{\sigma=1} \otimes_{\mathbb{Q}_p} B(\mathbb{F}) = N$. Moreover, (4.21.1) implies that, for any strongly divisible lattice Λ in N, $\Lambda^{\sigma=1}$ is a $\mathbb{Z}_p$-structure on Λ.

Let $\mathsf{Rep}^{\mathrm{crys}}(\Gamma)$ be the (Tannakian) category of crystalline representations of Γ. There are two fibre functors on $\mathsf{Rep}^{\mathrm{crys}}(\Gamma)$ over $\mathbb{Q}_p$, namely the forgetful functor and the functor $H \rightsquigarrow \mathbb{D}(H)^{\sigma=1}$. Assume that the torsor relating the two is trivial (cf. Wintenberger 1984, 4.2.5) and choose a trivialization. Then we can identify $\mathbb{D}(\mathbb{V}(H))$ with $H^{\sigma=1} \otimes B$ endowed with the filtration defined by μ_W and with ϕ acting as $x \mapsto \mu_W(p) \cdot \sigma x$.

Given Λ_p satisfying (4.16.3), define $\Lambda_{\mathrm{crys}} = \Lambda_p \otimes W$ and $\eta_{\mathrm{dR}} = \eta_p \otimes 1$. Then η_{dR} maps each t_i to s_i and it maps $V(W)$ onto Λ_{crys}. Note that these properties determine it uniquely up to conjugation by an element of $G(W)$. When we make a base change by $B \to \mathbb{C}$, we obtain a homomorphism

$$\eta_{\mathrm{dR}} \otimes 1 \colon V(\mathbb{C}) \longrightarrow \omega_{\mathrm{dR}}(M_{\mathbb{C}})$$

carrying each t_i into s_i. By (4.16.1), we already have such a map, namely, $\beta^{-1} \otimes 1$, which, moreover, has the property that it maps μ_x, some $x \in X$, to μ_M. Since the two maps differ by conjugation with an element of $G(\mathbb{C})$, this shows that $\eta_{\mathrm{dR}} \otimes 1$ maps $\mathfrak{c}(X)$ into the conjugacy class of μ_M. By definition, μ_M^{-1} splits the Hodge filtration on $\omega_{\mathrm{dR}}(M_{\mathbb{C}})$, and it follows that, after possibly replacing it with a $G(W)$-conjugate, η_{dR} will map $\mathrm{Filt}(\mu_0^{-1})$ to the Hodge filtration on $\omega_{\mathrm{dR}}(M)$.

Conversely, given Λ_{crys} we can define $\Lambda_p = (\Lambda_{\mathrm{crys}})^{\sigma=1}$.

Integral canonical models

As we saw in 4.16, in the case that $\mathrm{Sh}_p(G,X)$ is a fine moduli variety, its points with coordinates in B parametrize certain quadruples $(M, \mathfrak{s}, \eta^p, \Lambda_p)$. The existence of η^p implies that $\mathrm{Gal}(\bar{B}/B)$ acts trivially on $\omega_f^p(M)$, and this should imply that the motive has good reduction. In fact the whole quadruple should extend over W, and so we should have $\mathrm{Sh}_p(W) = \mathrm{Sh}_p(B)$. In Milne 1992, §2, this intuition is turned into the definition of a canonical model of $\mathrm{Sh}_p(G,X)$ over W. In order to achieve uniqueness, it is necessary to specify the points of the model, not just in W, but in very large, not necessarily Noetherian, W-algebras.

DEFINITION 4.24. A ***model*** of $\mathrm{Sh}_p(G,X)$ over $\mathcal{O}_v$ is a scheme S over $\mathcal{O}_v$ together with a continuous action of $G(\mathbb{A}_f^p)$ and a $G(\mathbb{A}_f^p)$-equivariant isomorphism

$$\gamma \colon S \otimes_{\mathcal{O}_v} E_v \to \mathrm{Sh}_p(G,X)_{E_v}$$

(of pro-varieties over E_v). Such a model is said to be ***smooth*** if there is a compact open subgroup K_0 of $G(\mathbb{A}_f^p)$ such that S_K is smooth over $\mathcal{O}_v$ for all $K \subset K_0$, and $S_{K'}$ is étale over S_K for all $K' \subset K \subset K_0$. Such a model is said to have the ***extension property*** if, for every regular scheme Y (not necessarily Noetherian)[27] over $\mathcal{O}_v$ such that Y_{E_v} is dense in Y, every E_v-morphism $Y_{E_v} \to S_{E_v}$ extends uniquely to an $\mathcal{O}_v$-morphism $Y \to S$. An ***integral canonical model*** of $\mathrm{Sh}_p(G,X)$ is a smooth model over $\mathcal{O}_v$ with the extension property.

CONJECTURE 4.25. *The variety* $\mathrm{Sh}_p(G,X)$ *always has an integral canonical model.*

In Milne 1992, the following results are proved:

(4.26) The integral canonical model of $\mathrm{Sh}_p(G,X)$, if it exists, is uniquely determined up to a unique isomorphism.

(4.27) The Siegel modular variety $\mathrm{Sh}_p(G(\psi),X(\psi))$ has an integral canonical model[28][29], namely, the moduli scheme constructed in Mumford 1965.

(4.28) Consider an inclusion $(G,X) \hookrightarrow (G',X')$ of pairs satisfying the axioms (SV1,2,3,6). Let K'_p be a hyperspecial subgroup of $G'(\mathbb{Q}_p)$, and assume that $K_p \overset{\text{def}}{=} K'_p \cap G(\mathbb{Q}_p)$ is hyperspecial in $G'(\mathbb{Q}_p)$. Then (cf. 3.24) there is a closed immersion

$$\mathrm{Sh}_p(G,X) \hookrightarrow \mathrm{Sh}_p(G',X')$$

defined over $E(G,X)$. If $\mathrm{Sh}_p(G',X')$ has a model S' over $\mathcal{O}_v$ with the extension property, then the closure of $\mathrm{Sh}_p(G,X)$ in S' also has the extension property (and hence will be canonical if it is smooth).

We offer two further remarks.

REMARK 4.29. Let $\mathrm{Sh}(G,X)$ be a Shimura variety of PEL-type, so that $\mathrm{Sh}_p(G,X)$ solves a moduli problem over $E(G,X)$ classifying isomorphism classes of triples consisting of a polarized abelian variety, an identification of the endomorphism algebra of the abelian variety with a fixed algebra, and a level structure, all satisfying certain conditions. (See Milne 1992, 1.1, for a precise definition.) There is a homomorphism $G \hookrightarrow G(\psi)$ sending X into $X(\psi)$, and hence a closed immersion $\mathrm{Sh}_p(G,X) \hookrightarrow \mathrm{Sh}_p(G(\psi),X(\psi))$ into the Siegel modular variety for any hyperspecial subgroup K'_p of $G(\psi)(\mathbb{Q}_p)$ such that $K_p = G(\mathbb{Q}_p) \cap K'_p$. It is more-or-less known that the closure of $\mathrm{Sh}_p(G,X)$ in the integral canonical model of $\mathrm{Sh}_p(G(\psi),X(\psi))$ represents a smooth functor, and therefore is itself smooth (see for example, Langlands and Rapoport 1987, 6.2; corrected in Kottwitz 1992, §5). Together with (4.28), this proves that $\mathrm{Sh}_p(G,X)$ has an integral canonical models.

REMARK 4.30. The validity of Conjecture 4.25 does not depend on $Z(G)$, at least if G^{der} is simply connected. To prove this, we need to make use of connected Shimura varieties (Deligne 1979; Milne 1990, II.1), and a descent theorem (Bosch et al. 1990, 6.2, Proposition C.1).

[27] Added 2017. Actually, there is no need to consider non Noetherian rings because $S = \varprojlim \mathrm{Sh}_K$ is Noetherian (Milne 1992, 3.4).

[28] The proof of this uses, in an essential way, Theorem V.6.8 of Faltings and Chai 1990. Recently I have learned that Gabber has a counterexample to this theorem, but the example appears to require ramification. Thus it is possible that the theorem of Faltings and Chai remains valid over the Witt vectors, at least for $p \neq 2$. If not, the definition of an integral canonical model will have to be modified. [Added 2017. In fact, Theorem 6.8 of Faltings and Chai does hold when $e < p-1$ (Vasiu and Zink, Doc. Math. 2010).]

[29] Added 2017. The example was found by Raynaud, reconstructed by Gabber, popularized by Ogus, and published by de Jong and Oort (J. Alg. Geom. 1997). Note that the last authors, when referring to my 1992 conjecture, cite the wrong paper and fail to acknowledge that I had earlier pointed out the error in this article and elsewhere.

The descent theorem says the following: The functor that attaches to an $\mathcal{O}_v$-scheme S the triple (S_1, S_2, θ) consisting of the E_v-scheme $S_1 \overset{\text{def}}{=} S \otimes_{\mathcal{O}_v} E_v$, the W-scheme $S_2 \overset{\text{def}}{=} S \otimes_{\mathcal{O}_v} W$, and the canonical isomorphism $\theta\colon S_1 \otimes_{E_v} B \to S_2 \otimes_W B$, is fully faithful. Its essential image consists of all triples (S_1, S_2, θ) that admit a quasi-affine open covering (in an obvious sense).

Now consider a system (G, X, K_p) as before, and suppose that G^{der} is simply connected. The composite of $h \in X$ with $G \to G^{\text{ab}}$ is independent of h—we write it h_X. The map $G \to G^{\text{ab}}$ defines a surjection

$$\text{Sh}(G, X) \to \text{Sh}(G^{\text{ab}}, h_X),$$

which, for simplicity, we assume identifies $\pi_0(\text{Sh}(G, X))$ with $\text{Sh}(G^{\text{ab}}, h_X)$ (in general, $\pi_0(\text{Sh}(G, X))$ will be a finite covering of $\text{Sh}(G^{\text{ab}}, h_X)$; see Deligne 1971b, 2.7.1). The inverse image of $e = [h^{\text{ab}}, 1]$ in $\text{Sh}(G, X)$ can be identified with the connected Shimura variety $\text{Sh}^0(G^{\text{der}}, X^+)$ for a suitable connected component X^+ of X.

On passing to the quotient by K_p we obtain a surjection

$$\text{Sh}_p(G, X) \to \text{Sh}_p(G^{\text{ab}}, h_X)$$

whose fibre over $[h_X, 1]$ we denote by $\text{Sh}^0_p(G^{\text{der}}, X^+)$. Because G is unramified at p, G^{ab} splits over B, and all the points of $\text{Sh}_p(G^{\text{ab}}, h_X)$ are rational over B (cf. Milne 1992, 2.16).

Now suppose $\text{Sh}_p(G, X)$ has an integral canonical model S over $\mathcal{O}_v$. From S_2 we obtain a model of the connected Shimura variety $\text{Sh}^0_p(G^{\text{der}}, X^+)$ over W that is smooth and has the extension property (in the sense of 4.24), and S_2 can be recovered from this integral canonical model by "induction", i.e., by translating it by elements of $G(\mathbb{A}^p_f) \times Z(\mathbb{Q}_p)$. Thus, by using the descent theorem, we see that S can be recovered from the integral canonical model of $\text{Sh}^0_p(G^{\text{der}}, X^+)$ over W.

Suppose now that $\text{Sh}^0_p(G^{\text{der}}, X^+)$ arises in the same way from a second variety $\text{Sh}_p(G_1, X_1)$. From the above discussion we see that, if $\text{Sh}_p(G_1, X_1)$ has an integral canonical model, then so also does $\text{Sh}_p(G, X)$.

EXAMPLE 4.31. Let B be a quaternion algebra over a totally real field F, split at the real primes v_i for $1 \le i \le r$ and nonsplit at the real primes v_i for $i > r$. Denote its canonical involution by $z \mapsto \bar{z}$. Let V be a free B-module endowed with a nondegenerate symmetric F-bilinear form Φ such that

$$\Phi(bx, y) = \Phi(x, \bar{b}y), \quad b \in B, \quad x, y \in V.$$

Assume that for $i > r$, the form defined by Φ on $V \otimes_{F, v_i} \mathbb{R}$ is positive-definite. Define G to be the reductive group over $\mathbb{Q}$ such that

$$G(\mathbb{Q}) = \{g \in \text{GL}_B(V) \mid \Phi(gx, gy) = \mu(g)\Phi(x, y), \text{ some } \mu(g) \in F^\times\}.$$

There is a unique $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ satisfying (SV1,2). The Shimura variety $\text{Sh}(G, X)$ is not a moduli variety if $r \ne [F\colon\mathbb{Q}]$ because its weight is not defined over $\mathbb{Q}$. However, there is a Shimura variety of PEL-type $\text{Sh}(G_*, X_*)$ such that $(G, X)^+ = (G_*, X_*)^+$ (see Deligne 1971b, §6). Therefore we can apply the preceding remarks to show that $\text{Sh}_p(G, X)$ has an integral canonical model.

Henceforth, we assume that $\text{Sh}_p(G, X)$ has an integral canonical model, which we again denote $\text{Sh}_p(G, X)$.

The points of $\mathrm{Sh}_p(G,X)$ *with coordinates in* W

From the definition of the integral canonical model, the points of $\mathrm{Sh}_p(G,X)$ with coordinates in W are the same as those with coordinates in B, but we shall need a second interpretation.

The following conditions on a motive M over B should be equivalent:

(4.32.1) the action of $\mathrm{Gal}(B^{\mathrm{al}}/B)$ on $\omega_f^p(M)$ is trivial;

(4.32.2) the filtered Dieudonné module $\omega_{\mathrm{dR}}(M)$ is admissible;

(4.32.3) M has good reduction.

We assume the equivalence of (4.32.1) and (4.32.2) for abelian motives, and use (4.32.3) to define good reduction. It is probably too optimistic to expect that the conditions imply that M extends to a motive over W constructed from smooth schemes over W. Rather, one expects that M will extend to a "log-smooth" motive over W whose reduction is smooth. If M is a motive over B with a $\mathbb{Z}_p$-integral structure, so that $\omega_{\mathrm{dR}}(M)$ is a W-module, then one expects it to be a strongly divisible lattice in $\omega_{\mathrm{dR}}(M)\otimes\mathbb{Q}$ when $M\otimes\mathbb{Q}$ has good reduction and the length of the filtration on $\omega_{\mathrm{dR}}(M)\otimes\mathbb{Q}$ is $<p$.

Define $\mathsf{Mot}^{\mathrm{ab}}(W)$ to be the subcategory of $\mathsf{Mot}^{\mathrm{ab}}(B)$ satisfying the conditions (4.32), and let $\mathsf{Mot}(\mathbb{F})$ be the category of motives over $\mathbb{F}$, defined as in (Milne 1994, §1). We shall need to assume that the étale and crystalline fibre functors are defined on $\mathsf{Mot}(\mathbb{F})$ and that there is a "reduction" functor of tensor categories

$$M \rightsquigarrow \bar{M}\colon \mathsf{Mot}^{\mathrm{ab}}(W)\to\mathsf{Mot}(\mathbb{F})$$

such that

(4.33.1) $\omega_f^p(\bar{M})=\omega_f^p(M)$; $\omega_{\mathrm{crys}}(\bar{M})=\omega_{\mathrm{dR}}(M)$;

(4.33.2) the functor $M\rightsquigarrow(\bar{M},\omega_{\mathrm{dR}}(M))$ is fully faithful (here $\omega_{\mathrm{dR}}(M)$ is regarded as a filtered Dieudonné B-module).

Of course, we also expect that the functor $M\rightsquigarrow\bar{M}$ and the isomorphisms implicit in (4.33.1) are compatible with those on abelian varieties.

With the above assumptions, we see that the points of $\mathrm{Sh}_p(G,X)$ with coordinates in W are in natural one-to-one correspondence with the set of isomorphism classes of quintuples $(M,\mathfrak{s},F,\eta^p,\Lambda_{\mathrm{crys}})$ consisting of a motive M over $\mathbb{F}$, a family $\mathfrak{s}$ of tensors on M, a filtration F on $\omega_{\mathrm{crys}}(M)$, an isomorphism $\eta^p\colon V(\mathbb{A}_f^p)\to\omega_f^p(M)$, and a lattice $\Lambda_{\mathrm{crys}}\subset\omega_{\mathrm{crys}}(M)$ satisfying the following condition:

(4.34) there is a quadruple $(\widetilde{M},\widetilde{\mathfrak{s}},\widetilde{\eta}^p,\widetilde{\Lambda}_{\mathrm{crys}})$ satisfying (4.16.1, 4.16.2, 4.22) that maps to $(M,\mathfrak{s},F,\eta^p,\Lambda_{\mathrm{crys}})$ under the reduction functor.

The points of $\mathrm{Sh}_p(G,X)$ *with coordinates in* $\mathbb{F}$

Because $\mathrm{Sh}_p(G,X)$ is smooth over W, the reduction map

$$\mathrm{Sh}_p(G,X)(W)\longrightarrow\mathrm{Sh}_p(G,X)(\mathbb{F})$$

is surjective. When we interpret $\mathrm{Sh}_p(G,X)(W)$ as in (4.34), this map should correspond to the reduction of motives. Assume this. Then the points of $\mathrm{Sh}_p(G,X)$ with coordinates in $\mathbb{F}$ are in natural one-to-one correspondence with the set of isomorphism classes of quadruples $(M,\mathfrak{s},\eta^p,\Lambda_{\mathrm{crys}})$ consisting of a motive M over $\mathbb{F}$, a family $\mathfrak{s}$ of tensors on M, an isomorphism $\eta^p\colon V(\mathbb{A}_f^p)\to\omega_f^p(M)$, and a lattice $\Lambda_{\mathrm{crys}}\subset\omega_{\mathrm{crys}}(M)$ satisfying the following condition:

(4.35) there exists a filtration F on $\omega_{\text{crys}}(M)$ and a quadruple $(\widetilde{M},\widetilde{\mathfrak{s}},\widetilde{\eta}^p,\widetilde{\Lambda}_{\text{crys}})$ satisfying (4.16.1, 4.16.2, 4.22) that maps to $(M,\mathfrak{s},F,\eta^p,\Lambda_{\text{crys}})$ under the reduction functor.

Call a pair $N=(M,\mathfrak{s})$ ***admissible*** if there exists a pair $(\eta^p,\Lambda_{\text{crys}})$ such that $(M,\mathfrak{s},\eta^p,\Lambda_{\text{crys}})$ satisfies the condition (4.35).

Fix N, and let $S(N)$ be the set of quadruples $(M,\mathfrak{s},\eta^p,\Lambda_{\text{crys}})$ such that $(M,\mathfrak{s})\approx N$. Then

$$\mathrm{Sh}_p(\mathbb{F})=\coprod S(N),$$

where the disjoint union is over a set of representatives for the isomorphism classes of the admissible pairs. The actions of $G(\mathbb{A}_f^p)$ and Φ preserve $S(N)$.

Let $I(N)=\mathrm{Aut}(M,\mathfrak{s})$. Define $X^p(N)$ to be the set of isomorphisms $\eta^p\colon V(\mathbb{A}_f^p)\to\omega_f^p(M)$ carrying each t_i to s_i, and define $X_p(N)$ to be the set of lattices Λ_{crys} in $\omega_{\text{crys}}(M)$ for which there exists a filtration F on $\omega_{\text{crys}}(M)$ such that $(M,\mathfrak{s},F,\Lambda_{\text{crys}})$ is the reduction of a triple $(\widetilde{M},\widetilde{\mathfrak{s}},\Lambda_{\text{crys}})$ satisfying (4.22). There is an action of $I(N)$ on $X^p(N)\times X_p(N)$ on the left, an action of $G(\mathbb{A}_f^p)$ on $X^p(N)$, an action of $Z(\mathbb{Q}_p)$ on $X_p(N)$ and an action of Φ on $X_p(N)$.

PROPOSITION 4.39. *With the above assumptions, there is a canonical bijection*

$$S(N)\simeq I(N)\backslash X^p(N)\times X_p(N)$$

compatible with the actions of $G(\mathbb{A}_f^p)$ *and* Φ.

PROOF. Obvious. ∎

Let $N=(M,\mathfrak{s})$ be admissible. Choose an isomorphism

$$\beta\colon\omega_{\text{crys}}(M)\longrightarrow V(B)$$

sending s_i to t_i for all i. The map

$$x\mapsto\beta\phi\beta^{-1}(\sigma^{-1}x)\colon V(B)\to V(B)$$

is linear, and it maps t_i to t_i for all i. Therefore it is multiplication by an element $b\in G(B)$. This b is the unique element of $G(B)$ such that

$$\beta\phi(y)=b\sigma\beta(y),\text{ all }y\in\omega_{\text{crys}}(M).$$

If we replace β with $g\circ\beta$, $g\in G(B)$, then b is replaced by its σ-conjugate $gb(\sigma g)^{-1}$.

Let $\Lambda_{\text{crys}}\in X_p(N)$. According to our assumption, there exists an isomorphism

$$\eta_p\colon V(B)\longrightarrow\omega_{\text{crys}}(M)$$

mapping each t_i to s_i, mapping $V(W)$ onto Λ_{crys}, and such that Λ_{crys} is strongly divisible for the filtration defined by μ_0^{-1}. The composite $\beta\circ\eta_p$ fixes each t_i, and hence is multiplication by $g\in G(B)$. The situation is summarized by the following diagram:

$$\begin{array}{ccccc}
V(B) & \xrightarrow{\eta_p} & \omega_{\text{crys}}(M) & \xrightarrow{\beta} & V(B)\\
t_i & \longleftrightarrow & s_i & \longleftrightarrow & t_i\\
V(W) & \twoheadrightarrow & \Lambda_{\text{crys}} & \twoheadrightarrow & gV(W)\\
g^{-1}b\sigma g & \longleftrightarrow & \phi & \longleftrightarrow & b\sigma\\
\mathrm{Filt}(\mu_0^{-1}) & \longleftrightarrow & F & \longleftrightarrow & \mathrm{Filt}(g\mu_0^{-1}).
\end{array}$$

The vector space $V(B)$ endowed with the σ-linear map $x \mapsto g^{-1}b\sigma(gx)$ and the filtration $\mathrm{Filt}(\mu_0^{-1})$ is a weakly admissible filtered module, with $V(W)$ as a strongly divisible module. Since μ_0^{-1} splits the filtration on $V(W)$, according to (4.21.1) this last condition means that

$$(g^{-1}b\sigma g)V(W) = \mu_0(p^{-1})V(W).$$

But the stabilizer in $G(B)$ of $V(W)$ is $G(W)$, and so

$$g^{-1}b\sigma g \in G(W)\cdot\mu_0(p^{-1})\cdot G(W).$$

PROPOSITION 4.40. *The map $\Lambda_{\mathrm{crys}} \mapsto g\cdot G(W)$ defines a bijection*

$$X_p(N) \longrightarrow \{g\cdot G(W) \in G(B)/G(W) \mid g^{-1}\cdot b\cdot\sigma g \in G(W)\cdot\mu_0(p^{-1})\cdot G(W)\}.$$

PROOF. Straightforward. ∎

We now restate our results in terms of the groupoid attached to the category of motives over $\mathbb{F}$. Let w be the extension of v to $\mathbb{Q}^{\mathrm{al}}$ induced by the inclusion $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{C}_p$, and let $\mathbb{Q}^w = \mathbb{Q}^{\mathrm{al}} \cap B$. The category $\mathsf{Mot}(\mathbb{F})$ has a fibre functor over $\mathbb{Q}^{\mathrm{al}}$ (Milne 1994), and the obstruction to it having a fibre functor over $\mathbb{Q}^w$ lies in $H^2(\mathbb{Q}^w, P)$, where P is the Weil number pro-torus (ibid. §2). But a theorem of Lang shows that $\mathbb{Q}^w$ is a C_1-field (Shatz 1972, p. 116, Theorem 27), and so $H^2(\mathbb{Q}^w, P) = 0$. Thus $\mathsf{Mot}(\mathbb{F})$ has a fibre functor over $\mathbb{Q}^w$. We choose one, and let $\mathfrak{M}$ be the corresponding $\mathbb{Q}^w/\mathbb{Q}$-groupoid. For each $\ell \neq p, \infty$, étale cohomology provides a fibre functor ω_ℓ over $\mathbb{Q}_\ell$, and correspondingly we obtain a morphism of groupoids

$$\zeta_\ell \colon \mathfrak{G}_\ell \longrightarrow \mathfrak{M}(\ell),$$

where $\mathfrak{G}_\ell$ is the trivial $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid and $\mathfrak{M}(\ell)$ is the $\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell$-groupoid obtained from $\mathfrak{M}$ by base change. Let $\mathfrak{G}_p$ be the $B/\mathbb{Q}_p$-groupoid attached to the category of isocrystals over $\mathbb{F}$ and the forgetful functor. Then ω_{crys} defines a morphism of groupoids

$$\zeta_p \colon \mathfrak{G}_p \longrightarrow \mathfrak{M}(p).$$

Finally, there is a morphism of groupoids over $\mathbb{R}$,

$$\zeta_\infty \colon \mathfrak{G}_\infty \longrightarrow \mathfrak{M}(\infty)$$

(Milne 1994, 3.29). Let $\mathfrak{G}_G$ denote the $\mathbb{Q}^w/\mathbb{Q}$-groupoid defined by G. With each homomorphism $\varphi \colon \mathfrak{M} \to \mathfrak{G}_G$ and representation of G, there is associated a motive $N(\varphi)$ over $\mathbb{F}$ endowed with a family of tensors, and we say that φ is ***admissible*** if $N(\varphi)$ is admissible in the above sense for one (hence every) faithful representation of G. For an admissible φ, define

$$I(\varphi) = \mathrm{Aut}(\varphi) \stackrel{\text{def}}{=} \{g \in G(\mathbb{Q}^w) \mid \mathrm{ad}(g)\circ\varphi = \varphi\}$$
$$X^p(\varphi) = \prod_{\ell \neq p} X_\ell(\varphi) \quad \text{(restricted product), where} \quad X_\ell(\varphi) = \mathrm{Isom}(\xi_\ell, \varphi(\ell)\circ\zeta_\ell).$$

Here $\varphi(\ell)$ is obtained from φ by base change, and ξ_ℓ is the obvious morphism $\mathfrak{G}_\ell \to \mathfrak{G}_G(\ell)$ (see Milne 1992, p. 186, for more details). Choose an $s(\sigma)$ in $\mathfrak{G}_p$ mapping to $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$, and let $(\varphi(p)\circ\zeta_p)(s(\sigma)) = (b, \sigma)$. Define

$$X_p(\varphi) = \{g\cdot G(W) \in G(B)/G(W) \mid g^{-1}\cdot b\cdot\sigma g \in G(W)\cdot\mu_0(p^{-1})\cdot G(W)\}.$$

As (ibid. p. 188) we can define an operator Φ on $X_p(\varphi)$. Let

$$S(\varphi) = I(\varphi)\backslash X^p(\varphi) \times X_p(\varphi),$$

with $G(\mathbb{A}_f^p)$ acting through its action on $X^p(\varphi)$, and Φ acting through its action on $X_p(\varphi)$.

THEOREM 4.41. *With the above assumptions, there is an isomorphism of sets with actions*

$$(\mathrm{Sh}_p(\mathbb{F}), \times, \Phi) \longrightarrow \coprod_{\varphi} (S(\varphi), \times(\varphi), \Phi(\varphi)),$$

where the disjoint union is over a set of representatives for the isomorphism classes of admissible homomorphisms $\mathfrak{M} \to \mathfrak{G}_G$.

PROOF. The choice of a faithful representation of G determines a bijection $\varphi \mapsto N(\varphi)$ from the set of isomorphism classes of admissible φ to the set of isomorphism classes of admissible pairs. Moreover, using (4.35), one sees that

$$(S(\varphi), \times(\varphi), \Phi(\varphi)) = (S(N), \times(N), \Phi(N)),$$

and so it follows from (4.36) and the discussion preceding (4.36) that there is a bijection $\mathrm{Sh}_p(\mathbb{F}) \to \coprod S(\varphi)$ compatible with the action of $G(\mathbb{A}_f^p)$. Checking that the actions of Φ agree is straightforward (it is similar to the proof of the last step of the proof of 3.21). ∎

Statement of the conjecture

We now drop all unproven assumptions, and state a conjecture. Let $(\mathfrak{P}, (\zeta_\ell))$ be a system as Milne 1994, 3.31, except that now $\mathfrak{P}$ is a $\mathbb{Q}^w/\mathbb{Q}$-groupoid (rather than a $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid).

Given a homomorphism $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$, we can define a set $S(\varphi)$ with an action of $G(\mathbb{A}_f^p)$ and a commuting action of a Frobenius element Φ exactly as in the last section (see also Milne 1994, §4). The conjecture will then state that there is an isomorphism of sets with operators

$$(\mathrm{Sh}_p(\mathbb{F}), \times, \Phi) \longrightarrow \coprod_{\varphi} (S(\varphi), \times(\varphi), \Phi(\varphi)),$$

where the disjoint union is over a set of representatives for the isomorphism classes of “admissible” homomorphisms $\varphi\colon \mathfrak{P} \to \mathfrak{G}_G$. The only remaining problem in stating the conjecture is to define “admissible”.

NECESSARY LOCAL CONDITIONS

There are some obvious necessary conditions for φ to be admissible.

(4.39_ℓ) The set $X_\ell(\varphi)$ is nonempty.

(4.39_p) The set $X_p(\varphi)$ is nonempty.

(4.39_∞) The homomorphism $\varphi(\infty) \circ \zeta_\infty\colon \mathfrak{G}_\infty \to \mathfrak{G}_G(\infty)$ of $\mathbb{C}/\mathbb{R}$-groupoids is isomorphic to that defined by X (see Milne 1992, 4.5).

THE CASE OF A TORUS

Consider a pair (T,h) satisfying the conditions (SV1,2,3,6) with T a torus. There is a unique homomorphism $\rho(h)\colon S\to T$ such that $\rho(h)_{\mathbb{R}}\circ h_{\mathrm{can}}=h$. As is explained in (Milne 1994, §4), there is a canonical homomorphism $\mathfrak{P}'\to\mathfrak{G}_S$, and we write φ_h for its composite with the map $\mathfrak{G}_S\to\mathfrak{G}_T$ defined by $\rho(h)$. Here $\mathfrak{P}'$ is the $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid obtained from $\mathfrak{P}$ by base change. Assume $T(\mathbb{Q}_p)$ has a hyperspecial subgroup $T(\mathbb{Z}_p)$. Then φ_h arises from a homomorphism $\mathfrak{P}\to\mathfrak{G}_T$.

PROPOSITION 4.43. *Let* $\mathrm{Sh}_p=\mathrm{Sh}_p(T,x)$*; then the sets with operators* $(\mathrm{Sh}_p(\mathbb{F}),\times,\Phi)$ *and* $(S(\varphi_h),\times(\varphi_h),\Phi(\varphi_h))$ *are isomorphic.*

PROOF. See Milne 1992, 4.2. ∎

The homomorphism φ_h satisfies the conditions (4.39) above, but unless unless T satisfies the Hasse principle for H^1 there will be other such homomorphisms. This suggests adding another condition.

(4.39_0) The composite of φ with the projection $\mathfrak{G}_G\to\mathfrak{G}_{G^{\mathrm{ab}}}$ is equal to φ_{h_X}.

STATEMENT OF THE CONJECTURE

In the above, we have found conditions that are surely necessary for φ to be admissible. We now provide one that is surely sufficient.

Let $(T,x)\subset(G,X)$ be a special pair. As above, we obtain a homomorphism $\varphi_x\colon\mathfrak{P}\to\mathfrak{G}_T\subset\mathfrak{G}_G$. Define $\varphi\colon\mathfrak{P}\to\mathfrak{G}_G$ to be ***special*** if it becomes isomorphic to φ_x for some special pair (T,x) when extended to $\mathfrak{P}'$. We should have the following implications:

$$\varphi \text{ special} \implies \varphi \text{ admissible} \implies \varphi \text{ satisfies (4.39).}$$

THEOREM 4.44. *If* G^{der} *is simply connected, then* $\varphi\colon\mathfrak{P}\to\mathfrak{G}_G$ *is special if and only if it satisfies the conditions (4.39).*

PROOF. See Langlands and Rapoport 1987, 5.3. ∎

In other words, when G^{der} is simply connected, the obvious necessary condition agrees with the obvious sufficient condition. Thus, when G^{der} is simply connected we can define φ to be admissible if it satisfies (4.39) or (equivalently) if it is special.

Unfortunately, the two notions is diverge when G^{der} is not simply connected. For reasons that we shall presently explain, we choose the second condition.

CONJECTURE 4.45. *There is an isomorphism of sets with operators*

$$(\mathrm{Sh}_p(\mathbb{F}),\times,\Phi)\to\coprod_{\varphi}(S(\varphi),\times(\varphi),\Phi(\varphi)),$$

where the φ *runs over a set of representatives for the isomorphism classes of special homomorphisms* $\varphi\colon\mathfrak{P}\to\mathfrak{G}_G$.

THEOREM 4.46. *If Conjecture 4.42 is true for all Shimura varieties defined by groups with simply connected derived groups, then it is true for all Shimura varieties.*

PROOF. See Milne 1992, 4.19. ∎

An example of Langlands and Rapoport shows that (4.43) is false if the disjoint union in (4.42) is taken over isomorphism classes of homomorphisms satisfying (4.39). It is for this reason that we use special homomorphisms in the statement of (4.42).

In the case that G^{der} is simply connected, Conjecture 4.42 is essentially Conjecture 5.e of Langlands and Rapoport 1987.

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