Review of: Shimura, Collected Papers, Vols I, II, III, IV

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When Weil arrived in Tokyo in 1955, planning to speak about his ideas on the extension to abelian varieties of the classical theory of complex multiplication, he was surprised to learn that two young Japanese mathematicians had also made decisive progress on this topic.[1] They were Shimura and Taniyama. While Weil wrote nothing on complex multiplication except for the report on his talk, Shimura and Taniyama published their results in a book in Japanese, which, after the premature death of Taniyama, was revised and published in English by Shimura. For a polarized abelian variety with many complex multiplications, the theory describes the action of the absolute Galois group of a certain reflex field on the moduli of the variety and its points of finite order, and it expresses the zeta function of the abelian variety in terms of Hecke L-series. Over the years Shimura found various improvements to these original results, which are included among the papers collected in these volumes.

Complex multiplication is the foundation stone of what has become known as the theory of canonical models. Each elliptic modular curve is defined in a natural way over a number field k (which depends on the curve). For analysts, the explanation for this is that the Fourier expansions at the cusps provide a k-structure on the spaces of modular functions and forms. For geometers, the explanation is that the curve is the solution of a moduli problem which is defined over k. In one of his most significant results, Shimura showed that quotients of the complex upper half plane by quaternionic congruence groups are also naturally defined over number fields, even when compact (hence without cusps) and even when they are not moduli varieties in any natural way.[2] As the fruit of a long series of investigations, he found a precise notion of a canonical model for the congruence quotients of bounded symmetric domains and proved that they exist for important families. Let G be a semisimple algebraic group over $\mathbb{Q}$ such the quotient of $G(\mathbb{R})$ by a maximal compact subgroup is a bounded symmetric domain X. Then each quotient $\Gamma \backslash X$ of X by a congruence subgroup Γ of $G(\mathbb{Q})$ has a model over a specific number field k_Γ. As Γ varies, these models are compatible, and the whole family is called the canonical model. It is characterized by reciprocity laws at the CM-points, and its definition requires a realization of G as the derived group of a reductive group.[3]

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[1]Weil, CW II, p541. He was in Tokyo for the famous International Symposium on Algebraic Number Theory at Tokyo and Nikko.

[2]This work is described in his article Notices Amer. Math. Soc. 43 (1996), no. 11, 1340–1347 (CW IV, p491).

[3]See Shimura's talk at the ICM 1970 (Nice) (CW II p400) for more precise statements and a statement of the "Shimura conjecture".

In his talk at the International Congress in 1978,[4] Shimura asked the following series of questions:

   I. Can one define the notion of arithmetic automorphic functions?

  II. Can one define the notion of arithmetic automorphic forms?

 III. Are (holomorphic) Eisenstein series arithmetic?

 IV. Is there any explicit way to construct arithmetic automorphic forms, similar to Eisenstein series, in the case of compact quotient?

  V. Supposing the answers to these questions are affirmative, is there any interpretation of the values of such explicit arithmetic automorphic forms at CM-points as [critical] values of zeta functions?

Question I is answered by the theory of canonical models: the model of $\Gamma \backslash X$ over k_Γ provides a k_Γ-structure on the space of automorphic functions for Γ. Question II asks whether there is (at least) a natural $\bar{\mathbb{Q}}$-structure on the holomorphic automorphic forms. Much of Shimura's work over the last twenty-five years has been directed towards answering these questions, especially question V. This has involved the study of the periods of abelian varieties, Eisenstein series, differential operators on bounded symmetric domains, and the notion of near holomorphy. (For a more extensive overview of Shimura's work, I recommend the article of H. Yoshida, Bull. AMS, 39 (2002), 441-448.)

The four volumes under review collect all the papers published by Shimura between 1954 and 2001 except for a few which are mainly expository. It also includes three articles not previously published[5] and two articles published only in mimeographed proceedings of the conferences, and hence not generally available.[6] Most papers are reproduced directly from the originals, but fifteen have been newly typeset (not without the introduction of new misprints), including three[7] that were re-typeset from the author's manuscripts because of errors introduced into the published versions by an "incompetent copyeditor and typesetter".

Over fifty pages of endnotes have been added. Most notes correct misprints or other minor errors, but some give more extended clarifications or complements to the papers. The origins of the conjecture on the modularity of elliptic curves are revisited in the endnotes to the papers [64e] and [89a] (and also in the article [96b] itself).

In the preface, Shimura writes: "Some of my recollections are included with the hope that they may help the reader have a better perspective. I have also mentioned the results in my later articles which supersede or are related to those in the article at issue. However, I decided not to mention the results of other later investigators, mainly in order to make my task easier". Thus, the endnotes help place the individual papers in the general context of Shimura's work, but not in any wider context. In fact, the work whose origins can be traced to that of Shimura is extensive.

Whereas reductive groups play a somewhat auxiliary role in Shimura's work, Deligne adopted them as the starting point in his 1969 Bourbaki report on Shimura's work. There is now a large body of work on what are called Shimura varieties, expressed in the language of abstract reductive groups (roots and weights) and Grothendieck algebraic geometry (schemes and motives). In this more general context, the existence of canonical models had been proved for all Shimura varieties, including those attached to the exceptional groups, by

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[4] CW III p148.

[5] 1968c, 2001b, 2001c

[6] 1963e, 1964e

[7] 1967c, 1978c, 1997b

1982,[8] and by 1986 the theory of automorphic vector bundles had yielded in complete generality a notion of the arithmeticity of holomorphic automorphic forms over the reflex field (or even $\mathbb{Q}$).[9] Moreover, by 1982 the main theorems of complex multiplication had been extended to all automorphisms of $\bar{\mathbb{Q}}$ (not just those fixing the reflex field).[10] Thus, by the mid 1980s, it was possible to ask some of the arithmeticity questions mentioned above over $\mathbb{Q}$.[11][12]

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[8]Deligne, Borovoi, Milne.

[9]Brylinski, Harris, Milne.

[10]Deligne, Langlands

[11]This paragraph was my attempt to briefly place Shimura's work in context. In fact, since about 1970 there have been two schools in the field, which I'll refer to as the Shimura school and the Deligne school. In terms of the number of published papers, the first is much larger than the second. [In 2025, the reverse is true – everyone working in the Langlands program uses Deligne's approach.]

I will describe what I see as the essential difference between the two schools. Initially one begins with a semisimple group G over $\mathbb{Q}$ with $G(\mathbb{R})$ acting on a hermitian symmetric domain D (satisfying certain conditions). As Shimura first understood, to get a canonical model one needs to realize G as the derived group of a reductive group. For Shimura the reductive group is auxiliary: given G he makes the most convenient choice for the reductive group. On the other hand, Deligne begins with the reductive group. Different choices of the reductive group for a given G give different canonical models, but they all give the same connected canonical model (in the sense of Deligne — it is an inverse system of connected varieties over $\bar{\mathbb{Q}}$ with an action of a big group). I think that for most of what he does, Shimura only needs the connected canonical model (and, in general, that's all his theory gives). Thus, except for special Shimura varieties (those given by his choice of the reductive group), his is intrinsically a $\bar{\mathbb{Q}}$-theory, whereas Deligne's is a $\mathbb{Q}$-theory. The challenge is to rewrite all of the work done by the Shimura school in Deligne's language. This will mean, for example, that when the Shimura school proves that some special value is ap where a is an algebraic number and p is a transcendental period, one should prove that a lies in an abelian extension of a specific field and describe how the Galois group acts (and, when the Shimura school obtains the finer result for special Shimura varieties, one should obtain it for general Shimura varieties).

[12]In his papers (including his survey papers), his books, and in the comments in his Collected Papers, Shimura ignores almost all work not done by himself or his students. Consequently, a young mathematician studying only his works will get an incorrect impression of what is known in the field.