Arithmetic Duality

J.S. Milne

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Abstract: In the 1950s and 1960s Tate proved some duality theorems in the Galois cohomology of finite modules and abelian varieties. As for most of Tate's work this has had a profound influence on mathematics with many applications and further developments. I'll discuss Tate's theorems and some of these developments.

These are the slides for my talk, except that I've added a rough transcript of what I said while the slides were being displayed. The talk was my response to a request from the organizers:

> We've chosen speakers to put aspects of Tate's work into perspective and point to the future so that the current and next generation can be inspired by them as much as our generations were. In particular, we'd be very grateful if you could speak on the theme of arithmetic duality.

Notation: I generally follow the notation I learned from Tate. For example, $X(\ell)$ is the ℓ-primary component of an abelian group X.

1 Local duality (Tate 1957)

Tate's duality theorems are probably his most cited results.[1] He discovered his first duality theorem while trying to understand the Weil–Châtelet group of an abelian variety.

The Weil–Châtelet group of an abelian variety, for example, of an elliptic curve, is the first Galois cohomology group of the variety. It also has a geometric interpretation as the principal homogeneous spaces of the variety.

For an abelian variety A over a field K, the Weil–Châtelet group

$$\mathrm{WC}(A,K) \stackrel{\text{def}}{=} \begin{cases} H^1(K,A) \stackrel{\text{def}}{=} H^1(\mathrm{Gal}(\bar{K}/K), A(\bar{K})), \quad \bar{K} = K^{\text{sep}} \\ \{\text{principal homogeneous spaces mod isomorphism}\}. \end{cases}$$

Châtelet was the first to recognize its importance for the diophantine study of elliptic curves, and Weil for the study of abelian varieties, so, in his 1957 Bourbaki talk, Tate named them Weil–Châtelet groups.

Except that they are torsion, almost nothing was known about the groups until Tate, in his Bourbaki talk, proved that, when K is a local field of characteristic zero, the Weil–Châtelet group (with its discrete topology) is dual to the group of rational points on the dual abelian variety (a compact group).

[1] I think this is correct, but it is difficult to document as there is no canonical source for the theorems.

When K is a local field of characteristic zero (nonarchimedean, finite residue field of characteristic p), Tate constructed an isomorphism

$$H^1(K, A) \xrightarrow{\simeq} A'(K)^*,$$

where A' is the dual abelian variety and $*$ means the Pontryagin dual.

Since $A'(K)$ was well-understood at the time, this tells us a great deal about $\mathrm{WC}(A, K)$.

The group $A'(K)$ contains a subgroup of finite index isomorphic to $\mathcal{O}_K^{\dim A}$ (Mattuck 1957).

For example, it showed that the non-p part of $\mathrm{WC}(A, K)$ is finite, and has a description in terms of the torsion subgroup of $A'(K)$. Lang and Tate had proved this earlier, and it was by thinking about this and investigating the elliptic curve case that Tate was led to his theorem.

Many readers will recognize the statement as being part of what we now call Tate local duality. It took Tate a little longer. In the final paragraph of his Bourbaki talk, almost as an afterthought, he noted that there are canonical pairings for all r, s, and that they should give dualities when $r + s = 1$.

Tate defined canonical pairings

$$H^r(K, A) \times H^s(K, A') \to H^{r+s+1}(K, \mathbb{G}_m) \stackrel{\text{def}}{=} H^{r+s+1}(\mathrm{Gal}(\bar{K}/K), \bar{K}^\times)$$

and showed they give a duality when $r + s = 1$,

$$H^r(K, A) \times H^{1-r}(K, A') \to H^2(K, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Once he had proved that, he was able to deduce, by a five-lemma argument, a 2 dimensional duality for the finite Galois modules[2] occurring as submodules of $A(K)$.

From the exact sequence

$$0 \to M \to A(K) \to B(K) \to 0$$

defined by an isogeny $A \to B$ and its dual,

$$0 \to M^D \to B'(K) \to A'(K) \to 0, \qquad M^D \stackrel{\text{def}}{=} \mathrm{Hom}(M, \bar{K}^\times)$$

Tate deduced (five-lemma)

$$\begin{array}{ccccccccc}
H^0(K,A) & \longrightarrow & H^0(K,B) & \longrightarrow & H^1(K,M) & \longrightarrow & H^1(K,A) & \longrightarrow & H^1(K,B) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
H^1(K,A')^* & \longrightarrow & H^1(K,B')^* & \longrightarrow & H^1(K,M^D)^* & \longrightarrow & H^0(K,A')^* & \longrightarrow & H^0(K,B')^*.
\end{array}$$

a duality

$$H^r(K, M) \times H^{2-r}(K, M^D) \to H^2(K, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

For a while, Tate thought this was a curious property of the Galois submodules of $A(K)$. Eventually, of course, he realized that all finite Galois modules have this property, and so obtained what we now call Tate local duality.[3]

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[2] By a finite Galois module, I mean a finite abelian group with a continuous action of the Galois group.

[3] Thus, the duality theorem for abelian varieties was proved before the (easier!) duality theorem for finite Galois modules, and even before a local duality theorem was available for elliptic curves.

There are compatible dualities

$$H^r(K,M)\times H^{2-r}(K,M^D)\to H^2(K,\mathbb{G}_m)\simeq \mathbb{Q}/\mathbb{Z},\quad \mathbb{G}_m\overset{\text{def}}{=}\mathrm{GL}_1$$
$$H^r(K,A)\times H^{1-r}(K,A')\to H^2(K,\mathbb{G}_m)\simeq \mathbb{Q}/\mathbb{Z},$$

M a finite Galois module, $M^D=\mathrm{Hom}(M,\bar{K}^\times)$, A an abelian variety, A' the dual abelian variety.

I should say a word about the dual abelian variety. Every abelian variety has a dual, which is an abelian variety of the same dimension, but not necessarily isomorphic. The dual of an elliptic curve is the curve itself. Usually, the dual of A is defined to classify translation invariant line bundles on A, but, as Weil observed, when you remove the zero-section of such line bundle, it acquires a group structure that makes it an extension of A by $\mathbb{G}_m$.

THE DUAL ABELIAN VARIETY

$$A'=\begin{cases}\mathrm{Pic}^0(A) & \text{[translation invariant line bundles]}\\ \mathrm{Ext}(A,\mathbb{G}_m) & [1\to\mathbb{G}_m\to E\to A\to 1].\end{cases}$$

In this way, A' can be identified with $\mathrm{Ext}(A,\mathbb{G}_m)$. This makes it easier to define the pairings. Indeed, when his collected works were published almost 60 years after he gave his Bourbaki talk, Tate added a note saying exactly that.

> In hindsight, the [cohomological] pairing for dual abelian varieties A and B is evident from the relation $B=\mathrm{Ext}(A,\mathbb{G}_m)$,...(Tate, Collected Works, I, p.127).

2 Global duality (Tate 1962 ICM)

Tate immediately recognized the importance of extending his local duality theorems to global fields. By 1960 he knew the statements he wanted, but not the proofs. By early 1962 he had the proofs, in time to announce his theorems at the 1962 ICM in Stockholm.[4]

One statement of his theorem is that there is a nine-term exact sequence, as below. To understand the sequence, note that the β's map the global Galois cohomology group into a product of the local groups. One would like to know the kernels and cokernels of these maps, but there is no simple expression for these. The best one can do is Tate's sequence.

[4]Poitou proved similar theorems for finite Galois modules at about the same time as Tate, and so the duality theorems are usually credited to both. Except that Serre alerted each of Poitou and Tate to the work of the other, they do not seem to have had any direct contact.

- K =global field; $\bar{K}$ = separable closure of K;
- M finite $\mathrm{Gal}(\bar{K}/K)$-module; $\mathrm{char}(K) \nmid [M]$;
- $M^D = \mathrm{Hom}(M, \bar{K}^\times)$; $*$=Pontryagin dual;
- $H^0(\mathbb{R}, M) = M^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}/(1+\iota)M$, $H^0(\mathbb{C}, M) = 0$.

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(K,M) & \xrightarrow{\beta^0} & \prod_v H^0(K_v,M) & \xrightarrow{\gamma^0} & H^2(K,M^D)^* & \\ & & & & & \downarrow & \\ & H^1(K,M^D)^* & \xleftarrow{\gamma^1} & \prod_v H^1(K_v,M) & \xleftarrow{\beta^1} & H^1(K,M) & \\ & \downarrow & & & & & \\ & H^2(K,M) & \xrightarrow{\beta^2} & \bigoplus_v H^2(K_v,M) & \xrightarrow{\gamma^2} & H^0(K,M^D)^* & \longrightarrow 0. \end{array}$$

For $r \geq 3$, $H^r(K,M) \simeq \bigoplus_{v \text{ real}} H^r(K_v,M)$.

Tate's 1963 proof

I do not think we know Tate's original proofs of his global duality theorems, but in a letter to Serre (25.04.63), he observed that the nine-term sequence can be obtained as an Ext-sequence.

The nine-term sequence is the $\mathrm{Ext}(M,-)$ sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Ext}^0_K(M,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^0_K(M,J) & \longrightarrow & \mathrm{Ext}^0_K(M,C) & \\ & & & & & \downarrow & \\ & \mathrm{Ext}^1_K(M,C) & \longleftarrow & \mathrm{Ext}^1_K(M,J) & \longleftarrow & \mathrm{Ext}^1_K(M,\mathbb{G}_m) & \\ & \downarrow & & & & & \\ & \mathrm{Ext}^2_K(M,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^2_K(M,J) & \longrightarrow & \mathrm{Ext}^2_K(M,C) & \longrightarrow 0 \end{array}$$

of the exact sequence of $\mathrm{Gal}(\bar{K}/K)$-modules

$$0 \to \mathbb{G}_m \to J \to C \to 0,$$

obtained as the direct limit of the sequences

$$0 \to L^\times \to (\text{idèles of } L) \to (\text{idèle classes of } L) \to 0,$$

where L runs over the finite extensions of K in $\bar{K}$. To make this true, we have to switch M and M^D, and modify the groups at the infinite primes.

In the spectral sequence below, the Exts on the right are in the category of Galois modules and those on the left in the category of abelian groups. From the divisibility of $\bar{K}^\times$, we see that the sequence collapses and identifies $\mathrm{Ext}^r_K(M,\mathbb{G}_m)$ with $H^r(K,M^D)$. This completes the easy third of Tate's proof.

There is a spectral sequence

$$H^r(K, \mathrm{Ext}^s_{\mathrm{AbGps}}(M, \bar{K}^\times)) \Rightarrow \mathrm{Ext}^{r+s}_K(M, \bar{K}^\times).$$

As $\bar{K}^\times$ is divisible by the primes $\neq p$, $\mathrm{Ext}^s_{\mathrm{AbGps}}(M, \bar{K}^\times) = 0$ for $s > 0$, and so

$$\mathrm{Ext}^r_K(M, \mathbb{G}_m) \simeq H^r(K, M^D).$$

Tate gave a detailed account of this proof in a letter to Springer (13.01.66), which was intended to be published in the Cassels–Fröhlich volume, but somehow did not make it. However, the letter was widely distributed and eventually published in Tate's collected works.

Global duality (variant)

We state a variant of Tate's global duality theorem in which the products over all primes of K are replaced by a direct sum over a finite set S of primes. The previous version can be obtained from this version by passing to a direct limit over the sets S.

- K =global field;
- S finite set of primes (including archimedean primes);
- K_S =maximal extension of K ramified only in S;
- M a finite G_S-module, $G_S = \mathrm{Gal}(K_S/K)$,
- $[M]$ not divisible by residue characteristic at any $v \notin S$;
- $H^r(K_S, M) = H^r(\mathrm{Gal}(K_S/K, M)$.

$$0 \longrightarrow H^0(K_S, M) \xrightarrow{\beta^0} \bigoplus_{v\in S} H^0(K_v, M) \xrightarrow{\gamma^0} H^2(K_S, M^D)^* \longrightarrow$$

$$\longrightarrow H^1(K_S, M) \xrightarrow{\beta^1} \bigoplus_{v\in S} H^1(K_v, M) \xrightarrow{\gamma^1} H^1(K_v, M^D)^* \longrightarrow$$

$$\longrightarrow H^2(K_S, M) \xrightarrow{\beta^2} \bigoplus_{v\in S} H^2(K_v, M) \xrightarrow{\gamma^2} H^0(K_S, M^D)^* \longrightarrow 0.$$

We now sketch a geometric derivation of the nine-term sequence in the function field case.

Étale duality over a curve

Consider a smooth complete curve X over a field k.

(a) When $k = \mathbb{C}$, $X(\mathbb{C})$ is a 2-dimensional manifold, so there is a 2-dimensional Poincaré duality theorem. When k is an arbitrary algebraically closed field, we still have a 2-dimensional duality theorem, but now in étale cohomology, provided we stick to finite sheaves prime to the characteristic of k.

(b) When k is a finite field, there is an obvious 1-dimensional duality theorem for finite Galois modules.

(c) When X is a smooth complete curve over finite field, the two dualities add to give a 3-dimensional duality theorem.

Let X be a complete smooth curve over a field k, and F a constructible locally free sheaf of $\mathbb{Z}/m\mathbb{Z}$-modules, m not divisible by p if $\mathrm{char}(k) = p \neq 0$. Let $F^\vee = \mathcal{H}om(F, \mathbb{G}_m)$.

(a) k algebraically closed. The pairing $F^\vee \times F \to \mathbb{G}_m$ defines a duality of finite groups

$$H^{2-r}(X, F^\vee) \times H^r(X, F) \to H^2(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

(b) k a finite field, M a $\mathrm{Gal}(\bar{k}/k)$-module, $pM = M$. Let $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. The pairing $M^\vee \times M \to \bar{k}^\times$ defines a duality of finite groups

$$H^{1-r}(k, M^\vee) \times H^r(k, M) \to H^1(k, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z}.$$

(c) X, k, F as in (a), but with k finite. The pairing $F^\vee \times F \to \mathbb{G}_m$ defines a duality of finite groups

$$H^{3-r}(X, F^\vee) \times H^r(X, F) \to H^3(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.$$

Etale duality ↔ Tate duality (char $p \neq 0$).

Now consider a smooth open curve U over a finite field. When we write the exact sequence relating the usual cohomology of U to its cohomology with compact support, and replace the latter with the group given by the duality theorem, we obtain Tate's nine-term exact sequence.

This gives a geometric explanation for the sequence, as well as a second proof.

Let X, k, F be as in (c), and let $j : U \hookrightarrow X$ be an open subscheme of X. We get the top row of the following diagram with $H^r_c(U, F) = H^r(X, j_!F)$, $S = X \smallsetminus U$, and $K_{(v)}$ =field of fractions of the henselization of $\mathcal{O}_{X,v}$. This essentially becomes Tate's 9-term sequence when we replace $H^r_c(U, F)$ with $H^{3-r}(U, F^\vee)^*$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^r_c(U,F) & \longrightarrow & H^r(U,F) & \longrightarrow & \bigoplus_{v\in S} H^r(K_{(v)},F) & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \\
 & & H^{3-r}(U,F^\vee)^* & & & & & & \\
 & & \| & & & & & & \\
\cdots & \longrightarrow & H^{3-r}(K_S,M^D)^* & \longrightarrow & H^r(K_S,M) & \longrightarrow & \bigoplus_{v\in S} H^r(K_v,M) & \longrightarrow & \cdots
\end{array}$$

$M \leftrightarrow F$ on U, $\quad M = F(U)$, $\quad G_S = \pi_1^{\text{ét}}(U)$.

Artin-Verdier duality (Woods Hole 1964)

Below, is the theorem as Artin and Verdier originally stated it. This gives a geometric explanation for Tate's nine-term sequence in the number field case, as well as a second proof (but one much more difficult than Tate's proof).

- K a number field, $X = \operatorname{Spec}(\mathcal{O}_K)$;
- $j : U \hookrightarrow X$ a nonempty open subset of X;
- for a sheaf F on U, let $H^r_c(U, F) = H^r(X, j_! F)$.

THEOREM 2.1 (ARTIN–VERDIER 1964). *For any constructible sheaf F on U, the Yoneda pairing*

$$\operatorname{Ext}^r_U(F, \mathbb{G}_m) \times H^{3-r}_c(U, F) \to H^3_c(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$$

is a nondegenerate pairing of finite groups, except possibly on the 2-torsion when K has a real prime.

- Note that there is no condition on primes.
- Can modify H^i_c so that the theorem also holds for 2.
- Have $H^r(U, \mathcal{E}xt^s(F, \mathbb{G}_m)) \Rightarrow \operatorname{Ext}^{r+s}(F, \mathbb{G}_m)$,...
- Can deduce Tate's global duality as before.

3 Global duality for abelian varieties

The group $A(K)$ of rational points on an abelian variety A over a global field K is finitely generated. It is easy to find the torsion subgroup of $A(K)$ (at least for elliptic curves) so the problem of computing $A(K)$ comes down to finding a set of generators for $A(K)$ modulo torsion. By computing, one obtains a lower bound; the general theory using Weil-Châtelet groups) gives an upper bound. Roughly speaking, the difference between the bounds is measured by the Tate–Shafarevich group. On the basis of calculations, Selmer found that the order of this group always seemed to be a square, which would be explained by its carrying an nondegenerate alternating form. Cassels proved that this was so in various cases, and finally for a general elliptic curve (assuming the group is finite). Tate proved a similar result for abelian varieties.

Let A be an abelian variety over a global field K and A' the dual abelian variety. The Tate–Shafarevich group of A is defined to be

$$Ш(K, A) = \operatorname{Ker}(H^1(K, A) \to \bigoplus\nolimits_{\text{all } v} H^1(K_v, A)).$$

Tate defined a bi-additive pairing

$$Ш(K, A)(l) \times Ш(K, A')(l) \to \mathbb{Q}/\mathbb{Z}$$

and showed that it is nondegenerate if $Ш(K, A)(l)$ is finite and $l \neq \text{char(K)}$.

- Proved for elliptic curves by Cassels 1959, 1962, 1964. Origin of duality on Ш is a conjecture of Selmer (1951, 1954) based on calculations for elliptic curves with $j = 0$.
- Proved for abelian varieties by Tate (statement, ICM 1962; proof outlined in letter to Serre, 28 July 1962).

Better, ignoring p components...and assuming Ш finite, we get dual exact sequences,

$$0 \to Ш(K,A) \to H^1(K,A) \to \bigoplus_{\text{all } v} H^1(K_v,A) \longrightarrow Б \longrightarrow 0$$

$$0 \leftarrow Ш(K,A') \leftarrow H^1(K,A)^* \longleftarrow \prod_{\text{all } v} A'(K_v) \longleftarrow A'(K)^\wedge \leftarrow 0,$$

where $A'(K)^\wedge$ is the profinite completion of $A'(K)$ and $A'(K_v)$ is to be replace by $\pi_0(A'(K_v))$ if v is archimedean.

4 Application: Isogeny invariance of BSD

> "It seemed to me that the natural setting for [the BSD conjectures] is abelian varieties of any dimension, defined over any global field." Tate CW, p.237.

CONJECTURE 4.1 (BSD, TATE SB 1966). *Let A be an abelian variety over a global field K. Then*

$$\lim_{s\to 1}\frac{L^*(A,s)}{(s-1)^{\mathrm{rk}(A(K))}} = \frac{[Ш(A)]\cdot D}{[A(K)_{\mathrm{tors}}][A'(K)_{\mathrm{tors}}]},$$

where A' is the dual abelian variety and D is the discriminant of the Néron-Tate height pair $A(K)\times A'(K)\to\mathbb{R}$.

One important application of the duality theorems is the isogeny invariance of the BSD conjecture.

THEOREM 4.2 (TATE, CASSELS FOR ELLIPTIC CURVES). *Let A and B be abelian varieties over a global field. If A and B are isogenous by an isogeny of degree prime to the characteristic, then BSD is true for both if it true for one.*

Proof uses

- Tate's global duality theorem for

$$M \overset{\text{def}}{=} \mathrm{Ker}(A(K_S)\to B(K_S)).$$

- Cassels-Tate duality (for A and B)

$$Ш(K,A)\times Ш(K,A')\to \mathbb{Q}/\mathbb{Z}$$

- Euler-Poincaré formula (Tate),

$$\frac{[H^0(K_S,M)][H^2(K_S,M)]}{[H^1(K_S,M)]} = \prod_{v \text{ arch}} \frac{[H^0(K_v,M)]}{|[M]|_v}.$$

More precisely, using the first two assertions, one finds that BSD for A and B are equivalent if and only if the third assertion is true, so Tate proved it (not without difficulty).

In the summer of 1967, I asked Tate how to prove the theorem, and my recollection is that he was able to write the proof down without looking anything up.

Tate (SB 1966; CW p. 227): The proof of compatibility with p-isogenies looks like an interesting problem.

Indeed.

5 Interlude: étale vs flat cohomology

Tate always worked with Galois cohomology. This entailed some restrictions, which we now remove.

- Étale topology: coverings are surjective families of flat unramified morphisms (plus a finiteness condition).
- Flat topology: coverings are surjective families of flat morphisms (plus a finiteness condition)
- For a field, étale cohomology = Galois cohomology.
- Flat cohomology of smooth group schemes = étale cohomology.
- Over a field of characteristic zero, all group schemes are smooth.
- Over a field of characteristic p, a finite group scheme is smooth if p does not divide its order.
- Over a field, smooth finite group scheme = étale group scheme "=" finite Galois module.

For a commutative group scheme N over a field K and a finite extension L of K, there is a natural definition of Čech cohomology groups $H^r(L/K,N)$. The Galois cohomology groups are obtained by passing to the direct limit over the *separable* L contained in a fixed algebraic closure of K, and the flat cohomology groups by passing to the limit over *all* L.

EXAMPLE 5.1. Let G be a commutative group scheme over a field K, and L a finite extension of K. From the system

$$L \underset{a\mapsto a\otimes 1}{\overset{a\mapsto 1\otimes a}{\rightrightarrows}} L\otimes_K L \;\;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\;\; L\otimes_K L\otimes_K L \;\;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\;\; \cdots$$

we get a complex,

$$G(L) \longrightarrow G(L\otimes_K L) \longrightarrow G(L\otimes_K L\otimes_K L) \longrightarrow \cdots$$

whose rth cohomology group we denote $H^r(L/K,G)$. Then

$$H^r_{\text{ét}}(K,G) = \varinjlim_{L\subset\bar{K},\, L \text{ separable over } K} H^r(L/K,G)$$

$$H^r_{\text{fl}}(K,G) = \varinjlim_{L\subset\bar{K}} H^r(L/K,G).$$

- If L/K is Galois, then $L\otimes_K L$ is a product of copies of L indexed by the Galois group.
- If L/K is inseparable, $L\otimes_K L$ may have nilpotents.

Abelian varieties in characteristic p

In the 1940s, André Weil developed a robust theory of algebraic varieties, including abelian varieties, over arbitrary fields. This theory had difficulty handling p-phenomena in characteristic p.[5] For example, in the algebraic geometry of that period, there are many maps that should be isomorphisms, but are only proved to be purely inseparable.

Cartier (1960) and Nishi (1959) independently extended Weil's theory of abelian varieties to cover p-phenomena in characteristic p. Let $\alpha: A \to B$ be an isogeny of abelian varieties over a field K and $\alpha': B' \to A'$ the dual isogeny. In the exact sequences

$$0 \to N \longrightarrow A \xrightarrow{\alpha} B \to 0$$
$$0 \to N^D \longrightarrow B' \xrightarrow{\alpha'} A' \to 0,$$

the finite group scheme N^D is the Cartier dual of N,

$$N^D \stackrel{\text{def}}{=} \mathcal{H}om(N, \mathbb{G}_m).$$

Moreover, the canonical map $A \to A''$ from A into its double dual is an isomorphism, and the second sequence can be obtained as the $\mathcal{E}xt(-, \mathbb{G}_m)$ sequence of the first.

6 Local flat duality

A student of Tate, Steve Shatz, took up the problem of extending Tate's local duality to local fields of characteristic $p \neq 0$. He succeeded in proving a flat duality theorem for finite group schemes in 1962, but the corresponding theorem for abelian varieties was not proved until almost 10 years later.

THEOREM 6.1 (SHATZ THESIS, 1962). *Let K be local field of characteristic p (finite residue field). Let N be a finite commutative group scheme over K, with Cartier dual N^D. For all r, the cup-product pairing*

$$H^r(K, N) \times H^{2-r}(K, N^D) \to H^2(K, \mathbb{G}_m) \simeq \mathbb{G}_m$$

is a perfect duality of locally compact groups.

THEOREM 6.2 (M 1970/72). *Let A be an abelian variety over a local field K and A' the dual abelian variety. For all r, Tate's pairing*

$$H^r(K, A) \times H^{1-r}(K, A') \to H^2(K, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$$

is a perfect duality (of locally compact groups).

The proof is based on Shatz's theorem. It passes to the case that A and A' have semistable reduction, and uses the structure of the Néron minimal models.

Note that the statement of Theorem 6.2 is exactly the same as that of Tate's theorem — in particular, the groups are Galois cohomology groups — except that it also holds for the p parts of the groups in characteristic p.

[5]Essentially because it didn't allow nilpotents.

7 Global flat duality

The following theorem, which has been widely used, was first stated by Mazur (1972, 7.2).

THEOREM 7.1. *Let U be a nonempty open subset of the spectrum of the ring of integers in a number field, and let N be a finite flat commutative group scheme N over U with Cartier dual N^D. For all r, the canonical pairing*

$$H_c^{3-r}(U,N)\times H^r(U,N^D)\to H_c^3(U,\mathbb{G}_m)\simeq\mathbb{Q}/\mathbb{Z}$$

is a perfect duality of finite groups (appropriate definition of H_c).

There are various accounts of the theorem in the literature, most recently in Demarche and Harari 2019, which also includes the case of an open curve over a finite field.

There are also global flat duality theorems for abelian varieties, which I shall skip.

8 Interlude: Harvard in the 1960s

I was a student at Harvard 1964–1967.

- Brauer and Zariski were still on the faculty, but so also were Hironaka, Mazur, Mumford, Tate. . . .
- Zariski students: Artin 1960, Hironaka 1960, Mumford 1961, . . .
- Paris – Cambridge axis.
- Visitors Grothendieck (1959-60, 1961-62, . . .), Serre (Fall 1964,. . .).
- Seminars 1962: Hironaka resolution; Artin étale cohomology; Tate; Grothendieck; Zariski; Kodaira; Thompson classification of minimal finite simple groups.
- Woods Hole conference 1964 (Tate conjecture, Artin-Verdier duality, Serre-Tate, . . .)
- Mumford course 1965/66.

Tate spent the academic year 1965/66 in Paris, during which he wrote his article with Shafarevich, proved an important case of the Tate conjecture, and, as I shall describe shortly, gave a Bourbaki seminar.

When he returned in the summer of 1966, I told him that I had been studying flat cohomology and he suggested that I should prove that the Tate-Shafarevich group[6] is finite.

9 Tate conjecture for a surface over a finite field

The following should be considered the first case of the Tate conjecture.

[6]Tate always called it the Shafarevich group, while I stubbonly stuck to "Tate-Shafarevich group", until one day we both switched to "Shah". Peace reigned.

Let X be a smooth complete surface over $\mathbb{F}_q$.

$$\zeta(X,s)=\frac{P_1(X,q^{-s})P_3(X,q^{-s})}{(1-q^{-s})P_2(X,q^{-s})(1-q^{2-s})},\quad P_i(X,T)\in\mathbb{Z}[T].$$

CONJECTURE 9.1 (TATE). *The order of the pole of $\zeta(X,s)$ at $s=1$ is the rank of the Néron–Severi group.*

Note that the order of the pole of $\zeta(X,s)$ at $s=1$ is equal to the order of zero of $P_2(X,q^{-s})$ at $s=1$.

According to Tate, just as the original BSD conjecture has a refined version, every Tate conjecture should have a refined version.

> The [Tate conjecture] must be formulated for schemes of finite type over $\mathbb{Z}$, for L-series as well as zeta's, and *most important* [the Tate conjecture] should get a refinement relating the highest coefficient of the principal part of ζ at the pole to a discriminant attached to the group of Néron-Severi type whose rank is the order of the pole and to the order of a Shafarevich or Brauer-type group, just as Birch and Swinnerton-Dyer are attempting to do in their special case.
> Tate, letter to Serre 11.06.63.

So what is the refined Tate conjecture for surfaces over finite fields? Following BSD, one could do some calculations and try to interpret the result. Instead Tate (in collaboration with Mike Artin) piggy-backed off the BSD conjecture.

10 Tate SB1966 (Séminaire Bourbaki Feb 1966)

Given a smooth complete surface X over a finite field k, Tate's idea was to map X to a curve C in such a way that the generic fibre $X_\eta\to\eta$ is smooth. Hence, X_η is a smooth curve over the global function field $K\overset{\text{def}}{=}k(C)$, and Tate's idea was to investigate what the BSD conjecture for the Jacobian of X_η said about X.

$$\begin{array}{ccccc} X & \longleftarrow & X_\eta & & A=\mathrm{Jac}(X_\eta) \\ \downarrow f & & \downarrow \text{generic fibre} & & \big| \\ C & \longleftarrow & \eta & & K=k(C) \end{array}$$

Base field $k=\mathbb{F}_q$ (finite)
X smooth projective surface
C smooth projective curve
f has smooth generic fibre X_η/K.

The result is summarized in the next slide.

CONJECTURE 10.1 (BSD).

$$\lim_{s\to 1}\frac{L^*(A,s)}{(s-1)^{\mathrm{rk}(A(K))}}=\frac{[Ш(A)]\cdot D}{[A(K)_{\mathrm{tors}}][A'(K)_{\mathrm{tors}}]},$$

CONJECTURE 10.2 (ARTIN-TATE).

$$\lim_{s\to 1}\frac{P_2(X,q^{-s})}{(1-q^{1-s})^{\mathrm{rk}(\mathrm{NS}(X)))}}=\frac{[\mathrm{Br}(X)]\cdot D}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tors}}]^2},$$

$\mathrm{Br}(X)$ *is the Brauer group of* X*,* D *is the discriminant of the intersection pairing on the divisors of* X*, and* $\alpha(X)\overset{\mathrm{def}}{=}\chi(X,\mathcal{O}_X)-1+\dim\mathrm{Pic}^0(X)$.

The terms of the two conjectures roughly correspond. For example, Artin showed that, for $\ell\neq p$, the ℓ-primary components of $Ш(A)$ and $\mathrm{Br}(X)$ differ by finite groups.

CONJECTURE 10.3 (d). *In the situation of the diagram, the two conjectures are equivalent.*

In the seminar, Tate stated four conjectures: (A) is the first form of BSD for abelian varieties over global fields, (B) is the full form of BSD (as above), (C) is what we now call the Artin–Tate conjecture, and (d) is the conjecture that, in the context of the above diagram, Conjectures (B) and (C) are equivalent. The last conjecture gets only a small "d" because, rather than being a deep conjecture, it is a conjectural relation between deep conjectures.

Theorems (Artin–Tate)

Tate's Bourbaki talk was not all conjectures. He also proved the theorems in the next slide (which he describes as joint with with Artin).

Let X be a smooth complete surface over $\mathbb{F}_q$, $q=p^a$. Let $\ell\neq p$.

THEOREM 10.4 (5.1). *There is a canonical skew-symmetric form*

$$\mathrm{Br}(X)(\ell)\times\mathrm{Br}(X)(\ell)\to\mathbb{Q}/\mathbb{Z}$$

that is nondegenerate if $\mathrm{Br}(X)(\ell)$ *is finite.*

THEOREM 10.5 (5.2). $\mathrm{Br}(X)(\ell)$ *is finite if and only if the Tate conjecture holds for* X*, in which case it has the order predicted by the Artin-Tate conjecture.*

Tate concluded his Bourbaki talk with the statement.

> The problem of proving analogs of theorems 5.1 and 5.2 for $\ell=p$ should furnish a good test for any p-adic cohomology theory, and might well serve as a guide for sorting out and unifying the various constructions that have been suggested: Serre's Witt vectors, Dwork's Banach spaces, the raisings via special affines of Washnitzer-Monsky, and Grothendieck's flat cohomology for μ_{p^n}.

Indeed, by the time we were able to prove the p-analogs of 5.1 and 5.2, we did know the "correct" p-adic cohomology theories. In the rest of the talk, I'll explain this and also how Conjecture d was proved.[7]

[7]While Tate was confident of these conjectures, not everyone was. Indeed, it was a leap to take a statement based on calculations concerning elliptic curves over $\mathbb{Q}$ and extend it all abelian varieties over global fields, including their p-phenomena in characteristic p. While I was still working on my thesis,

p analogue (product of two curves)

When Tate arrived back at Harvard, not long after giving his Bourbaki talk, and I told him that I had been studying flat cohomology, my thesis topic became clear: understand the p-part of the Artin-Tate conjecture, and, a related question, the p-part of the BSD conjecture over a global function field.

For a while I made no progress, but, at some point, Tate suggested that I look at an example where the conjecture predicted that the Brauer group is trivial, because it may be easier to prove that a group is trivial than to prove that it is finite. Indeed, in special cases, the Artin–Tate conjecture takes on a simple and explicit form.

For nonisogenous elliptic curves, E_1 and E_2 over $\mathbb{F}_q$, the Artin-Tate conjecture says that

$$[\mathrm{Br}(E_1 \times E_2)] = (N_1 - N_2)^2, \quad N_i = [E_i(\mathbb{F}_q)].$$

Note that this predicts that the order of the Brauer group is a square, as expected. Also that, while the Brauer group may be trivial, its order can't be zero, and so the equation predicts that two elliptic curves over a finite field with the same number of rational points must be isogenous. This can be considered the zeroth case of the Tate conjecture (essentially proved by Deuring in the 1930s).

For the case of the product of two elliptic curves, I eventually concluded that the key was a certain flat cohomology group.

Key to understanding the p-analog in the case $X = E_1 \times E_2$,

$$H^1_{\mathrm{fl}}(E_1, E_{2,p}), \quad E_{2,p} = \mathrm{Ker}(E_2 \xrightarrow{p} E_2).$$

When I explained this to Tate, I had no idea that anyone knew anything about the finite group scheme $E_p = \mathrm{Ker}(E \xrightarrow{p} E)$, but, in fact, Tate did. When he explained its structure to me I was able, on the spot, to obtain to prove the finiteness of $\mathrm{Br}(X)(p)$ in some cases.

Eventually, in my thesis (1967), I proved the p-analogs of the theorems 5.1 and 5.2 for the product of two curves. Since Tate had proved the Tate conjecture in that case, this gave the following theorem.

THEOREM 10.6 (TATE, ARTIN–TATE, MILNE). *The Artin–Tate conjecture holds for the product of two curves.*

A little later, I proved that the full BSD conjecture holds for constant abelian varieties over global fields — in particular, that their Tate-Shafarevich groups are finite. (An abelian variety is constant if it is defined by equations with coefficients in the field of constants). [8]

This is a interesting theorem, but not yet what I promised.

Key step in proof of p analogue: duality!

Although it seems trivial now, what gave me the most trouble in my thesis was proving a duality theorem for finite flat group schemes over a curve.

Tate got a letter from André Weil claiming an example of an elliptic curve over a global function field with infinite Tate-Shafarevich group, but by then I had already proved that the group was finite in the case considered by Weil.

[8]Weil's example was an elliptic curve with constant j-invariant. Thus, the curve need not be constant, but becomes constant after a finite extension. If a Tate–Shafarevich group becomes finite after a finite extension of the base field, then it was already finite.

THEOREM 10.7 (ARTIN-M 1976). *Let X be a smooth complete curve over a finite field k. Let N be a finite flat commutative group scheme over X, and let N^D be its Cartier dual. For all r, the cup-product pairing*

$$H^r(X,N) \times H^{3-r}(X,N^D) \to H^3(X,\mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$$

is a perfect duality.

For my thesis, I only needed the duality for the pairs (α_p,α_p) and $(\mathbb{Z}/p\mathbb{Z},\mu_p)$. Note that the pairing

$$(m,\zeta) \mapsto \zeta^m : \mathbb{Z}/p\mathbb{Z} \times \mu_p \to \mathbb{G}_m.$$

realizes each of $\mathbb{Z}/p\mathbb{Z}$ and μ_p as the Cartier dual of the other. Following is a sketch of the proof in this case.

Artin-Schreier sequence (exact on X_{et})

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{x\mapsto x^p - x} \mathcal{O}_X \longrightarrow 0.$$

Kummer sequence (exact on X_{fl}),

$$1 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{x\mapsto x^p} \mathbb{G}_m \longrightarrow 1.$$

Apply $f : X_{\text{fl}} \to X_{\text{et}}$ (identity map); deduce

$$R^i f_* \mu_p = \begin{cases} \mathcal{O}_X^\times/\mathcal{O}_X^{\times p} \overset{\text{def}}{=} \nu(1) & \text{if } i=1 \\ 0 & \text{otherwise,} \end{cases}$$

so

$$H^i_{\text{fl}}(X,\mu_p) = H^{i-1}_{\text{et}}(X,\nu(1)).$$

Now use

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{h\mapsto h^p} & \mathcal{O}_X^\times & \xrightarrow{h\mapsto dh/h} & \Omega^1_X & \xrightarrow{C-1} & \Omega^1_X & \longrightarrow & 0 \\ & & & & & \searrow \quad \nearrow & & & & & \\ & & & & & \nu(1) & & & & & \end{array}$$

Now have

1-dimensional duality	$\mathcal{O}_X$, Ω^1_X	Zariski topology
2-dimensional duality	$\mathbb{Z}/p\mathbb{Z}$, $\nu(1)$	étale topology
3-dimensiional duality	$\mathbb{Z}/p\mathbb{Z}$, μ_p	flat topology

The C in the above diagram is the Cartier operator.

For a smooth variety X over a perfect field k, Cartier (1957) showed that there is a (unique) family of maps

$$C : \Omega^r_{X/k,\text{closed}} \to \Omega^r_{X/k}$$

such that

- $C(\omega + \omega') = C(\omega) + C(\omega')$, $C(h^p\omega) = hC(\omega)$,
- $C(\omega \wedge \omega') = C(\omega) \wedge C(\omega')$,
- $C(\omega) = 0 \iff \omega$ is exact,
- $C(dh/h) = dh/h$.

For curves, the Cartier operator was defined in Tate 1952. [9]

11 Flat duality (Artin's conjecture 1974)

In an important article, Artin (1974) used flat cohomology to study supersingular $K3$ surfaces. This led him to conjecture a duality theorem in the flat cohomology of surfaces over fields of characteristic $p \neq 0$.

Let $\pi : X \to \operatorname{Spec} k$ be a smooth complete surface over a perfect field k of characteristic $p \neq 0$.

Rough form of conjecture: when k is algebraically closed, there is a 4-dimensional duality for the finite part of $H^i_{\text{fl}}(X, \mu_p)$ and a 5-dimensional duality for the vector space part.

Clearly this needs to be restated in terms of derived categories. Artin proved that the functor $R^r\pi_*\mu_{p^n}$ (flat cohomology) is represented by a group scheme of finite type over k. His conjecture concerned only these group schemes modulo infinitesimal group schemes

Precise form of conjecture: There is a canonical isomorphism

$$R\pi_*\mu_{p^n} \to R\operatorname{Hom}(R\pi_*\mu_{p^n}, \mathbb{Q}/\mathbb{Z})[-4]$$

in the derived category of the category of commutative group schemes over k modulo infinitesimal group schemes.

Proof of Artin's conjecture ($n = 1$)

In the proof of the flat duality theorem for curves, we saw that we should identify the flat cohomology of μ_p with the étale cohomology of the sheaf $\nu(1)$ shifted by 1. This idea works more generally.

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[9] In this early paper, Tate studied how the genus of a curve changes under extension of the base field. The reader may object that the genus doesn't change under base field extension. This is true in 2025, but in 1952 things were different. Consider a normal complete curve X over a field k. The curve X' obtained by extending the base field to k' does have the same genus as X, but it may no longer be normal, for example, its structure sheaf may acquire nilpotents. In 1952, by the extended curve one meant the associated normal curve, whose genus may indeed drop (but only by a multiple of $(p-1)/2$, as proved by Tate).

Let $\pi\colon X \to \operatorname{Spec} k$ be a smooth complete variety of dimension d over a perfect field k of characteristic p. Define a sheaf on $X_{\text{ét}}$ by

$$\nu(r) = \operatorname{Ker}(C - 1\colon \Omega^r_{X,\text{closed}} \to \Omega^r_X).$$

THEOREM 11.1 (M 1976). *There is a canonical isomorphism*

$$R\pi_*\nu(r) \to R\operatorname{Hom}(R\pi_*\nu(d-r), \mathbb{Z}/p\mathbb{Z})[-d]$$

in the derived category of the category of commutative group schemes killed by p modulo infinitesimal group schemes.

When $d = 2$, $r = 1$, this becomes Artin's conjecture for μ_p: we have

$$(X_{\text{fl}} \xrightarrow{f} X_{\text{et}} \xrightarrow{\pi^{\text{et}}} \operatorname{Spec} k) = \pi^{\text{fl}}$$

and

$$Rf_*\mu_p = \nu(1)[-1],$$

so

$$R\pi^{\text{fl}}_*\mu_p = R\pi^{\text{et}}_* Rf_*\mu_p = R\pi^{\text{et}}_*\nu(1)[-1].$$

Proof of Artin's conjecture (all n)

The defect of the above theorem is that it applies only to μ_p, not μ_{p^n}, because its proof depends on the sheaves of differentials, which are killed by p in characteristic p.

In 1974, I shared an office at the University of Michigan with Spencer Bloch. When I explained my problem to him he said that he had defined objects that were just like the sheaves of differentials, except killed by p^n, not p. Indeed, he had. This was the famous de Rham-Witt complex.

Bloch 1974/1976 constructed a projective system of complexes

$$W_n\mathcal{O}_X \xrightarrow{d} W_n\Omega^1_X \xrightarrow{d} W_n\Omega^2_X \to \cdots$$

of $W_n(\mathcal{O}_X)$-modules (de Rham-Witt complex).

Bloch defined the de Rham-Witt complex in order to relate K-theory to crystalline cohomology, but once he had introduced it, its importance was apparent, and it was soon developed further by others.[10] Not only does it give a new construction of crystalline cohomology, but it adds structure to it. For example, as mentioned earlier, Serre had studied the cohomology of the sheaf of Witt vectors on a variety, and had correctly concluded that it gives only part of the "good" cohomology. With the de Rham–Witt complex, it became possible to say exactly which part.

[10]Initially Illusie and Raynaud; more recently by Bhatt and Lurie.

Using the de Rham-Witt complex, it is possible to define sheaves $\nu_n(r)$ (killed only by p^n) and prove that there is a canonical isomorphism

$$R\pi_*\nu_n(r) \to R\,\mathrm{Hom}(R\pi_*\nu_n(d-r), \mathbb{Q}/\mathbb{Z})[-d].$$

When $d = 2$, this becomes the statement of Artin's conjecture for μ_{p^n}.

12 Conclusion

The analogs for $\ell = p$ of Theorems 5.1 and 5.2 of Tate SB1966

Using the sheaves $\nu_n(r)$, it became possible to complete the proof of the analogs for $\ell = p$ of the theorems in Tate's 1966 Bourbaki talk.

Let X be a smooth complete surface over a finite field k of characteristic p.

THEOREM 12.1 (M 1975). *Tate's skew-symmetric pairing*

$$\mathrm{Br}(X)(p) \times \mathrm{Br}(X)(p) \to \mathbb{Q}/\mathbb{Z}$$

is nondegenerate if $\mathrm{Br}(X)(p)$ *is finite.*

THEOREM 12.2 (M 1975, COMPLETING ARTIN–TATE 5.2). *The following are equivalent.*

(a) *The Tate conjecture holds for X.*
(b) *For some prime l ($l = p$ is allowed),* $\mathrm{Br}(X)(l)$ *is finite.*
(c) *The Artin-Tate conjecture holds for X (including the p part).*

Proof of Conjecture (d)

THEOREM 12.3 (KATO–TRIHAN 2003). *Let A be an abelian variety over a global function field K. The following are equivalent.*

(a) *The order of the zero of $L(s, A)$ at $s = 1$ is the rank of $A(K)$.*
(b) *For some prime l,* $\text{Ш}(A/K)(l)$ *is finite.*
(c) *The full BSD conjecture for A/K.*

PROOF. The proof uses global flat duality over a curve. □

We can now prove Tate's Conjecture d for the pair

$$\begin{array}{ccccc} X & \longleftarrow & X_\eta & \qquad & A = \mathrm{Jac}(X_\eta) \\ \downarrow f & & \downarrow \text{generic fibre} & & | \\ C & \longleftarrow & \eta & & K = k(C) \end{array}$$

Recall that Conjecture (d) says that, in the situation of the diagram,

statement (c) of 12.2 for X $\iff$ statement (c) of 12.3 for A.

Because of the equivalences in the theorem, it suffices to prove that

statement (b) of 12.2 for X $\iff$ statement (b) of 12.3 for A.

As noted earlier, Artin had proved this for $l \neq p$.

The "good" p-adic cohomology theories in characteristic p

Let X be a smooth complete variety over a field k of characteristic $p \neq 0$,

$$\begin{cases} \text{Weil cohomology:} & H^r_{\text{crys}}(X/W) \simeq H^r(X, W\Omega^\bullet_X) \\ \text{“}H^i_{\text{fl}}(X, \mu_{p^n}^{\otimes r})\text{”}: & H^{i-r}_{\text{ét}}(X, \nu_n(r)). \end{cases}$$

Note that the quotation marks can be removed with $r \leq 1$. The second definition may seem too ad hoc to be convincing, but there is a second description of it.

When we apply $R\Gamma$ to the de Rham–Witt complex of a variety, we get a complex of $W(k)$-modules, from which we can deduce the crystalline cohomology. The de Rham–Witt complex has extra structure, namely, an action of the Raynaud ring. When we remember this action, the same construction gives $\varprojlim_n H^{i-r}_{\text{ét}}(X, \nu_n(r))$ instead $H^r_{\text{crys}}(X/W)$.

When we regard $R\Gamma(W\Omega^\bullet_X)$ as an object in the triangulated category with t-structure $\mathsf{D}^+(W)$,

$$H^i_{\text{crys}}(X/W) \simeq \operatorname{Hom}_{\mathsf{D}^+(W)}(\mathbb{1}, R\Gamma(W\Omega^\bullet_X)[i]).$$

When we regard $R\Gamma(W\Omega^\bullet_X)$ as an object in the triangulated category with t-structure $\mathsf{D}^b_c(R)$ (R the Raynaud ring),

$$\text{“}\varprojlim_n H^i_{\text{fl}}(X, \mu_{p^n}^{\otimes r})\text{”} \simeq \operatorname{Hom}_{\mathsf{D}^b_c(R)}(\mathbb{1}, R\Gamma(W\Omega^\bullet_X)(r)[i]).$$

(M. and Ramachandran 2005).

The de Rham–Witt complex has operators F, V, d satisfying certain conditions. To say that an object has these operators satisfying the conditions is exactly to say that it has an action of the Raynaud ring.

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